# 13

# Martingales and Numeraires

## 13.1 The Martingale Property

The most important and profound concept that the reader may have gained from the material presented in this book so far is that of risk neutrality, which can be summarized as follows:

> Today's price of a (tradable) financial instrument is equal to the discounted expectation of its future price if this expectation is calculated with respect to the risk-neutral probability measure.

At this point, we recommend that the reader reviews Sects. 7.1.3, 7.2.1 and 9.2.3. We will now elaborate on the concept of risk neutrality.

The *risk-neutral probability* is an example of a *martingale measure*. Martingale measures are a specific class of probability measures satisfying the property—as we are about to see—described in Eq. 13.1. For an intuitive explanation of the term *measure*: probability distributions can be interpreted as *measures* since the expectation of a function $f(X)$ of a random variable $X$, having a distribution with density function $p$, can be interpreted as an integral with respect to a certain *integral measure*:

$$\mathrm{E}[f(X)] = \int f(x) \underbrace{p(x)dx}_{\text{Integral Measure}}$$



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By replacing the *Riemann integral measure* $dx$ with the *probability measure* $p(x)dx$, we obtain the expectation of the random variable $f(X)$ by integrating the function $f$ with respect to this probability measure.

Martingale theory is an intensively researched field of mathematics, and recognizing that the price process of a derivative can be interpreted as a martingale allows the application of martingale theory in the valuation of financial instruments. Because of its generality, the utilization of martingale theory requires very few of the assumptions listed in Chap. 4, namely Assumptions 1, 2, 3, 5. For sake of simplification, we assume that Assumption 4 from Chap. 4 is fulfilled, too. The methods presented here will not be discussed in complete mathematical detail, but will be motivated by our "experience" of the subject gained in previous sections.

The results in Sect. 9.2.3 are based on the somewhat surprising observation that the probability $p'$ of an upward move $u$ in the underlying price in the real world does not appear in Eqs. 9.8 or 9.13. Only the risk-neutral probability plays a role. This observation led to Eq. 9.20 and the interpretation of derivative prices as expectations taken with respect to a certain probability measure.

At this stage, we wish to generalize this concept. Consider again the derivation of Eq. 9.8. The specific type of the financial instrument $V$ does not come into play. The discussion took place in the context of options merely for simplicity's sake, making use of none of the properties particular to options. Hence, Eq. 9.20 holds for all financial instruments whose value is governed by the price of the underlying $S$. Furthermore, at no point in the derivation did the process $S$ have to have a particular form. This means that Eq. 9.20 holds for *arbitrary* instruments on *arbitrary* underlyings, in other words, for all general processes of the form 2.19. Such processes are much more general than the simple random walk given by Eq. 2.23, for example. This is of decisive importance in the analysis of term structure models. In fact, the generality of Eq. 9.20 goes still further. Using $B(T, T) = 1$, Eq. 9.20 can be written as

$$\frac{V(t)}{B(t, T)} = \mathrm{E}\left[\frac{V(T)}{B(T, T)}\right] ,$$

where we recall that the expectation at time $t$ has been taken with respect to the risk-neutral probability $p$.

Defining the *normalized* price as

$$Z(t) := \frac{V(t)}{B(t, T)} ,$$

we arrive at the elegant form

$$Z(t) = \mathrm{E}\left[Z(T)\right] \ . \tag{13.1}$$

In other words:

> The normalized price at time $t$ is given by the expectation (with respect to $p$) of the future normalized price.

This expresses the martingale property in its "purest" form. We can also say:

> The normalized price is a martingale (with respect to $p$).

*Normalizing* the price means nothing other than expressing the price of the instrument in units of zero bonds maturing at $T$ rather than in monetary units such as euros. Thus, the numerical price does not tell us how much the instrument costs in euros but how much the instrument costs in terms of zero bonds.

Having shown that the martingale property holds (within limits[1]) for *arbitrary financial instruments* on *arbitrary underlyings*, we now show that it also holds for *arbitrary normalizing factors*. This indicates the truly general character of the martingale property.

## 13.2 The Numeraire

In financial literature, the normalizing instrument is commonly referred to as *numeraire*. In our discussions here, we will frequently use the more intuitive expression *normalization*. We will now show that not only zero bonds but *arbitrary* (tradable) financial instruments may serve as normalizing factors, i.e., as numeraires. The numeraire used does not even have to refer to the underlying $S$ (the zero bond does not do so either). Let $S$ be the price of an underlying, $V$ the price of an arbitrary financial instrument on this underlying,

[1]The limits are that the underlying has to follow a general Ito process of the form 2.19 and that the financial instrument as well as the normalizing factor have to be traded instruments, see Sect. 14.4.

and $Y$ another arbitrary financial instrument (in our previous discussions, the zero bond was chosen to play the role of $Y$). As in the derivation of Eq. 9.8, let the time evolution of the price of the underlying be described by a *binomial tree* such as given in Eq. 9.2, for example.

We intend to construct a portfolio composed of $\alpha$ underlyings whose prices are given by the process $S$ and $\beta$ of the financial instruments with a price process given by $Y$:

$$\Pi(t) = \alpha_t S(t) + \beta_t Y(t) \ . \tag{13.2}$$

We require this portfolio to have a value equal to that of the derivative $V$ in all states of the world one time step later. Note that both the normalizing factor $Y$ and the underlying $S$ must be *tradable* since otherwise the "$\alpha$ underlyings at price $S$" or the "$\beta$ instruments at price $Y$" could be neither purchased nor sold on the market and the construction of such a portfolio would be impossible. Since the normalizing instrument $Y$ can, as will be shown, be chosen arbitrarily, there is no shortage of candidates for $Y$; we simply select any *tradable* instrument as the numeraire.

However, it is often the case that while a financial instrument on an underlying is tradable, the underlying *itself* is *not* tradable. This situation arises quite frequently. For example, the forward price $S(t, T)$ (see Eq. 6.1) is generally not tradable even if the associated spot price $S(t)$ is the price of a tradable instrument (such as a stock). Despite this fact, the forward price is often used as an underlying; the reader is referred to Eqs. 8.8 or 8.9 for examples. Also, as we will show in detail in Sect. 14.4, interest rates are not tradable either, in contrast to bonds which are financial instruments having the interest rate as an "underlying". In such cases, a second tradable instrument $U_S$ having $S$ as its underlying is chosen in addition to the numeraire instrument $Y$. The only restriction in the choice of this second instrument is that it must not be possible to construct $U_S$ by a portfolio consisting solely of the numeraire instrument (we need two truly "linearly independent" instruments).

A portfolio can now be constructed similar to Eq. 13.2 by replacing the non-tradable underlying $S$ with the tradable instrument $U_S$. A non-tradable underlying does not, in principle, complicate the situation as long as a tradable instrument on the underlying $U_S$ (which cannot be represented by the numeraire instrument) can be found.[2]

[2]If $S$ itself is tradable, it can, of course, be chosen as the (tradable) instrument $U_S$. If this is the case, we merely need to replace $U_S$ with $S$ in all pertinent equations derived in the following material.

$$\Pi(t) = \alpha_t U_S(t) + \beta_t Y(t) \; . \tag{13.3}$$

The portfolio thus constructed is required to have the same value as the derivative $V$ after proceeding by one time step. Hence

$$\begin{aligned} \alpha_t U_{S_u} + \beta_t Y_u &= \Pi_u \overset{!}{=} V_u \\ \alpha_t U_{S_d} + \beta_t Y_d &= \Pi_d \overset{!}{=} V_d \; . \end{aligned} \tag{13.4}$$

We end up with two equations for both cases $S(t + \delta t) = S_u$ and $S(t + \delta t) = S_d$.

The weights $\alpha$ and $\beta$ satisfying both of these equations can now be uniquely determined.[3]

$$\alpha_t = \frac{V_u Y_d - V_d Y_u}{U_{S_u} Y_d - U_{S_d} Y_u} \;, \quad \beta_t = \frac{V_d U_{S_u} - V_u U_{S_d}}{U_{S_u} Y_d - U_{S_d} Y_u} \; . \tag{13.5}$$

If in all events the derivative and the portfolio have the same value at time $t + \delta t$, their values must also be equal at time $t$. Otherwise an arbitrage opportunity would exist.

$$\begin{aligned} V(t) &= \Pi(t) = \alpha_t U_S(t) + \beta_t Y(t) \\ &= \frac{V_u Y_d - V_d Y_u}{U_{S_u} Y_d - U_{S_d} Y_u} U_S(t) + \frac{V_d U_{S_u} - V_u U_{S_d}}{U_{S_u} Y_d - U_{S_d} Y_u} Y(t) \; . \end{aligned} \tag{13.6}$$

It is exactly at this point that the assumption of an *arbitrage free market* enters into our discussion. As we continue with the derivation, we will clearly recognize how this assumption, together with the normalizing factor $Y$, uniquely determines the martingale probability $p$. Collecting terms with respect to the coefficients of $V_u$ and $V_d$, we obtain

$$V(t) = V_u \frac{Y_d U_S(t) - U_{S_d} Y(t)}{U_{S_u} Y_d - U_{S_d} Y_u} + V_d \frac{U_{S_u} Y(t) - Y_u U_S(t)}{U_{S_u} Y_d - U_{S_d} Y_u} \; .$$

