Chapter 3
How Technology Has Changed What It Means to Think Mathematically

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Early Mathematics

Assigning a start date to mathematics is an inescapably arbitrary act, as much as anything because there is considerable arbitrariness in declaring which particular activities are or are not counted as being mathematics.

Popular histories typically settle for the early development of counting systems. These are generally thought to have consisted of sticks or bones with tally marks etched into them. (Small piles of pebbles might have predated tally sticks, of course, but they would be impossible to identify confidently as such in an archeological dig.) The earliest tally stick that has been discovered is the Lebombo bone, found in Africa, which dates back to around 44,000 years ago. It has been hypothesized that the (evidently) human-carved tally marks on this bone were an early lunar calendar, since it has 29 tally marks (though it is missing one end, that had broken off, so the actual total could have been higher).

Whether the ability to keep track of sequential events by making tally marks deserves to be called mathematics is debatable. "Pre-numeric numeracy" might be a more appropriate term, though the seeming absurdity of that term does highlight the fact that you can count without having numbers, or even a sense of entities we might today call numbers.

Things become more definitive if you take the inventions of the positive counting numbers, as abstract entities in their own right, as the beginning of mathematics. The most current archeological evidence puts that development as occurring around 8000 years ago, give or take a millennium, in Sumeria (roughly, southern Iraq). Various generations of clay object tally systems led eventually to sophisticated schemas of iconic markings on clay tablets that I (and others) suggest we would today call numerals.

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To be sure, such an interpretation imposes a modern lens on a much earlier society, so it requires some justification. I recount the full story—pieced together from the archeological evidence—in my book *The Math Gene* (Devlin 2000: 48–49), but here is the general outline.

Initially, the Sumerians used small clay objects as tokens to represent goods, one shape of token for a jar of oil, another for a bale of wheat, another for a goat, and so on. A person's wealth at any one time was represented by the collection of tokens they had, which were kept in sealed clay envelopes held by a village elder (an early form of banking). When two individuals traded, they would go to the elder who would break open their envelopes and transfer tokens according to the transfer of goods, and then seal their "accounts" in fresh envelopes. In time, to facilitate easy checking of accounts prior to a trade, the elders adopted the practice of pressing each token onto the wet clay before placing it inside and then sealing the contents. The outside of each envelope thus carried markings that recorded the contents. The system thus had three components: actual goods, clay tokens that represented those goods (and hence were in one-to-one correspondence with the goods), and markings impressed on the clay exterior that represented the tokens inside (and hence were in one-to-one correspondence with the tokens).

The next step was the realization that there was no need for the clay tokens. In this case, the clay sheet did not have to be folded into an envelope. All you needed was a sheet of clay and one token of each kind to make the markings on the clay. In today's parlance, those markings would be called numerals (albeit, initially, one kind of numeral for each kind of good). Our present-day abstract numbers came into being as the mental ghosts of the tokens that used to be locked inside clay envelopes.

This perception definitely accords with our current concept of numbers, where numerals represent numbers and numbers count things in the world, though how the Sumerians thought of the process is not knowable to us. What we can say, however, is that if we look back in time to find a practice that accords with our current framework of

NUMERALS represent ABSTRACT NUMBERS represent OBJECTS

then the earliest known example is the Sumerian system of

MARKINGS IN CLAY represent CLAY TOKENS represent OBJECTS

When the clay tokens were eliminated, you arrive at a situation where the two frameworks are essentially the same.

A Sumerian might have said, "There used to be clay tokens in the middle." Today, we might say, "We postulate the existence of abstract entities called numbers in the middle." This modern-day mental shift of regarding the absence of some entity as the presence of some abstraction would surely have made no sense to the Sumerians 8000 years ago. So we cannot claim that the Sumerians had our modern concept of number. But from a functional perspective, that's exactly what they had.

Of course, having (counting) numbers is a far cry from having any form of arithmetic beyond the simple addition and subtraction that was implicit in their earlier manipulations of the clay tokens. So it barely counts as mathematics. Nevertheless, it provides a meaningful time stamp when mathematics first arose and what the first math comprised. (In Devlin 2000, I argue that the brain's *capability* to do mathematics was coevolved with the *capacity* for language, at least 70,000 years ago, but that's not the same as having a mental activity we can classify as mathematics.)

If you look for arithmetic (counting numbers with addition and multiplication) as the earliest genuine mathematics, the best current archeological evidence is the Ishango bone, found in Africa and dating back to around 20,000 BCE. The markings on this tally stick suggest some knowledge of multiplication.

From around 2000 BCE onwards, there is clear evidence of mathematics, with the Egyptians, the Babylonians, the Chinese, the Indians and the Greeks all developing some form of arithmetic, leaving behind multiplication tables inscribed on clay tablets or written on papyrus.

Around the same time, those ancient societies also developed early forms of geometry, extending mathematics from the recognition and study of patterns of number to include also the recognition and study of patterns of shape. In both cases, the driving force for these new ways of thinking was the solution of practical problems: trade and commerce in the case of arithmetic and land apportionment for geometry. (The word "geometry" comes from the Greek *geo metros*, meaning earth measurement.) The focus was primarily on computation—numerical computation and geometric computation, respectively, though in the case of geometry we usually refer to it in terms such as "procedural execution" or "construction" rather than "computation."

Then, starting with Thales of Miletus around 500 BCE, Greek mathematicians introduced the concept of mathematical proof, a process to establish the truth of a particular mathematical assertion, starting with a small collection of precisely stated assertions (called "axioms") and proceeding by the step-by-step application of precisely formulated logical rules of deduction. During the period from around 500 to 300 BCE, Greek mathematicians studied both arithmetic and geometry from this theoretical perspective, culminating in the publication of Euclid's famous work *Elements* around 350 BCE.

This development resulted in a classification of the discipline of mathematics into two broad categories that continues to this day: pure mathematics, where the emphasis is on establishing mathematical truth by means of formal (or at least rigorous) proofs, and applied mathematics, where the goal is to find answers to practical questions, those answers often, but by no means always, being numbers.

While that classification can be useful, it can also be misleading. For one thing, the two categories overlap massively. But more to the point, the distinction obscures the point that, whether or not the goal is to prove a theorem or to obtain an answer to a problem (say, solve an equation to obtain a numerical answer), what the mathematician actually does is compute—in the broader sense of that word mentioned earlier, which includes, in addition to step-by-step numerical calculation, processes such as step-by-step geometrical construction, step-by-step algebraic derivations, and step-by-step construction of a logical proof.

The Growth of Mathematics

By the time the nature of present-day mathematics was (essentially) established by the start of the Current Era, the scope of mathematics had already grown to encompass fractional arithmetic (quotients of counting numbers), integer arithmetic (positive and negative whole numbers), rational arithmetic (positive and negative fractions), real arithmetic (the concept of "real number" coming from measurement rather than counting), and trigonometry (combining geometry and real arithmetic).

In the ensuing two millennia, mathematics continued to expand still further, with new branches of the discipline being developed: algebra, probability theory, differential and integral calculus, mathematical logic, real analysis, complex analysis, differential equations, algebraic number theory, analytic number theory, topology, differential geometry, and more. (Several of these "new" branches had their origins much earlier; for instance, although historians typically ascribe the birth of calculus to Isaac Newton and Gottfried Leibniz in the seventeenth century, some of the key ideas were known to Archimedes around 250 BCE.)

Some of these domains are highly abstract, dealing with mathematical entities well beyond everyday cognitive experience. Nevertheless, regardless of whether the goal was to prove a theorem or calculate (in some manner) an answer, what mathematicians spent the bulk of their time doing was computation—developing and executing procedures of various kinds. Unless you were competent in executing computational procedures, you could not do mathematics. In fact, in the more recent times of systemic education, without mastery of calculation you could not obtain a credential in mathematics.

