The Prototype View of Concepts

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Abstract. Concepts are important and basic elements in human's cognition process. The formal concept gives a mathematical format of the classical view of concepts in which all instances of a concept share common properties. But in some situation this view is not consistent with human's understanding of concepts. The prototype view of concepts is more appropriate in our daily life. This view characters some analog categories as internally structured into a prototype (clearest cases, best examples of the category) and non-prototype members, with non-prototype members tending toward an order from better to poorer examples. The objective of this paper is to give a mathematical description of prototype view of concepts. Firstly, we give a similarity measurement of an object to another object in a formal context. Then based on this similarity measurement, the mathematical format of prototype view of concepts, named k-cutting concept, induced by one typical object is obtained. Finally, the properties of k-cutting concepts are studied. In addition to presenting theorems to summarize our results, we use some examples to illustrate the main ideas.

Keywords: Prototype view of concepts · Similarity measurement · k-cutting concepts · Object concepts

1 Introduction

Concepts are important and basic constituents in human's cognition process. Consequently, they are crucial in many psychological processes, such as categorization, inference, memory, learning, and decision-making. In philosophy, there are different views or structures of concepts. In classical view, a concept contains two parts, extension and intension. The extension is a group of objects belonging to the concept and the intension is a family of attributes characterizing the properties of the concept. The classical view holds that all instances of a concept share common properties, which are necessary and sufficient conditions for defining the concept. In order to apply the philosophical concept into data processing, Wille [22] proposed a new field, formal concept analysis (FCA), giving a mathematical format of the classical view of concepts.

FCA [22] shows a mathematical format of classical view of concepts, named formal concepts. A formal concept consists of a pair of an object set (extent) and

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an attribute set (intent). The objects in extent possess all the attributes in intent and the attributes in intent are possessed by all the objects in extent. Based on the partial order theory, Wille and Ganter [8] presented a lattice structure of formal concepts named a concept lattice which reveals hierarchical structure of concepts with respect to the generalization and the specialization of concepts. However, the formal concept is an all-or-none phenomenon. That is, if an object possesses all the attributes in the intent of a formal concept, it is definitely in the extent of this formal concept, but if an object does not possess all the attributes in the intent, even though it possesses most attributes in the intent, this object is definitely not in this formal concept. In other words, if two objects are in the extent of same concept, they must have same degree of typicality in this formal concept. That is, the objects in the extent of a concept are equally important in people's understanding of the concept. This view of concepts is mostly used in machine-oriented concept learning [1,10,12,25,27], but not always consistent with human's understanding of concepts. Classical formal concepts have been extended to other types, such as preconcepts [23], semiconcepts [24], protoconcepts [21], property oriented concepts [4], object oriented concepts [26], dual concepts [2,13], monotone concepts [3], RS-definable concepts [28] and three-way concepts [15–17].

There is increasing evidence that memberships of objects in semantic categories which are expressed by words of natural languages can be graded rather than all-or-none. Lakoff [11], Rosch [19] and Zadeh [29] argued that some natural categories are analog and must be represented logically in a manner which reflects their analog structure. Rosch [19] has further characterized some natural analog categories as internally structured into a prototype (clearest cased, best examples of the category) and non-prototype members, with non-prototype members tending toward an order from better to poorer examples. For example, *chair* is a more reasonable exemplar than *radio* of the concept *furniture*, or we can say that the *chair* has a larger membership than *radio* of the concept *furniture*. When we talk about color, *vermilion*, *fuchsia*, *pink*, *cerise*, *peach*, *garnet*, *cardinal*, *rose*, *wine* all belong to concept *red*. However, *rose* is more typical than *pink*. This kind of view of concepts are called prototype view of concepts.

In this paper, we try to give a mathematical representation of the prototype view of concepts [6,7]. Considering the cognitive process of recognizing concepts, we firstly choose an object as the prototype of a concept, which is the most typical object and can be a representative of this concept. Then the similarities between other objects and prototype are given according to a similarity measurement. Since the prototype is described by a group of attributes [9], the similarity measurement is defined based on the description of objects. The objects with high similarity to the prototype can be put into the concept. In order to quantitatively define high similarity, we preset a threshold k and the corresponding prototype view concepts are called k-cutting concepts. Since prototype o is the most typical object of this concept, the description of prototype o is regarded as the intent of this k-cutting concept and the objects whose similarity to prototype o is bigger than k are put into the extent of this k-cutting concept. Furthermore we study the properties of k-cutting concepts.

