# Chapter 1
# Introduction

Joseph Louis Lagrange was one of the greatest mathematicians of the eighteenth and early nineteenth centuries and he has left a remarkable legacy in both the fields of physics and mathematics. This volume begins by recounting the biographical highlights of his life and his contributions.

In the present chapter one of the cornerstones of the book in the form of the direction cosines and their relationship to the Euler angles is presented and elaborated upon. The direction cosines play an important role in the approach by Prof. Ranjan Vepa and are used extensively in Chap. 4.

## 1.1 Introductory Remarks

Joseph Louis Lagrange, originally Giuseppe Lodovico Lagrangia, was of French and Italian descent and was born in Turin in 1736 (see Rouse Ball [30, pp. 330–339]). Lagrange had originally intended to study law but while at college in Turin, he came across a tract by Halley which roused his enthusiasm for the analytical method. He thereupon applied himself to mathematics, and in his 17th year he became professor of mathematics in the royal military academy at Turin. Without assistance or guidance he entered upon a course of study which in 2 years placed him on a level with the greatest of his contemporaries. With the aid of his pupils he established a society which subsequently developed into the Turin Academy. Most of his earlier papers appear in the first five volumes of its transactions. At the age of 19 he communicated a general method of dealing with "isoperimetrical problems," known now as the calculus of variations to Euler. This commanded Euler's admiration, and the latter, for a time, courteously withheld some researches of his own on this subject from publication, so that the youthful Lagrange might complete his investigations and lay claim to being the first to posit the calculus of variations. Lagrange did quite as much as Euler towards the creation



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of the calculus of variations. The subject, as developed by Euler lacked an analytic foundation, and this Lagrange supplied. He separated the principles of this calculus from geometric considerations by which his predecessor had derived them. Euler had assumed as fixed, the limits of the integral, i.e. the extremities of the curve to be determined, but Lagrange removed this restriction and allowed all co-ordinates of the curve to vary at the same time. In 1766 Euler introduced the name "calculus of variations," and did much to improve this science along the lines marked out by Lagrange.

In the year 1766, Euler left Berlin for St. Petersburg, and he pointed to Lagrange as being the only man capable of filling his place. D'Alembert recommended him at the same time. Frederick the Great thereupon sent a message to Turin, expressing the wish of "the greatest king of Europe" to have "the greatest mathematician" at his court. Lagrange went to Berlin, and remained there for 20 years. Frederick the Great held him in high esteem, and frequently conversed with him on the advantages of perfect regularity of life. This led Lagrange to cultivate regular habits. He worked no longer each day than experience taught him he could, without breaking down. His papers were carefully thought out before he began writing, and when he wrote he did so without a single correction. During the 20 years in Berlin he crowded the transactions of the Berlin Academy with memoirs, and also wrote the epoch-making work called the Mécanique Analytique. The approach used by Lagrange will be the subject matter of this volume and will be presented in the subsequent chapters.

Newton's laws were formulated for a single particle and can be extended to systems of particles and rigid bodies. The equations of motion are expressed in terms of physical coordinates and forces, both quantities conveniently represented by vectors. For this reason, *Newtonian mechanics* is often referred to as *vectorial mechanics*. The main drawback of Newtonian mechanics is that it requires one free-body diagram for each of the masses in the system, thus necessitating the inclusion of reaction forces, the latter resulting from kinematical constraints ensuring that the individual bodies act together as a system. These reaction and constraint forces play the role of unknowns, which makes it necessary to work with a surplus of equations of motion, one additional equation for every unknown force. J.L. Lagrange reformulated Newton's Laws in a way that eliminates the need to calculate forces on isolated parts of a mechanical system. A different approach to mechanics, referred to as *analytical mechanics*, or *analytical dynamics*, considers the system as a whole, rather than the individual components separately, a process that excludes the reaction and constraint forces automatically. This approach, due to Lagrange, permits the formulation of problems of dynamics in terms of two scalar functions, the kinetic energy and the potential energy, and an infinitesimal expression, the virtual work performed by the non-conservative forces. Analytical mechanics represents a broader and more abstract approach, as the equations of motion are formulated in terms of generalized coordinates and generalized forces, which are not necessarily physical coordinates and forces, although in certain cases they can be chosen as such. Any convenient set of variables obeying *the constraints on a system* can be used to describe the motion. In this manner, the mathematical formulation is rendered independent of any special system of coordinates. There are only as many equations to solve as there are physically significant variables (see Meirovitch [24, pp. 262–263]).

