A Control Problem for Parabolic Systems with Incomplete Information

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Abstract. In this paper, abstract parabolic control systems in Hilbert space are considered. The state of the system is unknown, but there is an equation of measurement in discrete times. The initial state and disturbances are restricted by joint integral constraints. According to measurements, the information set is introduced that contains the true state of the system. This set includes all the states of the system that are compatible with the measurements. The preliminary aim of control consists in minimization of the terminal criterion depending of the information set. We suggest some statements of the problem based on the separation of control and observation processes. The optimal instants of transition from estimation to control are looked for as well. The approach is applied to distributed systems with partial derivatives and to systems with the deviation of time of retarded and neutral types. The approximation scheme are suggested and examples are considered.

Keywords: Control · Evolutionary systems · Information sets · Incomplete information

1 Introduction and Preliminaries

First of all we indicate that problems of control under incomplete information were investigated in many books and papers [3–8]. The authors use either the stochastic approach [7] or the minimax deterministic one going back to [3] and developed in subsequent works. We keep to the deterministic problem formulation in [3,4]. Similar formulations were used and modified in [9–11]. In this work, we continue and complement [12,13] trying to generalize some results from [14,15] on the case of infinite-dimensional systems. The algorithm of solution is developed and special cases are considered for parabolic and hyperbolic partial differential systems. Examples are examined. We consider also finite dimensional and numerical approximations for the problem.

1.1 Weak Solutions of Evolutionary Systems

Let V, H be two real Hilbert spaces with norms $\|\cdot\|$ and $|\cdot|$ respectively. Suppose that $V \subset H$, V is dense imbedded in H and separable, $|v| \leq \gamma\|v\|$

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for every $v \in V$. The last inequality means that the imbedding V into H is continuous and the dual space V^* contains $H^* = H$. The spaces H and H^* are identified. Let further $a(u, v)$ be a continuous, bilinear and coercive form on V, such that $a(v, v) \geq \alpha \|v\|^2$, $\forall v \in V$.

Let a function $f : [0, T] \to H$ be measurable and $\int_0^T \|f(t)\|^2 dt < \infty$. For every point $z_0 \in H$ there exists a unique continuous in H function $z(t) \in V$, $t > 0$, such that

$$d\langle z(t), v\rangle/dt + a(z(t), v) = \langle f(t), v\rangle, \quad \forall v \in V, \quad z(0) = z_0. \tag{1}$$

Here $z(t)$ is implicitly supposed to be weakly absolutely continuous (see [1]).

The form $a(u, v)$ defines a linear continuous operator $u \to Au \in V^*$ according to the equality $a(u, v) = \langle Au, v\rangle$. Define by $D(A)$ the set of all elements $h \in V$, for which $Ah \in H \subset V^*$. The operator $-A$ on H is an infinitesimal closed generator for some strongly continuous semigroup $S(t) : H \to H$ (see [1,2]). Besides the solution of (1) has a form

$$z(t) = S(t)z_0 + \int_0^t S(t - s)f(s)ds, \tag{2}$$

where the integral is understood in Bochner's sense [2]. Remark that the solution of (1) may be considered as a generalized solution of Cauchy problem

$$\dot{z} + Az = f(t), \quad z(0) = z_0 \in H. \tag{3}$$

The generalized solution of (3) exists, is unique and may be represented by (2). The solution $z(t)$ is weakly differentiable in H, i.e. the weak limit $\lim_{\delta \to 0}(z(t + \delta) - z(t))/\delta = dz(t)/dt$ there exists a.e. on $[0, T]$ in weak topology of H.

2 The System and Measurements

Consider a controlled system of the form

$$\dot{z} + Az = Bu(t) + C\xi(t), \quad z \in H. \tag{4}$$

Suppose that the operator A is defined by continuous bilinear form $a(u, v)$ given on a separable Hilbert space $V \subset H$; B and C are continuous linear operators from Hilbert spaces H_1 and H_2 to the H, respectively. Let $L_2(0, T, H_i)$ be the Hilbert space of weakly measurable functions $f(t) \in H_i$ such that $\int_0^T \|f(t)\|^2 dt \leq \infty$. According to Subsect. 1.1, an each pair of functions $u(\cdot) \in L_2(0, T; H_1)$ and $\xi(\cdot) \in L_2(0, T; H_2)$ along with an initial state $z_0 \in H$ defines a unique weak solution $z(t; z_0, u, \xi)$ of (4). This solution satisfies the equation

$$d\langle z(t), v\rangle/dt + a(z(t), v) = \langle Bu(t) + C\xi(t), v\rangle, \quad \forall v \in V, \quad z(0) = z_0,$$

and may be represented as

$$z(t) = S(t)z_0 + \int_0^t S(t - s)(Bu(s) + C\xi(s))ds. \tag{5}$$

In what follows the state $z(t)$ of (4) or (5) is unknown. The available information about it may be described as follows. Given a uniform partition $0 = t_0 < t_1 < \cdots < t_N = T$ of $[0, T]$, $t_i - t_{i-1} = T/N = \delta$, at the instants t_i a vector $y_i = Gz(t_{i-1}) + w_i$ is observed, where $G : H \to R^m$ is a finite-dimensional linear operator. Unknown disturbances $\xi(\cdot)$, the initial state z_0, and vectors w_i are restricted by the joint constraint

$$\|z_0\|^2_{P_0} + \int_0^T \|\xi(t)\|^2_Q dt + \sum_{i=1}^N \|w_i\|^2_R \leq 1. \tag{6}$$