The rest of the paper is organized as follows. Section 2 gives the basic notions in formal concept analysis. Then Sect. 3 presents the similarity measurement between two different objects and defines the k-cutting concept. Furthermore we show the properties of k-cutting concepts. Finally, this paper is concluded in Sect. 4.

2 Formal Concept Analysis

This section reviews basic notions in FCA. FCA, proposed by Wille in 1982 [22], gives a mathematical way to represent a concept with a pair of objects set (called the extent) and attributes set (called the intent). The data source of FCA is called formal context defined as follows [8,22].

Definition 1. *A formal context* $(OB, AT, \mathbf{I})$ *consists of two sets* OB *and* AT*, and a relation* $\mathbf{I}$ *between* OB *and* AT*. The elements of* OB *are called the objects and the elements of* AT *are called the attributes of the context. In order to express that an object* o *is in a relation* $\mathbf{I}$ *with an attribute* a*, we write* $o\mathbf{I}a$ *or* $(o, a) \in \mathbf{I}$ *and read it as "the object* o *has the attribute* a*".*

Based on the formal context, the set of attributes possessed by an object o and the set of objects possessing an attribute a are given as

$$\begin{aligned} o\mathbf{I}. &= \{a \in AT \mid o\mathbf{I}a\} \subseteq AT, \\ .\mathbf{I}a &= \{o \in OB \mid o\mathbf{I}a\} \subseteq OB. \end{aligned} \tag{1}$$

Actually, $o\mathbf{I}.$ can be regarded as the description of object o and $.\mathbf{I}a$ can be understood as a set of objects which can be described by attribute a or a set of representatives of description $\{a\}$. Given a formal context $(OB, AT, \mathbf{I})$, if for any $o \in OB$, we have $o\mathbf{I}. \neq \emptyset$, $o\mathbf{I}. \neq AT$, and for any $a \in AT$, we have $.\mathbf{I}a \neq \emptyset$, $.\mathbf{I}a \neq OB$, then the formal context $(OB, AT, \mathbf{I})$ is called canonical. If for any objects $o_1, o_2 \in OB$, from $o_1\mathbf{I}. = o_2\mathbf{I}.$, it always follows that $o_1 = o_2$ and, consequently, $.\mathbf{I}a_1 = .\mathbf{I}a_2$ implies $a_1 = a_2$ for all $a_1, a_2 \in AT$. We call this context a clarified formal context. In this paper, we suppose all formal contexts are canonical, clarified and finite. Based on the description of an object and the representatives of an attribute, a pair of operators called derivation operators are defined on an objects set $O \subseteq OB$ and an attributes set $A \subseteq AT$, respectively, in $(OB, AT, \mathbf{I})$ [8]:

$$\begin{aligned} O^* &= \{a \in AT \mid \forall o \in O(o\mathbf{I}a)\} = \{a \in AT \mid O \subseteq .\mathbf{I}a\} = \bigcap\{o\mathbf{I}. \mid o \in O\}, \\ A^* &= \{o \in OB \mid \forall a \in A(o\mathbf{I}a)\} = \{o \in OB \mid A \subseteq o\mathbf{I}.\} = \bigcap\{.\mathbf{I}a \mid a \in A\}. \end{aligned} \tag{2}$$

It is obvious to see that, for any object $o \in OB$ and any attribute $a \in AT$, it always follows $o\mathbf{I}. = \{o\}^*$ and $.\mathbf{I}a = \{a\}^*$. Then based on above derivation operators, a formal concept is obtained [8].

Definition 2. *A formal concept of the context* $(OB, AT, \mathbf{I})$ *is a pair* (O, A) *with* $O^* = A$ *and* $O = A^*$ $(O \subseteq OB, A \subseteq AT)$*. We call* O *the extent and* A *the intent of the formal concept* (O, A)*.*

The formal concepts of a formal context $(OB, AT, \mathbf{I})$ are ordered by

$$(O_1, A_1) \leq (O_2, A_2) \Leftrightarrow O_1 \subseteq O_2 \ (\Leftrightarrow A_1 \supseteq A_2). \tag{3}$$

All formal concepts of $(OB, AT, \mathbf{I})$ can form a complete lattice called the formal concept lattice of $(OB, AT, \mathbf{I})$, denoted by $L(OB, AT, \mathbf{I})$. The infimum and supremum are given by

$$\begin{aligned}(O_1, A_1) \wedge (O_2, A_2) &= (O_1 \cap O_2, (A_1 \cup A_2)^{**}),\\ (O_1, A_1) \vee (O_2, A_2) &= ((O_1 \cup O_2)^{**}, A_1 \cap A_2)\end{aligned} \tag{4}$$

In a formal context, there is a kind of important concept, named object concept [8].

