12

Hedging

The replication of derivatives with a portfolio consisting of underlyings and a bank account as, for example, in Eq. 9.3, can be used to *hedge* the derivative's risk resulting from the stochastic movement of its underlying (or conversely a derivative could be used to hedge such a portfolio). This is accomplished by going short in the portfolio and long in the derivative or vice versa. This idea can be extended to hedging against influences other than the underlying price, for example, changes in the volatility, interest rate, etc. Such concepts of safeguarding against a risk factor have already made their appearance in arbitrage arguments in previous chapters and will be presented in their general form in this chapter. In addition to the fundamental Assumptions 1, 2, 3, 4 and 5 from Chap. 4, continuous trading will also be assumed below, i.e., Assumption 6. We will allow the underlying to perform a general Ito process[1] of the Form 2.19 and assume that it pays a dividend yield q.

12.1 Hedging Derivatives with Spot Transactions

Consider first the change in the value V of a derivative resulting from the change in the price S of its underlying. V, as a function of the stochastic process S, is also a stochastic process. From Ito's lemma in the form of Eq. 2.22

[1]Thus, we are a more general here than Assumption 7 from Chap. 4.

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this process is explicitly

$$dV(S,t) = \left[\frac{\partial V}{\partial t} + \frac{1}{2}b^2(S,t)\frac{\partial^2 V}{\partial S^2}\right]dt + \frac{\partial V}{\partial S}dS(t)\ . \tag{12.1}$$

In a similar manner as in Sect. 7.1.1, we will now construct a portfolio which replicates (hedges) the derivative in Eq. 12.1. This portfolio will consist of a certain number Δ of underlyings and an amount g of money borrowed from or invested in the capital market. This portfolio is constructed to have the same value as the derivative,

$$V = \Delta S + g\ . \tag{12.2}$$

To ensure that the derivative is perfectly hedged for at least a short time span dt, it is required that the change in its value over dt to be the same as that of the derivative. The change in value of the derivative is given by Eq. 12.1. The value of the portfolio consisting of money and the underlying changes as already described in Eq. 9.3, for example. For small time intervals dt, B and $1/B$ can be approximates by $(1 - r\,dt)$ and $(1 + r\,dt)$, respectively, for all compounding conventions. The same approximation holds for B_q. Denoting the change in S by dS, the change in the value of the portfolio can be expressed by

$$\begin{aligned} d(\Delta\, S + g) &= \underbrace{\Delta(S + dS)(1 + q\,dt) + g(1 + r\,dt)}_{\text{new value}} - \underbrace{(\Delta\, S + g)}_{\text{old value}} \\ &= \Delta\, dS + (\Delta\, q\, S + r\, g)dt + \Delta\, q\, dS\, dt \\ &= \Delta\, dS + (\Delta\, q\, S + r\, g)dt + \ \cdots \end{aligned} \tag{12.3}$$

where in the last line only the first order terms in dS and dt are considered. To ensure that this change corresponds exactly to a change in the value of the derivative, the coefficients of dt and dS must be the same as those appearing in Eq. 12.1. Equating the coefficients of dS yields the well known result that the number of underlyings required for the hedge, called the *hedge ratio*, must be equal to the sensitivity of the derivative with respect to its underlying. Combined with Eq. 12.2 we can determine the amount of money in the bank account required for the hedge as well:

$$\Delta(t) = \frac{\partial V}{\partial S(t)} \ \Rightarrow\ g(t) = V - S(t)\frac{\partial V}{\partial S(t)}\ . \tag{12.4}$$

Thus, with information on the value V_s and its first partial derivative with respect to the underlying, it is possible to construct a portfolio hedging the derivative *perfectly* over the next time step. The partial derivatives of the price function are often referred to as the *sensitivities* (see Sect. 12.4).

We have yet to exploit the second condition, which is the equality of the coefficients of dt in Eqs. 12.1 and 12.3. This yields

$$\Delta S(t)q + rg = \frac{\partial V}{\partial t} + \frac{1}{2}b^2(S,t)\frac{\partial^2 V}{\partial S^2} \, .$$

Inserting Eq. 12.4 for Δ_S and g, we recognize the famous differential equation of Black and Scholes for derivatives on the spot price S on an underlying:

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{1}{2}b^2(S,t)\frac{\partial^2 V}{\partial S^2} = r\,V \, . \tag{12.5}$$

If we write (without loss of generality) the parameters a and b as in Eq. 7.1 we immediately see that Eq. 12.5 is exactly equal to Eq. 7.8.

As all other derivation of the Black-Scholes PDE, this most simple derivation of the *Black-Scholes differential equation*, does not specify the derivative under consideration in any way. No information about the payoff profile or any other property characterizing the derivative is necessary. Only the properties of the *underlying* are used, these being that the underlying earns a dividend yield q and that its price is governed by a stochastic process of the form 2.19. Thus, there is only *one* equation for the construction of the hedging portfolio and *one* differential equation for the price of *all* derivatives on this underlying. The different derivative instruments can be distinguished from one another solely on the basis of their respective initial and/or boundary conditions (for example, the payoff profile of an option at maturity or the fact that a forward contract is worth nothing at the time when the contract is entered into).

In effect, the value of the option is equivalent to the total cost of the hedge, i.e. the replication of the pay off profile by the carefully chosen portfolio (the replicating portfolio). The whole theory and practice of arbitrage free valuation of derivatives is based on this fundamental assumption that the value of an option is equivalent to the total cost to hedge it.

12.1.1 Hedging of Forwards and Futures

Forward and futures contracts belong to the most simple derivatives. Therefore, they will serve as an example how to apply Eqs. 12.4 and 12.5 to hedge derivatives with spot trades. We adopt here the commonly used convention of continuous compounding.

Consider first a forward contract. The value of a forward with delivery price K, assuming continuous compounding, is given by Eq. 6.5 as

$$f(t,T,K) = [S(t,T) - K]\,B(t,T) = e^{-q(T-t)}S(t) - e^{-r(T-t)}K\ .$$

The partial derivatives of this function appearing in Eqs. 12.4 and 12.5 are

$$\frac{\partial f}{\partial t} = q\,e^{-q(T-t)}S - re^{-r(T-t)}K\ , \quad \frac{\partial f}{\partial S} = e^{-q(T-t)}\ , \quad \frac{\partial^2 f}{\partial S^2} = 0\ .$$

Thus, Eq. 12.4 yields

$$\Delta(t) = e^{-q(T-t)}\ , \quad g(t) = -e^{-r(T-t)}K\ .$$

In order to hedge a (short) forward, $e^{-q(T-t)}$ underlyings must be purchased and an amount $e^{-r(T-t)}K$ of cash must be raised. Substituting the value of the forward and its derivative into Eq. 12.5 shows that the Black-Scholes differential equation is satisfied:

$$qe^{-q(T-t)}S - re^{-r(T-t)}K + (r-q)Se^{-q(T-t)} = re^{-q(T-t)}S - re^{-r(T-t)}K\ .$$