This dominance of computation was the case throughout the 2000-year development of mathematics up until the 1960s (of which more in due course).

As more and more new branches of mathematics were introduced, it was not just that the objects mathematicians computed on that changed; there were also changes in the way those objects were represented and in the manner in which the computations were carried out.

The most familiar new representation, and arguably by far the most significant in terms of broad impact, is the place value, Hindu-Arabic system for representing and computing with positive whole numbers using just ten symbols 0, 1,…, 9. Developed in India in the first few centuries of the Current Era, it was adopted and extended by Arabic- and Persian-speaking traders, who extended the numerical procedural rules (algorithms) for performing arithmetical calculations to include logical procedures. One of those logical procedures they called *al-jabr*, the Arabic term from which we get the modern Western name for that form of procedural reasoning: "algebra."

Today, we associate the word *algebra* with procedural, symbolic manipulation and reasoning, but that association is largely as a result of the invention of the printing press in the fifteenth century. Although the use of abstract symbols is as old as anything we would today call mathematics, until the fifteenth century, when mathematicians wrote up their work to be copied and distributed (on parchment or later paper), they wrote everything in natural language, with the only abstract symbols being numerals and symbols for the operations of basic arithmetic. This was the case for the many mathematics texts written in and around Baghdad in the ninth, tenth, and eleventh centuries, and the even greater number of books written in Italy (in particular) in the thirteenth and fourteenth centuries.

The reason why mathematics was written in prose was to ensure accuracy of any copies made. Books were duplicated by hand copying, by monks in monasteries in the case of the initial copies of a new work, thereafter by readers making their own copies. The most common way to learn mathematics or study a new mathematical technique was to borrow a copy of an appropriate book and slavishly make a copy of the manuscript, without pausing to understand it or work through the written examples. Then, after returning the original, the learner would slowly work through their newly created personal copy, writing symbolic expressions and drawing diagrams in the margins as they did so, in order to assist with their understanding. Since the 1960s, historians working in the archives in Italy have discovered hundreds of fourteenth-century manuscripts that were evidently created in that way.

Clearly, if a book made use of symbolic mathematical expressions, which would likely be unfamiliar to the monk or the learner making the copy, there would be a high likelihood of copying errors. And as anyone learning mathematics quickly discovers, just one symbolic error can cause a beginner significant difficulties. To avoid this, authors of mathematics books spelled out everything in words and numerals. Even the first ever algebra textbook, written by the Persian mathematician al-Khwarizmi in the ninth century, contains no symbolic equations.

With the introduction of the printing press, however, the situation changed dramatically. Because of demand, mathematics texts were among the very first books to be put into print. With printed books, the process of learning mathematics from a text changed from writing symbolic expressions in the margin to help understand the prose as you progressed through the text, to writing prose remarks and short notes in the margin to elucidate the printed symbolic expressions.

In other words, the cognitive challenge of distilling a prose description of a problem and its solution down to the bare structure and logic (going from concrete to abstract) changed to be the very opposite: taking a symbolic representation of a problem and its solution and creating a mental image—turning the symbols into a story (going from abstract to concrete).

The ability to accurately reproduce symbolic mathematical expressions—and diagrams—that came with the printing press not only changed mathematics learning, it also greatly accelerated the growth of mathematics. The steady development of new branches of mathematics (the algebra, probability theory, differential and integral calculus, mathematical logic, real analysis, complex analysis, differential equations, algebraic number theory, analytic number theory, topology, differential geometry I listed earlier, and others) involved an overall increase in abstraction.

For example, arithmetic and geometry begin with the abstraction of patterns in the world (number and shape, respectively); number theory studies patterns of numbers (patterns of mathematical abstractions); algebra (high school algebra, that is) looks at patterns of arithmetic (patterns across mathematical procedures); and so on.

Such is the complexity and the degree of abstraction of the majority of mathematical patterns studied over the past several centuries that to use anything other than symbolic notation would be prohibitively cumbersome. And so the more recent development of mathematics has involved a steady increase in the use of abstract notations.

The introduction of symbolic mathematics in its modern form is generally credited to the French mathematician Francois Viète in the sixteenth century.

The Nineteenth-Century Mathematical Revolution

During the nineteenth century, mathematicians tackled problems of ever greater complexity, and in so doing they occasionally found that their intuitions were inadequate to guide their work. Counterintuitive (and occasionally paradoxical) results made them realize that some of the methods they had developed to solve important, real-world problems—particularly where calculus was involved—had consequences they could not explain. For instance, the Banach-Tarski theorem says that you can, in principle, take a sphere and cut it up in such a way that you can reassemble it to form two identical spheres each the same size as the original one. Because the mathematics is correct, the Banach-Tarski result had to be accepted as a fact, even though it defies our imagination.

Faced with such "paradoxes," mathematicians had to accept that there are occasions when certainty is achieved only through the mathematics itself. In order to have confidence in discoveries made by way of mathematics, but not verifiable by other means, they had to be sure that the definitions of the mathematical entities and concepts the reasoning depends on are sound and unambiguous, and that the mathematical reasoning itself is correct. To achieve this end, they turned the methods of mathematics inwards, and used them to examine the subject itself.

By the middle of the nineteenth century, this introspection culminated in the adoption of a new and different conception of mathematics, where the primary focus was no longer on performing calculations or computing answers, but formulating and understanding abstract concepts and relationships.

Led by pioneering mathematicians such as Lejeune Dirichlet, Richard Dedekind, Bernhard Riemann, and David Hilbert, there was a shift in emphasis from doing to understanding. Mathematical objects were no longer thought of as given primarily by formulas, but rather as carriers of conceptual properties. Proving something was no longer a matter of transforming terms in accordance with rules—a form of calculation—but a process of logical deduction from concepts. [To be sure, it is possible to view the process of logical deduction as another form of calculation. When you do so you arrive at the branch of mathematics known as formal logic. Indeed, you can do it in using abstract symbols, which results in the subject known as symbolic logic. But this is a side issue for another day.]

In terms of the mechanics of doing mathematics, things did not outwardly appear to have changed; indeed, the entire shift came about as a result of turning those very mechanics inwards onto the abstract entities of mathematics itself. But in the minds of mathematicians, things *had* changed. By the start of the twentieth century, mathematics was primarily about understanding, not calculation.

For example, prior to the nineteenth century, mathematicians were used to the fact that a formula such as $y = x^2 + 3x - 5$ specifies a function that produces a new number y from any given number x. Then Dirichlet said to forget the formula and concentrate on what the function does in terms of input-output behavior. A function, according to Dirichlet, is any rule that produces new numbers from old. The rule does not have to be specified by an algebraic formula. In fact, there's no reason to restrict attention to numbers. A function can be any rule that takes objects of one kind and produces new objects from them. This definition legitimizes functions such as the one defined on real numbers by the rule

$$\text{If } x \text{ is rational, set } f(x) = 0; \text{if } x \text{ is irrational, set } f(x) = 1:$$

For such a function, the notion of "calculating values of the function" makes no sense. It is not possible to graph the function. The questions mathematicians asked about abstract functions, not specified by a formula, focused on their behavior. For example, does the function have the property that when you present it with different starting values it always produces different answers? (This property is called *injectivity*.)

This abstract, conceptual approach was particularly fruitful in the development of the new subject called real analysis—the rigorous underpinnings of calculus, which had been a mathematical Holy Grail since calculus was invented by Isaac Newton and Gottfried Leibniz in the seventeenth century. In real analysis, mathematicians studied the properties of continuity and differentiability of functions as abstract concepts in their own right. French and German mathematicians developed the "epsilon-delta definitions" of continuity and differentiability, that to this day cost each new generation of advanced calculus mathematics students considerable effort to master.