Definition 3. *Let* $(OB, AT, \mathbf{I})$ *be a formal context,* (o^{**}, o^*) *is a formal concept for all* $o \in OB$*, which is called an object concept. Here, for convenience, we write* o^* *instead of* $\{o\}^*$ *for any* $o \in OB$*.*

The object concept (o^{**}, o^*) can be understood as a concept induced by object o, which means the object o is a typical object (prototype) of concept (o^{**}, o^*). Specifically, the description (intent) of concept (o^{**}, o^*) is the description of object o and the extent of this concept is a set of objects which can be described by the description of object o. In order to show the importance of the object concept, the notion of join-dense is recalled in next definition [8].

Definition 4. *Let* P *be an ordered set and let* $Q \subseteq P$*. Then* Q *is called join-dense in* P *if for every element* $a \in P$ *there is a subset* A *of* Q *such that* $a = \vee_P A$*.*

Following theorem shows that any formal concept can be constructed based on a set of object concepts, so the object concepts can be regarded as the fundamental elements in concept construction [8].

Theorem 1. *Let* $(OB, AT, \mathbf{I})$ *be a formal context and* $L(OB, AT, \mathbf{I})$ *the associated complete lattice of concepts. Then the set of all the object concepts is join-dense in* $L(OB, AT, \mathbf{I})$*. Specifically, for a formal concept* (O, A)*,*

$$\bigvee \{(o^{**}, o^*) \mid o \in O\} = (O, A) \tag{5}$$

holds.

Finally, we give an example to illustrate the definitions and theorems presented in this section.

Table 1. A formal context $(OB, AT, \mathbf{I})$

OB	a	b	c	d
o_1	1	0	0	1
o_2	0	1	0	1
o_3	1	1	1	0
o_4	0	1	1	0

Example 1. Table 1 is a formal context $(OB, AT, \mathbf{I})$ with four objects $OB = \{o_1, o_2, o_3, o_4\}$ and four attributes $AT = \{a, b, c, d\}$. The description of every object and the representatives of every attribute are as follows:

$$o_1\mathbf{I}. = \{a, d\},\ o_2\mathbf{I}. = \{b, d\},\ o_3\mathbf{I}. = \{a, b, c\},\ o_4\mathbf{I}. = \{b, c\}.$$
$$.\mathbf{I}a = \{o_1, o_3\},\ .\mathbf{I}b = \{o_2, o_3, o_4\},\ .\mathbf{I}c = \{o_3, o_4\},\ .\mathbf{I}d = \{o_1, o_2\}.$$

We can see that for any object $o_i \in OB$, its description is neither whole attribute set AT nor the empty set. Also, for any attribute in AT, its representatives set is neither whole object set OB nor the empty set. Thus the formal context $(OB, AT, \mathbf{I})$ is canonical. Moreover, for any two different objects, their descriptions are different, and for any two different attributes, their representatives sets are different. Thus the formal context $(OB, AT, \mathbf{I})$ is clarified.

The formal concept lattice of context $(OB, AT, \mathbf{I})$ is shown in Fig. 1. The object concepts are: $(o_1^{**}, o_1^*) = (o_1, ad)$, $(o_2^{**}, o_2^*) = (o_2, bd)$, $(o_3^{**}, o_3^*) = (o_3, abc)$, $(o_4^{**}, o_4^*) = (o_3o_4, bc)$. After calculation, we have

$$\begin{aligned}
&(o_2o_3o_4, b) = (o_2, bd) \vee (o_3, abc) \vee (o_3o_4, bc),\\
&(o_1o_3, a) = (o_1, ad) \vee (o_3, abc),\\
&(o_1o_2, d) = (o_1, ad) \vee (o_2, bd),\\
&(OB, \emptyset) = (o_1, ad) \vee (o_2, bd) \vee (o_3, abc) \vee (o_3o_4, bc).
\end{aligned}$$

That is, any formal concept can be constructed by joining a set of object concepts. Thus, the set of all object concepts is join-dense in formal concept lattice.