Now consider a futures contract. From a valuation perspective, the essential difference compared to a forward contract is the obligatory maintenance of a margin account. As a consequence, every price change of the futures contract is reflected by an equivalent change of the margin account balance. The margin account is similar to a bank account. The buyer or seller of the futures contract receives interest on the margin account, but has no direct access to the money in the account. At the trade day of the futures contract, the buyer/seller must transfer an initial margin on the account. If the futures price raises or falls later, the margin account balance raises or falls by the same amount. If the margin account balance falls below a certain level, fresh money must be transferred to the account, otherwise the whole position will be closed out. On the other hand, if the margin account balance raise beyond a certain level, money could be transferred back if the balance exceeds certain trigger levels. This is called variation margin. Neglecting default risk, it can be assumed that the money for initial and variation margin can be borrowed for the risk-free interest rate. Also, the margin account itself is assumed to pay interest based on the risk-free interest.[2] We will denote the total cash amount (which can be negative or positive) as $g_F(t)$.

Let's consider the evolution of the total value of the futures position over time. This development is completely reflected in the development of the function $g_F(t)$. Contrary to a forward contract, a futures contract does not define a fixed strike price to be paid at maturity in exchange of the underlying. Instead, the buyer of the futures contract will have to pay at maturity the then current spot price, while the difference of forward and future spot price at trade date is at maturity identical to the balance of the variation margin (plus accumulated interest). We need the following derivatives of $g_F(t)$:

$$\frac{\partial g_F}{\partial t} = -(r-q)e^{(r-q)(T-t)}S + rg_F \ , \quad \frac{\partial g_F}{\partial S} = e^{(r-q)(T-t)} \ , \quad \frac{\partial^2 g_F}{\partial S^2} = 0 \ .$$

Inserting these in Eq. 12.4 yields

$$\Delta(t) = e^{(r-q)(T-t)} \ , \quad g(t) = g_F(t) \ .$$

Therefore, hedging a (short) futures contract requires to buy $e^{(r-q)(T-t)}$ underlyings and to borrow an amount of $-g(t) = g_F(T)$ currency units. Inserting the value of the Portfolio $g_F(t)$ into Eq. 12.5 shows that $g_F(t)$ fulfills the Black-Scholes differential equation:

$$-(r-q)e^{(r-q)(T-t)}S + rg_F + (r-q)e^{(r-q)(T-t)}S = rg_F \ .$$

The mindful reader may wonder, in which way $g_F(t)$ differs from the fair value $F(S,t,T,K)$ of the futures contract, which itself does not fulfill the Black-Scholes equation. According to Eq. 6.6 the fair value of a futures contract is

$$F(S,t,T,K) = e^{(r-q)(T-t)}S(t) - K \ ,$$

[2]Reality, as often, is more complex. E.g. the actual margin interest rates may differ from the risk-free interest rate or maybe floored at zero, while the risk-free interest rate might be allowed to become negative.

where K is the futures price at trade date. In contrast to $g_F(t)$ the fair value does not contain the accumulated interest, since they have already been realized.

12.2 Hedging Derivatives with Forward Contracts

The portfolio for the synthetic derivative is now to be constructed using *forward* contracts instead of spot transactions. We cannot merely replace the spot price with the forward price in Eq. 12.3 since the forward price itself is not tradable. Forward and futures contracts on the other hand are tradable.

12.2.1 Hedging with Forwards

Since futures satisfy the Black Scholes differential equation only upon conclusion of the contract, we will avoid this difficulty at this point and begin by using forwards to construct the synthetic derivative. The hedging portfolio is then

$$\Delta_f(t) f(t, T', K) + g(t) = \Delta_f(t) \left[S(t, T') - K\right] B(t, T') + g(t) \text{ with } T' \geq T > t\ ,$$

where the subscript f indicates that the hedge is accomplished with forwards. T denotes the maturity of the derivative and T' the maturity of the forward contract. Note that the condition $T' \geq T$ is not strictly necessary for our considerations, but it is convenient since it ensures that the hedge does not "vanish" before the maturity T of the derivative. If $T' < T$ it would be required to roll over the hedge into another forward contract (with a later maturity) at time T' or earlier.

In the time span dt this portfolio changes its value as a result of changes in the forward price and the discount factor as follows:

$$\begin{aligned} d(\Delta_f f + g) &= \underbrace{\Delta_f \left[S(t, T') + dS(t, T') - K\right] B(t, T')(1 + r dt) + g(t)(1 + r dt)}_{\text{new value}} \\ &\quad - \underbrace{\Delta_f \left[S(t, T') - K\right] B(t, T') + g(t)}_{\text{old value}} \\ &= \Delta_f B(t, T') dS(t, T') + \Delta_f \left[S(t, T') - K\right] B(t, T') r\, dt + g r\, dt + \cdots \\ &= \Delta_f B(t, T') \frac{\partial S(t, T')}{\partial S(t)} dS(t) + \Delta_f(t) f(t, T', K) r\, dt + g\, r\, dt + \cdots\ , \end{aligned} \tag{12.6}$$

where again we consider only changes which are of linear order in dt or dS. In the last equation, the change in the forward price $dS(t, T')$ was rewritten in terms of a change in the spot price to facilitate the comparison of the coefficients with the associated coefficients in Eq. 12.1 for the change in the price of the derivative. Equating the coefficients of dS in the above mentioned expressions now yields:

$$\Delta_f(t) = \frac{1}{B(t, T')} \frac{\partial V/\partial S(t)}{\partial S(t, T')/\partial S(t)} = \frac{1}{B(t, T')} \frac{\partial V}{\partial S(t, T')} .$$

Using Eq. 6.1 for the dividend yield, we can calculate the derivative of the forward price with respect to the spot price and thus obtain the following expression for the hedge ratio:

$$\Delta_f(t) = \frac{1}{B_q(t, T)} \frac{\partial V}{\partial S(t)} = \frac{1}{B(t, T')} \frac{\partial V}{\partial S(t, T')} . \tag{12.7}$$

In other words, the number of forwards required to hedge the derivative is equal to the sensitivity of the derivative with respect to the spot price compounded at the dividend yield up to maturity of the derivative, or equivalently, is equal to the sensitivity with respect to the forward price compounded at the risk-free rate up to maturity of the hedging instrument (the forward).

Since by construction, the value of the replicating portfolio is equal to the value of the derivative we can determine the amount of cash needed for the hedge:

$$g(t) = V - f(S, t, T', K)\, \Delta_f(t) . \tag{12.8}$$

We have thus completely determined the hedging portfolio using solely information derived from the price of the derivative and its sensitivities.