Again, in the 1850s, Riemann defined a complex function by its property of differentiability, rather than a formula, which he regarded as secondary.

The residue classes defined by the Karl Friedrich Gauss were a forerunner of the approach, now standard, whereby a mathematical structure is defined as a set endowed with certain operations, whose behaviors are specified by axioms.

Taking his lead from Gauss, Dedekind examined the new concepts of ring, field, and ideal, each of which was defined as a collection of objects endowed with certain operations.

And so on, continuing to this day.

Like most revolutions, the nineteenth-century shift in focus had its origins in times long before the main protagonists came on the scene. The Greeks had certainly shown an interest in mathematics as a conceptual endeavor, not just calculation; and in the seventeenth century, calculus co-inventor Gottfried Leibniz thought deeply about both approaches. But for the most part, until the nineteenth century, mathematics was viewed primarily as a collection of procedures for solving problems. To twentieth-century (and today's) mathematicians, however, brought up entirely with the postrevolutionary conception of mathematics, what in the nineteenth century was a revolutionary new conception of mathematics is simply taken to be what mathematics is. The revolution may have been quiet, and to a large extent forgotten, but it was complete and far reaching.

Outside the professional mathematical community, however, there were few signs of a revolution at all. For most scientists, engineers, and others who make use of mathematical methods in their daily work, things continued much as before, and that remains the same today. Computation (and getting the right answer) remains just as important as ever, and even more widely used than at any time in history.

Mathematics in the Digital Age

If we view the development of Hindu-Arabic arithmetic as the first revolutionary change in the way mathematics is done, then the second change in mathematics praxis of comparable magnitude would be the introduction of symbolic mathematics in the sixteenth century—facilitated in large part by the introduction of the printing press a century earlier.

I would argue that there has been just one further shift in praxis that qualifies as a major revolution. It began in the 1960s with the introduction of the electronic calculator followed by the graphing calculator, and culminated with the appearance of computer algebra systems (*Mathematica*, *Maple*, and others) running on personal computers, in the late 1980s.

For the entire history of mathematics up until the computer age, you had to be good at calculation to get into mathematics, including (in more recent times) acquiring qualifications in the subject, and you had to be good at calculation in order to do or apply mathematics. [By calculation, I mean the execution of any procedure or algorithm.] Moreover, prior to the digital age, if you developed or used mathematics in your career, almost all your time was spent doing calculations.

That is why most people, even to this day, think that mathematics essentially *is* calculation. Yet it is not, and many mathematicians from the ancient Greeks onwards were aware of the distinction, though even they spent most of their time calculating. But the ready availability of first computers and then electronic calculators in the 1960s removed the need for humans to perform numerical calculations.

Because of the electronic calculator, when I arrived at university to study mathematics in 1965, I did not need to make use of the fluency at arithmetic I had developed through many years of school education. (Indeed, over the ensuing decades my arithmetic prowess gradually lost its edge through under-use.) On the other hand, I did have to spend a great deal of my undergraduate career as a mathematics major mastering a whole range of algorithms and techniques for performing a variety of different kinds of numerical and symbolic calculations, geometric reasoning, algebraic reasoning, and equation solving. I had to. In order to solve many problems, I had to be able to crank the algorithmic and procedural handles. There was no other way. There were no machines to do it for me the way the calculator in my pocket performed arithmetic calculations for me (faster, with virtually no errors, and for far more—and larger—numbers than I could handle in my head or with paper and pencil).

That remained the case for the early part of my career as a mathematician. But then, in October 1987, Steven Wolfram released the first version of his massive computer algebra package *Mathematica*. The name "computer algebra system" was an inherited baggage from early attempts to automate mathematical calculation, which totally under-represents what Wolfram's program can do. Quite simply, it can execute pretty well any mathematical procedure, in any branch of mathematics.

Soon after, Canadian developers released *Maple*, and a number of other products came out that do similar things. These products did for almost all of mathematics what the electronic calculator did for arithmetic: they made it obsolete as a human skill (other than for educational purposes, of which more later).

For the first time in history, being able to perform calculations was no longer a necessary requirement for using mathematics. This highlighted the distinction, always there but invisible to most people, between the routine parts of using mathematics (executing procedures) and the creative parts. (I'll discuss later the uses of systems like *Mathematica* in pure mathematics, i.e., the formulation and proof of theorems.)

For a few years, products like *Mathematica* and *Maple* were used mainly in university departments of mathematics, physics, and engineering. They were expensive and challenging to use, and ran only on upper end personal computers. But with the release of *Wolfram Alpha* in 2009, the power of *Mathematica* became available in a cloud-based application that could be accessed (for free) from any PC, tablet, or smartphone. Moreover, *Wolfram Alpha* had a simple user interface that makes it possible to execute pretty well any mathematical procedure with as much ease as using an electronic calculator.

The simplest way to get a sense of how *Alpha* works is simply access it with a Web browser and explore for a while. The point relevant to this essay is that it makes it possible for people to use mathematics without having expertise in any particular topic or procedure. (I'll come later to exactly what knowledge is required to do this.)

The arrival of *Wolfram Alpha* has changed forever the way people can use mathematics. More than that, it has made it possible for people who cannot (or believe they cannot) execute formal mathematical procedures—for example, solving a quadratic equation, to take a particularly simple case—to make effective use of mathematics. Today, having a mastery of calculation is no longer the price anyone has to pay to use mathematics.

To help people understand what it is like to use mathematics in today's world, I often draw an analogy with the world of music. To be a mathematician in the pre-*Alpha* era was akin to mastering many instruments in an orchestra. You had to master the arithmetic instrument, the geometry instrument, the trigonometry instrument, the algebra instrument, the calculus instrument, and so on. The more mathematical instruments you mastered, the greater your power as a mathematician. But using mathematics today is more akin to being a conductor of the orchestra. To conduct that orchestra well, you have to know what all of those instruments are capable of, and you surely need to gain experience with a number of them, at least one of them fairly well (ideally more than one). But there is no need to be world class in any of them. The instruments are what "do all the work." As conductor, you have to know how and when to make them work together, deciding which one(s) to use for each purpose as you progress through the symphony.

Actually, a symphony orchestra is too big for the analogy to work for any one math problem; it's more like a small ensemble. But you get the picture. And for sure, there are enough different mathematical tools out there that they definitely constitute an orchestra, and a large one at that. Indeed, *Wolfram Alpha* alone is orchestra scaled, since it encompasses all the mathematical methods that are typically taught at universities at undergraduate and graduate levels—and a lot that are not.

Clearly, with mathematics being done that way, the experience of using mathematics is very different than it was throughout the entire previous history of mathematics. And gone is the need to be good at any kind of calculation. Mathematicians today do not need to be able to calculate quickly or accurately; indeed, they never do that. The detailed execution of any formal procedure or algorithm is now done by machines. They do it considerably faster than humans ever could, and they make far fewer errors (essentially none). Moreover, they do it with far bigger data sets. For example, mathematics students of my generation learned how to solve linear equations and handle matrices and determinants for two, three, and maybe four variables, and if required could go beyond that to five or six or so, maybe a bit higher. But today, many optimization problems solved routinely by computer packages have thousands or millions of variables. No human could ever cope with that.