Here and further we use the notation $\|u\|^2_F = \langle u, Fu\rangle$ for a self-adjoint positive and coercive operator F; $\langle \cdot, \cdot \rangle$ is an inner product in the corresponding space. The operators P_0, Q, and the matrix R are supposed to be similar to F. Besides, we have a constraint on the control $u(\cdot)$:

$$\int_0^T \|u(t)\|^2_F dt \leq 1. \tag{7}$$

2.1 Transformation to a Discrete-Time System

System (5) with measurements and controls may be represented in a discrete-time form

$$\begin{gathered} z_i = Sz_{i-1} + \eta_i + \xi_i, \quad \text{where} \quad S = S(\delta), \quad z_i = z(t_i), \\ \eta_i = \int_{t_{i-1}}^{t_i} S(t_i - s)Bu(s)ds, \quad \xi_i = \int_{t_{i-1}}^{t_i} S(t_i - s)C\xi(s)ds, \\ y_i = Gz_{i-1} + w_i, \quad i \in 1 : N. \end{gathered} \tag{8}$$

Let us derive constraints on parameters in (8). Denote by ξ_i^N the set of elements $\{\xi_i, \ldots, \xi_N\}$. The symbol w_i^N has the same meaning. If $i = 1$, we write $\xi^N = \xi_1^N$. Find first the support function (see, for example, [6]) of all the parameters $\{z_0, \xi^N, w^N\}$ according to constraints (6). Let $\chi_A(s)$ be a characteristic function. We have

$$\begin{aligned} &\max_{z_0, \xi(\cdot), w^N} \left\{ \langle k, z_0\rangle + \sum_{i=1}^N \Big(\langle l_i, \xi_i\rangle + \langle m_i, w_i\rangle \Big) \right\} \\ &= \max_{z_0, \xi(\cdot), w^N} \left\{ \langle k, z_0\rangle + \int_0^T \left\langle \sum_{i=1}^N \chi_{[t_{i-1}, t_i]}(s) C^* S^*(t_i - s) l_i, \xi(s) \right\rangle ds + \sum_{i=1}^N \langle m_i, w_i\rangle \right\} \\ &= \sqrt{\langle k, P_0^{-1} k\rangle + \sum_{i=1}^N \Big(\langle l_i, \mathbf{C} l_i\rangle + \langle m_i, R^{-1} m_i\rangle \Big)}, \end{aligned}$$

where the self-adjoint positive operator $\mathbf{C}$ is defined as

$$\mathbf{C}l = \int_0^{\delta} S(\delta - s)CQ^{-1}C^*S^*(\delta - s)lds$$
$$= \int_{t_{i-1}}^{t_i} S(t_i - s)CQ^{-1}C^*S^*(t_i - s)lds.$$

that does not depend on i. Doing the same with the control, we obtain

$$\max_{u(\cdot)} \left\{ \sum_{i=1}^{N} \langle l_i, \eta_i \rangle \right\} = \max_{u(\cdot)} \left\{ \int_0^T \left\langle \sum_{i=1}^{N} \chi_{[t_{i-1},t_i]}(s)B^*S^*(t_i - s)l_i, u(s) \right\rangle ds \right\}$$
$$= \sqrt{\sum_{i=1}^{N} \Big(\langle l_i, \mathbf{B}l_i \rangle \Big)},$$

where the self-adjoint positive operator $\mathbf{B}$ is defined as

$$\mathbf{B}l = \int_0^{\delta} S(\delta - s)BF^{-1}B^*S^*(\delta - s)lds. \tag{9}$$

Now defining $\mathbb{B} = \mathbf{B}^{1/2}$ and $\mathbb{C} = \mathbf{C}^{1/2}$ we come to the conclusion.

Lemma 1. *The discrete-time system* (8) *with constraints* (6), (7) *is equivalent to the system*

$$z_i = Sz_{i-1} + \mathbb{B}u_i + \mathbb{C}v_i, \quad \textit{with constraints} \tag{10}$$
$$\sum_{i=1}^{N} \|u_i\|^2 \le 1, \quad \|z_0\|^2_{P_0} + \sum_{i=1}^{N} \left(\|v_i\|^2 + \|w_i\|^2_R \right) \le 1,$$
$$y_i = G_i z_{i-1} + w_i, \quad i \in 1 : N.$$

Proof. It follows from the fact that the support functions of the sets $\{\mathbb{B}u^N\}$ and $\{z_0, \mathbb{C}v^N, w^N\}$ coincide with functions found above. □

Note that the states z_i of system (10) are not the approximations of $z(t_i)$. We have the equality $z_i = z(t_i)$ under some parameters in the systems.