Before we establish a mathematical expression obtained from comparison of the coefficients of dt in Eqs. 12.1 and 12.6, we change the coordinates in Eq. 12.1 to transform the second derivative of the price with respect to *spot* price $S(t)$ into a derivative with respect to the *forward* price $S(t, T')$. This is accomplished using 6.1.

$$\begin{aligned}
\frac{\partial}{\partial S(t)} &= \frac{\partial S(t, T')}{\partial S(t)} \frac{\partial}{\partial S(t, T')} = \frac{B_q(t, T')}{B(t, T')} \frac{\partial}{\partial\, S(t, T')} \\
\frac{\partial^2}{\partial S^2(t)} &= \frac{\partial}{\partial S(t)} \frac{B_q(t, T')}{B(t, T')} \frac{\partial}{\partial S(t, T')} = \frac{B_q^2(t, T')}{B^2(t, T')} \frac{\partial^2}{\partial S^2(t, T')} .
\end{aligned} \tag{12.9}$$

Equating now the coefficient of dt and using Eq. 12.8 for g yields

$$\frac{\partial V}{\partial t} + \frac{1}{2} b^2(S,t) \frac{B_q^2(t,T')}{B^2(t,T')} \frac{\partial^2 V}{\partial S^2(t,T')} = rV \ .$$

This is the Black-Scholes differential equation for derivatives on the forward price of an underlying.

To put this in a more familiar form we rewrite (without loss of generality) the parameter b as in Eq. 7.1 and use Eq. 6.1 to introduce the forward price into the parameter b, i.e.,

$$b(S,t) =: \sigma(S,t)S(t) = \sigma(S,t) \frac{B(t,T')}{B_q(t,T')} S(t,T') \ . \tag{12.10}$$

With this substitution we arrive at the well-known form of the Black-Scholes differential equation for derivative on the forward price of an underlying:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2(t,T') \frac{\partial^2 V}{\partial S^2(t,T')} = r\,V \ . \tag{12.11}$$

As in Sect. 12.1, only *one* equation is needed for the construction of the hedging portfolio with forward contracts, and there is only *one* differential equation for *all* derivatives on the forward price. Different derivatives can be distinguished from one another only through their respective initial and/or boundary conditions.

Note the absence of a first derivative term with respect to the forward price. Equation 12.5 (which is the corresponding Black-Scholes PDE for derivatives on the *spot* price of an underlying) would look quite similar to Eq. 12.11 for an underlying which earns a dividend yield q being exactly equal to the risk free rate r. In fact, both equations would look identical if this underlying would have $S(t,T')$ as its *spot* price. Herein lies the deeper explanation for the "recipe" for pricing options on futures as given in Eq. 8.8: a derivative on the forward price can be priced as a derivative on the spot price of an (artificial) underlying whose spot price is equal to $S(t,T')$ and whose dividend yield equals the risk-free rate.

12.2.2 Hedging with Futures

If futures are used to construct a synthetic derivative, we have to deal with the additional complexity that the margin account balance depends on when and how many futures contracts have been bought or sold. Since we assume that the margin account is compounded with the same interest rate as the risk-free cash amount borrowed or lend $g(t)$, we consider the margin account as part of the total cash balance or bank account. Then, we need to consider the futures contract's price (which equals the forward price) changes, only. As hedging instrument, we use a futures contract with fixed maturity T.[3] Since the futures price equals the forward price $S(t, T')$, we get

$$\Delta_F(t)S(t, T') + g(t) \text{ with } T' \geq T > t .$$

Changes in the futures contract price and the bank account in the time interval dt cause the value of the portfolio to change as follows

$$\begin{aligned} d(\Delta_F S(t, T') + g) &= \underbrace{\Delta_F \left[S(t, T') + dS(t, T')\right] + g\,(1 + r dt)}_{\text{new value}} - \underbrace{\Delta_F \left[S(t, T')\right] + g}_{\text{old value}} \\ &= \Delta_F\, dS(t, T') + gr\, dt + \cdots \\ &= \Delta_F \frac{\partial S(t, T')}{\partial S(t)} dS(t) + gr\, dt + \cdots , \end{aligned} \tag{12.12}$$

where again we consider only the parameter changes of linear order in dt or dS, neglecting all higher order terms. In the last equation, the change in the futures price has been replaced by the corresponding expression with respect to a change in the spot price, facilitating a direct comparison with the coefficients in Eq. 12.1 for the change in the price of the derivative. Setting the coefficients of dS in both expressions equal yields

$$\Delta_F(t) = \frac{\partial V/\partial S(t)}{\partial S(t, T')/\partial S(t)} = \frac{\partial V}{\partial S(t, T')} .$$

[3] Here again the condition $T' \geq T$ is not strictly necessary but convenient since it ensures that the hedge does not "vanish" before maturity T of the derivative. Again, $T' < T$ would require to roll over the hedge into another futures contract (with a later maturity) at time T' or earlier.

Equation 6.1 for the forward price of an underlying earning a dividend yield q gives the partial derivative of V_S with respect to the spot price.

$$\Delta_F(t) = \frac{B(t,T')}{B_q(t,T')}\frac{\partial V}{\partial S(t)} = \frac{\partial V}{\partial S(t,T')} \; . \tag{12.13}$$

In other words, the number of futures required to hedge a derivative is equal to the sensitivity of the derivative with respect to the *spot* price compounded at the rate obtained by using the risk-free rate reduced by the dividend yield. Or equivalently, the number of futures needed equals the sensitivity with respect to the *forward* price.

By definition, the value of the hedging portfolio must be equal to the value of the derivative. It follows that the amount of money needed for the hedge is given by

$$g(t) = V - F_S(t,T',K)\Delta_F(t) \; . \tag{12.14}$$

We have thus completely determined the hedging portfolio using solely the information derived from the value of the derivative instrument and its sensitivities.

12.3 Hedge-Ratios for Arbitrary Combinations of Financial Instruments

An analogous approach to those described above can be taken to derive a formula for the number of an *arbitrary* hedge instrument required to replicate an *arbitrary* derivative. The number of hedging instruments is always equal to the sensitivity of the derivative with respect to this hedging instrument. Intuitively, if the value h of a hedge instrument changes by the amount dh, then the value V of the derivative changes by the amount dV. The quotient dV/dh is precisely the number of hedging instruments needed to compensate for the change dV in the derivative. For very small changes, this quotient approaches the differential quotient. Any instrument can assume the role of either the derivative or the hedging instrument. Since the value obtained by differentiating a derivative with respect to an arbitrary hedging instrument cannot, in general, be calculated directly, a common reference variable, usually the spot price $S(t)$, is introduced with respect to which both the partial derivative of the value as well as the partial derivative of the hedging instrument

Table 12.1 Sensitivities for options and forward contracts with respect to the spot price $S(t)$. x and x' are defined as in Eqs. 8.7 and 8.8