So does any mathematics student have to be able to handle any kind of linear equation, matrix, or determinant, and if so to what extent? Mathematics educators are still assessing the pedagogic implications of the digital revolution in mathematical praxis, but the general consensus for that particular example is that mastery of 2 and 3 variables is sufficient, with the learner being able to get correct answers in the two-variable case and solve three-variable examples without worrying too much if they make slips. Of course, making an error when dealing with a real-world problem can be a big deal, sometimes having catastrophic consequences, but the computer system that actually executes the procedure won't make that mistake. The goal of mathematics teaching today is not execution; it is understanding. The conductor of any orchestra, musical or mathematical, has to have a deep understanding of what each instrument does, what it is capable of, and when and how to make use of it, but mastery of an instrument is not necessary.

Notice that it is not mathematics that has changed in the digital age, though there have been changes in the form of new branches of mathematics that resulted from the growth of computer technology (fractal geometry, for example). That caveat aside, however, what has changed is the way people use it. Since mathematics itself is largely unchanged, to understand a new mathematical result is essentially the same challenge it always was. It is mathematical praxis that has changed. And with that change in praxis has come a change—or rather, there is an emerging process of change—in what it takes to become a mathematician.

Being able to calculate quickly, efficiently, and accurately used to be essential; now it is not required. In place of that skillset (which took most people considerable time and effort to master, with many dropping by the wayside in the process) is a new set of skills. Those new skills are in fact much closer to those in the humanities or the creative arts than most people yet realize (or in some cases are willing to contemplate). [In fact, my personal view is that they are now practically indistinguishable, but that's for future generations to judge. Mathematics, I would argue, is no longer a special case. From the perspective of mathematical cognition, I believe that the modifier "mathematical" is no longer necessary; it's just (human) cognition. What distinguishes mathematical praxis is the *what* to which human cognition is applied. That's all.]

Whatever childhood (or adult) experiences arouse an individual's interest in pursuing mathematics, being able to master the art of calculation (i.e., executing any formal procedure except in the rudimentary form required to gain sufficient understanding) is no longer a prerequisite. To be sure, you have to be intrigued by the very idea of formally specified abstractions and context-free reasoning. Not everyone will see mathematics as having appeal. But then, few among us can see the attraction in everything our fellow humans decide to pursue either. From the human perspective, it's not so much that today's digital mathematical tools have added something to the discipline; rather, they have removed what for many was an obstacle.

Experimental Mathematics

So far, my focus has been on the use of mathematics in the world. That may be unusual in a mathematical commentary (which this essay is), but using mathematics to solve real-world problems is what the vast majority of mathematicians do. Admittedly, in many cases such individuals don't call themselves mathematicians, since that word tends to be reserved for the few who focus on pure mathematics (as I did for the first 20 years of my career). The essence of pure mathematics was captured perfectly by Euclid in his famous geometry and number theory text *Elements*, written around 350 BCE: the formulation and proof of precise statements (theorems) about mathematical abstractions.

By and large, it's fair to say that, for most pure mathematicians, the core activities today are much the same as they have always been. The most important tool remains paper and pencil, or perhaps a blackboard. (Mathematicians overwhelmingly prefer a chalkboard to a white board, for ease of frequent erasing—the outsider's perception that mathematicians hardly ever make mistakes is as far from the truth as could be; pure mathematicians engaged in research make errors all the time. Errors frequently lead to new ideas.)

In fact, paper-and-pencil math was the key even for the famous, first major inroad of computer technology into pure mathematics: Kenneth Appel and Wolfgang Haken's 1976 proof of the four color theorem. (For any map drawn on a plane, four colors suffice to color the regions so no two with a stretch of common border are colored the same.) Their proof was obtained by familiar paper-and-pencil-assisted mental reasoning, with a twist that their argument left them having to check that 1,936 different possible (specific) configurations of adjacent regions (mini-maps) could be so colored.

Had they been faced with just three or four special cases, or maybe even a dozen or so, Appel and Haken would surely have done everything by hand. But almost 2,000 cases was far too big a task. (The problem of finding a coloring for each one was also time consuming; the method was simply to examine all possible combinations of colors and see if one worked.) Instead, they wrote a computer program to go through all those configurations and find (by exhaustive search) an admissible coloring for each one. When, after over a thousand hours of computing (using 1976 technology), the program had generated colorings of all the special mini-maps, the four color theorem, first conjectured 124 years earlier in 1852, was declared proven.

With the Appel and Haken case, the computer was not really doing any of the logical reasoning. The two mathematicians simply outsourced to a computer a mundane task that could have done by hand, were it not for the number of cases involved. (The number of cases necessary to examine was later reduced to 1,476. A later proof by another team required only 633 special configurations to be examined; but that is still too many for a human to do.)

Well, that last paragraph is not entirely true; at least, it's not the whole truth. There is another aspect to the story that should be included. Viewing their proof as a classical mental construction, with a computer being used only to cope with a large amount of data, is valid if you focus only on the final proof. In terms of process, Appel and Haken actually used the computer as an experimental tool to help them arrive at the set of special configurations they used for the final search. That aspect of their work, often overlooked, proved to be an early instance of what is now regarded as a whole new area of mathematical research: experimental mathematics (Borwein and Devlin 2008).

Experimental mathematics is the name generally given to the use of a computer to run computations—sometimes no more than trial-and-error tests—to look for patterns, to identify particular numbers and sequences, and to gather evidence in support of specific mathematical assertions that may themselves arise by computational means, including search.

But the truth is mathematicians have always engaged in experimental activities. Had the early mathematicians in ancient Greece and elsewhere had access to computers, it is likely that the word "experimental" in the phrase "experimental mathematics" would be superfluous; the kinds of activities or processes that make a particular mathematical activity "experimental" would be viewed simply as mathematics.

True, the carefully crafted image of mathematics presented in published papers and textbooks gives no indication of "experiments." Mathematicians' published works consist of precise statements of true facts, established by logical proofs, based upon axioms (which may be, but more often are not, stated in the work). But if you examine the private notebooks of practically any of the mathematical greats, you will find page after page of exploratory calculations, trial-and-error experimentation (symbolic or numeric), guesses formulated, hypotheses examined, and so forth. Famous mathematicians such as Pierre De Fermat, Carl Friedrich Gauss, Leonhard Euler, and Bernhard Riemann spent many hours of their lives carrying out (mental) calculations in order to ascertain "possible truths," many but not all of which they subsequently went on to prove.

Indeed, the experimental part of mathematics is precisely what mathematicians enjoy! As Gauss wrote to his colleague Janos Bolyai in 1808, "It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment."

Gauss was very clearly an "experimental mathematician" of the first order. For example, his numerical analysis—while still a child—of the density of prime numbers led him to formulate what is now known as the prime number theorem, a result not proved conclusively until 1896, more than 100 years after the young genius made his experimental discovery.

It was when mathematicians started using computers to carry out the exploratory work that the massive role played by calculation and experimentation came to the fore. What makes modern experimental mathematics different (as an enterprise) from the classical conception and practice of mathematics is that the experimental process is regarded not as a precursor to a proof, to be relegated to private notebooks and perhaps studied for historical purposes only after a proof has been obtained. Rather, experimentation is viewed as a significant part of the mathematical enterprise in its own right, to be published, considered by others, and (of particular importance) contributing to our overall mathematical knowledge.

In particular, this shift in perception gives an epistemological status to assertions that, while supported by a considerable body of experimental results, have not yet been formally proved, and in some cases may never be proved.

On the other hand it may also happen that an experimental process actually yields a formal proof. For example, if a computation determines that a certain parameter p, known to be an integer, lies between 2.5 and 3.784, that amounts to a rigorous proof that $p = 3$. There have been instances of this. (See Borwein and Devlin (2008) cited earlier.) More common has been when insights gained by an experimental investigation have been sufficient for mathematicians to develop classical proofs. This has happened a number of times with proofs in number theory, theory of minimal surfaces, geometry, and other areas (op. cit.).