3 Estimation for Discrete-Time Evolutionary Systems

For system (10) the *information set* $\mathcal{Z}_j(y, u)$ (see [4]) is defined as follows.

Definition 1. The set $\mathcal{Z}_j(y, u) \subset H$ is said to be *informational* if it consists of all vectors z_j for which there exist elements z_0, v_i, w_i, such that Eq. (10) are fulfilled for all $i \in 1 : j$, constraints in (10) hold, and measurements $y_i = Gz_{i-1} + w_i$ are valid for all $i \in 1 : j$.

Introduce the linear operator $\mathbb{S}(z, v) = Sz + \mathbb{C}v$. The representation of $\mathcal{Z}_i(y, u)$ is given by

Theorem 1. *The information set is the ellipsoid* $\mathcal{Z}_i(y,u) = \{z : \|z-\hat{z}_i\|^2_{P_i}+h_i \leq 1\}$ *with parameters given by the formulas*

$$\begin{aligned} P_i^{-1} &= SJ_i^{-1}S^* + \mathbf{C}, \quad J_i = P_{i-1} + G^*RG, \\ \hat{z}_i &= \mathbb{B}u_i + S\check{z}_i, \quad \check{z}_i = \hat{z}_{i-1} + J_i^{-1}G^*R(y_i - G\hat{z}_{i-1}), \\ \hat{z}_0 &= 0, \quad h_i = h_{i-1} + \|y_i - G\hat{z}_{i-1}\|^2_{\mathcal{G}_i}, \\ h_0 &= 0, \quad \mathcal{G}_i^{-1} = GP_{i-1}^{-1}G^* + R^{-1}. \end{aligned} \tag{11}$$

The sum $\|z-\hat{z}_i\|^2_{P_i}+h_i$ *is a minimum of relation* $\|z_0\|^2_{P_0}+\sum_{j=1}^{i}\left(\|v_j\|^2+\|w_j\|^2_R\right)$ *under the assumption that parameters* z_0, v_j, w_j *submit the boundary condition* $z_i = z$ *due to Eq.* (10).

Proof. Theorem 1 may be proved by induction. Let $u_i = 0$ and $F_i(z,v) = \|v\|^2 + \|y_i - Gz\|^2_R$. Introduce some axillary sets and functions:

$$\begin{aligned} &\mathcal{V}_i(y) = \{(z,v) \in H \times H : V_{i-1}(z) + F_i(z,v) \leq 1\}, \\ &\mathcal{Z}_i(y) = \mathbb{S}\mathcal{V}_i(y), \quad V_0(z) = \|z\|^2_{P_0}, \quad i \in 1:N, \\ &V_i(z_i) = \begin{cases} \min\limits_{(z,v)\in\mathcal{V}_i(y)} \{V_{i-1}(z) + F_i(z,v) : z_i = \mathbb{S}(z,v)\}, & z_i \in \mathcal{Z}_i(y), \\ 2, & z_i \notin \mathcal{Z}_i(y). \end{cases} \end{aligned} \tag{12}$$

The set $\mathcal{V}_i(y)$ is said to be *compartible with signal* at the instant i, the set $\mathcal{Z}_i(y)$ is *informational* at the instant i. So, the sets $\mathcal{Z}_i(y)$ are images of $\mathcal{V}_i(y)$ according to (10). Let the signal y^N be realized under the elements z_0^*, v_i^*, w_i^*, $i \in 1:N$. Then the constraints in (10) are fulfilled with these elements. We assert that sets $\mathcal{V}_i(y)$ and $\mathcal{Z}_i(y)$ are not empty for all $i \in 1:N$. The function $V_i(z_i)$ is equal to the minimum of functional $\tilde{F}_i(z_0,v^i,y) = \|z_0\|^2_{P_0}+\sum_{j=1}^{i} F_j(z_{j-1},v_j)$ over all the elements z_0, v^i, satisfying to (10) and the boundary condition $z_i = \mathbb{S}(z_{i-1},v_i)$. The informational sets $\mathcal{Z}_i(y)$ are expressed by the inequality $\mathcal{Z}_i(y) = \{z \in H : V_i(z) \leq 1\}$. Note that the functional $\tilde{F}_i(z_0^*,v^{i*},y) \leq 1$ for all $i \in 1:N$. Therefore, the pair $(z_{i-1}^*,v_i^*) \in \mathcal{V}_i(y)$ and the element $z_i^* \in \mathcal{Z}_i(y)$ $\forall i$. The sets in (12) are not empty. The relation $\mathcal{Z}_i(y) = \{z \in H : V_i(z) \leq 1\}$ is obvious for $i = 1$. Indeed, we have

$$\begin{gathered} \mathcal{V}_1(y) = \left\{(z,v) : \|z\|^2_{P_0} + \|v\|^2 + \|y_1 - Gz\|^2_R = \|z - \check{z}_1\|^2_{J_1} + \|v\|^2 + h_1 \leq 1\right\}, \\ \mathcal{Z}_1(y) = \mathbb{S}\mathcal{V}_1(y) = \left\{z : \|z-\hat{z}_1\|^2_{P_1} + h_1 = V_1(z) \leq 1\right\}. \end{gathered}$$