Underlying (Spot)	1	
Futures	$B_q(t,T)B(t,T)^{-1}$	
Forward	$B_q(t,T)$	
	Call	Put
Option on spot price	$B_q(t,T)\,\mathrm{N}(x)$	$-B_q(t,T)\,\mathrm{N}(-x)$
Option on futures	$B_q(t,T')\frac{B(t,T)}{B(t,T')}\mathrm{N}(x')$	$-B_q(t,T')\frac{B(t,T)}{B(t,T')}\mathrm{N}(-x')$
Option on forward	$B_q(t,T')\mathrm{N}(x')$	$-B_q(t,T')\mathrm{N}(-x')$

is known. The equation for this general *hedge ratio* is therefore

$$\Delta_{\substack{\text{Derivate } V \text{ with}\\ \text{Hedging Instrument } h}} = \frac{\partial V}{\partial h} = \frac{\partial V}{\partial S}\left[\frac{\partial h}{\partial S}\right]^{-1} . \qquad (12.15)$$

Consequently, the sensitivities of all financial instruments with respect to the spot price must be available if the hedge ratio is to be calculated for an *arbitrary* combination of these instruments. In general, though, it is not recommended to use hedging instruments which introduce additional risk factors the original trade is not sensitive to (e.g. additional foreign exchange risk). The sensitivities already dealt with in the last section, i.e., the sensitivities of the option, forward and spot transactions with respect to the spot price are listed in Table 12.1 (for the Black-Scholes world). For forward contracts, they follow from Eqs. 6.1, 6.5 and 6.6, which were derived using arbitrage arguments. For options, the sensitivities follow directly from the Black-Scholes equations 8.7, 8.8 and 8.9.

Readers wishing to reproduce these results themselves are advised to use the following property[4] of $\mathrm{N}'(x-\sigma\sqrt{T-t})$:

$$\mathrm{N}'(x-\sigma\sqrt{T-t}) = \frac{B_q(t,T)\,S(t)}{B(t,T)\,K}\mathrm{N}'(x) . \qquad (12.16)$$

[4]To prove Eq. 12.16 we first write

$$\mathrm{N}'(x-\sigma\sqrt{T-t}) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\sigma\sqrt{T-t})^2} = \underbrace{\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}}_{\mathrm{N}'(x)}e^{x\sigma\sqrt{T-t}-\frac{1}{2}\sigma^2(T-t)}$$

The relation is now obtained by substituting the definition of x, Eq. 8.5, into the second exponential term.

Here, N′ denotes the derivative of the cumulative standard normal distribution with respect to its argument:

$$\mathrm{N}'(y) \equiv \frac{d}{dy}\mathrm{N}(y) = \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}$$

The entries in Table 12.1 together with Eq. 12.1 provide all the information needed to determine the hedge ratio for every conceivable combination of these financial instruments in the Black-Scholes world. We now present three examples to illustrate this.

a) Hedging a put on the spot price with futures:

$$\begin{aligned}\Delta_{\substack{\text{Put on spot}\\ \text{with futures}}} &= \frac{\partial V}{\partial S}\left[\frac{\partial h}{\partial S}\right]^{-1} = -B_q(t,T)\,\mathrm{N}(-x)\left[\frac{B_q(t,T')}{B(t,T')}\right]^{-1} \\ &= -B(t,T')\frac{B_q(t,T)}{B_q(t,T')}\mathrm{N}(-x)\;.\end{aligned}$$

A long put on the spot can thus be *replicated* by a *short* position consisting of

$$B(t,T')\mathrm{N}(-x)/B_q(T,T'\,|\,t)$$

futures. Or interpreted as a hedge, a long put can be *hedged* by a *long* position in this number of futures.

b) Hedging a put on a future with forward contracts:

$$\begin{aligned}\Delta_{\substack{\text{Put on futures}\\ \text{with forward}}} &= \frac{\partial V}{\partial S}\left[\frac{\partial h}{\partial S}\right]^{-1} = -B_q(t,T')\frac{B(t,T)}{B(t,T')}\mathrm{N}(-x')\left[B_q(t,T')\right]^{-1} \\ &= -\frac{B(t,T)}{B(t,T')}\mathrm{N}(-x')\;.\end{aligned}$$

A long put on the future can be *replicated* with a *short* position of

$$\mathrm{N}(-x')/B(T,T'\,|\,t)$$

forwards. Or interpreted as a hedge, a long put can be *hedged* by a *long* position in this number of forwards.

c) Hedging a call on the spot with puts on a future:

$$\Delta_{\substack{\text{Call on spot with}\\\text{put on futures}}} = \frac{\partial V}{\partial S}\left[\frac{\partial h}{\partial S}\right]^{-1} = B_q(t,T)\,\mathrm{N}(x)\left[-B_q(t,T')\frac{B(t,T)}{B(t,T')}\mathrm{N}(-x')\right]^{-1}$$

$$= -\frac{B(t,T')}{B(t,T)}\frac{B_q(t,T)}{B_q(T,T')}\frac{\mathrm{N}(x)}{\mathrm{N}(-x')}\,.$$

A long call on the spot can thus be *replicated* by a *short* position in

$$\frac{B(T,T'\,|\,t)\mathrm{N}(x)}{B_q(T,T'|\,t)\mathrm{N}(-x')}$$

puts on the future. Or interpreted as a hedge, the long call can be *hedged* by a *long* position in this number of puts on the future.

12.4 "Greek" Risk Management with Sensitivities

12.4.1 Sensitivities and a Portfolio's Change in Value

The *sensitivity* of a financial instrument with respect to a parameter is defined as a variable that, when multiplied by a (small) parameter change, yields the change in the value of the instrument resulting from this parameter change.

Price change = sensitivity × parameter change

The sensitivity is thus the derivative of the price with respect to the parameter. Since options depend on several such parameters such as the price of the underlying, the time to maturity, the volatility, the dividend yield of the underlying and the risk-free market rate, the sensitivities listed in Table 12.2 are of particular interest.

The first of these sensitivities, namely Δ, has already been used in Sect. 12.1 to construct a replicating portfolio. In Eq. 12.15 the hedge ratio Δ was defined in greater generality as the sensitivity of the derivative with respect to the price change of the hedging instrument. In the special case where the hedging instrument is the underlying itself, this general hedging ratio and the delta listed in Table 12.2 agree.