In view of our goal of finding a representation of the value of the derivative normalized with respect to the numeraire instrument $Y$, we rewrite the above

[3]E.g., $\alpha$ may be determined by multiplying the first equation with $Y_d$ and the second equation with $Y_u$ and subtraction of the resulting equations. Similar, $\beta$ could be determined by multiplying the first equation with $U_{S_d}$ and the second equation with $U_{S_u}$ and, again, subtraction of the resulting equations.

equation in the form

$$\frac{V(t)}{Y(t)} = \frac{V_u}{Y_u}\frac{Y_d U_S(t)/Y(t) - U_{S_d}}{U_{S_u}Y_d/Y_u - U_{S_d}} + \frac{V_d}{Y_d}\frac{U_{S_u} - Y_u U_S(t)/Y(t)}{U_{S_u} - U_{S_d}Y_u/Y_d}\,.$$

This equation can now be written as

$$Z(t) = Z_u p_u + Z_d p_d \tag{13.7}$$

by defining the normalized prices as

$$Z_S(t) := \frac{V(t)}{Y(t)}\;,\quad Z_S^{u,d}(t+\delta t) := \frac{V_{S_{u,d}}(t+\delta t)}{Y_{u,d}(t+\delta t)}$$

and the "probabilities" as

$$\begin{aligned} p_u &:= \frac{Y_d\frac{U_S}{Y} - U_{S_d}}{U_{S_u}\frac{Y_d}{Y_u} - U_{S_d}} = \frac{Y_u Y_d U_S - Y_u Y U_{S_d}}{Y_d Y U_{S_u} - Y_u Y U_{S_d}} = \frac{\frac{U_S}{Y} - \frac{U_{S_d}}{Y_d}}{\frac{U_{S_u}}{Y_u} - \frac{U_{S_d}}{Y_d}} \\ p_d &:= \frac{U_{S_u} - Y_u\frac{U_S}{Y}}{U_{S_u} - U_{S_d}\frac{Y_u}{Y_d}} = \frac{Y_d Y U_{S_u} - Y_d Y_u U_S}{Y_d Y U_{S_u} - Y_u Y U_{S_d}} = \frac{\frac{U_{S_u}}{Y_u} - \frac{U_S}{Y}}{\frac{U_{S_u}}{Y_u} - \frac{U_{S_d}}{Y_d}}\,. \end{aligned} \tag{13.8}$$

where the last expressions are obtained by dividing both the numerator and denominator by $Y_u Y_d Y$. Note that these $p$ are independent of the derivative $V$. They depend explicitly only on the normalizing factor $Y$ and the instrument $U_S$. The form of these functions arose from the assumption of an arbitrage-free market as expressed in Eq. 13.6. *If* $p_u$ and $p_d$ could actually be interpreted as probabilities, Eq. 13.7 would have the form indicated in Eq. 13.1. Before showing that this is the case, we make the following remark on the computation of $p$ in practice: Instead of expressing $p$ in terms of $U_S$ and $Y$, as just presented, Eq. 13.7 can be used to write $p$ as a function of $Z$ (and thus of $V$ and $Y$). Making use of the equality $p_d = 1 - p_u$ yields

$$p_u = \frac{Z(t) - Z_d}{Z_u - Z_d}\,. \tag{13.9}$$

This is a method frequently employed in explicitly computing martingale probabilities in practice. This expression has the disadvantage that the independence of $p$ on the derivative $V$ is not immediately recognizable.

Before we can interpret $p$ as a probability, it remains to show that $p$ does in fact satisfy all requisite properties. The following conditions, holding for all probabilities, must be checked[4]:

$$p_u + p_d = 1 \quad , \quad p_u \geq 0 \quad , \quad p_d \geq 0 \tag{13.10}$$

Using simple algebra, it follows immediately from the explicit representation in Eq. 13.8 that $p_u + p_d = 1$ holds. In order to recognize the implications of the other two conditions, note that $p_u$ and $p_d$ have a *common* denominator $\frac{U_{S_u}}{Y_u} - \frac{U_{S_d}}{Y_d}$. This factor is greater than zero if and only if the normalized price of $U_S$ in the "down state" is smaller than in the "up state". This is not necessarily always the case,[5] since "up" and "down" are defined by the *un*normalized *underlying* price $S$ ($S_u > S_d$ by definition) and not by the normalized price of the instrument $U_S$. If such should be the case, i.e., if the *denominator* should be less than zero, *both numerators* must be less than zero as well. Just as both numerators must be greater then zero if the denominator is greater than zero. In summary,

$$\begin{gathered} \frac{U_{S_u}}{Y_u} > \frac{U_S}{Y} > \frac{U_{S_d}}{Y_d} \quad \text{for} \quad S_u > S > S_d \\ \text{or} \\ \frac{U_{S_u}}{Y_u} < \frac{U_S}{Y} < \frac{U_{S_d}}{Y_d} \quad \text{for} \quad S_u > S > S_d \end{gathered} \tag{13.11}$$

must hold. The normalized price of the instrument $U_S$ must therefore be a strictly monotone function of the underlying price. If this is not the case, we are immediately presented with an arbitrage opportunity. Let us assume for instance, that $\frac{U_{S_u}}{Y_u} < \frac{U_S}{Y} > \frac{U_{S_d}}{Y_d}$. This market inefficiency could be exploited by selling (short selling) the instrument $U_S$ at time $t$ and using the proceeds to purchase $a = U_S/Y$ of the instrument $Y$. This is always possible as both $Y$ and $U_S$ are tradable instruments. This portfolio has a value at time $t$ of

$$-U_S + aY = -U_S + (\frac{U_S}{Y})Y = 0$$

[4] If all three of these conditions hold, it follows immediately that $p_u \leq 1$ and $p_d \leq 1$ as well.

[5] Even when $S$ is tradable, allowing $U_S$ to be replaced by the underlying, there are several common instruments that violate this condition when used as a normalizing instrument. For example, the value $C$ of a plain vanilla call on $S$ increases *faster* (in percentage terms) than $S$ itself, implying for the quotient $\frac{S}{C(S)}$ that $\frac{S_2}{C(S_2)} < \frac{S_1}{C(S_1)}$ for $S_2 > S_1$.

One time step later, the portfolio's value is, in all events $u$ and $d$, positive since

$$-U_{S_u} + aY_u = -U_{S_u} + (\frac{U_S}{Y})Y_u = Y_u \underbrace{\left[-\frac{U_{S_u}}{Y_u} + \frac{U_S}{Y}\right]}_{>0} > 0$$

$$-U_{S_d} + aY_d = -U_{S_d} + (\frac{U_S}{Y})Y_d = Y_d \underbrace{\left[-\frac{U_{S_d}}{Y_d} + \frac{U_S}{Y}\right]}_{>0} > 0$$

This strategy leads to a certain profit without placing investment capital at risk. The fact that the value of the portfolio is positive in both possible states $u$ and $d$ can be directly attributed to the assumption that $\frac{U_S}{Y}$ is greater than both $\frac{U_{S_d}}{Y_d}$ and $\frac{U_{S_u}}{Y_u}$, thereby violating the condition in Eq. 13.11. Conversely, if $\frac{U_{S_u}}{Y_u} > \frac{U_S}{Y} < \frac{U_{S_d}}{Y_d}$, an analogous arbitrage opportunity arises by following the strategy of going long in $U_S$ and short in $a = U_S/Y$ of the instrument $Y$.

It follows immediately from these arbitrage considerations that, if the market is arbitrage free, the condition in Eq. 13.11 is *automatically* satisfied by every tradable financial instrument playing the role of the numeraire and every tradable instrument $U_S$ on the underlying and, in consequence, need not be verified in practice. $p_u$ and $p_d$ in Eq. 13.8 are therefore actually probabilities (they satisfy all the properties in 13.10), implying that the normalized price $Z = V/Y$ (and thus the normalized prices of all tradable instruments on $S$) is a martingale. Or conversely, if it were possible to find a tradable instrument $U_S$ on $S$ whose price, normalized with respect to numeraire instrument $Y$, is not a martingale, a portfolio consisting of instruments $U_S$ and $Y$ could be constructed, making arbitrage possible.

## 13.3 Self-financing Portfolio Strategies

The discussion given above was restricted to one single discrete time step. The extension to arbitrarily many discrete time steps is completely analogous. As described in Chap. 9, we obtain at every node of the tree a replicating portfolio consisting of the numeraire instrument and a financial instrument on the underlying (or the underlying itself if it is tradable[6]) in order to replicate both possible derivative prices in the next step. Such a replication of the derivative $V$ through a portfolio composed of $U_S$ and $Y$ (or $S$ and $Y$, if $S$ is tradable) corresponds to a *hedge* of $V$ (for more on this subject, see Chap. 12). As we move from one time step to the next, this hedge is constantly adjusted by adjusting the value of the weights $\alpha_t$ and $\beta_t$ of the (financial instrument on the) underlying and the numeraire.

Two conditions must be fulfilled to make this adjustment. First, it must be possible to specify the weights at the beginning of a time step and they must not depend on later realizations of the spot price. That is, the weights $\alpha_t$ and $\beta_t$ required to replicate exactly the derivative one time step later at $t + \delta t$ in all cases need to be known at time $t$. In other words, it is predictable (previsble) at time $t$ which weights $\alpha_t$ and $\beta_t$ will replicate the derivative at time $t + \delta t$. Such a process, which's value at one time step later is known at any time step is called *previsable process*. The second condition demands that the portfolio is self financing. This implies that the required adjustments of the weights by selling or buying $U_S$ must be reflected by similar buying or selling an equal amount of the numeraire $Y$. In other words, at no time money must be injected to or withdrawn from the hedge portfolio of underlying and numeraire. Such a strategy is called *self-financing*.