In terms of the topic of this essay, mathematical praxis, the acceptance of experimental mathematics as a recognized branch of pure mathematics provides us a clear instance of an area of pure mathematics where praxis has been changed by computer technology.

Among the daily activities of an experimental mathematicians are:

- Symbolic computation using a computer algebra system such as *Mathematica* or *Maple*
- Data visualization methods
- Integer-relation methods, such as the PSLQ algorithm
- High-precision integer and floating-point arithmetic
- High-precision numerical evaluation of integrals and summation of infinite series
- Iterative approximations to continuous functions
- Identification of functions based on graph characteristics

In terms of my earlier orchestra analogy, where mathematicians in the past spent many hours carrying out hand calculations, the pure mathematician working in an experimental fashion today is simply a mathematician who conducts an ensemble of a particular set of computational instruments. In the case of experimental mathematics, the computer revolution certainly changed how some pure mathematicians work; moreover it did so in essentially the same way it did for applied mathematicians.

Mathematics Education

Given the significant change to the way mathematics is done in the world today, how do math educators prepare their students for life in that world?

So far, many don't. By and large, math classes around the world today operate in much the same way they did in medieval times, often using what are essentially the same textbooks, albeit with computers and other digital technologies sometimes playing an auxiliary role. To a large extent, this is because of resistance to change among some teachers, and (often strong) opposition to change from parents and education administrators who are not familiar with the degree to which the mathematical landscape has been transformed.

This was illustrated dramatically in the United States by strong opposition to the Common Core State Standards, released in 2009 to provide guidelines as to what skills were required for today's world. While poor implementation of the standards—by education boards and the developers and suppliers of textbooks and other educational materials—can fairly be blamed for some of the complaints, the pushback went well beyond that, to opposition toward the basic principles of modern mathematics the standards are based on in order to prepare future citizens for life in the modern world. Why was this? What was being missed?

Once again, an everyday analogy might be helpful here. Florence Cathedral, completed in 1436, took 140 years to build. It is universally acknowledged to be one of the world's most beautiful large buildings. So too is Sydney Opera House, completed in 1973. Yet, for all it has comparable size, it took a mere 14 years to build. How was it possible to build Sydney Opera House ten times faster than Florence Cathedral?

After all, the basic principles of large building construction are essentially the same. The laws of physics did not change. Aesthetic principles are broadly the same.

What changed, of course, are the tools available. Late-twentieth-century architects and construction engineers had very different tools at their disposal from their forebears in the thirteenth to fifteenth centuries. With different tools available, they needed very different skillsets. Whereas medieval builders had to do many things by hand—or at the very least using hand tools—modern builders "conduct an orchestra of different construction tools." Different tools require different skillsets.

Analogously, until the final decade of the twentieth century, mathematics educators had to ensure that students graduated with basic number skills and the ability to perform mathematical reasoning using (in particular) those number skills. Using those basic number skills required good calculation skills, with a premium put on speed and accuracy. But with today's digital tools, the need for calculation has been removed. Instead, today's graduate needs to be able to make good, efficient, constructive, and accurate use of the vast array of mathematical tools now available. Different tools require different skillsets.

Being able to use those new mathematical tools does not require training in any particular one of them (which is just as well, since they evolve and change with considerable rapidity). Rather it requires understanding the basic concepts and principles of mathematics that underpin them.

The key word in that last sentence is *understanding*. For example, in the days when only people could perform calculations, it was important that arithmetical algorithms were as efficient as possible. What are nowadays called the "classical algorithms" of arithmetic were developed and honed over many centuries to do just that. The brilliance of the Hindu-Arabic number representation is that it facilitated the creation of such algorithms. It was not necessary to understand numbers in any deep way—indeed, studies showed that few people really understood the place-value system—or to understand how the algorithm works or why it was constructed the way it was. You just had to master the (few) basic rules and apply them carefully; something that practically anyone could achieve, given sufficient repetitive practice. It was, to all intents and purposes, mindless arithmetic—sometimes amusingly rendered with intentional ambiguity as "meaningless arithmetic." [ASIDE: In terms of *computation*, the shift from the use of an abacus (either the European marked board and pebbles or the Chinese beads-on-wires equivalent) to written Hindu-Arabic arithmetic was largely a wash; one learned mechanical procedure was replaced by another. The real *benefit* of the written system—and it was a huge benefit—was that the written algorithms left an audit trail, enabling anyone to check, and perhaps correct, a calculation after it was completed.]

In contrast to mindless arithmetic, the arithmetic algorithms used in the more progressive schools today have been designed to optimize not speed or accuracy, but understanding (of numbers, the place value system, and the basic ideas of arithmetic). Not surprisingly, those algorithms are, from a getting-the-answer perspective, nothing like as efficient as the classical algorithms that twentieth-century students had to master. That's one of the reasons parents and education administrators opposed the Common Core, which supported the use of algorithms optimized for understanding. But in so doing, they were missing the key point. Today, we have ubiquitous, cheap machines that do the calculations for us. In fact, we don't need to buy specialized machines at all. The smartphone we carry around with us serves that purpose by way of cheap, if not cost-free, apps. A crucially important mathematical skill today is the ability to "conduct the orchestra of those calculation tools."

One of the core skillsets that mathematics educators identified is *number sense*. The most frequently cited definition of this notion is due to Gersten and Chard (1999):

> Number sense "refers to a child's fluidity and flexibility with numbers, the sense of what numbers mean and an ability to perform mental mathematics and to look at the world and make comparisons."

Interestingly, the notion of number sense was originally formulated as a way for educators to help students with special needs master the basic arithmetic that, at the time, still dominated mathematics instruction. With (belated) recognition that the need for calculation had faded away with the increasing availability of computational devices, however, educators began to recognize the relevance, and power, of the notion in navigating the world of twenty-first-century mathematics. [It may be instructive to recast Gersten and Chard's definition in terms of music, to see what is required to be a good orchestral conductor.]

How do you acquire that high-level number sense? The answer is the same way people always did: through lots of practice. What is different is that instead of the practice being structured to achieve speed and accuracy, the goal is to produce understanding. That requires reflective practice, not the rote repetition that can, at least in some people, result in fast, accurate computation—albeit not remotely as fast or accurate as a free app on your smartphone!

The change from society's need for calculation skill to the new need of the higher order number sense may seem revolutionary, and indeed it is. But it is at heart just today's iteration of a series of revolutions that have occurred throughout mathematics' history. Other skills that are essential for today's mathematics developer or user are the ability to recognize and construct logically sound arguments (and recognize unsound ones); the ability to make smooth, efficient use of the digital tools that are available (conducting the orchestra); and increasingly the ability to work well in teams. Since mathematics began, mathematicians have calculated and reasoned logically with the basic building blocks of the time. Today's procedures (that have to be executed) turn into tomorrow's basic entities (on which you operate).

For instance, in the ninth century, the Arabic-Persian-speaking traders around Baghdad developed a new, and in many instances more efficient, way to do arithmetic calculations at scale, by using logical reasoning rather than arithmetic. Their new system quickly became known as *al-jabr*, after one of the techniques they developed to solve equations.

Whereas arithmetic operated on numbers, algebra (as we now call it) is a form of calculation that (essentially) operates on classes of numbers. (That's where the variables come in.) When the sixteenth-century French mathematician François Viète introduced symbolic algebra, those classes of numbers were the new building blocks, on which it was possible to study the operations of arithmetic, and more general forms of operations.

In each case, advances in mathematics were introduced to make mathematics more easy to use and to increase its application.

The rise of modern science (starting with Galileo in the seventeenth century) and later the Industrial Revolution in the nineteenth century led to still more impetus to develop new mathematical concepts and techniques, though some of those developments were geared more toward particular groups of professionals.