Here we use the known inverse operator formula $R - RG(P+G^*RG)^{-1}G^*R = (R^{-1}+GP^{-1}G^*)^{-1}$. Let the relation $\mathcal{Z}_{i-1}(y) = \{x \in H : V_{i-1}(x) \leq 1\}$ be valid and formulas (11), (12), $i \geq 2$, be fulfilled for $i-1$. Now, from (12) it follows that the inclusion $z_i \in \mathcal{Z}_i(y)$ results in the existence of pair $(z_{i-1},v_i) \in \mathcal{V}_i(y)$, for which $z_i = \mathbb{S}(z_{i-1},v_i)$. Therefore, $V_i(z_i) \leq 1$. Conversely, if the last inequality is valid, then by definition there exists a pair such that $z_i = \mathbb{S}(z_{i-1},v_i) \in \mathbb{S}\mathcal{V}_i(y) = \mathcal{Z}_i(y)$. Moving back in indexes, we obtain that the inclusion $z \in \mathcal{Z}_i(y)$ is equivalent to the existence of the set (z_0,v^i), for which $\tilde{F}_i(z_0,v^i,y) \leq 1$ and

$z = \mathbb{S}(z_{i-1}, v_i)$ under Eq. (10). So, we get $\min_{z_0, v^i} \tilde{F}_i(z_0, v^i, y) = V_i(z)$ under the boundary condition $z = \mathbb{S}(z_{i-1}, v_i)$. Suppose that $V_{i-1}(z) = \|z - \hat{z}_{i-1}\|^2_{P_{i-1}} + h_{i-1}$, $i \geq 2$. Then

$$\begin{aligned} \mathcal{V}_i(y) &= \big\{(z, v) : \|z - \hat{z}_{i-1}\|^2_{P_{i-1}} + h_{i-1} + \|v\|^2 + \|y_i - Gz\|^2_R \\ &\qquad = \|z - \check{z}_i\|^2_{J_i} + \|v\|^2 + h_i \leq 1\big\}, \\ \mathcal{Z}_i(y) &= \mathbb{S}\mathcal{V}_i(y) = \big\{z : \|z - \hat{z}_i\|^2_{P_i} + h_i = V_i(z) \leq 1\big\}. \end{aligned}$$

We see that values $y_i - G\hat{z}_{i-1}$ and h_i do not depend on controls u_i. Therefore, the values $\mathbb{B}u_i$ are added additively only for the second equality in (11). □

4 Problem Formulation and General Solution

We are going to formulate a problem in which processes of estimation and control are separate in time. At first the estimation is provided under given control and we get the information set $\mathcal{Z}_i(y, u)$. After that the minimax off-line procedure is realized. Our main control problem consists in finding of the instant i of finishing observation and passing to the new control on the rest of time.

4.1 Minimax Off-Line Control

From now on we introduce the other *compatible set* $\mathbf{V}_i(y, u)$ of uncertain parameters consisting of all pairs (z_i, v^N_{i+1}) that are compatible with the signal y^i. The projection $\mathrm{proj}_H \mathbf{V}_i(y, u)$ of the compatible set on H coincides with the information set $\mathcal{Z}_i(y, u)$. This new compatible set is defined by the formula

$$\mathbf{V}_i(y, u) = \left\{ (z, v^N_{i+1}) : \|z - \hat{z}_i\|^2_{P_i} + \sum_{j=i+1}^{N} \|v_j\|^2 \leq 1 - h_i \right\},$$

where parameters are given in (11). Let $\tilde{u} = u^N_{i+1}$ be some controls and $\mathcal{Z}_N(\tilde{u} \mid \mathbf{V}_i(y, u))$ be the attainability domain of first equation in (10) with respect to $\mathbf{V}_i(y, u)$ under given further controls $\tilde{u}$. Consider some functional $\Phi(\mathcal{Z})$ that defined on all bounded sets $\mathcal{Z} \subset H$. The primary objective of controls consists in minimization of the cost $\Phi(\mathcal{Z}_N(y, u))$ that depends on the information set. At the initial instant we choose optimal control $u^{N,0}$ that solves the problem $\Phi(\mathcal{Z}_N(u^N \mid \mathbf{V}_0)) \to \min_{u^N} = r_0$ and after that it is corrected. Here $\mathbf{V}_0 = \Big\{(z, v^N) : \|z\|^2_{P_0} + \sum_{j=1}^{N} \|v_j\|^2 \leq 1\Big\}$ and the measurements are not taken into account.