To ensure that the first condition is met, we consider again Eq. 13.5:

$$\alpha_t = \frac{V_u Y_d - V_d Y_u}{U_{S_u} Y_d - U_{S_d} Y_u}, \quad \beta_t = \frac{V_d U_{S_u} - V_u U_{S_d}}{U_{S_u} Y_d - U_{S_d} Y_u} .$$

The terms $V_{u,d}$, $Y_{u,d}$ and $U_{S_u,S_d}$ depend on $S(t)$ (since then $S_u$ resp. $S_d$ are determined), but don't depend on $S(t + \delta t)$, i.e., the actual realization of $S$ at time $t + \delta t$. Therefore, it is a previsible process and the first condition is met.

Next, we consider the time development of the hedge-portfolio $\Pi(t)$ and the derivative $V(t)$ to verify the second condition. At the start time $t$ we have by construction

$$V(t) = \Pi_t(t) = \alpha_t U_S(t) + \beta_t Y(t) .$$

[6]In Chap. 9, the normalizing instrument $Y$ was taken to be the zero bond $B$ with a face value of $B(T, T) = 1$ and weight $\beta_t = g(t)$. The value of this zero bond changed from $t$ to $t + \delta t$ from $Y = B(t, T)$ to $Y_u = Y_d = B(t, T)B(t)^{-1}$ with $B(t) = B(t, t + dt)$. The weight $\alpha_t$ of the underlying was denoted by $\Delta$.

The index $t$ of $\Pi_t(t)$ indicates here that we consider the hedge portfolio with parameters $\alpha_t$ and $\beta_t$, as determined at time $t$. One time step late, we have

$$\Pi_t(t+\delta t) = \alpha_t U_S(t+\delta t) + \beta_t Y(t+\delta t) \ .$$

Now we determine new values for $\alpha$ and $\beta$ belonging to time $t+\delta t$ and the hedge portfolio will be re-adjusted. Thus, we get for the new portfolio

$$V(t+\delta t) = \Pi_{t+\delta t}(t+\delta t) = \alpha_{t+\delta t} U_S(t+\delta t) + \beta_{t+\delta t} Y(t+\delta t) \ .$$

Our hedging strategy is self-financing if and only if the portfolio value before and after the re-adjustment is the same:

$$\Pi_{t+\delta t}(t+\delta t) = \Pi_t(t+\delta t) = \alpha_t U_S(t+\delta t) + \beta_t Y(t+\delta t) \ . \qquad (13.12)$$

Since $\alpha_t$ and $\beta_t$ are determined according to Eq. 13.4, it follows immediately that Eq. 13.12 is also true in all possible scenarios of our binomial world ($S(t+\delta t) = S_u$ or $S(t+\delta t) = S_d$). Therefore, the strategy is self-financing by construction.

It is worthwhile to consider this fact from another point of view which allows us to quickly verify whether or not a portfolio is self financing. The total difference $\delta\Pi(t)$ in the value of a portfolio for a certain trading strategy over a time span $\delta t$ equals the difference in the value at time $t+\delta t$ of the portfolio set up at time $t+\delta t$ and the value at time $t$ of the portfolio set up at time $t$, thus

$$\delta\Pi(t) = \Pi_{t+\delta t}(t+\delta t) - \Pi_t(t) = \underbrace{\Pi_t(t+\delta t) - \Pi_t(t)}_{\delta\Pi^{\text{Market}}} + \underbrace{\Pi_{t+\delta t}(t+\delta t) - \Pi_t(t+\delta t)}_{\delta\Pi^{\text{Trading}}} .$$

We arrive at the second equation by simply inserting a zero in the form $0 = \Pi_t(t+\delta t) - \Pi_t(t+\delta t)$. It is now easy to recognize both components contributing to the total change in $\Pi$: $\delta\Pi^{\text{Market}}$ is the change in value of the portfolio resulting from changes in the market without having adjusted the positions in the portfolio. $\delta\Pi^{\text{Trading}}$ is the difference at time $t+\delta t$ between the value of the new portfolio and that of the old portfolio, i.e., the value change resulting solely from trading. For the strategy to be self financing, $\delta\Pi^{\text{Trading}}$ must equal zero since the value of the old portfolio must provide exactly the funds necessary to finance the new portfolio. Consider for example a portfolio composed of two instruments having the form $\Pi(t) = \alpha_t U_S(t) + \beta_t Y(t)$ as above:

$$\delta\Pi^{\text{Market}} = \Pi_t(t+\delta t) - \Pi_t(t) = \alpha_t \underbrace{[U_S(t+\delta t) - U_S(t)]}_{\delta U_S(t)} + \beta_t \underbrace{[Y(t+\delta t) - Y(t)]}_{\delta Y(t)}$$

$$\begin{aligned}\delta\Pi^{\text{Trading}} &= \Pi_{t+\delta t}(t+\delta t) - \Pi_t(t) \\ &= U_S(t+\delta t)\underbrace{[\alpha(t+\delta t) - \alpha_t]}_{\delta\alpha_t} + Y(t+\delta t)\underbrace{[\beta(t+\delta t) - \beta_t]}_{\delta\beta(t)} ,\end{aligned}$$

that is,

$$\begin{aligned}\delta\Pi(t) &= \alpha_t \delta U_S(t) + \beta_t \delta Y(t) + U_S(t+\delta t)\delta\alpha_t + Y(t+\delta t)\delta\beta_t \\ &= \alpha_t \delta U_S(t) + \beta_t \delta Y(t) \quad \text{for a self financing strategy.}\end{aligned} \tag{13.13}$$

This implies that the strategy is self financing if and only if the *total* change in the portfolio's value from one adjustment period to the next can be explained exclusively by *market* changes. This holds for infinitesimal time steps as well:

$$\begin{aligned}&\Pi(t) = \alpha_t U_S(t) + \beta_t Y(t) \quad \text{self financing strategy} \\ \Longleftrightarrow \quad & d\Pi(t) = \alpha_t dU_S(t) + \beta_t dY(t) .\end{aligned} \tag{13.14}$$

From this point of view, we again consider the structure of Eq. 13.13. The change $\delta\Pi$ over the next time step is composed of the change $\delta U$ and the change $\delta Y$. At time $t$, neither the value of $\delta U$ nor of $\delta Y$ are known (we cannot even say if these are "upward" or "downward"), but the *coefficients* $\alpha_t$ and $\beta_t$ controlling the influence of these changes on $\delta\Pi$ are known at time $t$.

Compare this with the general Ito process given in Eq. 2.19. The coefficients $a(S, t)$ and $b(S, t)$ which control the next step in the process are also already known at time $t$. These coefficients $a$ and $b$ are also *previsible processes*.

## 13.4 Generalization to Continuous Time

The profound and important concepts presented above can be summarized as follows:

- For any arbitrary underlying $S$ which follows a stochastic process of the general form indicated in Eq. 2.19,

- and any tradable instrument $U$ with $S$ as its underlying (or $S$ itself if it is tradable)
- and any other arbitrary, tradable, financial instrument $Y$ (*numeraire*),
- the assumption of an arbitrage-free market implies the existence of a (numeraire-dependent) unique probability *measure* $p$,
- such that the current, normalized price $Z = V/Y$ (also called the *relative price*) of an arbitrary, tradable, financial instrument $V$ on $S$
- equals the expectation of the future normalized price and thus, $Z$ is a martingale with respect to $p$.
- This statement holds because a *self financing* portfolio strategy with *previsible* weights can be followed which replicates (*hedges*) the price of the financial instrument $V$ at all times.

The expectation is taken at time $t$ with respect to the $Y$-dependent probability measure $p$. This dependence is given explicitly by Eq. 13.8 (in the context of a binomial tree over one time step). To emphasize this dependence, the expectation is often equipped with the subscript $t$ and the superscript $Y$:

$$Z(t) = \mathrm{E}_t^Y\left[Z(u)\right] \quad \forall u \geq t\ . \tag{13.15}$$

The intuitive interpretation of Eq. 13.15 is that the *expected* change in the normalized price of the tradable instrument is zero with respect to this probability measure, in other words, the normalized price has *no drift.* This implies that in this measure such a normalized price process has the form given in Eq. 2.19 with no drift term:

$$dZ(t) = b_Z(S,t)\, d\widetilde{W} \quad \text{mit} \quad d\widetilde{W} \sim X\sqrt{dt},\ X \sim \mathrm{N}(0,1)\ . \tag{13.16}$$

Here, $d\widetilde{W}$ is a Brownian motion and $b_Z$ a (*previsible*) process which is different for each different instrument.

The material presented thus far has been restricted to *discrete* time steps. But the above statements hold in continuous time as well. To see this, several fundamental theorems from stochastic analysis are required. In the following, the insights gained in the study of discrete processes will be carried over to the continuous case and the necessary theorems from stochastic analysis will be used (without proof) when needed. In addition to extending the results already obtained to the time continuous case, the object of the following discussion is to provide a deeper understanding of the general approach to pricing derivatives, in particular, an understanding of role of the *drift* of an underlying.