Calculus provides a good example. In differential calculus, functions are no longer viewed as rules that you execute to yield new numbers from old numbers, but higher level objects on which you operate to produce new functions from old functions, new building blocks on which to reason.

Today, entire computations can be treated as mental building blocks. If and when those computations are run (on a machine), you may end up with a number, a graph, or some other output. But until then, the (human) mathematician reasons about them as entities in their own right. (It does not necessarily feel that way, but functionally that's what is going on.)

To conclude this section, I'll present a simple arithmetical puzzle to illustrate the kind of mathematical thinking processes that today's more progressive mathematics teachers sometimes use to help their students develop. Because of its simplicity, it's easy to miss the key issues, but for all that simplicity it captures the spirit of how today's mathematicians work.

The puzzle is of the kind you often find in cheap puzzle books or on puzzle websites. In this case, however, my goal in presenting it is not for you to get the right answer. Rather, it is for you to solve it *as quickly as you can*—ideally *instantaneously*. The reason is to try to get some insight into what the human brain can do with ease, so that educational emphasis can be put on enhancing the brain's capacity to do mathematics when working in the "orchestral conducting" fashion of today's professionals (rather than wasting time trying to train the brain to perform calculations, which your smartphone app can do much faster and more accurately).

Here then is the puzzle:

> PROBLEM: A bat and a ball cost $1.10. The bat costs $1 more than the ball. How much does the ball cost on its own? (There is no special pricing deal.)

How did you do? The most common answer people give *instantly* to this problem is that the ball costs 10¢. That answer is wrong (and many realize that is the case soon after their mind has jumped to that wrong number). What leads many astray is that the problem is carefully worded to run afoul of what under normal circumstances is an excellent strategy. [So if you got it wrong, you probably did so because you are a good thinker with some well-developed problem-solving strategies—problem-solving "heuristics" is the official term, and I'll get to those momentarily. So take heart. You are well placed to do just fine in twenty-first-century mathematical thinking. You simply need to develop your heuristics to the next level.]

Here is, most likely, what your mind did to get to that 10¢ answer. As you read through the problem statement and came to that key phrase "cost more," your mind said, “I will need to subtract.” You then took note of the data: those two figures \$1.10 and \$1. So, without hesitation, you subtracted \$1 from \$1.10 (the smaller from the larger, since you knew the answer has to be positive), getting 10¢.

Notice you did not really perform any calculation. The numbers are particularly simple ones. Almost certainly, you retrieved from memory the fact that if you take a dollar from a dollar-ten, you are left with 10¢. You might even have visualized those amounts of money in your hand. Notice too that you understood the mathematical concepts involved. Indeed, that was why the wording of the problem led you astray! What you did is apply a heuristic you have acquired over many financial transactions and most likely a substantial number of arithmetic quiz questions in elementary school. In fact, the timed tests in schools actively encourage such a “pattern recognition” approach. For the simple reason that it is fast and usually works!

We can, therefore, formulate a hypothesis as to why you “solved’ the problem the way you did. You had developed a heuristic (identify the arithmetic operation involved and then plug in the data) that is (a) fast, (b) requires no effort, and (c) usually works. This approach is a smart one in that it uses something the human brain is remarkably good at—pattern recognition—and avoids something our minds find difficult and requiring effort to master (namely, arithmetic calculation).

Of course, primed by the context in which I presented this particular problem, you probably expected there to be a catch. So, after letting your mind jump to the 10¢ answer, you likely took a second stab at it (or, if you were anxious about “getting a wrong answer,” made this your first solution) by applying an algorithm you had learned at school. Namely, you reasoned as follows:

Let x = cost of bat and y = cost of the ball. Then, we can translate the problem into symbolic form as the two equations: $x + y = 1.10$, $x = y + 1$.
Eliminate x from the two equations by algebra, to give $1.10 - y = y + 1$.
Transform this by algebra to give $0.10 = 2y$.
Thus, dividing both sides by 2, you conclude that y = 5¢.
And this time, you get the correct answer.

You may, in fact, have been able to carry out this procedure in your head. When I was at school, I could do algebraic manipulations far more complicated than this in my head, at speed. But, truth be told, since I started outsourcing arithmetic to machines over three decades ago, I have lost that skill, and now have to write down the equations and solve them on paper. (This is a confirmation, if any were needed, that arithmetic calculations do not come naturally to the human brain. Over the years, as my mental arithmetic skills have declined, my pattern recognition abilities have not diminished, but on the contrary have dramatically improved, as I learned—automatically, through exposure—to recognize evermore fine-grained distinctions.)

Whether or not you can do the calculation in your head, it is of course entirely formulaic and routine. Unlike the first method I looked at (a heuristic that is fast and usually right), this method is an algorithmic procedure, it is slow (much slower than the first method, even when the algebraic reasoning is carried out in your head), but it always works. It is also an approach that can be executed by a machine. True, for such a simple example, it's quicker to do it by hand on the back of an envelope, but as a general rule it makes no sense to waste the time of a human brain following an algorithmic procedure, not least because even with simple examples it is familiarly easy to make a small error that leads to an incorrect answer.

But there is another way to solve the problem. It is typical of the ways professional mathematicians vocalize their solutions when asked to do so. Like the first method we looked at, it is a heuristic, hence instinctive and fast, but unlike the first heuristic method it always works.

This third method requires looking beyond the words, and beyond the symbols in the case of a problem presented symbolically, to the quantities represented. Though I (and likely other mathematicians) don't visualize it quite this way (in my case it is more of a vague sense of size), Fig. 3.1 more or less captures what the pros do.

As we read the problem, we form a mental sense of the two quantities, the cost of the ball on its own and the cost of the bat plus ball, together with the stated relation between them, namely that the latter is $1 more than the former. From that mental image, where we *see* that the $1.10 total consists of three pieces, one of which has size $1 and the other two of which are equal, we simply "read off" the fact that the ball costs 5¢. No calculation, no algorithm. Pure pattern recognition.

This solution is an example of number sense in action, the critical twenty-first-century arithmetic skill I discussed earlier. It's hard to imagine how a computer system could solve the problem that way.

The acclaimed Australian (pure) mathematician Terrence Tao has called those three ways of solving the bat-and-ball problem, respectively, pre-rigorous thinking, rigorous thinking, and post-rigorous thinking. In a post in his blog *What's new* (https://terrytao.wordpress.com), titled "There's more to mathematics than rigour and proofs," in which he introduced those terms, he was discussing the way professional mathematicians solve abstract problems in pure mathematics. The formal, symbolic, rigorous description you see in papers and books comes primarily at the end, he notes, to check that the solution is logically correct, or at various intermediate points to make those checks along the way. But the key thinking is post-rigorous.

In the case of solving real-world problems, the pros almost always turn to technology to handle any algebraic deductions. In contrast, though pure mathematicians

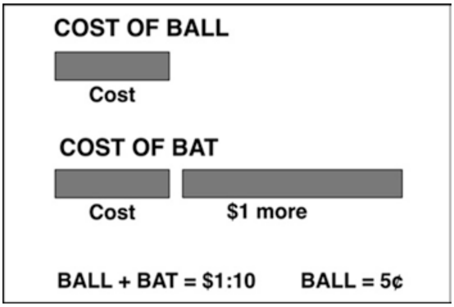

Fig. 3.1 A "professional's" mental representation of the bat-and-ball problem

sometimes do use those technology products as well, they often find it much quicker, and perhaps more fruitful in terms of gaining key insights, to do the algebraic work by hand. But in all cases, they go beyond the numerals and the symbols and reason with the semantic entities those linguistic elements represent.