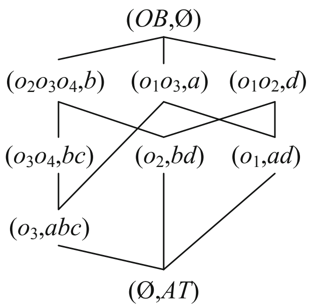

Fig. 1. The formal concept lattice $L(OB, AT, \mathbf{I})$

3 The Prototype View of Concept

Section 2 shows the importance of object concepts in concept construction. However, the definition of object concepts is too strict. According to Definition 3, the extent of an object concept (o^{**}, o^*) is a set of objects which can be fully described by the description of object o. Actually, the semantic concept in our daily life is based on a typical object (prototype), but the extent of semantic concept is not required to be fully described by the description of the typical object. Based on the similarity [5,18,20] to the typical object, the typicality of objects in extent can be defined. In this section, we give a mathematical way to represent the semantic concept and discuss its properties. Firstly, the similarity measurement of one object to another is shown in Sect. 3.1.

3.1 Similarity Measurement Between Two Objects

In a formal context $(OB, AT, \mathbf{I})$, an object o can be described by a set of attributes $o\mathbf{I}.$ (called description of object o). And if two objects have same description, they can be regarded as same one [8]. Thus, in order to measure the similarity of object o_i to object o, we only need to measure the similarity between the descriptions of these two objects. The more similar the descriptions of two objects are, the more similar the two objects are.

Definition 5. *Let $(OB, AT, \mathbf{I})$ be a formal context and o be a reference object. For any object $o_i \in OB$, the similarity measurement of o_i to o is defined as*

$$Sim(o_i, o) = \frac{|o_i\mathbf{I}. \cap o\mathbf{I}.|}{|o\mathbf{I}.|}. \tag{6}$$

The range of the value of this similarity measurement is $0 \leq Sim(o_i, o) \leq 1$. The closer the similarity is to 1, the more similar object o_i is to object o; the closer the similarity is to 0, the less similar object o_i is to object o. In other words, the more attributes in description of object o can be used to describe object o_i, the more similar object o_i is to object o. Now let us consider the similarity of objects in the extent of an object concept (o^{**}, o^*) to the object o.

Proposition 1. *Let $(OB, AT, \mathbf{I})$ be a formal context and (o^{**}, o^*) be an object concept induced by object o. Then the value of similarity measurement of any object in o^{**} to object o is* 1*.*

Proof. Suppose (o^{**}, o^*) is an object concept of formal context $(OB, AT, \mathbf{I})$ and $o_i \in o^{**}$ is an object from the extent of this concept. Since $o_i \in o^{**}$, according to the properties of operator $*$, we have $o^{***} \subseteq o_i^*$ and $o^* = o^{***}$. Thus, we obtain $o^* \subseteq o_i^*$. That is, $o\mathbf{I}. \subseteq o_i\mathbf{I}$. Hence we have $o_i\mathbf{I}. \cap o\mathbf{I}. = o\mathbf{I}$. Thus, $Sim(o_i, o) = \frac{|o_i\mathbf{I}. \cap o\mathbf{I}.|}{|o\mathbf{I}.|} = \frac{|o\mathbf{I}.|}{|o\mathbf{I}.|} = 1$.

By Proposition 1, the extent o^{**} of object concept (o^{**}, o^*) consists of objects whose value of similarity measurement to object o is 1. If we regard object o as a typical object (prototype) of an semantic concept, in order to get all the objects in this prototype view of concepts, we should not only consider objects with similarity value of 1, but also objects with similarity value less than 1.

We use a simple example to illustrate the basic notions and ideas introduced so far.

Example 2. (Continued with Example 1) We set object o_3 as a reference object. In the following, we will compute the similarity of each object in OB to reference object o_3.

$$\begin{aligned}
Sim(o_1, o_3) &= \frac{|o_1\mathbf{I}.\ \cap\ o_3\mathbf{I}.|}{|o_3\mathbf{I}.|} = \frac{|\{a,d\} \cap \{a,b,c\}|}{|\{a,b,c\}|} = \frac{1}{3},\\
Sim(o_2, o_3) &= \frac{|o_2\mathbf{I}.\ \cap\ o_3\mathbf{I}.|}{|o_3\mathbf{I}.|} = \frac{|\{b,d\} \cap \{a,b,c\}|}{|\{a,b,c\}|} = \frac{1}{3},\\
Sim(o_3, o_3) &= \frac{|o_3\mathbf{I}.\ \cap\ o_3\mathbf{I}.|}{|o_3\mathbf{I}.|} = \frac{|\{a,b,c\} \cap \{a,b,c\}|}{|\{a,b,c\}|} = 1,\\
Sim(o_4, o_3) &= \frac{|o_4\mathbf{I}.\ \cap\ o_3\mathbf{I}.|}{|o_3\mathbf{I}.|} = \frac{|\{b,c\} \cap \{a,b,c\}|}{|\{a,b,c\}|} = \frac{2}{3}.
\end{aligned}$$

