Chapter 5
Conclusions

In this book an attempt was made to review the basics of Newtonian mechanics (see Chap. 2), and introduce some of the key concepts involved in formulating Lagrangian dynamics such as *virtual work, kinetic energy, the principle of d'Alembert for dynamical systems, the mathematics of conservative forces, generalized coordinates, generalized forces, constraints, both holonomic and non-holonomic, the extended Hamilton's principle, etc.* (Chaps. 2 and 3). The treatment of a particular class of non-holonomic constraints, where the quasi-velocities can be modeled as linear functions of the time derivatives of the generalized coordinates, is dealt with in Chap. 4 by methods introduced by Whittaker [52], Meirovitch [23], and Cameron and Book [8]. While this approach is sound, it is also somewhat cumbersome as was demonstrated in examples 1 and 2 of Sect. 4.3.The method of Prof. Ranjan Vepa in Sects. 4.4 and 4.5 is more suited for deriving equations of motion. His scheme is based upon transforming the Lagrangian, which is a function of the generalized coordinates and generalized velocities, that is, $\mathcal{L}(q, \dot{q})$, into the Lagrangian containing both the quasi-coordinates and quasi-velocities, that is, $\overline{\mathcal{L}}(\Gamma, \dot{\Gamma})$, where Γ is the vector of quasi-coordinates and $\dot{\Gamma}$ is the vector of quasi-velocities, respectively. The advantage of this approach is that the derivation of the equations of motion turns out to be far less cumbersome. Throughout the text, examples have been presented to illustrate the concepts involved. Although the presentation is mathematically sound, the approach taken in this text was intermediate and did not cover many important topics such as the calculus of variations as exemplified by Cornelius Lanczos' book "The Variational Principles of Mechanics" [19].There is a more comprehensive and more mathematically oriented (and perhaps more advanced) treatment of the subject of mechanics in the form of Arnold's book "Mathematical Methods of Classical Mechanics" [2], with topics ranging from Lagrangian mechanics, variational calculus, Lagrangian mechanics on manifolds, differential forms, Lie algebras of vector fields, and so on. As mentioned in Sect. 3.2, a more exhaustive approach to the subject of non-holonomic systems, their characterization, identification, and control based on the following topics found

A. W. Pila, *Introduction To Lagrangian Dynamics*,
https://doi.org/10.1007/978-3-030-22378-6_5

in differential geometry and related subject matter such as Lie groups, Lie algebras, etc. would include the following topics among others:

- Manifolds, Differentiable manifolds, manifolds and maps
- Tangent vectors, spaces, vector fields
- Fiber bundles
- Differential k-forms
- Exterior derivatives
- Jacobi–Lie brackets, Lie groups
- Vector fields and flows
- Lie brackets and Frobenius' theorem, the Lie algebra associated with a Lie group, actions of Lie groups, Canonical coordinates on a Lie group
- Tangent spaces and tangent maps
- Cotangent spaces and cotangent maps
- Differential forms
- The exponential map
- The geometry of the Euclidean group, metric properties of SE(3), volume forms on SE(3)
- Lie groups and robot kinematics

The interested reader is encouraged to pursue these topics in greater detail by referring to the works by Murray et al. "A Mathematical Introduction to Robotic Manipulation" [25], Bullo and Lewis "Geometric Control of Mechanical Systems Modeling, Analysis, and Design for Simple Mechanical Control Systems" [7], Bloch et al. "Nonholonomic Mechanics and Control" [5], Soltakhanov et al. "Mechanics of Non-Holonomic Systems—A New Class of Control Systems" [35], to name but a few. The area of robotics has also borrowed heavily from these advanced mathematical methods, for example, Siciliano et al. in the "Springer Handbook of Robotics—2nd Ed." [33] and Siciliano et al. in the text "Robotics—Modelling, Planning and Control" [34], both discuss in detail the subject of non-holonomic trajectory planning. In addition, the control of robotic systems is replete with variational approaches in the form of optimal control theory, such as appears, for instance, in the book by Bloch et al. "Nonholonomic Mechanics and Control" [5] and the work of Soltakhanov et al. "Mechanics of Non-Holonomic Systems—A New Class of Control Systems [35]."