We therefore consider a very general (not necessarily tradable) underlying $S$, which, *in the real world*, is governed by an Ito process satisfying Eq. 2.19, i.e.

$$dS(t) = a(S,t)\,dt + b(S,t)\,dW \quad \text{with} \quad dW \sim X\sqrt{dt},\ X \sim \mathrm{N}(0,1)\ . \tag{13.17}$$

Let $U(S,t)$ denote the price of a tradable, financial instrument with an underlying $S$. The process for $U$ *in the real world,* according to Ito's lemma, is given by Eq. 2.21 as

$$\begin{aligned} dU(S,t) &= a_U(S,t)dt + \frac{\partial U}{\partial S}\,b(S,t)\,dW \quad \text{with} \\ a_U(S,t) &:= \frac{\partial U}{\partial S}a\,(S,t) + \frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}b(S,t)^2\ , \end{aligned} \tag{13.18}$$

Here $a_U(S,t)$ denotes the drift of $U$.

Furthermore, we select an arbitrary, tradable instrument $Y$ as the numeraire instrument. Note, however, that the choice of the numeraire $Y$ is not completely arbitrary. It has always been tacitly assumed that our market is driven by just one *single* random factor (*one-factor model*), namely the *Brownian motion* $dW$ in Eq. 2.19. The numeraire instrument may indeed be any arbitrary, deterministic or stochastic instrument, but if it has a stochastic component, it must be driven by *the same* random walk as the underlying $S$. If not, the resulting model would be a *multi-factor model,* and in consequence could not be completely "spanned" by the two instruments $U$ and $Y$. Analogous to Eq. 2.19, the most general process describing the numeraire instrument $Y$ satisfies

$$dY(t) = m(Y,t)\,dt + n(Y,t)\,dW \quad \text{with} \quad dW \sim X\sqrt{dt},\ X \sim \mathrm{N}(0,1) \tag{13.19}$$

with (*previsible*) processes $m$ and $n$ and the same random walk $dW$, which drives the random component of the underlying $S$ in Eq. 13.17.

Motivated by our experience with Sect. 13.1, we seek a probability measure with respect to which the prices of all tradable instruments (which depend on no stochastic factors other than the Brownian motion $dW$ in Eq. 13.17) normalized with the numeraire instrument $Y$ are martingales. We are as yet quite far from attaining this goal. We start by aiming at the short-term goal of finding a measure for which the normalized price

$$Z(S,t) := \frac{U(S,t)}{Y(t)} \tag{13.20}$$

of a *single* selected instrument $U$ is a martingale. The product rule establishes the following equation for the process $Z$[7]:

$$dZ = d\left[Y^{-1}U\right] = U\,d\left[Y^{-1}\right] + Y^{-1}dU + dU\,d\left[Y^{-1}\right] \tag{13.21}$$

The differential $dU$ was already specified above. The differential of $f(Y) := Y^{-1}$ is obtained through an application of Ito's lemma, Eq. 2.21, under consideration of $\partial f/\partial Y = -1/Y^2$, $\partial^2 f/\partial Y^2 = 2/Y^3$ and $\partial f/\partial t = 0$. Simple substitution gives

$$d\left[Y^{-1}\right] = \left[-\frac{1}{Y^2}m + \frac{1}{Y^3}n^2\right]dt - \frac{1}{Y^2}n\,dW$$

The last term in Eq. 13.21 appears since the product of the two differentials contains not only higher order terms but also a term $\sim dW^2$ which is *linear* in $dt$ (see Eq. 2.20), explicitly:

$$\begin{aligned} dU\,d\left[Y^{-1}\right] &= \left(a_U dt + \frac{\partial U}{\partial S}b\,dW\right)\frac{1}{Y^2}\left(\left[-m + n^2/Y\right]dt - n\,dW\right) \\ &= -\frac{\partial U}{\partial S}b\frac{n}{Y^2}\underbrace{(dW)^2}_{\sim dt} + \mathcal{O}\left(dt\,dW\right) \ . \end{aligned}$$

This effect clearly stems from the fact that both $dU$ and $dY$ are no ordinary but *stochastic* differentials.[8] Altogether $dZ$ becomes

$$dZ = \frac{U}{Y}\left(\left[\frac{n^2}{Y^2} - \frac{m}{Y}\right]dt - \frac{n}{Y}dW\right) + \frac{a_U}{Y}dt + \frac{\partial U}{\partial S}\frac{b}{Y}dW - \frac{\partial U}{\partial S}\frac{bn}{Y^2}dt$$

[7]We suppress the arguments of $U$, $Y$, $a$, $b$, $m$ and $n$ in order to keep the notation simple. The arguments of these variables are always those as given in Eqs. 13.17 and 13.19.

[8]Equation 13.21 can be proven formally by applying Ito's lemma (in the version for two stochastic variables) to the function $f(U,Y) = U\,Y^{-1}$.

$$= \left(\frac{b}{Y}\frac{\partial U}{\partial S} - \frac{n}{Y}\frac{U}{Y}\right) dW + \left(\frac{a_U}{Y} + \left[\frac{n^2}{Y^2} - \frac{m}{Y}\right]\frac{U}{Y} - \frac{bn}{Y^2}\frac{\partial U}{\partial S}\right) dt$$

$$= \left(\frac{b}{Y}\frac{\partial U}{\partial S} - \frac{n}{Y}\frac{U}{Y}\right) \left\{ dW + \frac{a_U + \left[\frac{n^2}{Y^2} - \frac{m}{Y}\right] U - b\,\frac{n}{Y}\frac{\partial U}{\partial S}}{b\frac{\partial U}{\partial S} - n\frac{U}{Y}} dt \right\} , \tag{13.22}$$

where the coefficient of $dW$ is factored out "by force" in the final step. We seek a probability measure with respect to which $Z$ is a martingale, or in other words, a process of the form specified in Eq. 13.16. The process $dZ$ would have the desired form if a measure existed with respect to which

$$d\widetilde{W} := dW + \frac{a_U + \left[\frac{n^2}{Y^2} - \frac{m}{Y}\right] U - b\,\frac{n}{Y}\frac{\partial U}{\partial S}}{b\frac{\partial U}{\partial S} - n\frac{U}{Y}} dt \tag{13.23}$$

were a standard Brownian motion, i.e.,

$$d\widetilde{W} \sim X\sqrt{dt}\ \text{mit} X \sim N(0, 1) .$$

Stochastic analysis delivers just such a theorem, namely the famous Girsanov Theorem:

**Theorem 1 (Girsanov)** *Let $W(t)$ be a Brownian motion with respect to a probability measure $\mathcal{P}$, and $\gamma(t)$ a previsible process which (for some future time $T$) satisfies the boundedness condition*

$$\mathrm{E}_{\mathcal{P}}\left[\exp\left(\frac{1}{2}\int_0^T \gamma(t)dt\right)\right] < \infty$$

*Then there exists a measure $\mathcal{Q}$, equivalent*[9] *to $\mathcal{P}$, with respect to which*

$$\widetilde{W}(t) = W(t) + \int_0^t \gamma(s)ds$$

[9]Two probability measures are called equivalent if they agree exactly on what is possible and what is impossible. I.e. an event is impossible (probability zero) in one probability measure if and only if it is impossible in all equivalent probability measures.

*is a Brownian motion. This implies that*

$$dW(t) + \gamma(t)dt = d\widetilde{W}(t) \sim X\sqrt{dt} \text{ with } X \sim N(0,1)$$

*Conversely, in the measure Q the original process $W(t)$ is a Brownian motion with an additional drift component, $-\gamma(t)$: $dW(t) = d\widetilde{W}(t) - \gamma(t)dt$*

In order to apply the theorem, the coefficient of $dt$ in Eq. 13.23 must be identified with the process $\gamma(t)$ in the Girsanov Theorem. We begin by observing that this coefficient depends only on variables which can be evaluated at time $t$ and as such can itself be determined at time $t$. This implies that it is previsible. Proceeding under the assumption that the technical boundedness condition in the theorem is satisfied (this will always be the case in our models), the theorem provides a measure with respect to which $d\widetilde{W}$ is in fact a simple Brownian motion. $dZ$ in Eq. 13.22 then has a form as in Eq. 13.16 with

$$b_Z(S,t) = \frac{b}{Y}\frac{\partial U}{\partial S} - \frac{n}{Y}\frac{U}{Y} \,. \tag{13.24}$$

Can we now conclude that $Z$ is a martingale with respect to this measure? Does the absence of a drift in Eq. 13.16 directly imply[10] the martingale property in Eq. 13.1? Again, stochastic analysis provides theorems which ensures (if certain technical conditions are satisfied, see [11, page 79], for example) that this is the case. The measure for which $dZ$ has the form 13.16 is thus a martingale measure.