## 1.2 Direction Cosines and Euler Angles of Rotation

The relationship between direction cosines and Euler angles is presented as background material to be used in the subsequent portions of this text. This chapter has been adopted from Wells [51, pp. 139–141 and Appendix A, pp. 343–344]. The direction cosines $l, m, n$ of line $Ob$, relative to axes $X, Y, Z$ are just $l = x/r, m = y/r, n = z/r$, where $x, y, z$ are the $X, Y, Z$ coordinates of the tip of $r$, where $r = \sqrt{(x^2 + y^2 + z^2)}$ (see Fig. 1.1). It then follows that $(x^2 + y^2 + z^2)/r^2 = (x^2 + y^2 + z^2)/(x^2 + y^2 + z^2) = 1$.

Assuming that coordinates $X_1, Y_1, Z_1$ form an inertial coordinate frame, while coordinates $X, Y, Z$ are attached to a translating and rotating body, the angles between the $X$ coordinate and coordinates $X_1, Y_1, Z_1$ are $\theta_{11}, \theta_{12}, \theta_{13}$, respectively. Hence the direction cosines between coordinate $X$ and coordinates $X_1, Y_1, Z_1$ are $\alpha_{11} = \cos\theta_{11}, \alpha_{12} = \cos\theta_{12}, \alpha_{13} = \cos\theta_{13}$, respectively (see Fig. 1.2). The same relationships between the $X$ coordinate and coordinates $X_1, Y_1, Z_1$ exist as for line $Ob$, that is:

$$\alpha_{11}^2 + \alpha_{12}^2 + \alpha_{13}^2 = 1 \tag{1.1}$$

We can similarly show that the direction cosines between coordinate $Y$ and $X_1, Y_1, Z_1$, that is $\alpha_{21}, \alpha_{22}, \alpha_{23}$ and between coordinate $Z$ and $X_1, Y_1, Z_1$, that is