Table 12.2 Definitions of the "Greeks"

Symbol	Name	Definition	Interpretation
Δ	Delta	$\frac{\partial V}{\partial S}$	Price change resulting from change of underlying spot price
Γ	Gamma	$\frac{\partial^2 V}{\partial S^2}$	Change of delta resulting from change of underlying spot price
Ω	Omega	$\frac{S}{V}\frac{\partial V}{\partial S}$	Relative price change (in %) resulting from relative change (in %) of underlying spot price
Ψ	Vega	$\frac{\partial V}{\partial \sigma}$	Price change resulting from change of volatility of underlying
Θ	Theta	$\frac{\partial V}{\partial t}$	Price change resulting from change of time to maturity
ρ	Rho	$\frac{\partial V}{\partial r}$	Price change resulting from change of the risk-free rate
ρ_q	Rho$_q$	$\frac{\partial V}{\partial q}$	Price change resulting from change of dividend yield of underlying

Any method used to calculate the price of an option can be used to calculate the sensitivities, even if the explicit expressions for the sensitivities cannot be obtained directly. The price model must simply be applied twice: the first time to calculate the price with the current parameters and a second time for a valuation with one of the parameters slightly changed. The difference between these two option values divided by the parameter change yields an approximation for the sensitivity of the option with respect to that parameter. Approximating the second derivative (as for gamma, for example) requires the price to be calculated with respect to three different parameter values. In summary: as soon as a pricing method is available we can calculate sensitivities by approximating the differential quotients in Table 12.2 with difference quotients as follows:

$$
\begin{aligned}
\text{Delta} &\approx \frac{V(S+\delta S)-V(S)}{\delta S}, \quad \text{Gamma} \approx \frac{V(S+\delta S)-2V(S)+V(S-\delta S)}{\delta S^2} \\
\text{Vega} &\approx \frac{V(\sigma+\delta\sigma)-V(\sigma)}{\delta\sigma}, \quad \text{Theta} \approx \frac{V(t+\delta t)-V(t)}{\delta t} \\
\text{Rho}_r &\approx \frac{V(r+\delta r)-V(r)}{\delta r}, \quad \text{Rho}_q \approx \frac{V(q+\delta q)-V(q)}{\delta q}
\end{aligned}
\tag{12.17}
$$

The risk of an instrument (or a portfolio) can be measured by its reaction to changes in the parameters influencing its price. In other words, the more sensitive it is to parameter changes, the "riskier" the instrument is. For this reason, the sensitivities are also known as *risk ratios*. A commonly used method of risk management is to control these sensitivities, i.e., to set limits on or targets for their values, for instance. The possibilities and limitations of this kind of risk management become clear, when we consider the Taylor series representation for small parameter changes: The value V depends on all of the above listed parameters: $V = V(t, S, \sigma, q, r)$. A change in value is thus

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial \sigma}d\sigma + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial q}dq + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 + \cdots \\ &= \Theta\, dt + \Delta\, dS + \Psi\, d\sigma + \rho\, dr + \rho_q\, dq + \frac{1}{2}\Gamma\, dS^2 + \cdots \end{aligned}$$

The coefficients of all linear parameter changes are given by sensitivities. A linear approximation holds only for small parameter changes. This is especially true for highly non-linear instruments. Because of the particularly strong influence of the underlying price, the effect of its quadratic change has been included in the above expression resulting in the appearance of the sensitivity Γ.

All risk variables in Table 12.2 (with the exception of the variable Ω) have the important property that the sensitivity of a portfolio with respect to a risk factor is equal to the sum of the sensitivities of its component instruments to this risk factor, i.e., the sensitivities are *linear*. For a portfolio consisting of n instruments of type A and m of type B we can therefore write

$$\Delta(n\,A + m\,B) = n\,\Delta(A) + m\,\Delta(B) \quad \text{likewise for } \Gamma, \Theta, \Psi, \rho, \rho_q \tag{12.18}$$

This property makes it possible to make a portfolio *delta-neutral*, for example: given n, $\Delta(A)$ and $\Delta(B)$, it is possible to make the portfolio's delta, $\Delta(nA + mB)$, equal to zero, by choosing $m = -n\Delta(A)/\Delta(B)$. This is called a *delta hedge*. The portfolio is then (in linear approximation) insensitive to small changes in the price of the underlying S. In the special case that the instrument B is the underlying itself we have $\Delta(B) = 1$. This is reflected in the result given in Sect. 12.1 that the number of underlyings needed to hedge a particular instrument is obtained by taking the derivative of the value of the instrument with respect to the underlying. We can now observe that the hedge methods introduced in Sect. 12.1 were delta hedges and represent only in linear approximation a safeguard against the risk resulting from a change in the underlying. This is no limitation under the (idealized) assumption of continuous trading with zero transaction cost, though.

Likewise, a portfolio can be constructed which is gamma, theta, vega or rho neutral. In general, for Plain Vanilla options, Delta and Vega are hedged at least.[5] Even *several* sensitivities of the portfolio can be neutralized *simultaneously* but more than one hedging instrument is then required to do so. In general, we need at least as many *different* hedging instruments as the number of sensitivities to be neutralized. For example, if a portfolio consisting of n instruments of type A is to be delta and gamma hedged, we would require two hedging instruments B_1 and B_2. The condition that the delta and the gamma of the portfolio be equal to zero means that

$$\Delta(n\,A + m_1\,B_1 + m_2\,B_2) = n\,\Delta(A) + m_1\,\Delta(B_1) + m_2\,\Delta(B_2) = 0$$
$$\Gamma(n\,A + m_1\,B_1 + m_2\,B_2) = n\,\Gamma(A) + m_1\,\Gamma(B_1) + m_2\,\Gamma(B_2) = 0$$

Solving this system of equations yields the number of hedging instruments m_1 and m_2 for the delta and gamma hedged portfolio:

$$m_1 = -n\frac{\Gamma(B_2)\Delta(A) - \Delta(B_2)\Gamma(A)}{\Gamma(B_2)\Delta(B_1) - \Delta(B_2)\Gamma(B_1)}\,,\; m_2 = -n\frac{\Gamma(B_1)\Delta(A) - \Delta(B_1)\Gamma(A)}{\Gamma(B_1)\Delta(B_2) - \Delta(B_1)\Gamma(B_2)}$$

From the Black-Scholes differential equation 12.5 we can obtain a relation, which the different sensitivities of *all* financial instruments *must* satisfy. Simply replace the partial derivatives in Eq. 12.5 by the corresponding sensitivities to obtain

$$\Theta + (r - q)S\Delta + \frac{1}{2}b^2\Gamma = r\,V\ .$$

One consequence of this equation is, for example, the following statement: The change value of a delta and gamma neutral portfolio per time (i.e., Θ) is equal to its current value multiplied by the risk-free interest rate.

[5]This is common practice even in cases in which the option is priced by means of Black-Scholes or some other model that assumes deterministic volatility. For Hedging, it is assumed in contrast to the model assumptions that the underlying volatility might indeed change. The practice proofs that this improves hedge efficiency, leading to the conclusion that the model with deterministic volatility is insufficient. Such a hedge is also called *out of model hedge*.