We have thus attained our first goal by finding a measure with respect to which the normalized price of one selected instrument $U$ (or for $S$ if $S$ should be tradable) is a martingale. The only requirement made of the instrument $U$ is that it be tradable. Thus, for every arbitrary, tradable financial instrument (with $S$ as an underlying) there exists a martingale measure. This measure could, at this point in the discussion, be different for each instrument $U$, in other words, it may be dependent on our choice of $U$ (just as it depends on the choice of numeraire instrument $Y$). It remains to show that the normalized price of *every* tradable instrument (with $S$ as its underlying) is a martingale with respect to *the same* probability measure. In the discrete case, the essential point was that a self financing strategy for a portfolio consisting of $U$ and $Y$

[10] We have already shown the reverse implication above.

exists which replicates $V$ exactly. We will need another important theorem from stochastic analysis to establish this for the time continuous case:

**Theorem 2 (Martingale Representation)** *If $Z$ is a martingale with respect to the probability measure $\mathcal{P}$ with a volatility which is non-zero almost everywhere with probability $P = 1$, i.e., if $Z$ follows a stochastic process satisfying*

$$dZ = b_Z(t)dW \quad with \quad P\left[b_Z(t) \neq 0\right] = 1 \quad \forall t$$

*with a previsible process $b_Z(t)$, and if there exists in this measure another martingale $X$, then there exists a previsible process $\alpha(t)$ such that*

$$dX = \alpha(t)dZ$$

*Or equivalently in integral form*

$$X(t) = X(0) + \int_0^t \alpha(s)dZ(s)$$

*The process $\alpha(t)$ is unique. Furthermore, $\alpha$ and $b_Z$ together satisfy the boundedness condition*

$$\mathrm{E}\left[\exp\left(\frac{1}{2}\int_0^T \alpha^2(t)b_Z^2(t)dt\right)\right] < \infty \, .$$

This theorem states intuitively that (if the volatility is non-zero), two martingales differ at most by a previsible process. This implies that any martingale can be represented by any other martingale and a previsible process.

As yet, we only have one martingale in our measure, namely $Z$, the normalized price of $U$. In order to apply the theorem, we need a second martingale. And since we wish to gather information about the price of another arbitrary financial instrument $V$, we must construct a second martingale from this instrument $V$. We do this with the help of yet another quite simple theorem:

**Theorem 3 (Tower Law)** *For any arbitrary function $V$, depending on events occurring up to some specified future time $T > t$, the expectation at time $t$ of $V(T)$ with respect to any arbitrary probability measure $\mathcal{P}$,*

$$E(t) := E_t^{\mathcal{P}}\left[V(T)\right]$$

*is a martingale with respect to $\mathcal{P}$, explicitly*

$$E(t) = E_t^{\mathcal{P}}[E(u)] \quad \forall u > t$$

It is easy to see that this theorem is true: substituting the definition of $E$ into the claim that $E(t)$ is a martingale reads

$$\mathrm{E}_t^{\mathcal{P}}[V(T)] = \mathrm{E}_t^{\mathcal{P}}\left[\mathrm{E}_u^{\mathcal{P}}[V(T)]\right] \quad \forall u > t \tag{13.25}$$

This implies that in taking the expectation at time $u$ and subsequently taking the expectation *of this expectation* at an earlier time $t$ we arrive at the same result as if we had directly taken the expectation with respect to the earlier time $t$ in the first place. The reader should become familiar with this idea by verifying it using the binomial or trinomial trees presented in Figs. 9.1 or 9.2.

The payoff profile $V(T)$ of a financial instrument with maturity $T$ is a function depending only on events (values of the underlying process $S$) occurring up to time $T$. The Tower Law states that the *expectation* of this payoff profile is a martingale with respect to *every* probability measure, in particular with respect to the martingale measure of $Z$ from Eq. 13.20. Thus, we have found two processes which are martingales with respect to this measure, namely $Z(t)$ and $\mathrm{E}_t^Y[V(T)]$.

But we want more. We want the (appropriately normalized) price $V(t)$ itself to be a martingale, not merely the expectation of the payoff profile $V(T)$. In the discrete case (and in the continuous case for $U$ as well), this was accomplished by considering the *normalized* prices. We therefore consider instead of $V$ the payoff profile normalized with $Y$, $V(T)/Y(T)$. The expectation (taken at time $t$ with respect to the martingale measure of $Z$) of this function

$$E(t) := \mathrm{E}_t^Y\left[\frac{V(T)}{Y(T)}\right]$$

is, because of the Tower Law, also a martingale, which we denote by $E(t)$ in what follows. Furthermore, the payoff of $V$ at maturity $t = T$ is exactly replicated by the product $Y(t)E(t)$ since

$$Y(T)E(T) = Y(T)\mathrm{E}_T^Y\left[\frac{V(T)}{Y(T)}\right] = Y(T)\frac{V(T)}{Y(T)} = V(T) \tag{13.26}$$

It is now clear how the existence of a replicating portfolio can be established through an application of the martingale representation theorem: the martingale in question is $E(t)$, the expectation of the normalized derivative price at maturity, and as our second martingale we take $Z(t)$ from Eq. 13.20, the normalized price of the initially selected tradable instrument $U$. The martingale representation theorem now states that (if the volatility of $Z$ is always non-zero) the process $E(t)$ differs from the process $Z(t)$ only by a previsible process $\alpha_t$:

$$dE = \alpha_t dZ \tag{13.27}$$

We use this previsible process now to construct a portfolio consisting of $\alpha_t$ of the instrument $U$ and $\beta_t$ of the numeraire instrument $Y$ as was done in Eq. 13.3. This is always possible since $\alpha_t$ is previsible by the Martingale Representation Theorem and both $U$ and $Y$ are tradable.

$$\Pi(t) = \alpha_t U(t) + \beta_t Y(t) \ . \tag{13.28}$$

This portfolio should equal $Y(t)E(t)$ for all times $t \leq T$ since, according to Eq. 13.26, it then replicates the payoff of $V$ upon its maturity exactly, i.e., when $t = T$. From this condition, we can derive the number $\beta_t$ of numeraire instruments required for the replicating portfolio:

$$\begin{aligned} Y(t)E(t) = \Pi(t) &= \alpha_t U(t) + \beta_t Y(t) \\ \iff \qquad \beta_t &= E(t) - \alpha_t \frac{U(t)}{Y(t)} = E(t) - \alpha_t Z(t) \ . \end{aligned} \tag{13.29}$$

We have thus established the existence of a replicating portfolio. It remains to show that this portfolio is *self financing*, because only if no injection or withdrawal of capital is required throughout the lifetime of the derivative can we deduce the equality of the portfolio's value and the value of the instrument $V$. To this end, we consider the total change in the value of the portfolio in light of Eq. 13.14:

$$\begin{aligned} d\Pi &= d(YE) \\ &= EdY + YdE + dEdY \\ &= EdY + Y\alpha dZ + \alpha dZdY \\ &= [\beta + \alpha Z]\,dY + \alpha YdZ + \alpha dZdY \end{aligned}$$

$$= \alpha\,[ZdY + YdZ + dZdY] + \beta dY$$
$$= \alpha\, d\underbrace{(ZY)}_{U} + \beta\, dY\,,$$

where in the second equality the (stochastic) product rule for stochastic integrals has been applied, in the third equality the Martingale Representation Theorem in form of Eq. 13.27, and in the fourth, Eq. 13.29 in the form $E(t) = \beta_t + \alpha_t Z(t)$ has been used. In the last equation the (stochastic) product rule has been applied again. The equation now states that the total change in the portfolio defined in Eq. 13.28 results solely from the change in price of the instruments $U$ and $Y$ and *not* from any adjustment of the positions $\alpha$ or $\beta$:

$$d\Pi(t) = \alpha_t dU(t) + \beta_t dY(t)\,.$$

Thus, via Eq. 13.14 the portfolio is self financing. The value of the portfolio is by construction $\Pi(t) = Y(t)E(t)$ for all times. According to Eq. 13.26, this replicates the payoff profile $V(T)$ exactly at time $T$. The value of the portfolio must therefore equal that of the derivative for all previous times as well:

$$V(t) = \Pi(t) = Y(t)E(t) = Y(t)\mathrm{E}_t^Y\left[\frac{V(T)}{Y(T)}\right] \tag{13.30}$$

and thus

$$\frac{V(t)}{Y(t)} = \mathrm{E}_t^Y\left[\frac{V(T)}{Y(T)}\right] \tag{13.31}$$

Therefore, the normalized price of the tradable financial instrument $V$ is a martingale in the *same* probability measure with respect to which the normalized price of the instrument $U$ is a martingale. Since $V$ was chosen arbitrarily, this implies that the normalized price of *all* tradable instruments are martingales with respect to the *same* probability measure.

Furthermore, the process $\alpha_t$ can be calculated explicitly. Equation 13.27 states that $\alpha_t$ is the change of $E$ per change in $Z$ or in other words the derivative of $E$ with respect to $Z$. Using Eqs. 13.20 and 13.30, both $Z$ and $E$ can be expressed in terms of the prices of tradable instruments (known at time $t$)

$$\alpha_t = \frac{\partial E(t)}{\partial Z(t)} = \frac{\partial\,[V(t)/Y(t)]}{\partial\,[U(t)/Y(t)]}\,. \tag{13.32}$$

The process $\alpha(t)$ corresponds to the sensitivity $\Delta$ introduced in Chap. 12. It follows that Eq. 13.29 can be applied to calculate $\beta(t)$ at time $t$ explicitly as well.

$$\beta_t = E(t) - \alpha_t Z(t) = \frac{V(S,t)}{Y(t)} - \frac{U(S,t)}{Y(t)} \frac{\partial\,[V(S,t)/Y(t)]}{\partial\,[U(S,t)/Y(t)]} .$$

We have thus accomplished our goal, having shown that the normalized prices $V/Y$ of all tradable instruments are martingales with respect to the martingale measure of $Z = U/Y$. The only question remaining is whether this measure is unique or whether several such measures may exist. To answer this question we apply yet another theorem from stochastic analysis which states that for *complete markets*[11] the martingale measure obtained above is unique. The theorem [89] is stated explicitly here[12]:

**Theorem 4 (Harrison-Pliska)** *A market consisting of financial instruments and a numeraire instrument is arbitrage free if and only if there exists a measure, equivalent to the real world measure, with respect to which the prices of all financial instruments normalized with the numeraire instrument are martingales. This measure is unique if and only if the market is complete.*

**Summary**

At this stage, it is helpful to summarize what has been done in this section. The summary corresponds to the summary at the beginning of the Sect. 13.4 (which was done for discrete time steps).