At any instant $i = 1, \ldots, N$ we solve the auxiliary control problem

$$\Phi(\mathcal{Z}_N(u \mid \mathbf{V}_i(y, u^0))) \to \min_{u \in \mathbf{U}_i(u^0)} = r_i(y, u^0), \tag{13}$$

where $u^0 = u^{N,0}$ is a control chosen at initial instant; $\mathbf{U}_i(u^0)$ is a set of controls after the instant i, i.e. $\mathbf{U}_i(u^0) = \big\{u^N_{i+1} : \sum_{j=i+1}^{N} \|u_j\|^2 \leq 1 - \sum_{j=1}^{i} \|u^0_j\|^2\big\}$. Suppose that there exists at least one optimal control $u^{N,i}_{i+1}$ in problem (13).

4.2 Finding of the Observation Stopping Time

Now we explain how to find the instant i of finishing observation and passing to the new optimal control $u_{i+1}^{N,i}$ of problem (13) on the rest of time. To do the choice we compare the value $r_i(y, u^0)$ with value of forecasting

$$r_i(s, y^i, u^s) = \max_{y_{i+1}^s \in Y_{s,i}(y^i, u^s)} r_s(y, u), \tag{14}$$

where $Y_{s,i}(y^i, u^s) = \{y_{i+1}^s\}$ is a set of all possible continuations of signal y^i up to the instant $s > i$. The value (14) is the worst result of control if the system is located in the position $\{y^i, u^i\}$ and up to the instant s the control u_{i+1}^s is used. We set $r_i(i, y^i, u^i) = r_i(y, u)$. Our problem can be repeated [14,15]. Introduce one more value $\underline{r}_i(y, u) = \min_{s \in i:N} r_i(s, y^i, u^s)$. Let us be already located in position $\{y^i, u^i\}$, where u^i is a part of control u^N previously found. In this case, we verify the condition $\underline{r}_i(y, u) < r_i(y, u)$, $(i \in 1 : N-1)$. If this holds, then the control u_{i+1}^N does not change. Otherwise, we pass to the new control $u_{t+1}^{N,t}$, delivering the minimum in (13). So, the first instant i such that

$$\underline{r}_i(y, u) \geq r_i(y, u), \quad \text{where} \quad i \in 1 : N-1, \tag{15}$$

we call the *observation stopping time*. In this instant i the observation is stopped and we pass the optimal off-line control in problem (13).

Consider some particular cases. Let $u = u^{N,0}$. If $\underline{r}_1(y, u) \geq r_1(y, u)$, then the observation is stopped at first instant. From the other hand, suppose that relations (15) are not valid for all $i \in 1 : N-1$ and $\sum_{i=1}^N \|u_i^0\|^2 < 1$. In this case, the observation continues all the time, but the resource of control is not exhausted at the last instant N. Therefore, we can solve the minimax problem $\Phi(\mathcal{Z}_N(y, u)) \to \min_{u_N}$, $\|u_N\|^2 \leq 1 - \sum_{i=1}^N \|u_i^0\|^2$, and regard optimal $\tilde{u}_N$ as an additional control action at the last instant.

4.3 An Algorithm of Repeated Correction

If we can continue observation after any stopping time, then the following algorithm of repeated correction can be proposed.

1. We find the value r_0 and optimal control $u^{N,0}$ before any observations.
2. At $i = 1$ we decide if this control has to be changed, i.e. if the value $\underline{r}_1(y, u^{N,0}) < r_1(y, u^{1,0})$ then the control $u^{N,0}$ should be kept. Otherwise, we pass to the new control $u_2^{N,1}$, delivering the minimum in (13).
3. In position $\{y^i, u^i\}$, where u^i is a part of control u^N previously found, we verify the condition (15), where $i \in 1 : N-1$. If this holds, then we pass to the optimal control $u_{i+1}^{N,i}$, delivering the minimum in (13).
4. In any case, if at the last instant N the inequality $\sum_{i=1}^N \|u_i\|^2 < 1$ is obtained, we solve the minimax problem $\Phi(\mathcal{Z}_N(y, u)) \to \min_{u_N}$, $\|u_N\|^2 \leq 1 - \sum_{i=1}^N \|u_i\|^2$, and regard optimal $\tilde{u}_N$ as an additional control action at the last instant.

According to the algorithm, we obtain the sequence $\{\tau_1, \tau_2, \dots\}$ of instants where control has been changed. This sequence depends on the signal. In particular, the sequence may be empty when observations are bad for control, or it may coincide with the set $1 : N-1$, when, on the contrary, the observations give essential information. The values $r_i = r_{\tau_i}(y,u)$ form the nonincreasing sequence. Here the strong inequalities $r_i > r_{i+1}$ hold if $\tau_{i+1} - \tau_i \geq 2$. In the case $\tau_{i+1} - \tau_i = 1$ the strong inequality $r_i > r_{i+1}$ holds if and only if the signal $y_{\tau_{i+1}}$ is not the worst.

Instead of inequality (15) at every instant $i < N$, we may check the simpler condition $r_t(t+1, y^t, u^{t+1}) < r_t(y,u)$. If it is fulfilled, then the control u_{t+1}^N does not change. Otherwise, we pass to the new control $u_{t+1}^{N,t}$ in problem (13).