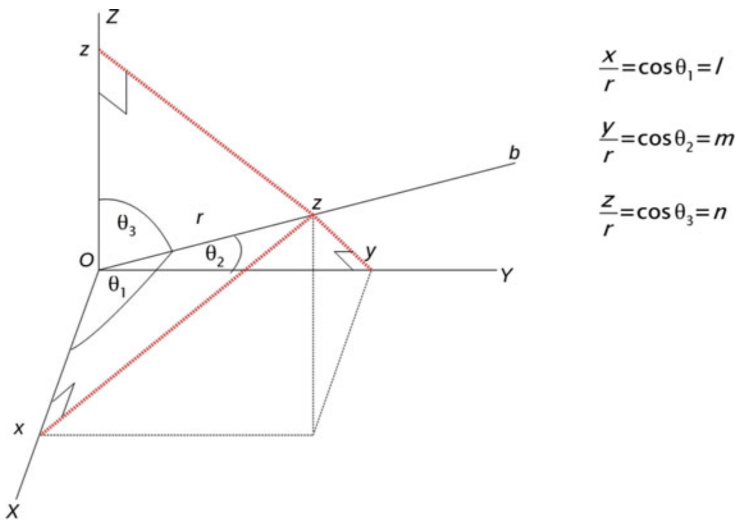


**Fig. 1.1** Definition of direction cosines $l, m, n$

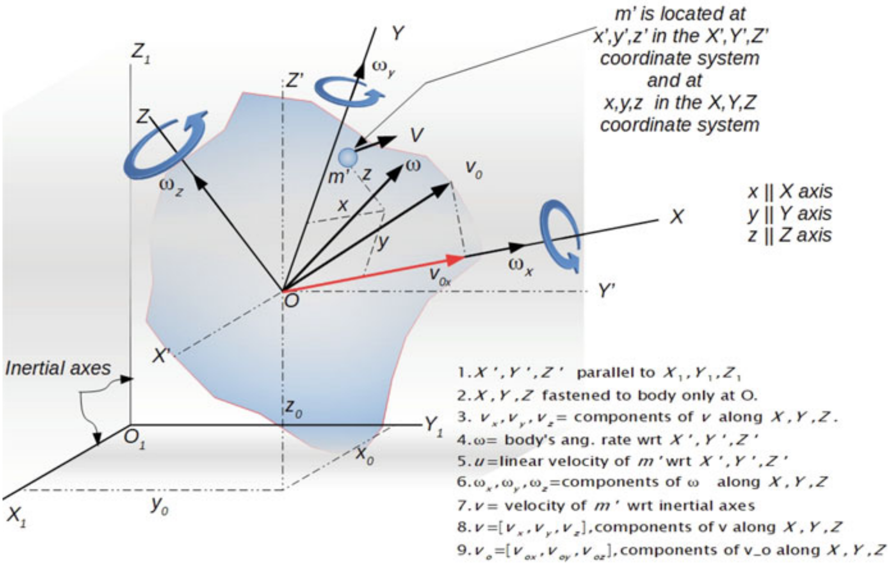


**Fig. 1.2** Body translating and rotating, while $X$, $Y$, $Z$ frame rotates about $O$ relative to the body

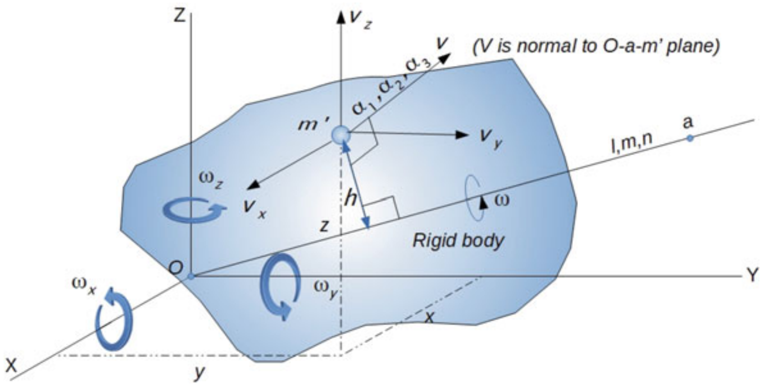


**Fig. 1.3** Sketch of body, fixed at $O$, but free to rotate in any manner about this point

$\alpha_{31}$, $\alpha_{32}$, $\alpha_{33}$, respectively, obey the same relationship as in Eq. 1.1 or:

$$\alpha_{21}^2 + \alpha_{22}^2 + \alpha_{23}^2 = 1;\ \ \alpha_{31}^2 + \alpha_{32}^2 + \alpha_{33}^2 = 1 \tag{1.2}$$

Consider that the body in Fig. 1.3 is fixed at $O$, but is free to rotate in an arbitrary and random fashion about this point. All quantities under consideration will be measured relative to the **fixed inertial axis system** $X$, $Y$, $Z$. At a given instant of time, the body is undergoing rotation about some line $Oa$ with an angular velocity of $\omega$. As a consequence of this rotation, the mass particle $m'$ possesses a linear velocity $v$

normal to the $Oa - m'$ plane, of magnitude $v = \omega h$, where $h$ is the normal distance from $m'$ to the rotating $Oa$ axis. The axis of rotation $Oa$ has direction cosines with respect to the fixed coordinate system $X, Y, Z$ of $l, m, n$, respectively. Similarly, the velocity vector $v$ has direction cosines $\alpha_1, \alpha_2, \alpha_3$ with respect to the $X, Y, Z$ axes and its components along the $X, Y, Z$ axes are, respectively, $v_x$, $v_y$, and $v_z$. From the discussion related to Fig. 1.1 above, it follows that: $\alpha_1 = v_x/v, \alpha_2 = v_y/v, \alpha_3 = v_z/v$. Similarly, $\omega$ is also composed of components along the $X, Y, Z$ axes, that is: $\omega = [\omega_x, \omega_y, \omega_z]$. The direction cosines may then be shown to be: $l = \omega_x/\omega, m = \omega_y/\omega, n = \omega_z/\omega$. Recall that $\omega$ is directed along the line $Oa$. Now the velocity of the mass particle $m'$ may be written in the form: $v = \omega \times r$, where $r = x\hat{i} + y\hat{j} + z\hat{k}$ and $\omega = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$. Performing the above vector multiplication results in:

$$v_x = \omega_y z - \omega_z y; \;\; v_y = \omega_z x - \omega_x z; \;\; v_z = \omega_x y - \omega_y x \tag{1.3}$$