- We select a tradable instrument $Y$ as the numeraire and another tradable instrument $U$ which has $S$ as an underlying (if $S$ itself is tradable, $S$ can be chosen directly).
- We then find the probability measure for which $Z = U/Y$ is a martingale. The *Girsanov-Theorem* guarantees that this is always possible via a suitable drift transformation as long as a technical boundedness condition is satisfied.
- The *Martingale Representation Theorem* and the *Tower Law* enable the construction of a *self financing portfolio* composed of the instruments $U$ and $Y$ which replicates the payoff profile at maturity of any arbitrary, tradable

[11]A market is called *complete* if there exists a replicating portfolio for each financial instrument in the market.

[12]As always, the term "if and only if" means that one follows from the other *and vice versa*.

instrument $V$ having $S$ as an underlying. The value of this replicating portfolio is given by Eq. 13.30 where the expectation is taken with respect to the martingale measure of $Z = U/Y$.

- This portfolio must be equal to the value of the derivative $V(t)$ for all times before maturity if the market is arbitrage free. This means that, according to Eq. 13.31, the normalized price $V/Y$ with respect to the martingale measure of $Z$ is likewise a martingale. Thus, having obtained (via Girsanov) a martingale measure for $Z$, the normalized price $V/Y$ of all other tradable instruments are martingales with respect to this same measure.
- Finally, the *Harrison-Pliska Theorem* states that this measure is unique in complete markets: in complete markets there exists for each numeraire instrument *one single* measure with respect to which all tradable instruments normalized with this numeraire are martingales.

## 13.5 The Drift

With respect to the martingale measure, the price processes of all instruments normalized by the numeraire instrument $Y$ are drift-free. The *expected* changes in the normalized prices of tradable instruments are thus exactly equal to zero. What can we say about the process of the *underlying* with respect to this measure? The model Eq. 2.19 was set up to describe the underlying in the *real* world. As we have seen however, the *valuation* of financial instruments is accomplished in a world governed by the probability with respect to which normalized prices of tradable instruments are martingales, i.e., processes of the form indicated in Eq. 13.16. Hence, it is important to know how the *underlying* process is transformed when the real probability measure is transformed into the martingale measure.

Not only do we know that a martingale measure exists (and is unique in a complete market) for $Z = U/Y$ with respect to which $Z$ can be represented by a process of the form in Eq. 13.16. We also know from Eqs. 13.23 and 13.24 the explicit relationships between the variables in the real world and the world governed by the martingale measure. We now make use of Eq. 13.23 in particular, to express the underlying $S$ as given by the process 13.17 with respect to the Brownian motion $dW$ in the real world in terms of the Brownian motion $d\widetilde{W}$ with respect to the martingale measure (using the explicit form for $a_u$ from Eq. 13.18):

$$\begin{aligned} dS &= a\,dt + b\,dW \\ &= adt + b\left\{ d\widetilde{W} - \frac{a_u + \left[\frac{n^2}{Y^2} - \frac{m}{Y}\right]U - b\,\frac{n}{Y}\frac{\partial U}{\partial S}}{b\frac{\partial U}{\partial S} - n\frac{U}{Y}} dt \right\} \\ &= adt - \frac{\frac{\partial U}{\partial S}a + \frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}b^2 + \left[\frac{n^2}{Y^2} - \frac{m}{Y}\right]U - b\,\frac{n}{Y}\frac{\partial U}{\partial S}}{\frac{\partial U}{\partial S} - \frac{n}{b}\frac{U}{Y}} dt + bd\widetilde{W} \\ &= \frac{b\,\frac{n}{Y}\frac{\partial U}{\partial S} - a\frac{n}{b}\frac{U}{Y} - \frac{\partial U}{\partial t} - \frac{1}{2}\frac{\partial^2 U}{\partial S^2}b^2 - \left[\frac{n^2}{Y^2} - \frac{m}{Y}\right]U}{\frac{\partial U}{\partial S} - \frac{n}{b}\frac{U}{Y}} dt + bd\widetilde{W}\;. \end{aligned} \tag{13.33}$$

This equation explicitly specifies the underlying process with respect to the martingale measure.

Note that in the transition from the real world measure to the martingale measure, only the drift of the underlying has changed and *not* the volatility; the coefficient of the Brownian motion remains $b(S, t)$. This is not merely coincidence but a natural consequence of the Girsanov Theorem which intuitively states that a transformation between two equivalent probability measures effects nothing more than a change in the drift.

We should further note that the Harrison-Pliska Theorem states that the martingale measure in complete markets (we will from now on always assume that the market is complete, if nothing else is explicitly stated) is *unique.* This implies that the drift of $S$, used in the valuation of financial instruments on $S$ is *unique* as well, up to the choice of the numeraire instrument $Y$.

In the last equality in 13.33 the drift $a(S, t)$ in the real world has been written to share a common denominator with the second term $\sim dt$. In doing so, we observe that the terms $\sim \frac{\partial U}{\partial S}a$ cancel each other and the drift in the real world enters into the equation corresponding to the martingale measure only in the form of $\frac{n}{b}\frac{U}{Y}a$. For numeraire instruments $Y$ satisfying Eq. 13.19 with $n(Y, t) = 0\ \forall Y, t$ , i.e., for numeraires with processes of the form

$$dY(t) = m(Y, t)\,dt \tag{13.34}$$

the drift $a$ of the *real* world disappears completely for the martingale measure associated with $Y$! Regardless of the drift $a(S,t)$ chosen in the model, $a(S,t)$ is completely irrelevant for the valuation of financial instruments if the numeraire chosen is a process of the form specified in Eq. 13.34.

Equation 13.34 by no means implies that $Y$ must be deterministic, since $m(Y,t)$ is not assumed to be deterministic but only *previsible*.[13] This means that at time $t$ the evolution of $Y$ is only known for the step *immediately following* $t$, but not for later steps.

As has already been mentioned on more than one occasion, the choice of instrument to be used as the normalizing factor (the *numeraire*) is arbitrary but this choice significantly affects whether a specific problem can be solved elegantly or awkwardly. This is analogous to the selection of a suitable system of coordinates when solving problems in physics, for example. We see from Eq. 13.33 that an appropriate choice of numeraire can simplify calculations substantially. The numeraire should always be chosen to be of the form 13.34 for some previsible process $m$. This is always possible in practice and all numeraire instruments in this book satisfy this property.[14] For such numeraires, $m/Y$ in Eq. 13.33 is precisely the *yield of the numeraire instrument*, since Eq. 13.34 obviously implies:

$$m = \frac{dY}{dt} \quad \Longrightarrow \quad \frac{m}{Y} = \frac{1}{Y}\frac{dY}{dt} = \frac{d\ln Y}{dt} \, . \tag{13.35}$$

Using this equation and $n = 0$, the drift transformation equation 13.23 becomes

$$\begin{aligned} d\widetilde{W} &= dW + \frac{a_U - \frac{d\ln Y}{dt}U}{b\frac{\partial U}{\partial S}}dt \\ &= dW + \frac{1}{b}\left[a_U\left(\frac{\partial U}{\partial S}\right)^{-1} - \frac{d\ln Y}{dt}\left(\frac{\partial \ln U}{\partial S}\right)^{-1}\right]dt \end{aligned} \tag{13.36}$$

[13] A previsible process is a *stochastic* process whose current value can be determined from information available at the previous time step. Intuitively, it is a stochastic process "shifted back" one step in time.

[14] This will be shown below explicitly for all numeraire instruments used.

and the underlying process in Eq. 13.33 reduces to

$$dS(t) = \widetilde{a}(S,t)\,dt + b(S,t)d\widetilde{W} \qquad \text{mit}$$
$$\widetilde{a}(S,t) = \left(\frac{\partial U(S,t)}{\partial S}\right)^{-1}\left[U(S,t)\frac{d\ln Y(t)}{dt} - \frac{\partial U(S,t)}{\partial t} - \frac{b^2(S,t)}{2}\frac{\partial^2 U(S,t)}{\partial S^2}\right]. \tag{13.37}$$

Only the tradable instrument $U$ and the numeraire $Y$ (and their respective derivatives) and the "volatility" $b(S,t)$ of the underlying process $S$ appear in this expression. As already noted, the real underlying drift $a(S,t)$ has disappeared completely.