Then, we check the correctness of Proposition 1. In Example 1, we get four object concepts (o_1, ad), (o_2, bd), (o_3, abc) and (o_3o_4, bc) inducing by objects o_1, o_2, o_3 and o_4, respectively. The similarities of objects in extent of object concepts to the objects inducing these concepts are computed as follows:

$$\begin{aligned}
Sim(o_1, o_1) &= \frac{|o_1\mathbf{I}.\ \cap\ o_1\mathbf{I}.|}{|o_1\mathbf{I}.|} = \frac{|\{a,d\} \cap \{a,d\}|}{|\{a,d\}|} = 1,\\
Sim(o_2, o_2) &= \frac{|o_2\mathbf{I}.\ \cap\ o_2\mathbf{I}.|}{|o_2\mathbf{I}.|} = \frac{|\{b,d\} \cap \{b,d\}|}{|\{b,d\}|} = 1,\\
Sim(o_3, o_3) &= \frac{|o_3\mathbf{I}.\ \cap\ o_3\mathbf{I}.|}{|o_3\mathbf{I}.|} = \frac{|\{a,b,c\} \cap \{a,b,c\}|}{|\{a,b,c\}|} = 1,\\
&\begin{cases} Sim(o_3, o_4) = \frac{|o_3\mathbf{I}.\cap o_4\mathbf{I}.|}{|o_4\mathbf{I}.|} = \frac{|\{a,b,c\}\cap\{b,c\}|}{|\{b,c\}|} = \frac{|\{b,c\}|}{|\{b,c\}|} = 1,\\ Sim(o_4, o_4) = \frac{|o_4\mathbf{I}.\cap o_4\mathbf{I}.|}{|o_4\mathbf{I}.|} = \frac{|\{b,c\}\cap\{b,c\}|}{|\{b,c\}|} = 1. \end{cases}
\end{aligned}$$

The computation results are consistent to Proposition 1. That is, the value of similarity of objects in extent of object concept to the object inducing this concept is 1.

3.2 The *k*-cutting Concept Induced by One Typical Object

The classical view of concepts holds that all instances of a concept share common properties that are necessary and sufficient conditions for defining the concept [14]. However, in our daily life, semantic category in our nature language is not an all-or-none phenomenon. For example, we know that a *chair* is a more reasonable exemplar of the category *furniture* than a *radio*. In other words, the *chair* is more typical than *radio* in the category *furniture*. This is contrary to the assumption that categories are necessarily logical, bounded entities, Rosch [19] has characterized some natural analog categories as internally structured into a prototype (clearest cases, best example of the category) and nonprototype members, with nonprototype members tending toward an order from better to poorer example. Based on the results of Rosch's study, we summarize the process of human to recognize a semantic concept as follows:

step 1: *Pick up the typical object (prototype) of the concept*;

step 2: *Calculate the characterized attributes (description) of the typical object*;

step 3: *Calculate the similarity of each object to the typical object*;

step 4: *Put the objects with high similarity into the extent of concept.*

The above steps are just a qualitative description of process to obtain semantic concepts. If we want to express this process in a mathematical way, some quantitative index is needed. For example, in step 4, *high* is a qualitative description. In order to determine which object has high similarity to the typical object, we give a preset threshold k. If the similarity measurement of an object to typical object is bigger than k, this object can be regarded as being highly similar to typical object. Thus, this object can be put into the extent and the corresponding concept is called the k-cutting concept. Since the objects in one extent belong to the same concept, they should possess some common attributes with each other. Thus we strict that k should satisfy $k > \frac{1}{2}$. The following use of k satisfies these settings.

The above process is easy for us to understand, but it is hard to give a mathematical definition of k-cutting concept directly. In the following, we show the mathematical definition of the k-cutting concept. Firstly, a pair of k-cutting derivation operators are given as follows.

Definition 6. *Let $(OB, AT, \mathbf{I})$ be a formal context. A pair of k-cutting derivation operators ($k > \frac{1}{2}$) for objects set $O \subseteq OB$ and attributes set $A \subseteq AT$ are defined as:*

$$\begin{aligned} O^{*k} &= \{a \in AT \mid |a^* \cap O| \geq k \cdot |O|\}, \\ A^{*k} &= \{o \in OB \mid |o^* \cap A| \geq k \cdot |A|\} \end{aligned} \tag{7}$$

In Definition 6, the attribute shared by more than $k \cdot |O|$ objects in O belongs to attributes set O^{*k}; the object possessing more than $k \cdot |A|$ attributes in A belongs to objects set A^{*k}. In the following, we present the properties of k-cutting derivation operators.