5 A Special Case of the Terminal Cost

Let the terminal functional has the form $\Phi(\mathcal{Z}) = \max_{z\in\mathcal{Z}} \|\Delta z\|$, where $\Delta : H \to R^k$ is a linear finite-dimensional operator and $\|\cdot\|$ is the Euclidean norm. In this case, we can obtain formulas (13)–(15) in more detail.

First of all we describe all the continuations of the signal.

Lemma 2. *A signal y_{i+1}^s is a continuation of the signal y^i iff there exists a sequence φ_{i+1}^s such that $\sum\limits_{j=i+1}^{s} \|\varphi_i\|^2_{\mathcal{G}_i} \leq 1 - h_i$, and $\hat{z}_j = \mathbb{B}u_j + S(\hat{z}_{j-1} + J_j^{-1}G^*R\varphi_j)$, $y_j = G\hat{z}_{j-1} + \varphi_j$, for $j \in i+1 : s$.*

This lemma follows from Eq. (12). Below we use vectors $l \in R^k$ as column-vectors and the symbol l' is used for row-vector. Then we have the relation

$$r_i(y,u) = \max_{l'l\leq 1}\Bigg\{\gamma_i(l)\hat{z}_i - \Bigg(1 - \sum_{j=1}^{i}\|u_i\|^2\Bigg)^{1/2}\Bigg(\sum_{j=i+1}^{N}\gamma_j(l)\mathbf{B}\gamma_i^*(l)\Bigg)^{1/2} + (1-h_i)^{1/2}\left(\pi_0(i)(1-l'l) + l'\Delta P_{N,i}\Delta^* l\right)^{1/2}\Bigg\}, \tag{16}$$

where $\gamma_j(l) = \gamma_{j+1}(l)S$, $\gamma_N(l) = l'\Delta$; $P_{j,i} = SP_{j-1,i}S^* + \mathbf{C}$, $P_{i,i} = P_i^{-1}$; $\pi_0(i) = \max_{l'l\leq 1} l'\Delta P_{N,i}\Delta^* l$. Using Lemma 2, we obtain

$$r_i(s, y^i, u^i) = \max_{l'l\leq 1}\Bigg\{\gamma_i(l)\hat{z}_i + \sum_{j=i+1}^{s}\gamma_j(l)\mathbb{B}u_j - \Bigg(1 - \sum_{i=1}^{s}\|u_i\|^2\Bigg)^{1/2} \cdot\Bigg(\sum_{j=s+1}^{N}\gamma_j(l)\mathbf{B}\gamma_j^*(l)\Bigg)^{1/2} + (1-h_i)^{1/2}(\pi_0(s)(1-l'l) + l'\Delta P_{N,i}\Delta^* l)^{1/2}\Bigg\}. \tag{17}$$

Formulas (16)–(17) are established similarly to [4,9]. In addition, let us note that optimal control is on the formula

$$u_j^0 = -\mathbb{B}\gamma_j^*(l^0)\Bigg(1 - \sum_{i=1}^{j}\|u_i\|^2\Bigg)^{1/2}\Bigg(\sum_{i=j+1}^{N}\gamma_i(l^0)\mathbf{B}\gamma_i^*(l^0)\Bigg)^{-1/2}, \quad j > i,$$

where l^0 is a maximizer in formula (16) which does not convert the corresponding sum into zero.

6 A Finite-Dimensional Approximation

Let us return to general relations in Sect. 1, where V is a separable Hilbert space and $a(u,v)$ is a bilinear form with properties:

$$a(v,v) \geq \alpha\|v\|^2, \quad a(u,v) \leq \beta\|u\|\|v\|. \tag{18}$$

Given finite-dimensional subspace $\mathcal{F} \subset V$, define Ritz's projector $\Pi : V \to \mathcal{F}$ as $a(v, u - \Pi u) = 0,\ \forall v \in \mathcal{F}$ (see [16]). The following estimate holds:

$$\|u - \Pi u\| \leq \beta d(u,\mathcal{F})/\alpha, \quad \text{where} \quad d(u,\mathcal{F}) = \min_{v\in\mathcal{F}} \|u - v\|. \tag{19}$$

Consider an increasing sequence $\mathcal{F}^n$ of finite-dimensional subspaces $\mathcal{F}^n \subset \mathcal{F}^{n+1} \subset V$ such that the distance $d(u,\mathcal{F}^n) \to 0$ as $n \to \infty\ \forall u \in V$. Such a sequence is called *complete*. The proof of following lemma may be found in [16] or somewhere.