Since the prices of financial instruments must obviously be independent of the method used to compute them, the following theorem holds irrespective of the choice of numeraire:

**Theorem 5** *Suppose there exists a numeraire $Y$ in an arbitrage-free market satisfying Eq. 13.34 with a previsible process $m(Y,t)$. Then the drift of the underlying in the real world is irrelevant to the prices of financial instruments. Arbitrage freedom alone determines the prices of financial instruments and not the expectation of the market with respect to the evolution of the underlying.*

In this context, it is interesting to consider the behavior of the *non-*normalized process of the tradable instrument $U$ in the martingale measure. The process 13.18 in the real world is transformed via Eq. 13.36 to

$$\begin{aligned}
dU &= a_U dt + \frac{\partial U}{\partial S} b\,dW \\
&= a_U dt + \frac{\partial U}{\partial S} b\left\{d\widetilde{W} - \frac{1}{b}\left[a_U\left(\frac{\partial U}{\partial S}\right)^{-1} - \frac{d\ln Y}{dt}\left(\frac{\partial \ln U}{\partial S}\right)^{-1}\right]dt\right\} \\
&= a_U dt + \frac{\partial U}{\partial S} b\,d\widetilde{W} - \left[a_U - U\frac{d\ln Y}{dt}\right]dt\ ,
\end{aligned}$$

and thus

$$dU(S,t) = \frac{d\ln Y(t)}{dt}U(S,t)dt + b(S,t)\frac{\partial U(S,t)}{\partial S}d\widetilde{W}\ . \tag{13.38}$$

The drift of a tradable instrument in the martingale measure for a numeraire of the form specified in Eq. 13.34 is thus simply the product of the price of the instrument and the yield of the numeraire!

**Theorem 6** *The expected yield (defined as the expected logarithmic price change per time) of a tradable financial instrument in the martingale measure with a numeraire of the form 13.34 is always equal to the yield of the numeraire instrument.*

$$E_t^Y\left[\frac{dU(S,t)}{U(S,t)}\right] = \frac{d\ln Y(t)}{dt}dt\ . \tag{13.39}$$

In Eqs. 13.38 and 13.39, a tradable instrument is denoted by the letter $U$ but these properties naturally hold for all tradable instrument since, as discussed in detail in the previous section, all tradable instruments are martingales with respect to the same probability measure. The instrument $U$ is not essentially different from other tradable instruments in the market.

## 13.6 The Market Price of Risk

As emphasized several times previously, all tradable instruments $U$ in a complete, arbitrage-free market (normalized with respect to a selected numeraire) have the same martingale measure. This implies that $d\widetilde{W}$ in Eq. 13.23 is always *the same* Brownian motion for this measure. Since the Brownian motion $dW$ of the underlying in the real world is not dependent on the specific financial instrument either, this must hold for the difference $d\widetilde{W} - dW$ as well. The change in drift from $dW$ to $d\widetilde{W}$ in Eq. 13.23 must be *the same* for every tradable instrument $U$. This implies that for two arbitrary tradable instruments, $U_1(S,t)$ and $U_2(S,t)$ the following must hold:

$$\frac{a_{U_1} + \left[\frac{n^2}{Y^2} - \frac{m}{Y}\right]U_1 - b\,\frac{n}{Y}\frac{\partial U_1}{\partial S}}{b\frac{\partial U_1}{\partial S} - n\frac{U_1}{Y}} = \frac{a_{U_2} + \left[\frac{n^2}{Y^2} - \frac{m}{Y}\right]U_2 - b\,\frac{n}{Y}\frac{\partial U_2}{\partial S}}{b\frac{\partial U_2}{\partial S} - n\frac{U_2}{Y}}\ , \tag{13.40}$$

where $a_{U_i}$ denotes the drift of $U_i$ in the *real* world in accordance with Eq. 13.18. The existence of a unique measure with respect to which all normalized, tradable instruments are martingales satisfying Eq. 13.16 can be formulated equivalently as follows: regardless of the appearance of the drift of the financial instrument in the real world, the combination of this drift with other characteristics of the financial instrument and the numeraire as specified in Eq. 13.40 must be the same for *all* financial instruments. This combination has its own name; it is known as the *market price of risk*.[15] The market price of risk $\gamma_U$ for an instrument $U$ is defined by

$$\begin{aligned}\gamma_U(t) &:= \frac{a_U + \left[\frac{n^2}{Y^2} - \frac{m}{Y}\right] U - b\,\frac{n}{Y}\frac{\partial U}{\partial S}}{b\frac{\partial U}{\partial S} - n\frac{U}{Y}} \\ &= \frac{1}{b}\left[a_U\left(\frac{\partial U}{\partial S}\right)^{-1} - \frac{d\ln Y}{dt}\left(\frac{\partial \ln U}{\partial S}\right)^{-1}\right]\end{aligned} \tag{13.41}$$

with $a_U$ as in Eq. 13.18. This expression simplifies further if the numeraire fulfills Eq. 13.34:

$$\gamma_U(t) = \frac{1}{b}\left[a_U\left(\frac{\partial U}{\partial S}\right)^{-1} - \frac{d\ln Y}{dt}\left(\frac{\partial \ln U}{\partial S}\right)^{-1}\right].$$

From this definition and Eq. 13.40 it is now immediate that the following theorem holds.

**Theorem 7** *The market price of risk is identical for all tradable instruments in a complete, arbitrage-free market.*

Comparing the definition in Eq. 13.41 with Eq. 13.23 and Eq. 13.22 immediately gives

**Theorem 8** *The previsible process $\gamma(t)$ in the Girsanov Theorem effecting the drift transformation for the transition from the probability measure in the real world to the martingale measure is the market price of risk.*

Let us consider the process given by Eq. 13.18 representing a tradable instrument $U$ in the real world from this point of view. The drift $a_U$ in the real world can by Definition 13.41 be expressed in terms of the market price

[15] The motivation for this name will become clear further below, when we look at certain special cases.

of risk:

$$a_U = \gamma_U\, b \frac{\partial U}{\partial S} + U \frac{d \ln Y}{dt} \,.$$

Substituting this expression into Eq. 13.18 yields the process for a tradable financial instrument in the *real* world, expressed in terms of the market price of risk:

$$dU(S,t) = \left[ \gamma_U(t)\, b(s,t) \frac{\partial U(S,t)}{\partial S} + \frac{d \ln Y(t)}{dt} U(S,t) \right] dt + b(S,t) \frac{\partial U(S,t)}{\partial S} dW \,. \tag{13.42}$$

The *valuation* of the financial instrument is accomplished not in the real world but in that governed by the martingale measure. In the martingale measure, the instrument $U$ is a process satisfying Eq. 13.38. Comparing this process with Eq. 13.42 directly yields the following "recipe":

**Theorem 9** *Setting the market price of risk equal to zero in the expression for the stochastic process (more explicitly in the differential equation which is satisfied by this process) which governs the financial instrument in the real world immediately yields the stochastic process (i.e., the differential equation) which is to be applied in the valuation of this instrument.*

## 13.7 Tradable Underlyings

The equations in the previous section appear relatively complicated because we have assumed throughout that the underlying $S$ is not necessarily tradable. If the underlying is in fact tradable, it can be used in place of the instrument $U$ directly with the consequence that

$$U = S \Longrightarrow \frac{\partial U}{\partial S} = 1 \;,\quad \frac{\partial^2 U}{\partial S^2} = 0 = \frac{\partial U}{\partial t} = 0 \,. \tag{13.43}$$

The general equation 13.33 for this special case reduces to

$$dS = \frac{b\,\frac{n}{Y} - a\frac{n}{b}\frac{S}{Y} - \left[\frac{n^2}{Y^2} - \frac{m}{Y}\right] S}{1 - \frac{n}{b}\frac{S}{Y}} dt + b\,d\widetilde{W} \,.$$

The corresponding Eq. 13.37 for a more appropriate choice of numeraire instrument satisfying Eq. 13.34 further reduces to

$$dS(t) = S(t)\frac{d\ln Y(t)}{dt}dt + b(S,t)d\widetilde{W} \ , \tag{13.44}$$

which, of course, agrees with Eq. 13.38, this equation holding for every tradable instrument. The expectation of the yield (defined as the expected relative price change per time) of a tradable underlying in the martingale measure is thus (as for every tradable instrument) always equal to the yield of the numeraire instrument.

The market price of risk $\gamma_S(t)$ for a tradable underlying is obtained from Eq. 13.41 with 13.43 as

$$\gamma_S(t) = \frac{a + \left[\frac{n^2}{Y^2} - \frac{m}{Y}\right]S - b\frac{n}{Y}}{b - n\frac{S}{Y}} \ , \tag{13.45}$$

where $a = a(S,t)$ is the underlying drift in the real world (see Eq. 13.17). For a well-chosen numeraire of the form 13.34 we get:

$$\gamma_S(t) = \frac{a(S,t)}{b(S,t)} - \frac{S(t)}{b(S,t)}\frac{d\ln Y}{dt} \ .$$

With this, the process Eq. 13.17 in the real world (in consistence with Eq. 13.42) becomes

$$dS(t) = \left[b(S,t)\gamma_S(t) + \frac{d\ln Y(t)}{dt}S(t)\right]dt + b(S,t)\,dW \ . \tag{13.46}$$

Comparison with Eq. 13.44 again yields the "recipe" (which holds for all tradable instruments and hence for $S$ as well) that the market price of risk must simply be set equal to zero for valuation purposes.