However, $\alpha_1 = v_x/v = v_x/\omega h$. This implies that:

$$\alpha_1 = \frac{\omega_y z - \omega_z y}{\omega h} = \frac{\omega_y z}{\omega h} - \frac{\omega_z y}{\omega h} = \frac{mz - ny}{h} \tag{1.4}$$

due to the fact that $l = \omega_x/\omega$ , $m = \omega_y/\omega$, and $n = \omega_z/\omega$. Hence the direction cosines $\alpha_1, \alpha_2$, and $\alpha_3$ may be written as:

$$\begin{aligned}
\alpha_1 = v_x/v &= \frac{\omega_y z - \omega_z y}{\omega h} = \frac{\omega_y z}{\omega h} - \frac{\omega_z y}{\omega h} = \frac{mz - ny}{h} \\
\alpha_2 = v_y/v &= \frac{\omega_z x - \omega_x z}{\omega h} = \frac{\omega_z x}{\omega h} - \frac{\omega_x z}{\omega h} = \frac{nx - lz}{h} \\
\alpha_3 = v_z/v &= \frac{\omega_x y - \omega_y x}{\omega h} = \frac{\omega_x y}{\omega h} - \frac{\omega_y x}{\omega h} = \frac{ly - mx}{h}
\end{aligned} \tag{1.5}$$

The body in Fig. 1.2 is assumed to be rotating and translating with respect to the inertial coordinate frame $X_1, Y_1, Z_1$. The $X, Y, Z$ coordinate system, with its origin attached to the rigid body, at $O$, rotates in a random fashion relative to the body. The $X', Y', Z'$ axes whose origin is also located at $O$ remain parallel to the inertial axes $X_1, Y_1, Z_1$. The coordinates of $m'$ with respect to the $X, Y, Z$ and $X', Y', Z'$ axes, respectively, are: $x, y, z$ and $x', y', z'$.

Letting $\omega$ represent the angular velocity of the body while $u$ stands for the linear velocity of $m'$, each measured relative to $X', Y', Z'$, the components of the vectors $\omega$ and $u$, along the $X', Y', Z'$ axes are designated as $\omega'_x, \omega'_y, \omega'_z$ and $u'_x, u'_y, u'_z$, respectively. Then akin to the fact established earlier that $v_x = \omega_y z - \omega_z y; \;\; v_y = \omega_z x - \omega_x z; \;\; v_z = \omega_x y - \omega_y x$, we have: $u'_x = \omega'_y z' - \omega'_z y'; \;\; u'_y = \omega'_z x' - \omega'_x z'; \;\; u'_z = \omega'_x y' - \omega'_y x'$. Allowing $u_x, u_y, u_z$ to be the components of $u$ along the momentary positions of the $X, Y, Z$ axis frame, we can write:

$$u_x = u'_x\alpha_{11} + u'_y\alpha_{12} + u'_z\alpha_{13}$$
$$u_y = u'_x\alpha_{21} + u'_y\alpha_{22} + u'_z\alpha_{23}$$
$$u_z = u'_x\alpha_{31} + u'_y\alpha_{32} + u'_z\alpha_{33} \tag{1.6}$$

where $\alpha_{11}, \alpha_{12}, \alpha_{13}$ are the direction cosines of $X$ relative to $X', Y', Z'$, $\alpha_{21}, \alpha_{22}, \alpha_{23}$ are the direction cosines of $Y$ relative to $X'$, $Y'$, $Z'$, and $\alpha_{31}, \alpha_{32}, \alpha_{33}$ are the direction cosines of $Z$ relative to $X'$, $Y'$, $Z'$. Equation 1.6 may be understood by taking the partial derivative of $u_x$ with respect to $u'_x$, which results in: $\frac{\partial u_x}{\partial u'_x} = \alpha_{11}$. In other words, the cosine of the angle between $u_x$ and $u'_x$ is the same as the cosine of the angle between the $X$ axis and the $X'$ axis. This statement also holds for the cosine of the angle between the $X$ and the $Y'$ axes, etc. Another interpretation of the above equation is that $u_x$ is the sum of the geometric projections onto the $X$ axis of the velocities $u'_x$, $u'_y$ and $u'_z$. A similar situation holds for $u_y$ and $u_z$. Thus we have:

$$u_x = (\omega'_y z' - \omega'_z y')\alpha_{11} + (\omega'_z x' - \omega'_x z')\alpha_{12} + (\omega'_x y' - \omega'_y x')\alpha_{13}$$
$$u_y = (\omega'_y z' - \omega'_z y')\alpha_{21} + (\omega'_z x' - \omega'_x z')\alpha_{22} + (\omega'_x y' - \omega'_y x')\alpha_{23}$$
$$u_z = (\omega'_y z' - \omega'_z y')\alpha_{31} + (\omega'_z x' - \omega'_x z')\alpha_{32} + (\omega'_x y' - \omega'_y x')\alpha_{33} \tag{1.7}$$