Property 1. Let $(OB, AT, \mathbf{I})$ be a formal context. The following properties hold for any objects sets $O, O_1, O_2 \subseteq OB$ and attributes sets $A, A_1, A_2 \subseteq AT$:

$$
\begin{aligned}
&G1.\ \emptyset^{*k} = AT,\ \text{when } \emptyset \subseteq OB,\\
&\quad\ \ \emptyset^{*k} = OB,\ \text{when } \emptyset \subseteq AT;\\
&G2.\ O^{*1} = O^{*},\quad A^{*1} = A^{*};\\
&G3.\ O^{*k} = O^{*} \text{ when } O \text{ is a singleton set},\\
&\quad\ \ A^{*k} = A^{*} \text{ when } A \text{ is a singleton set};\\
&G4.\ k \le h \Rightarrow O^{*h} \subseteq O^{*k},\\
&\quad\ \ k \le h \Rightarrow A^{*h} \subseteq A^{*k};\\
&G5.\ O^{*k} = \cup_{k \le k_i} O^{*k_i} = \cap_{k_j \le k} O^{*k_j},\\
&\quad\ \ A^{*k} = \cup_{k \le k_i} A^{*k_i} = \cap_{k_j \le k} A^{*k_j};\\
&G6.\ O \subseteq O^{**k},\quad A \subseteq A^{**k}.
\end{aligned}
$$

Proof. The results in $G1$ and $G2$ are obvious.

$G3$. If object set O is a singleton set, then there exists an object $o_i \in OB$ satisfying $O = \{o_i\}$. Since $\{o_i\}$ is a singleton set, the result of $|a^* \cap \{o_i\}|$, for any $a \in AT$, is either 0 or 1. According to Definition 6, for any $k > \frac{1}{2}$, we have $O^{*k} = \{a \in AT \mid |a^* \cap O| \ge k \cdot |O|\} = \{a \in AT \mid |a^* \cap \{o_i\}| \ge k|\{o_i\}|\} = \{a \in AT \mid |a^* \cap \{o_i\}| \ge k\}$. That is, if attribute $a \in O^{*k}$, then $|a^* \cap \{o_i\}| \ge k$. That means $|a^* \cap \{o_i\}| = 1$. Thus $O^{*k} = \{a \in AT \mid |a^* \cap \{o_i\}| = 1\} = \{a \in AT \mid \{o_i\} \subseteq a^*\} = \{o_i\}^* = O^*$. The rest part can be proved similarly.
$G4$. For any $a \in O^{*h}$, according to Definition 6, we have $|a^* \cap O| \ge h \cdot |O|$. Since $k \le h$, we have $h \cdot |O| \ge k \cdot |O|$. Thus, $|a^* \cap O| \ge h \cdot |O| \ge k \cdot |O|$. That is, $a \in O^{*k}$. Because of the arbitrariness of attribute a, we obtain $O^{*h} \subseteq O^{*k}$. The formula $k \le h \Rightarrow A^{*h} \subseteq A^{*k}$ can be proved similarly.
$G5$. From property $G4$, for any $k_i \ge k$, we have $O^{*k_i} \subseteq O^{*k}$. Hence, we obtain $\cup_{k \le k_i} O^{*k_i} \subseteq O^{*k}$. Also, since $k \le k$, we can get $O^{*k} \subseteq \cup_{k \le k_i} O^{*k_i}$. Thus, we obtain $O^{*k} = \cup_{k \le k_i} O^{*k_i}$. Analogously, from property $G4$, for any $k_j \le k$, we have $O^{*k} \subseteq O^{*k_j}$. Hence, we obtain $O^{*k} \subseteq \cap_{k_j \le k} O^{*k_j}$. Also, since $k \le k$, we can get $\cap_{k_j \le k} O^{*k_j} \subseteq O^{*k}$. Thus, we obtain $O^{*k} = \cap_{k_j \le k} O^{*k_j}$. The rest part $A^{*k} = \cup_{k \le k_i} A^{*k_i} = \cap_{k_j \le k} A^{*k_j}$ can be proved similarly.
$G6$. For any $o_i \in O$, from property of operator $*$, we have $O^* \subseteq o_i^*$. Consequently, we obtain $o_i^* \cap O^* = O^*$. Thus, $|o_i^* \cap O^*| = |O^*| \ge k|O^*|$ holds no matter what value k has. According to Definition 6, we can get $o_i \in O^{**k}$. The rest part $A \subseteq A^{**k}$ can be proved similarly.