Note that these considerations are valid for a still very general case: In our treatment, it has only been assumed that the underlying is tradable and that the numeraire instrument is of the form 13.34.

## 13.8 Applications in the Black-Scholes World

Special cases of the general material discussed above have been encountered in Sects. 7.1.3, 7.2.1 and 9.2.3 in various "disguises". The irrelevance of the real world drift became most apparent in Sect. 9.3. There, the requirement that the expectation of the underlying, computed using its stochastic process, must equal the expectation of the underlying resulting from the martingale property[16] led to Eq. 9.25, i.e., an explicit specification of the drift to be used in the valuation. The volatility $\sigma$, in contrast, was subject to no such condition and remained the same.

Equation 9.25 holds for a very special case, namely when *(a)* the underlying behaves as in Eq. 2.23 and *(b)* is tradable, *(c)* the numeraire is given by $Y(t) = B(t, T)$ and *(d)* interest rates and volatilities are constant; in short, in the Black-Scholes world. Let us therefore apply the general results of the above sections to this special situation as an example.

Firstly, a zero bond $B(t, T)$ is chosen as the numeraire instrument maturing at some arbitrary future time $T$. The process for $B$ in the real world is of the form 13.34. For continuous compounding, we have explicitly

$$dY \equiv dB(t,T) = \frac{dB(t,T)}{dt}dt = rB(t,T)dt \Longrightarrow \frac{d\ln Y(t)}{dt} = r\ . \tag{13.47}$$

The special process, Eq. 2.23 (or Eq. 2.24) corresponds to the general process, Eq. 13.17 with the parameters

$$\begin{aligned} a(S,t) &= \widetilde{\mu}S(t) = \left(\mu + \frac{\sigma^2}{2}\right)S(t) \\ b(S,t) &= \sigma S(t)\ . \end{aligned}$$

Since the underlying is tradable, Eq. 13.44 can be applied directly to obtain the underlying process in the martingale measure

$$dS(t) = rS(t)\,dt + \sigma S(t)d\widetilde{W}\ .$$

[16]There, instead of the general formulation "expectation with respect to the martingale measure", the expression "risk neutral expectation" was used.

This process and not 2.23 is to be used in the valuation of the financial instrument. Comparison of this process with Eq. 2.23 shows that the choice

$$\mu = r - \sigma^2/2 \quad \text{or equivalently} \quad \widetilde{\mu} = r \tag{13.48}$$

for the drift transforms the real world process directly to the process to be used for pricing, in agreement with Eq. 9.25 (dividend yield equal to zero). The market price of risk of the underlying is simply

$$\gamma_S(t) = \frac{a(S,t) - rS(t)}{b(S,t)} = \frac{\widetilde{\mu} - r}{\sigma} \tag{13.49}$$

in this special case and is exactly equal to zero for $\widetilde{\mu} = r$. In fact "setting the market price of risk equal to zero" is equivalent to "choosing the correct drift for pricing".

Equation 13.49 provides the motivation for the name "market price of risk". In the special case considered here the underlying drift in the real world is simply $a(S,t) = \widetilde{\mu}S(t)$. The expectation of the underlying-*yield* is thus $a(S,t)/S(t) = \widetilde{\mu}$. This implies that $\widetilde{\mu} - r$ represents the excess yield above the risk-free rate, which is expected from the underlying in the real world. If the volatility $\sigma$ is viewed as a measure of the risk of the underlying, then the market price of risk $\gamma_S$ (at least in the context of this special case) can be interpreted as the excess yield above the risk-free rate *per risk unit* $\sigma$ which the market expects from the underlying. This is, so to speak, the price (in the form of an excess yield above the risk-free rate) which the market demands for the risk of investing in the underlying. In the real world, the market is by no means risk neutral, but rather expects higher yields for higher risks; the market price of risk illustrates this clearly.

Note that the market price of risk in Eq. 13.49 is identical to the so called *Sharpe Ratio* [138, 139] heavily used in asset management and portfolio optimization since more than 50 years. Very generally, the Sharpe Ratio is defined as the expected excess return (above the risk free rate) of an investment divided by the investment risk (measured as its volatility).

$$\text{Sharpe Ratio} \equiv \frac{R - r_f}{\sigma}$$

Our case above corresponds to an investment in a single risky asset, namely in the risk factor $S$. As we have shown in Eq. 2.32, the drift $\widetilde{\mu}$ appearing in Eq. 13.49 is the expected return for linear compounding. Since in asset management returns are usually defined as relative (as opposed to logarithmic) price changes, linear compounding is indeed applicable (see Eq. 2.31). Thus, $\widetilde{\mu}$ in Eq. 13.49 exactly corresponds to the expected return $R$ used in portfolio management for an investment in $S$. And therefore the Sharpe Ratio and the market price of risk are the same thing, see also [49].

To conclude this section we will now show that we can actually use all of this information about martingales and drifts to really calculate something. We consider below only payoff profiles of path-independent instruments. These are payoff profiles which depend solely on the value of the underlying at maturity $T$ and not on the path taken by the process $S$ between $t$ and $T$. For such processes

$$V(S,T) = V(S(T),T) \ .$$

holds. Therefore we only need the distribution of $S$ at time $T$ (and not the distribution of all paths of $S$ between $t$ and $T$). Choosing the zero bond as the numeraire instrument, $Y(t) = B(t,T)$, Eq. 13.31 for the price $V$ of a financial instrument becomes

$$V(S,t) = Y(t)\mathrm{E}_t^Y\left[\frac{V(S,T)}{Y(T)}\right] = B(t,T)\mathrm{E}_t^Y\left[V(S(T),T)\right] \ .$$

since $B(T,T) = 1$. We again model the underlying with the simple process in Eq. 2.23. The associated underlying process over a *finite* time interval of length $\delta t = T - t$, i.e., the solution of the stochastic differential equation 2.23 for $S$, has already been given in Eq. 2.28 for an arbitrary drift $\mu$, and thus for an arbitrary probability measure, namely

$$\begin{aligned} S(T) &= S(t)\exp\left[\mu(T-t) + \sigma W_{T-t}\right] \quad \text{mit} \ \ W_{T-t} \sim \mathrm{N}(0, T-t) \\ &= S(t)\exp\left[x\right] \quad \text{with} \ \ x \sim \mathrm{N}\left(\mu(T-t), \sigma^2(T-t)\right) \ . \end{aligned} \tag{13.50}$$

The distribution of $S(T)$ is therefore $S(t)$ multiplied by the exponential of the normal distribution with expectation $\mu(T-t)$ and variance $\sigma^2(T-t)$. Using this, the expectation of the function

$$V(S(T),T) = V(S(t)e^x, T) =: g(x)$$

can be computed explicitly:

$$\mathrm{E}_t\left[V(S(T),T)\right] =: \mathrm{E}_t\left[g(x)\right] = \int_{-\infty}^{\infty} g(x)p(x)dx$$

with the probability density of the normal distribution (see Eq. A.46)

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left[-\frac{(x-\mu(T-t))^2}{2\sigma^2(T-t)}\right] .$$

Thus

$$\mathrm{E}_t\left[V(S(T),T)\right] = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \int_{-\infty}^{\infty} V\left(S(t)e^x, T\right) \exp\left[-\frac{(x-\mu(T-t))^2}{2\sigma^2(T-t)}\right] dx .$$

This is the expectation of the payoff profile with respect to a probability measure associated with $\mu$, for instance with respect to the probability measure in the real world if $\mu$ represents the drift in the real world. To allow the use of this expectation for the *valuation* of the instrument $V$, it must be computed with respect to the (in a complete market, unique) *martingale* measure. This is accomplished through the choice of drift in accordance with Eq. 13.48. The price of $V$ then becomes

$$\begin{aligned} V(S,t) &= B(t,T)\mathrm{E}_t^Y\left[V(S(T),T)\right] \\ &= \frac{B(t,T)}{\sqrt{2\pi\sigma^2(T-t)}} \int_{-\infty}^{\infty} V\left(S(t)e^x, T\right) \exp\left[-\frac{\left[x-\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right]^2}{2\sigma^2(T-t)}\right] dx . \end{aligned} \quad (13.51)$$

This equation holds in complete generality for every financial instrument with a non-path dependent payoff profile $V(S(T),T)$ on an underlying $S$ of the form specified in Eq. 2.23.

To be more specific, we now use the payoff profile of a plain vanilla call as an example:

$$V(S(T),T) = \max\{S(T)-K, 0\} = \max\left\{e^x S(t) - K, 0\right\} .$$

with $x$ as defined in Eq. 13.50. This payoff profile is only non-zero when $e^x S(t) \geq K$, or equivalently when $x \geq \ln(K/S(t))$. Equation 13.51 for the price of the call is then

$$V(S,t) = \frac{B(t,T)}{\sqrt{2\pi\sigma^2(T-t)}} \int_{\ln(K/S(t))}^{\infty} \left[e^x S(t) - K\right] \exp\left[ -\frac{\left(x - \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)^2}{2\sigma^2(T-t)} \right] dx \; .$$

This (with $\mu = r - \sigma^2/2$) is in complete agreement with Eq. 8.4, for example. In connection with Eq. 8.4 it has been demonstrated how this integral can be computed explicitly. The result is the famous Black-Scholes equation 8.6.