The relationship between the $X'$ coordinate relative to $X$, $Y$, $Z$ coordinates may similarly be shown to be of the form: $x' = x\alpha_{11} + y\alpha_{21} + z\alpha_{31}$ where $\alpha_{11}, \alpha_{21}, \alpha_{31}$ are the direction cosines of $X'$ relative to $X$, $Y$, $Z$. We may show that for all three coordinates $X'$, $Y'$, $Z'$ relative to the $X$, $Y$, $Z$ coordinates, the relationship is the following:

$$x' = x\alpha_{11} + y\alpha_{21} + z\alpha_{31}$$
$$y' = x\alpha_{12} + y\alpha_{22} + z\alpha_{32}$$
$$z' = x\alpha_{13} + y\alpha_{23} + z\alpha_{33} \tag{1.8}$$

where $\alpha_{11}, \alpha_{21}, \alpha_{31}$ are the direction cosines of $X'$ relative to $X$, $Y$, $Z$, $\alpha_{12}, \alpha_{22}, \alpha_{32}$ are the direction cosines of $Y'$ relative to $X$, $Y$, $Z$, and $\alpha_{13}, \alpha_{23}, \alpha_{33}$ are the direction cosines of $Z'$ relative to $X$, $Y$, $Z$. Similarly, for angular rates $\omega'_x$, $\omega'_x\omega'_x$, we have:

$$\omega'_x = \omega_x\alpha_{11} + \omega_y\alpha_{21} + \omega_z\alpha_{31}$$
$$\omega'_y = \omega_x\alpha_{12} + \omega_y\alpha_{22} + \omega_z\alpha_{32}$$
$$\omega'_z = \omega_x\alpha_{13} + \omega_y\alpha_{23} + \omega_z\alpha_{33} \tag{1.9}$$

Using the identities:

$$
\begin{aligned}
u_x &= (\omega'_y z' - \omega'_z y')\alpha_{11} + (\omega'_z x' - \omega'_x z')\alpha_{12} + (\omega'_x y' - \omega'_y x')\alpha_{13} \\
u_y &= (\omega'_y z' - \omega'_z y')\alpha_{21} + (\omega'_z x' - \omega'_x z')\alpha_{22} + (\omega'_x y' - \omega'_y x')\alpha_{23} \\
u_z &= (\omega'_y z' - \omega'_z y')\alpha_{31} + (\omega'_z x' - \omega'_x z')\alpha_{32} + (\omega'_x y' - \omega'_y x')\alpha_{33}
\end{aligned} \tag{1.10}
$$

and the values for $x'$, $y'z'$ and $\omega'_x$, $\omega'_y$, $\omega'_z$ in Eqs. 1.8 and 1.9, we have:

$$
\begin{aligned}
u_x &= (\omega'_y z' - \omega'_z y')\alpha_{11} + (\omega'_z x' - \omega'_x z')\alpha_{12} + (\omega'_x y' - \omega'_y x')\alpha_{13} \\
\Rightarrow u_x &= (\omega'_y [x\alpha_{13} + y\alpha_{23} + z\alpha_{33}] - \omega'_z [x\alpha_{12} + y\alpha_{22} + z\alpha_{32}])\alpha_{11} \\
&\quad + (\omega'_z [x\alpha_{11} + y\alpha_{21} + z\alpha_{31}] - \omega'_x [x\alpha_{13} + y\alpha_{23} + z\alpha_{33}])\alpha_{12} \\
&\quad + (\omega'_x [x\alpha_{12} + y\alpha_{22} + z\alpha_{32}] - \omega'_y [x\alpha_{11} + y\alpha_{21} + z\alpha_{31}])\alpha_{13}
\end{aligned} \tag{1.11}
$$