Lemma 3. *Let $u : [0,T] \to V$ be a continuous function and $\mathcal{F}^n$ be a complete sequence of finite-dimensional subspaces. Then the real function $\|u(t) - \Pi^n u(t)\|$ tends to zero uniformly in $t \in [0,T]$, where $\Pi^n : V \to \mathcal{F}^n$ is the Ritz projector.*

Let H be another Hilbert space and let the space $V \subset H$ be densely imbedded in H as in Sect. 1. The linear operator A with a dense domain $D(A) \subset V$ has been defined as $a(u,v) = \langle Au, v\rangle_H,\ \forall v \in V$. The dual operator A^* is defined by the relation $a(u,v) = \langle u, A^*v\rangle_H,\ \forall u \in V$. The operator $-A^*$ is a infinitesimal generator for the semigroup $S^*(t)$ (see, for example, [17]). In addition, the function $\psi(t) = S^*(t)\psi$, where $\psi \in H$, is defined a weak solution of equation

$$d\langle v, \psi(t)\rangle_H/dt + a(v,\psi(t)) = 0 \ \ \forall v \in V, \quad \psi(0) = \psi.$$

This equation is similar to (1). Let us remind that the inclusion $z_0 \in D(A)$ implies $z(t) = S(t)z_0 \in D(A)$ for all $t \geq 0$ and

$$dz(t)/dt + Az(t) = 0, \tag{20}$$

i.e. $z(t)$ is a strong solution of Eq. (20).

Suppose that the increasing sequence $\mathcal{F}^n \subset V$ of finite-dimensional subspaces is complete. Consider the problem

$$d\langle z^n(t), v^n\rangle/dt + a(z^n(t), v^n) = \langle f(t), v^n\rangle \ \ \forall v^n \in \mathcal{F}^n, \quad z^n(0) = z^n, \tag{21}$$

where one needs to find a function $z^n(t) \in \mathcal{F}^n$. The problem (21) is called the *Galerkin-type finite-dimensional approximation* of problem (1). We need the following

Theorem 2 **([18]).** *Let $z^n \to z$ in the space H as $n \to \infty$. Then the solution $z^n(t)$ of problem (21) uniformly converges on $[0,T]$ to the solution $z(t)$ of problem (1) in the space H.*

Let $e_1, \dots, e_n$ be a basis in the space $\mathcal{F}^n$. We set

$$z^n(t) = \sum_{j=1}^{n} q^j(t)e_j, \quad z^n = \sum_{j=1}^{n} q^j e_j.$$

A finite-dimensional approximation of problems in Sect. 4 with respect to the complete sequence $\mathcal{F}^n$ of subspaces is as follows. Problem (21) is equivalent to the solution of differential equations in matrix form:

$$M\dot{q} + Kq = \mathbf{f}(t), \quad q_0 = [q^1; \dots; q^n], \quad \mathbf{f}(t) = [\langle f(t), e_1\rangle; \dots; \langle f(t), e_n\rangle],$$

where M ($\det M \neq 0$) and K have elements $\langle e_i, e_j\rangle$ and $a(e_i, e_j)$ respectively. The solution of the system for our problems may be written similarly to (5):

$$q(t) = S^n(t)q_0 + \int_0^t S^n(t-s)(B^n\mathbf{u}(s) + C^n\mathbf{v}(s))ds, \tag{22}$$

where $S^n(t) = \exp(-M^{-1}Kt)$ is the transition matrix having $n \times n$-dimension, $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are n-dimensional measurable functions. Matrices B^n and C^n have the similar structure and represent a multiplication of matrix M^{-1} and the square root of matrices with elements $\langle e_i, BF^{-1}B^*e_j\rangle$ and $\langle e_i, CQ^{-1}C^*e_j\rangle$ respectively. Constraints (6) and (7) are transformed to

$$\|q_0\|^2_{P_0^n} + \int_0^T \|\mathbf{v}(s)\|^2 ds + \sum_{i=1}^{N} \|w_i\|^2_R \leq \nu^n, \quad \int_0^T \|\mathbf{u}(s)\|^2 ds \leq \mu^n. \tag{23}$$

Measurement equation from (8) has the form

$$y_i = G^n q(t_i) + w_i, \quad G^n = [Ge_1, \dots, Ge_n] \in R^{k\times n}. \tag{24}$$

Problems of Sects. 4 and 5 may be solved for relations (22)–(24) as described above.

Let us explain the appearance of numbers μ^n and ν^n in constraints (23). The matter is that the system (10) is infinite-dimensional and, therefore, the signal y^N of this system in some cases can not be realized in finite-dimensional approximation (22)–(24) if we set $\mu^n = 1$, $\nu^n = 1$. But under some $\mu^n > 1$, $\nu^n > 1$ the finite-dimensional formulas like (13)–(17) are valid. Moreover, we get

Theorem 3. *There exist sequences $\mu^n \downarrow 1$, $\nu^n \downarrow 1$ as $n \to \infty$ such that formulas like* (11)–(17) *for finite-dimensional approximation* (22)–(24) *hold and $r_i^n(y, u) \to r_i(y, u)$, $r_i^n(s, y^i, u^i) \to r_i(s, y^i, u^i)$ as $n \to \infty$ in relations* (16), (17).

In the general case, it is hard to obtain the estimates of velocity for convergence $\mu^n \downarrow 1$, $\nu^n \downarrow 1$ with respect to parameters α, β in (18), (19).