Table 12.3 Examples of Omega and Delta

	strike = 70 EUR	strike = 90 EUR
Price	10.18 EUR	0.04 EUR
Delta	0.992	0.024
Omega	7.79	47.41

12.4.2 Omega and Beta

In contrast to the delta which gives the *absolute* price change of the derivative (in monetary units) resulting from an *absolute* change of one monetary unit in the underlying, *omega* is the *relative* price change of the derivative (in %) resulting from a *relative* change in the underlying[6]:

$$\Omega \equiv \frac{[V(S+dS) - V(S)]/|V(S)|}{dS/S} = \frac{S}{|V(S)|}\Delta$$

In situations where a trader is instructed to invest a specific *amount* of money rather than in a certain *number* of instruments, this allows a significantly better assessment of the risk than the (much more prominent) delta. A simple example will serve to clarify this point.

Assume that 10,000 euros are to be invested either in a call with a strike price of 90 euros on an underlying paying no dividend with a spot price of 80 euros or in a call on the same underlying with strike price of 70 euros. Suppose the annualized volatility of this underlying is 20%. Both options mature in 30 days. At a risk-free rate of 3% per annum, Eq. 8.7 (implemented in the Excel workbook BlackScholesModel.xls on the accompanying website [50]) gives the prices and risk ratios shown in Table 12.3.

The sensitivity delta is almost 50 times greater for the call with a strike price of 70 euros than for the call with the strike price of 90 euros. Is the call with a strike of 70 euros riskier? If we have the choice of buying a certain *number* of the strike-70 calls or the same number of strike-90 calls then yes, since we could lose a lot more with a certain number of the strike-70 calls than with the same number of strike-90 calls. But this is solely because the *price* of the strike-70 call is so much higher than the price of the strike-90 call! The higher risk is entirely due to the significantly larger investment: if we invest more euros we can lose more euros.

[6]The absolute value of $V(S)$ in this definition has the effect that omega always has the same sign as delta, even for a position with a negative present value.

But if the *same amount* of money is to be invested, a portfolio with strike-90 calls (250,250 of these calls can be purchased for 10,000 euros) is significantly riskier than a portfolio with strike-70 calls (only 982 of these calls can be purchased for 10,000 euros). This risk is impressively quantified by omega, also called the *elasticity*. For the strike-70 call portfolio, a 1.60 euros (=2%) decline in the underlying's price results in a loss of value of 7.79 * 2 %, or 1,558 euros. However, the strike-90 call portfolio loses 47.41*2% of its value, a sum of 9,482 euros. Investing in the strike-90 calls, a relatively small change in the price of the underlying (2%) almost wipes out the entire investment. On the other hand, a slight increase in the price of the underlying can just as easily double the value of the strike-90 call portfolio. Thus, the volatility of the option equals omega times the volatility of the underlying

$$\sigma_{\text{Option}} = \Omega\, \sigma_{\text{Underlying}} \tag{12.19}$$

A similar relation can also be shown to hold for the expected return of an option (in the real world). The market requires, as a compensation for the risk of an investment, a higher expected return of the security under consideration compared to the risk-free rate. Since the risk (as represented by the volatility) of an option on that security is omega times as high as the risk of the security itself, the compensation for taking the risk of investing in the option must be omega times as high as well:

$$\mu_{\text{Option}} - r = \Omega\, (\mu_{\text{Underlying}} - r) \tag{12.20}$$

We can now establish a connection with the *CAPM* (*capital asset pricing model*). The variable β (*beta*) in the CAPM relates the risk premium of a security to the risk premium of a portfolio representing the entire market (with the corresponding diversification). *Indices* such as the *DAX* or the *Dow Jones* are constructed to represent such entire markets. To be more specific: the beta of a security with a price S with respect to an index with a price I is defined as the covariance of the price with the index divided by the variance of the index (see Eq. A.20 in Appendix A):

$$\beta_{SI} = \frac{\text{cov}\,[S, I]}{\text{var}\,[I]} = \frac{\sigma_S}{\sigma_I}\rho_{SI} \tag{12.21}$$

This variable can be interpreted as follows: should the value I of the index increase by 1%, then the price S of the security will (on average) increase by beta %. The relative change of S is thus beta times the relative change in I.

$$\beta_{SI} = \frac{\text{relative change in } S \text{ resulting from a change in } I}{\text{relative change in } I} \tag{12.22}$$
$$= \frac{(dS)/S}{(dI)/I} = \frac{I}{S}\frac{dS}{dI}$$

This corresponds to the definition of omega if S is interpreted as a financial instrument having the index I as an underlying. Within the capital asset pricing model the risk premium for investing in S is shown to be

$$\mu_S - r = \beta_{S,I}\,(\mu_I - r) \tag{12.23}$$

Substituting this for the risk premium of the underlying S in Eq. 12.20 gives

$$\mu_{\text{Option}} - r = \Omega\beta_{S,I}\,(\mu_I - r)$$

Thus, the risk premium for investing in the option is omega times beta multiplied by the risk premium for investing in the entire market. We could therefore also use Eq. 12.23 to define a beta for the option as well.

$$\beta_{\text{Option}} = \Omega\,\beta_{\text{Underlying}} \tag{12.24}$$

Volatility, risk premium and the beta of an option are thus obtained by simply multiplying the corresponding variables for the underlying by the elasticity omega.

12.4.3 Summation of Sensitivities of Different Underlyings

The summation of sensitivities presented in Sect. 12.4.1 is only possible if they are sensitivities of instruments on the same underlying. It is often the case, however, that a portfolio consists of instruments on several different underlyings. The net delta of such a portfolio is *not* simply the sum of the deltas of the individual instruments. For example, in a chemical portfolio with options on Bayer and BASF stocks, the delta of the Bayer option with respect to the Bayer stock price cannot be added to the delta of the BASF option with respect to the BASF stock price to obtain the portfolio's delta. We must first answer the question: Which stock should we chose as the reference underlying in calculating the sensitivity of our portfolio in the above example? Bayer or BASF?