One of the big questions facing mathematics teachers today is how do we best teach students to be good post-rigorous mathematical thinkers.

In the days when the only way to acquire the ability to use mathematics to solve real-world problems involved mastering a wide range of algorithmic procedures, becoming a mathematical problem solver frequently resulted in becoming a post-rigorous thinker automatically. But with the range of tools available to us today, there is a good reason to assume that, with the right kinds of educational experiences, we can significantly shorten (though almost certainly not eliminate) the learning path from pre-rigorous, through rigorous thinking, to post-rigorous mathematical thinking. The goal is for learners to acquire enough effective heuristics.

To a considerable extent, those heuristics are not about "doing math" in the traditional sense. Rather, they are focused on making efficient and effective use of the many sources of information available to us today. But before anyone throws away their university-level textbooks, it's important to be aware that the intermediate step of mastering some degree of rigorous thinking is probably essential.

Post-rigorous thinking is almost certainly something that emerges from repeated practice at rigorous thinking. (See, for example, Willingham 2010.) Any increased efficiency in the education process will undoubtedly come from teaching the formal methods in a manner optimized for understanding, as opposed to optimized for attaining procedural efficiency, as it was in the days when we had to do everything by hand. Stay tuned!

Figure 3.2 provides a graphical summary of Tao's categorization of the three kinds of mathematical thinking we can bring to problem-solving.

In addition to providing a perspective on the three phases each one of us has to go through to become a proficient mathematical (real-world) problem solver, Tao's classification also provides an excellent summary of three historical stages of mathematical thinking as it has evolved over the past 10,000 years or so, from the invention of numbers in Sumeria, where the mathematical thinking of the time was accessible to all, through three millennia of formal mathematics development, where many people were never able to understand it or make effective use of it, and now into the third phase, where, because of technology, mathematical thinking can once again, I believe, be accessible to all.

As noted above, we do not know the degree to which people have to master rigorous thinking to become good post-rigorous thinkers, but Willingham (2010) and others present evidence to suggest that stage cannot be bypassed. Still, given today's technological toolkit, including search, social media, online resources like Wolfram Alpha and Khan Academy, and a wide array of online courses, it is surely possible to master much of the rigorous thinking you need "on the job," in the course of working on meaningful, and hence motivational and rewarding, real-world problems.

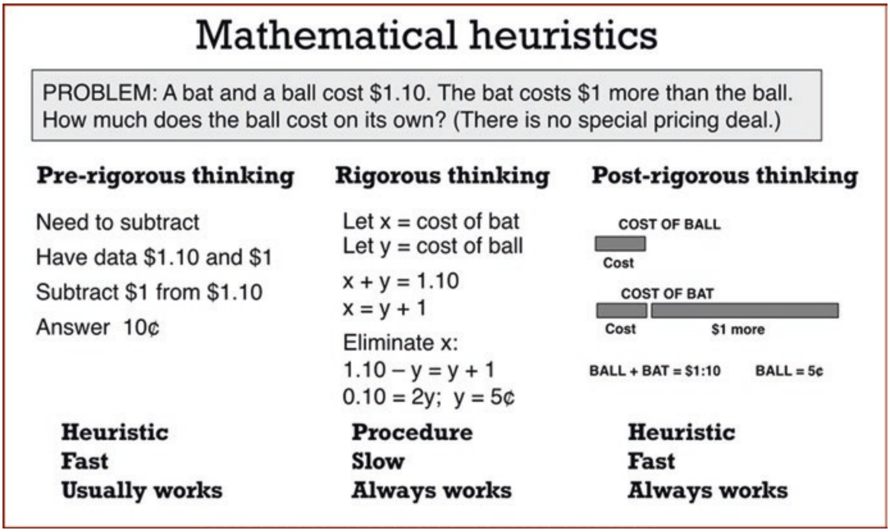

Fig. 3.2 Tao's categorization

This is not to say that there is no further need for teachers. Far from it. Very few people are able to become good mathematical thinkers on their own. Newtons and Ramanujans, both of whom achieved great things with just a few books to guide them, are extremely rare. The vast majority of us need the guidance and feedback of a good teacher.

But, whereas the process of doing mathematics was, until a quarter century ago, dependent on being able to perform calculations of various kinds, a skillset that the brain does not find naturally and requires considerable training and practice, given the readily accessible calculation tools at our disposal, mathematical praxis today consists largely of using the brain in a manner it finds far more natural: analogical reasoning, rather than the logical reasoning previously required.

The Symbol Barrier

Heuristics-driven, post-rigorous thinking is—or at least, should be—the goal of today's mathematics educators, in order for tomorrow's mathematics users to be able to make full and good use of all the available technology tools. Perhaps then, digital technologies themselves can provide new ways to develop (or help develop) those new skillsets. That, in fact, has been the focus of much of my own research over the past few years. The approach I have taken goes back to some groundbreaking social science research conducted almost 30 years ago.

In the early 1990s, three researchers, Terezinha Nunes (at the University of London, UK), Analucia Dias Schliemann, and David William Carraher (both of the

Federal University of Pernambuco in Recife, Brazil) embarked on an anthropological study in the street markets of Recife. With concealed tape recorders, they posed as ordinary market shoppers, seeking out stalls being staffed by young children (between 8 and 14 years of age, it turned out). At each stall, they presented the young stallholder with a transaction designed to test a particular arithmetical skill. The purpose of the research was to compare traditional instruction (which all the young market traders had been receiving in school since the age of 6) with learned practices in context. In many cases, they made purchases that presented the children with problems of considerable complexity.

What they found was that the children got the correct answer 98% of the time. "Obviously, these were not ordinary children," you might imagine, but you'd be wrong. There was more to the study. Posing as shoppers and recording the transactions was only the first part. About a week after they had "tested" the children at their stalls, the three researchers went back to the subjects and asked each of them to take a pencil-and-paper test that included exactly the same arithmetic problems that had been presented to them in the context of purchases the week before, but expressed in the familiar classroom form, using symbols.

The investigators were careful to give this second test in as nonthreatening a way as possible. It was administered in a one-on-one setting, either at the original location or in the subject's home, and the questions were presented in written form and verbally. The subjects were provided with paper and pencil, and were asked to write their answer and whatever working they wished to put down. They were also asked to speak their reasoning aloud as they went along. Although the children's arithmetic had been close to flawless when they were at their market stalls—just over 98% correct despite doing the calculations in their heads, and despite all of the potentially distracting noise and bustle of the street market—when presented with the same problems in the form of a straightforward symbolic arithmetic test, their average score plummeted to a staggeringly low 37%.

The children were absolute number wizards when they were at their market stalls, but virtual dunces when presented with the same arithmetic problems presented in a typical school format. The researchers were so impressed—and intrigued—by the children's market stall performances that they gave it a special name: they called it "*street mathematics.*"

As you might imagine, when the three scholars published their findings (Nunes et al. 1993) it created a considerable stir. Many other teams of researchers around the world carried out similar investigations, with target groups of adults as well as children, and obtained comparable results. When ordinary people are faced with doing everyday math regularly as part of their everyday lives, they rapidly achieve a high level of proficiency (typically hitting that 98% mark). Yet their performance drops to the 35–40% range when presented with the same problems in symbolic form.

It is simply not the case that ordinary people cannot do everyday math. Rather, they cannot do *symbolic* everyday math. In fact, for most people, it's not accurate to say that the problems they are presented in paper-and-pencil format are "the same as" the ones they solve fluently in a real-life setting. When you read the transcripts of the ways they solve the problems in the two settings, you realize that they are doing completely different things. (Nunes and her colleagues present some of those transcripts in their book.) Only someone who has mastery of symbolic mathematics can recognize the problems encountered in the two contexts as being "the same."