Then based on the k-cutting derivation operators, the definition of k-cutting concept induced by one prototype (we will simply call it k-cutting concept if there is no confusion) is given as follows.

Definition 7. *Let* $(OB, AT, \mathbf{I})$ *be a formal context. The* k*-cutting concept induced by one typical object* o *is defined as* $(\hat{o}^{**k}, o^*)$*.* $\hat{o}^{**k}$ *and* o^* *are called extent and intent of* k*-cutting concept* $(\hat{o}^{**k}, o^*)$*. Here,* $\hat{o}^{**k} = (o_i, m(o_i))$*,* $o_i \in o^{**k}$ *is a set of objects in* o^{**k} *accompanied with a membership value.*

Specifically, the element of $\hat{o}^{**k}$ is an object-membership pair $(o_i, m(o_i))$, in which $o_i \in o^{**k}$ and $m(o_i)$ is the membership of object o_i belonging to the k-cutting concept $(\hat{o}^{**k}, o^*)$. The membership can be measured in different ways, and the most common way is using the similarity measurement value of object o_i to typical object o. In following analysis, for convenience, we can regard o^{**k} instead of $\hat{o}^{**k}$ as an extent of k-cutting concept $(\hat{o}^{**k}, o^*)$. That is, we can rewrite k-cutting concept $(\hat{o}^{**k}, o^*)$ as (o^{**k}, o^*).

From Definition 7, the intent of the k-cutting concept (o^{**k}, o^*) is a set of attributes which is the description of typical object o and the extent of the concept is a set of object-membership pairs in which the description of object contains more than $k \cdot |o^*|$ attributes in description of o. The set of all k-cutting concepts induced by one typical object in formal context $(OB, AT, \mathbf{I})$ is denoted by $OCC_k(OB, AT, \mathbf{I})$. Now we check the similarity of any object in k-cutting concept given in Definition 7 to verify its rationality.

Theorem 2. *Let $(OB, AT, \mathbf{I})$ be a formal context and (o^{**k}, o^*) is a k-cutting concept. An object $o_i \in OB$ belongs to o^{**k} if and only if the similarity of o_i to o is bigger than k. That is, $Sim(o_i, o) \geq k$.*

Proof. According to Definition 6, we have $o_i \in o^{**k}$ is equivalent to $|o_i^* \cap o^*| \geq k \cdot |o^*|$. Since we assumed that the formal context in this paper is canonical, we have $o^* \neq \emptyset$, that is, $|o^*| \neq 0$. Thus, both sides of the inequality $|o_i^* \cap o^*| \geq k \cdot |o^*|$ can be divided by $|o^*|$. The result is $\frac{|o_i^* \cap o^*|}{|o^*|} \geq k$. That is, $Sim(o_i, o) \geq k$. Thus, $o_i \in o^{**k}$ holds if and only if $Sim(o_i, o) \geq k$ holds.

Remark 1. The higher the value of similarity the object has, the closer it is to the typical object. However, the similarity measure can not be used to decide whether or not the object is a prototype or typical object. That is, for some object, the value of similarity measurement is 1, but it is not a prototype of this concept, since it has more attributes than the attributes in intent.

At the beginning of Sect. 3.2, we discussed that since all the objects in one extent belong to a same concept, they should possess some common attributes with each other. Hence, we restrict the value of $k > \frac{1}{2}$. The following proposition shows that the restriction of k guarantees the existence of common attributes of a concept.

Proposition 2. *Let (o^{**k}, o^*) be a k-cutting concept. If $k > \frac{1}{2}$, then, for any $o_1, o_2 \in o^{**k}$, we have $o_1^* \cap o_2^* \cap o^* \neq \emptyset$.*

Proof. Because of $o_1, o_2 \in o^{**k}$, based on Definition 6, we have $|o_1^* \cap o^*| \geq k \cdot |o^*|$ and $|o_2^* \cap o^*| \geq k \cdot |o^*|$. Since we suppose $k > \frac{1}{2}$, the formulas $|o_1^* \cap o^*| > \frac{|o^*|}{2}$ and $|o_2^* \cap o^*| > \frac{|o^*|}{2}$ can be obtained. Thus, $|o_1^* \cap (o_2^* \cap o^*)| = |(o_1^* \cap o^*) \cap (o_2^* \cap o^*)| = |(o_1^* \cap o^*)| + |(o_2^* \cap o^*)| - |(o_1^* \cap o^*) \cup (o_2^* \cap o^*)| > \frac{|o^*|}{2} + \frac{|o^*|}{2} - |(o_1^* \cap o^*) \cup (o_2^* \cap o^*)| > |o^*| - |o^*| = 0$. That is, $o_1^* \cap o_2^* \cap o^* \neq \emptyset$.