Having chosen a reference underlying S_i from among all the instruments S_k in the portfolio with respect to which the sensitivities of the entire portfolio are to be calculated, we must then proceed by expressing the sensitivities of the remaining underlyings in terms of the sensitivity of this chosen reference. The Betas of the S_k with respect to the S_i can be used in the calculation of such an expression. From the volatilities of the prices and the pair wise correlation between the underlyings, we can use Eqs. 12.21 or A.20 to calculate the necessary Betas, even when neither of the prices is an index. From the sensitivity Δ_k of an instrument with value V_k on the underlying S_k we can then deduce the sensitivity Δ_i of this instrument with respect to *another* underlying S_i by using Eq. 12.22:

$$\Delta_i = \frac{dV_k}{dS_i} = \frac{dS_k}{dS_i}\frac{\partial V_k}{\partial S_k} = \frac{dS_k}{dS_i}\Delta_k = \frac{S_k}{S_i}\beta_{ki}\Delta_k = \frac{S_k\sigma_k}{S_i\sigma_i}\rho_{ki}\Delta_k\ . \qquad (12.25)$$

In the second to last step, Eq. 12.22 in the form "$dS/dI = \beta_{SI}S/I$" is applied while in the last step we make use of Eq. 12.21. Having in this manner transformed the sensitivities with respect to the underlyings S_k into sensitivities with respect to the selected reference underlying S_i, we can now simply add them up to calculate the total sensitivity of the portfolio since they now refer to the same underlying. The sensitivity of the entire portfolio with respect to the reference underlying is thus given by

$$\Delta_i^{\text{total}} = \frac{1}{S_i}\sum_k S_k\,\beta_{ik}\,\Delta_k = \frac{1}{S_i\sigma_i}\sum_k S_k\,\sigma_k\,\rho_{ik}\,\Delta_k\ . \qquad (12.26)$$

This reference underlying can (but need not) be an index on a relevant business sector, for example. Such a combination of securities on different underlyings makes sense when the underlyings are highly correlated. The correlation of the underlyings, or at least the correlation with the selected reference underlying, must be known for the following two reasons:

- They are necessary for the transformation of the deltas in accordance with Eq. 12.25.
- Sound decisions as to a reasonable choice of the above mentioned combination of securities can only be made on the basis of the correlations.

12.5 Computation of the Greek Risk Variables

Having introduced the *Greeks* and their applications in the last section, we will now give a short discussion on their determination using the most important pricing methods.

12.5.1 Sensitivities in the Binomial Model

In order to obtain the price changes in the binomial model resulting from a change in the underlying's spot price, under the assumption that all other parameters (including the time) remain the same, we require several different spot prices for the same time. This is the case for all times except at the beginning of the tree. To obtain more than one spot price at time t, we start the tree at least one time step earlier for each order of differentiation desired. The highest order derivative with respect to S which will be needed is that for gamma, i.e., the second derivative. Thus we let the tree start two time steps in the past, i.e., at time $t - 2dt$. In doing so, we generate (in a recombinant tree) three different price evolutions uuS, udS and ddS. The tree should be constructed in such a way that the middle price evolution udS represents the actual existing spot price $S(t)$. This implies that the tree must start at $S(t - 2dt) = e^{-2\mu dt}S(t)$ if we wish to apply the procedure given by Eq. 9.30. If Eq. 9.31 is to be utilized instead, the tree should begin with $S(t-2dt) = S(t)$. Let $V(t)$ denote the value of any given financial derivative (or portfolio) on the spot price S. For a binomial tree beginning two time steps before the present time t, the following sensitivities can be calculated *directly* within the same tree:

$$
\begin{aligned}
\Delta &\approx \frac{V^{uu}(t) - V^{dd}(t)}{(u^2 - d^2)S(t - 2\delta t)} \approx \frac{V^{uu}(t) - V^{ud}(t)}{u(u-d)S(t - 2\delta t)} \approx \frac{V^{ud}(t) - V^{dd}(t)}{d(u-d)S(t - 2\delta t)} \\
\Omega &= \frac{u\,d\,S(t - 2\delta t)}{V^{ud}(t)}\Delta \\
\Gamma &= 2\frac{d\,V^{uu}(t) - (u+d)V^{ud}(t) + uV^{dd}(t)}{ud(u-d)(u^2-d^2)S^2(t - 2\delta t)} \\
\Theta &= \frac{V^{uudd}(t + 2\delta t) - V(t - 2\delta t)}{4\delta t} \approx \frac{V^{uudd}(t + 2\delta t) - V^{ud}(t)}{2\delta t} \\
&\approx \frac{V^{ud}(t) - V(t - 2\delta t)}{2\delta t}\,.
\end{aligned}
\tag{12.27}
$$

Other sensitivities such as vega and rho can only be determined via Eq. 12.17: we have to calculate the price for a slightly modified volatility or risk-free rate, respectively, with a completely new tree, subsequently dividing the difference between the old and the new option price by the respective change in the parameter to obtain the desired sensitivity.

For the first-order partial derivatives of the price, namely delta, omega and theta, the binomial model provides three different choices: a symmetric derivative, an asymmetric derivative for which the price change with respect to an increasing parameter value is used, and an asymmetric derivative for which the price change with respect to a declining parameter value is used. All these alternatives are displayed in Eq. 12.27.

We have previously introduced a somewhat different equation for delta (see Eq. 9.8). The difference is that in the earlier expressions for delta, the time was not assumed to be strictly constant but changed by dt. The price change induced by this time change is compensated for by the other form of the delta so that both methods are correct up to the order of precision obtained by using the binomial model.

The equation for Theta is, strictly speaking, only correct for the procedure given by Eq. 9.31 since in this procedure, the derivative prices used all contain the *same* underlying price. In the procedure given by Eq. 9.30, however, not only does the time change but, because of the drift appearing in u and d, the underlying S changes as well. Consequently, a change in the option price resulting *solely* because of a time change is not obtained. This is one (more) reason why Eq. 9.31 is used almost exclusively in this text.

The expression for gamma can be obtained from the following argument: gamma is by definition the change of delta per change of the underlying. At time t we have two deltas at our disposal, these being the two unsymmetrical derivatives in Eq. 12.27

$$\Delta_u \approx \frac{V^{uu}(t) - V^{ud}(t)}{u(u-d)S(t-2\delta t)} \quad , \quad \Delta_d \approx \frac{V^{ud}(t) - V^{dd}(t)}{d(u-d)S(t-2\delta t)} \ .$$

Δ_u is the delta between the upper and middle node at time t. Similarly Δ_d is the delta between the middle and lower node. The difference between these two deltas will serve as the delta change needed to calculate gamma. In addition we need the values of the underlying belonging to those two deltas since gamma is the delta change per corresponding underlying change. As the underlying value belonging to Δ_u we use the average (denoted by $\overline{S}_u$) of the underlying values at the upper and middle nodes. Similarly we use the average (denoted by $\overline{S}_d$) of the underlying values at the lower and middle nodes as the underlying value belonging to Δ_d.

$$\overline{S}_u = \frac{1}{2}[uu + ud]\,S(t - 2\delta t) = \frac{1}{2}u(u + d)S(t - 2\delta t)$$
$$\overline{S}_d = \frac{1}{2}[dd + ud]\,S(t - 2\delta t) = \frac{1}{2}d(u + d)S(t - 2\delta t)$$

With respect to *these* underlying values Δ_u and Δ_d are even *symmetric* derivatives. Gamma is now simply the delta difference divided by the difference in underlying values:

$$\begin{aligned}
\Gamma &= \frac{\Delta_u - \Delta_d}{\overline{S}_u - \overline{S}_d} \\
&= \frac{1}{\overline{S}_u - \overline{S}_d}\frac{\left[V^{uu}(t) - V^{ud}(t)\right]/u - \left[V^{ud}(t) - V^{dd}(t)\right]/d}{(u - d)S(t - 2\delta t)} \\
&= \frac{1}{\overline{S}_u - \overline{S}_d}\frac{d\left[V^{uu}(t) - V^{ud}(t)\right] - u\left[V^{ud}(t) - V^{dd}(t)\right]}{ud(u - d)S(t - 2\delta t)} \\
&= \frac{2}{(u - d)(u + d)S(t - 2\delta t)}\frac{dV^{uu}(t) - (u + d)V^{ud}(t) + uV^{dd}(t)}{ud(u - d)S(t - 2\delta t)}\ .
\end{aligned}$$

This corresponds exactly to the gamma in Eq. 12.27.