It turns out that the coefficient which multiplies $x$ is zero, or $\frac{\partial u_x}{\partial x} = 0$. This may be seen from the following expression:

$$
\begin{aligned}
\frac{\partial u_x}{\partial x} &= \alpha_{13}(\alpha_{12}[\alpha_{11}\omega_x + \alpha_{21}\omega_y + \alpha_{31}\omega_z] - \alpha_{11}[\alpha_{12}\omega_x + \alpha_{22}\omega_y + \alpha_{32}\omega_z]) \\
&\quad - \alpha_{12}(\alpha_{13}[\alpha_{11}\omega_x + \alpha_{21}\omega_y + \alpha_{31}\omega_z] - \alpha_{11}[\alpha_{13}\omega_x + \alpha_{23}\omega_y + \alpha_{33}\omega_z]) \\
&\quad + \alpha_{11}(\alpha_{13}[\alpha_{12}\omega_x + \alpha_{22}\omega_y + \alpha_{32}\omega_z] - \alpha_{12}[\alpha_{13}\omega_x + \alpha_{23}\omega_y + \alpha_{33}\omega_z]) \\
&= (\alpha_{13}\alpha_{12} - \alpha_{12}\alpha_{13})[\alpha_{11}\omega_x + \alpha_{21}\omega_y + \alpha_{31}\omega_z] \\
&\quad + (\alpha_{11}\alpha_{13} - \alpha_{13}\alpha_{11})[\alpha_{12}\omega_x + \alpha_{22}\omega_y + \alpha_{32}\omega_z] \\
&\quad + (\alpha_{11}\alpha_{12} - \alpha_{12}\alpha_{11})[\alpha_{13}\omega_x + \alpha_{23}\omega_y + \alpha_{33}\omega_z] = 0
\end{aligned}
$$

thus implying that $u_x$ is of the form:

$$
\begin{aligned}
u_x &= \alpha_{11}\alpha_{23}\alpha_{32}\omega_z y - \alpha_{11}\alpha_{22}\alpha_{33}\omega_z y + \alpha_{12}\alpha_{21}\alpha_{33}\omega_z y - \alpha_{12}\alpha_{23}\alpha_{31}\omega_z y \\
&\quad - \alpha_{13}\alpha_{21}\alpha_{32}\omega_z y + \alpha_{13}\alpha_{22}\alpha_{31}\omega_z y + \alpha_{11}\alpha_{22}\alpha_{33}\omega_y z - \alpha_{11}\alpha_{23}\alpha_{32}\omega_y z \\
&\quad - \alpha_{12}\alpha_{21}\alpha_{33}\omega_y z + \alpha_{12}\alpha_{23}\alpha_{31}\omega_y z + \alpha_{13}\alpha_{21}\alpha_{32}\omega_y z - \alpha_{13}\alpha_{22}\alpha_{31}\omega_y z \\
&= (\omega_y z - \omega_z y)[\alpha_{11}\alpha_{22}\alpha_{33} - \alpha_{12}\alpha_{21}\alpha_{33}] \\
&\quad + (\omega_y z - \omega_z y)[\alpha_{13}\alpha_{21}\alpha_{32} - \alpha_{11}\alpha_{23}\alpha_{32}] \\
&\quad + (\omega_y z - \omega_z y)[\alpha_{12}\alpha_{23}\alpha_{31} - \alpha_{13}\alpha_{22}\alpha_{31}]
\end{aligned}
$$

which may be simplified as follows:

$$u_x = (\omega_y z - \omega_z y)\left(\underbrace{(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})}_{=\alpha_{33}}\alpha_{33} + \underbrace{(\alpha_{13}\alpha_{21} - \alpha_{11}\alpha_{23})}_{=\alpha_{32}}\alpha_{32} + \underbrace{(\alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22})}_{=\alpha_{31}}\alpha_{31})\right) = \underbrace{\left(\alpha_{31}^2 + \alpha_{32}^2 + \alpha_{33}^2\right)}_{=1}(\omega_y z - \omega_z y) \tag{1.12}$$

since $u_x = \omega_y z - \omega_z y$. Similarly $u_y$ and $u_z$ are:

$$u_y = (\omega_z x - \omega_x z)\left(\underbrace{(\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32})}_{=\alpha_{23}}\alpha_{23} + \underbrace{(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31})}_{=\alpha_{22}}\alpha_{22} + \underbrace{(\alpha_{13}\alpha_{32} - \alpha_{12}\alpha_{33})}_{=\alpha_{21}}\alpha_{21})\right) = \underbrace{\left(\alpha_{23}^2 + \alpha_{22}^2 + \alpha_{21}^2\right)}_{=1}(\omega_z x - \omega_x z)$$

$$u_z = (\omega_y x - \omega_x y)\left(\underbrace{(\alpha_{22}\alpha_{31} - \alpha_{21}\alpha_{32})}_{=\alpha_{13}}\alpha_{13} + \underbrace{(\alpha_{21}\alpha_{33} - \alpha_{23}\alpha_{31})}_{=\alpha_{12}}\alpha_{12} + \underbrace{(\alpha_{23}\alpha_{32} - \alpha_{33}\alpha_{22})}_{=\alpha_{11}}\alpha_{11})\right) = \underbrace{\left(\alpha_{13}^2 + \alpha_{12}^2 + \alpha_{11}^2\right)}_{=1}(\omega_y z - \omega_z y)$$

The identities for $\alpha_{31}$, $\alpha_{32}$, and $\alpha_{33}$ appear in Wells' book [51, pp. 343] and will be developed in the sequel.

Let $i_1, i_2, i_3$ be the orthogonal unit vectors along the $X_1, Y_1, Z_1$ axes, respectively, and $e_1, e_2, e_3$ be the orthogonal unit vectors along the $X, Y, Z$ axes. The direction cosines between the $i_1$ and $e_1, e_2$, and $e_3$ unit vectors are accordingly: $\alpha_{11}$, $\alpha_{21}$ and $\alpha_{31}$. The $i_1$, $i_2$, and $i_3$ vectors may then be written in terms of the $e_1$, $e_2$, and $e_3$ vectors and the corresponding direction cosines between the two systems of unit vectors as follows:

$$\begin{aligned} i_1 &= \alpha_{11}e_1 + \alpha_{21}e_2 + \alpha_{31}e_3 \\ i_2 &= \alpha_{12}e_1 + \alpha_{22}e_2 + \alpha_{32}e_3 \\ i_3 &= \alpha_{13}e_1 + \alpha_{23}e_2 + \alpha_{33}e_3 \end{aligned} \tag{1.13}$$

Similarly, the $e_1$, $e_2$, and $e_3$ unit vectors may be expressed in terms of the $i_1, i_2, i_3$ unit vectors and the corresponding direction cosines between the two systems of unit orthogonal vectors as follows:

$$
\begin{aligned}
e_1 &= \alpha_{11} i_1 + \alpha_{12} i_2 + \alpha_{13} i_3 \\
e_2 &= \alpha_{21} i_1 + \alpha_{22} i_2 + \alpha_{23} i_3 \\
e_3 &= \alpha_{31} i_1 + \alpha_{32} i_2 + \alpha_{33} i_3
\end{aligned} \tag{1.14}
$$

Since the unit vectors are orthogonal we have: $e_1 \cdot e_2 = 0$; $e_1 \cdot e_3 = 0$; $e_2 \cdot e_3 = 0$; $e_1 \cdot e_1 = 1$; $e_2 \cdot e_2 = 1$; $e_3 \cdot e_3 = 1$. Similarly for the $i_1, i_2, i_3$ orthogonal unit vectors we have: $i_1 \cdot i_2 = 0$; $i_1 \cdot i_3 = 0$; $i_2 \cdot i_3 = 0$; $i_1 \cdot i_1 = 1$; $i_2 \cdot i_2 = 1$; $i_3 \cdot i_3 = 1$. The dot products of the vectors $i_1 \cdot i_1$, $i_2 \cdot i_2$ and $i_3 \cdot i_3$ will yield the following:

$$
\begin{aligned}
i_1 \cdot i_1 &= (\alpha_{11} e_1 + \alpha_{21} e_2 + \alpha_{31} e_3) \cdot (\alpha_{11} e_1 + \alpha_{21} e_2 + \alpha_{31} e_3) \\
&\Rightarrow 1 = \alpha_{11}^2 + \alpha_{21}^2 + \alpha_{31}^2 \\
i_2 \cdot i_2 &= (\alpha_{12} e_1 + \alpha_{22} e_2 + \alpha_{32} e_3) \cdot (\alpha_{12} e_1 + \alpha_{22} e_2 + \alpha_{32} e_3) \\
&\Rightarrow 1 = \alpha_{12}^2 + \alpha_{22}^2 + \alpha_{32}^2 \\
i_3 \cdot i_3 &= (\alpha_{13} e_1 + \alpha_{23} e_2 + \alpha_{33} e_3) \cdot (\alpha_{13} e_1 + \alpha_{23} e_2 + \alpha_{33} e_3) \\
&\Rightarrow 1 = \alpha_{13}^2 + \alpha_{23}^2 + \alpha_{33}^2
\end{aligned} \tag{1.15}
$$