The result $o_1^* \cap o_2^* \cap o^* \neq \emptyset$ in Proposition 2 can be rewritten as $(o_1^* \cap o^*) \cap (o_2^* \cap o^*) \neq \emptyset$. This proposition shows that in order to let objects in the same concept have common attributes, the value of k should satisfy $k > \frac{1}{2}$. These common attributes are the most important characters of the concept, since they can reflect the commonness of objects in extent.

We continue with Example 2 to demonstrate the ideas of k-cutting concept induced by one typical object and to verify the correctness of Theorem 2 and Proposition 2.

Example 3. According to Definition 7, the $\frac{2}{3}$-cutting concepts induced by typical object o_1, o_2, o_3, and o_4 are $(\{(o_1, 1)\}, ad)$, $(\{(o_2, 1)\}, bd)$, $(\{(o_3, 1), (o_4, \frac{2}{3})\}, abc)$, and $(\{(o_3, 1), (o_4, 1)\}, bc)$. Compared with the classical object concept induced by object o_3 whose extent only contains object o_3, the $\frac{2}{3}$-cutting concepts induced by object o_3 contains objects o_3 and o_4. And these two objects have different memberships. Since object o_3 is the prototype of this concept, its membership is 1. The membership of object o_4 is $\frac{2}{3}$. The $\frac{2}{3}$-cutting concept can be regarded as a more general concept than the classical concept. The object in k-cutting concept induced by object o does not need to possess all the attributes in description of object o. We only restrict that the description of any object in extent of k-cutting concept contains more than $k \cdot |o|$ attributes in the description of prototype o.

We use $(\{(o_3, 1), (o_4, \frac{2}{3})\}, abc)$, the $\frac{2}{3}$-cutting concept induced by typical object o_3, as an example to show the correctness of Theorem 2. According to the similarity measurement calculated in Example 2, we have $Sim(o_1, o_3) = \frac{1}{3} < \frac{2}{3}$, $Sim(o_2, o_3) = \frac{1}{3} < \frac{2}{3}$, $Sim(o_3, o_3) = 1 \geq \frac{2}{3}$ and $Sim(o_4, o_3) = \frac{2}{3} \geq \frac{2}{3}$. Based on Theorem 2, only objects o_3 and o_4 belong to the extent of $\frac{2}{3}$-cutting concept $(\{(o_3, 1), (o_4, \frac{2}{3})\}, abc)$, which is consistent with our calculation by Definition 7. Also, from $(\{(o_3, 1), (o_4, 1)\}, bc)$, the $\frac{2}{3}$-cutting concept induced by typical object o_4, we can see that the similarity of object o_3 to typical object o_4 is 1, but object o_3 is not the typical object of this concept, since its description is $\{a, b, c\}$ which is bigger than $\{b, c\}$, description of typical object o_4.

We will check the correctness of Proposition 2 in the following. Since $k = \frac{2}{3} > \frac{1}{2}$, every two objects in $o_3^{**\frac{2}{3}}$ should have common attributes. Based on Table 1, we have $o_3^{**\frac{2}{3}} = \{o_3, o_4\}$, and we can calculate $(o_3^* \cap o_3^*) \cap (o_4^* \cap o_3^*) = \{a, b, c\} \cap \{b, c\} = \{b, c\} \neq \emptyset$. The results are consistent to Proposition 2.

4 Conclusion

In formal concept analysis, the formal concept is a mathematical formation of the classical view of concept and reflects a semantic meaning “commonly possessing”. However, in our daily life, the prototype view of concepts is more common and just reflects the meaning of “mostly possessing”. In this paper, we discussed the similarity between two objects and defined the mathematical formation of the prototype of concepts, named k-cutting concepts. Moreover, the properties of this newly proposed concept are studied and its rationality is discussed.

The results of this paper suggest several future research topics. It is interesting to investigate the structure of k-cutting concepts and the k-cutting concepts can be generalized as the k-cutting concepts induced by a group of typical objects.

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