6.1 An Application to Heat Equation

Let the controlled system be described by the equations

$$z_t = z_{xx} + u(t)f(x), \quad x \in [0,l], \quad t \geq 0, \text{ with boundary conditions} \tag{25}$$
$$z_x(t,0) = z(t,0), \quad z_x(t,l) = -z(t,l).$$

Here $f(x)$ is a smooth function on $[0,l]$, $u(t)$ is a control. This system describe the heat process for the uniform bar. In our situation $H_1 = R$, $C = 0$, $V = H^1(0,l)$, $H = L_2(0,l)$ where $H^1(0,l)$ is the Sobolev space with parameter $k = 1$. The operator $B : R \to L_2(0,l)$ has the form $Bu = uf(x)$. Dual operator $B^* : L_2(0,l) \to R$ is written as $B^*\phi = \int_0^l f(x)\phi(x)dx$, $\phi \in L_2(0,l)$. The weak form of considered system is obtained by the multiplication of (25) by $\phi \in H^1(0,l)$ with subsequent integration on $[0,l]$ using boundary conditions. The form $a(\phi,\psi)$ may be written as

$$a(\phi,\psi) = \int_0^l \dot{\phi}(x)\dot{\psi}(x)dx + \phi(l)\psi(l) + \phi(0)\psi(0).$$

The coercivity follows from Friedrich's inequality. So, relation (1) for system (25) looks like

$$\partial \int_0^l z(t,x)\phi(x)dx/\partial t + a(z(t,\cdot),\phi(\cdot)) = u(t)\int_0^l f(x)\phi(x)dx$$

for all $\phi \in H^1(0,l)$, $z(0,x) = z_0(x)$.

Let us divide the segment $[0,l]$ by n subsegments of length l/n. Let x_i, $i \in 0:n$, be the points of partition. For the space $\mathcal{F}^n$ we consider piecewise-linear functions $e_i(x)$, for which $e_i(x_i) = 1$ and $e_i(x_j) = 0$ if $i \neq j$. The sequence of finite-dimensional subspaces $\mathcal{F}^n$ with basis $e_i(x)$, $i \in 0:n$, is complete. Therefore, we can perform the approximation. Suppose that measurement equations are of the form

$$y_i = \int_0^l b(x)z(t_{i-1},x)dx + w_i, \quad i \in 1:N, \quad \text{where} \quad b(\cdot) \in L_2(0,l).$$

Consider the $(n+1)\times(n+1)$-matrices M with elements $M_{ij} = \int_0^l e_i(x)e_j(x)dx$ and K with elements $K_{ij} = a(e_i,e_j)$. The M is a three-diagonal symmetric matrix, where $M_{00} = M_{nn} = l/(3n)$ and other diagonal elements are equal to $2l/(3n)$. The secondary diagonal elements are equal to $l/(6n)$. The K is also a three-diagonal symmetric matrix, where $K_{00} = K_{nn} = n/l + 1$ and other diagonal elements are equal to $2n/l$. The secondary diagonal elements of K are equal to $-n/l$. If $f(x) \equiv 1$, then we obtain the finite-dimensional system

$$M\dot{q} + Kq = u(t)\mathbf{f}, \quad \text{where} \quad \mathbf{f} = l[1;2;\ldots;2;1]/(2n) \in R^{n+1}. \tag{26}$$

Let $b(x) \equiv 1$. Then measurement Eq. (24) has the form

$$y_i = G^n q(t_{i-1}) + w_i \quad \text{where} \quad G^n = \mathbf{f}'.$$

Suppose that initial constraints (6), (7) may be written as

$$\int_0^l z^2(0,x)dx + \sum_{i=1}^N w_i^2 \le 1, \quad \int_0^T u^2(t)dt \le 1.$$

It follows from this that constraints (23) are:

$$\|q_0\|_M^2 + \sum_{i=1}^N w_i^2 \le \mu^n, \quad \int_0^T u^2(t)dt \le 1.$$

We need not to increase the constraints for $u(\cdot)$, but we do it for q_0 and w_i in order to include the sequence y^N in the scope. After that we need to convert the continuous system (26) to discrete one of the type (9), (10). Many solved examples of such a finite-dimensional problems where considered in [12,13,19].

7 Conclusion

We considered a control problem with incomplete information for abstract parabolic control systems in Hilbert space. Information about the system state are known in discrete instants. According to measurements, the information set was introduced that contained the true state of the system. This set included all the states of the system that were compatible with the measurements. For the terminal criterion depending of the information set, we suggested some statements of the problem based on the separation of control and observation processes. The optimal instants of transition from estimation to control were looked for as well. The approach was applied to distributed systems with partial derivatives. The approximation scheme was suggested and example with heat equation was considered. In this research some aspects demand more detailed study. For example, we need to obtain the estimates for values μ^n, ν^n, and convergence speed for parameters in Theorem 3. It is interesting to expand the approach to the case of continuous measurements.

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