In my 2011 book *Mathematics Education for a New Era* (2011), I referred to the problem Nunes et al. discovered as the "symbol barrier." Much of my work since that book was published has been to try to develop technological learning tools that set out to break the symbol barrier, by presenting mathematical puzzles (in mathematics education language, they are complex performance tasks) in a manner similar to the kinds of mental representations that arose in my above discussion of post-rigorous thinking for the solution to the bat-and-ball puzzle.

To do that, I contacted some colleagues I had met while consulting for an educational technology company, and together we co-founded a small development studio (subsequently named BrainQuake) to design and build such tools.

Though each of BrainQuake's puzzles (three have been released to date) is built around particular mathematical concepts (integer arithmetic, linear growth, and proportional reasoning, respectively, for the first three puzzles we created), they are not designed to teach or provide practice in the basic skills on which they are built (though engaging with the tools undoubtedly does provide additional practice in those requisite skills). Rather, the goal is to develop number sense and general problem-solving ability.

Because the primary target audience is middle-school mathematics students, the mathematical puzzles we developed are presented as challenges in a video game (called *Wuzzit Trouble*), to maximize engagement, but that aspect is not relevant to this discussion. What is relevant is that they provide an alternative, more learner-friendly interface to mathematical thinking and (multistep) problem-solving than do the traditional symbolic presentations.

The manipulable digital objects in BrainQuake's learning products provide direct representations of mathematical concepts, breaking the symbol barrier. Students (players) solve puzzles entirely within the application itself, by manipulating digital objects, instead of writing and manipulating symbols on a page. The (multistep) solutions students have to develop to solve all but the most elementary puzzles are logically identical to the steps they would carry out to solve the puzzle in classical symbolic form. But the experience of doing so is dramatically different. So much so, that hundreds of thousands of children in the age range of 14–16 have, for instance, successfully solved systems of simultaneous linear equations in up to four unknowns, subject to optimizing their solution to meet various constraints on the solution. See Fig. 3.3.

Figure 3.3 shows two representations of the same problem. On the right is a classical symbolic representation of a problem requiring the student to solve a system of simultaneous linear equations in two unknowns, subject to various constraints. The student is also asked to try to find solutions that are optimal in two ways (parts 2 and 3 to the question). On the left, the same problem is presented as a mechanical puzzle dressed up as a quest to free a caged creature (a *Wuzzit*) from a trap, by rotating, one at a time, two small cogs to turn the large wheel. When the player turns the

THE DEEP CONCEPTUAL MATH IN *WUZZIT TROUBLE*

Same problem, different representations

1. Collect the keys to free the Wuzzit

2. For maximum stars, use the least number of moves.

3. For maximal points, collect the bonus items before you pick up the last key.

1. Solve the system of equations

$$\begin{array}{ll} 4x_1 + 7y_1 = z_1 & (\text{mod } 65) \\ 4x_2 + 7y_2 = z_2 - z_1 & (\text{mod } 65) \\ 4x_3 + 7y_3 = z_3 - z_2 & (\text{mod } 65) \\ \dots \quad \dots \quad \dots & \\ 4x_n + 7y_n = z_n - z_{n-1} & (\text{mod } 65) \end{array}$$

subject to the constraints

$$0 \le x_i, y_i \le 5, \;\; x_i y_i = 0, \;\; 1 \le i \le n$$

so that 4, 11, 18 are members of the orbit set

$$\begin{array}{l} \{4i \mid 1 \le i \le x_1\} \cup \{7i \mid 1 \le i \le y_1\} \cup \\ \{z_1 + 4i \mid 1 \le i \le x_2\} \cup \{z_1 + 7i \mid 1 \le i \le y_2\} \cup \\ \{z_2 + 4i \mid 1 \le i \le x_3\} \cup \{z_2 + 7i \mid 1 \le i \le y_3\} \cup \\ \dots \quad \dots \quad \dots \\ \{z_{n-1} + 4i \mid 1 \le i \le x_n\} \cup \{z_{n-1} + 7i \mid 1 \le i \le y_n\} \end{array}$$

2. For bonus points, solve the system with n minimal.

3. For honor points, ensure that one of 4, 18 occurs in the final component of the orbit.

Fig. 3.3 Breaking the symbol barrier

cogs to rotate wheel to bring one or more of the three items to land beneath the origin marker at the top, the player acquires the item. Acquisition of both keys frees the Wuzzit and the puzzle is solved. (The equations have been solved.) Maximum stars are awarded if the player solves it in the fewest possible number of turns (part 2 of the question). Part 3 asks the player to collect the bonus item on the wheel before the last key is acquired.

To be sure, the system of equations on the right is not a standard one. Rather, it is precisely the system of equations that corresponds to solving the puzzle on the left. But the purpose of the puzzle is not to develop the ability to solve systems of symbolic linear equation; the goal is to develop number sense. In this case, the solution of systems of linear equations is simply the mathematical topic chosen as a vehicle to do that. [BrainQuake has produced another version of the puzzle that is stripped of the game features but carries the gears mechanism and the corresponding symbolic equation representations side by side, so the student can see both develop in tandem, thereby explicitly linking the two representations.]

The *Wuzzit Trouble* puzzles have from one to four drive cogs, which means that the mechanism provides a mechanical representation of systems of linear equations in up to four unknowns. See Fig. 3.4.

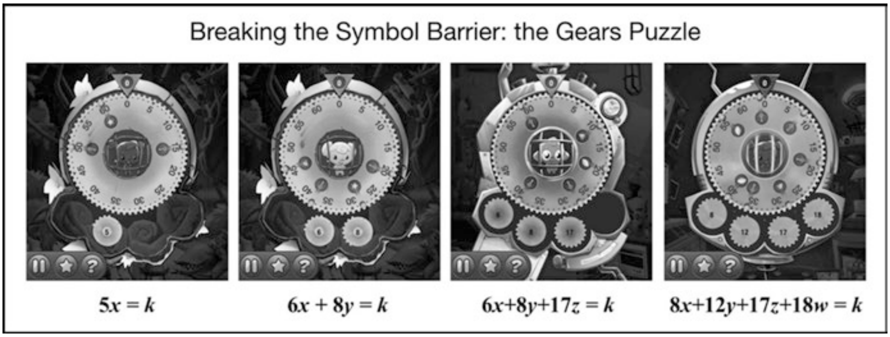

Fig. 3.4 The *Wuzzit Trouble* gears puzzle

Two independent university classroom studies (one in the United States, the other in Finland: Pope and Mangram 2015; Kiili et al. 2015) showed that use of the game *Wuzzit Trouble* for as little as 120 min of play spread over 4 weeks in 10-min bursts at the end of math class produced significant improvements in student number sense, as measured by a written pre- and post-test in the first study, and by both a written test and another digital math game as pre- and post-evaluations in the second. Thus, we know that this approach works.

[BrainQuake is one of a handful of educational technology developers that have adopted this approach. Other products of note are *DragonBox Algebra*, *MotionMath*, and MIND Research Institute's *ST Math*.]

The use of alternative, nonsymbolic representations clearly provides an alternative approach to developing number sense, breaking the symbol barrier that can cause so many problems for learners. Of course, for students who wish to go on to further study or a career in STEM, number sense alone is not sufficient. There remains the problem of leveraging the problem-solving skills acquired in a nonsymbolic fashion to master the traditional symbolic representations, which is a necessary skill for STEM areas. This process is known as "concreteness fading," and has already been studied by others (e.g., Goldstone and Son 2005). It is a special case of education's notorious transfer problem. Technology can help, and as already mentioned BrainQuake has started to develop such tools. But at present this is still work in progress, after completion of which efficacy studies will have to be conducted.

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