A detailed demonstration on how to determine the sensitivities of an option portfolio using the binomial model is provided in the Excel workbook BinomialTree.xls.

12.5.2 Sensitivities in the Black-Scholes Model

Since the Black-Scholes model provides analytic solutions for option prices, the sensitivities can be determined immediately by calculating the required partial derivatives directly. The sensitivities of a plain vanilla call and put on an underlying with spot price S earning a yield q will serve as an example and are given below. The expressions appearing here arise from differentiating[7] the

[7] Only the *explicit* dependence and not the *implicit* dependence of the option on the parameters under consideration play a role when calculating these partial derivatives. In determining theta, for example, the time dependence of S should not be taken into consideration, since this dependence is *implicit* for the option price. This means that in taking the partial derivatives, S, t, r, q and σ are to be viewed as independent parameters.

function in Eq. 8.7 and exploiting the relation in Eq. 12.16. The sensitivities of options on futures and forwards can be calculated similarly by differentiating the functions given in Eqs. 8.8 and 8.9.

$$
\begin{aligned}
\Delta_{\text{Call}} &= B_q(t,T)\,\mathrm{N}(x) \quad , \quad \Delta_{\text{Put}} = B_q(t,T)\;[\mathrm{N}(x)-1] \\
\Gamma_{\text{Call}} &= \frac{B_q(t,T)\,\mathrm{N}'(x)}{S(t)\sigma\sqrt{T-t}} = \Gamma_{\text{Put}} \\
\Theta_{\text{Call}} &= -\frac{B_q(t,T)S(t)\,\mathrm{N}'(x)\sigma}{2\sqrt{T-t}} + S(t)\mathrm{N}(x)\frac{\partial B_q(t,T)}{\partial t} - K\mathrm{N}(x-\sigma\sqrt{T-t})\frac{\partial B(t,T)}{\partial t} \\
\Theta_{\text{Put}} &= -\frac{B_q(t,T)S(t)\,\mathrm{N}'(x)\sigma}{2\sqrt{T-t}} - S(t)\mathrm{N}(-x)\frac{\partial B_q(t,T)}{\partial t} \\
&\quad + K\mathrm{N}(-x+\sigma\sqrt{T-t})\frac{\partial B(t,T)}{\partial t} \\
\Psi_{\text{Call}} &= B_q(t,T)S(t)\,\mathrm{N}'(x)\sqrt{T-t} = \Psi_{\text{Put}} \\
\rho_{\text{Call}} &= -K\mathrm{N}(x-\sigma\sqrt{T-t})\frac{\partial B(t,T)}{\partial r} \quad , \quad \rho_{\text{Put}} = K\mathrm{N}(-x+\sigma\sqrt{T-t})\frac{\partial B(t,T)}{\partial t} \\
\rho_{q_{\text{Call}}} &= S(t)\mathrm{N}(x)\frac{\partial B_q(t,T)}{\partial q} \quad , \quad \rho_{q_{\text{Put}}} = -S(t)\mathrm{N}(-x)\frac{\partial B_q(t,T)}{\partial q}\ .
\end{aligned}
\tag{12.28}
$$

Here, x is as defined in Eq. 8.5 and N' again denotes the derivative of the cumulative standard normal distribution with respect to its argument:

$$
\mathrm{N}'(x) = \frac{d\mathrm{N}(x)}{dx} = \frac{e^{-x^2/2}}{\sqrt{2\pi}}
$$

The Excel workbooks BLACKSCHOLES.XLS and STRADDLE.XLS demonstrate these sensitivities. In BLACKSCHOLES.XLS, the hedging of a portfolio consisting of puts, calls and underlyings is demonstrated and in STRADDLE.XLS, the behavior of the sensitivities of a straddle as a function of the underlying is described.

12.5.3 Sensitivities by Means of Finite Difference Methods

In principle, every numerical algorithm used to compute the price of an option can also be applied for the determination of the sensitivities or Greeks. All numerical procedures for computing the sensitivities are in essence based on the same fundamental idea displayed in Eq. 12.17. Two option prices are computed. These prices differ since one of the risk factors used in their determination was slightly different. The sensitivity of the option price with respect to that particular risk factor is then the difference between the two option prices divided by the difference in the risk factor.

With finite difference methods—as with binomial trees—some sensitivities can be calculated (as finite differences) *directly on the grid* used for pricing without having to go through the lengthy procedure based on Eq. 12.17. In fact, a finite difference scheme does nothing other than calculate (approximations for) the derivatives with respect to S and t at each time step. The relevant formulas have already been presented in Chap. 10. Explicitly, Eq. 10.8 directly yields *delta*, Eq. 10.9 *gamma*, and Eq. 10.11 *Theta.* Calculating Greeks directly on the grid has proved to yield results which are significantly more exact[8] than procedures based on Eq. 12.17. Hence, the Greeks should be calculated on the grid whenever possible.

Vega and *rho,* however, cannot be calculated directly from an (S, t)-grid and so for these sensitivities we have to rely on Eq. 12.17, i.e., the calculation must be run through twice, in each case with a slight change in the respective parameter.

The determination of the sensitivities of an options portfolio using finite difference methods is shown in the Excel workbook FiniteDifferenceMethod.xlsm from the download section [50].

12.5.4 Sensitivities by Means of Monte Carlo Simulations

When using Monte Carlo simulations in determining the sensitivities, the computations are always performed in accordance with Eq. 12.17. This means the two simulations are performed with slightly different values for the parameter under consideration but with *exactly the same random numbers*.

The difference between the two option prices divided by the difference in the parameter value gives an approximation of the desired sensitivity. Differentiating twice, as for gamma, for example, is accomplished in accordance with Eq. 12.17 with three different simulations for three different parameter values. Each simulation must be performed *with the same random numbers*. The computation of the sensitivities by means of Monte Carlo simulations is demonstrated in full detail for an option portfolio in the Excel workbook MonteCarloSimulation.xlsm from the download section [50].

[8] For instance, we might otherwise observe that the sensitivities oscillate as a function of the spot price of the underlying.