Similarly the dot products of the vectors $i_1 \cdot i_2$, $i_1 \cdot i_2$ and $i_2 \cdot i_3$ result in:

$$
\begin{aligned}
i_1 \cdot i_2 &= (\alpha_{11} e_1 + \alpha_{21} e_2 + \alpha_{31} e_3) \cdot (\alpha_{12} e_1 + \alpha_{22} e_2 + \alpha_{32} e_3) \\
&\Rightarrow 0 = \alpha_{11}\alpha_{12} + \alpha_{21}\alpha_{22} + \alpha_{31}\alpha_{31} \\
i_1 \cdot i_3 &= (\alpha_{11} e_1 + \alpha_{21} e_2 + \alpha_{31} e_3) \cdot (\alpha_{13} e_1 + \alpha_{23} e_2 + \alpha_{33} e_3) \\
&\Rightarrow 0 = \alpha_{11}\alpha_{13} + \alpha_{21}\alpha_{23} + \alpha_{31}\alpha_{33} \\
i_2 \cdot i_3 &= (\alpha_{12} e_1 + \alpha_{22} e_2 + \alpha_{32} e_3) \cdot (\alpha_{13} e_1 + \alpha_{23} e_2 + \alpha_{33} e_3) \\
&\Rightarrow 0 = \alpha_{12}\alpha_{13} + \alpha_{22}\alpha_{23} + \alpha_{32}\alpha_{33}
\end{aligned} \tag{1.16}
$$

The same procedure is employed on the $e_1$, $e_2$, and $e_3$ vectors, that is:

$$
\begin{aligned}
e_1 \cdot e_1 &= (\alpha_{11} i_1 + \alpha_{12} i_2 + \alpha_{13} i_3) \cdot (\alpha_{11} i_1 + \alpha_{12} i_2 + \alpha_{13} i_3) \\
&\Rightarrow 1 = \alpha_{11}^2 + \alpha_{12}^2 + \alpha_{13}^2 \\
e_2 \cdot e_2 &= (\alpha_{21} i_1 + \alpha_{22} i_2 + \alpha_{23} i_3) \cdot (\alpha_{21} i_1 + \alpha_{22} i_2 + \alpha_{23} i_3) \\
&\Rightarrow 1 = \alpha_{21}^2 + \alpha_{22}^2 + \alpha_{23}^2
\end{aligned}
$$

$$
\begin{aligned}
e_3 \cdot e_3 &= (\alpha_{31} i_1 + \alpha_{32} i_2 + \alpha_{33} i_3) \cdot (\alpha_{31} i_1 + \alpha_{32} i_2 + \alpha_{33} i_3) \\
&\Rightarrow 1 = \alpha_{31}^2 + \alpha_{32}^2 + \alpha_{33}^2 \\
e_1 \cdot e_2 &= (\alpha_{11} i_1 + \alpha_{12} i_2 + \alpha_{13} i_3) \cdot (\alpha_{21} i_1 + \alpha_{22} i_2 + \alpha_{23} i_3) \\
&\Rightarrow 0 = \alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{23} \\
e_1 \cdot e_3 &= (\alpha_{11} i_1 + \alpha_{12} i_2 + \alpha_{13} i_3) \cdot (\alpha_{31} i_1 + \alpha_{32} i_2 + \alpha_{33} i_3) \\
&\Rightarrow 0 = \alpha_{11}\alpha_{31} + \alpha_{12}\alpha_{32} + \alpha_{13}\alpha_{33} \\
e_2 \cdot e_3 &= (\alpha_{21} i_1 + \alpha_{22} i_2 + \alpha_{23} i_3) \cdot (\alpha_{31} i_1 + \alpha_{32} i_2 + \alpha_{33} i_3) \\
&\Rightarrow 0 = \alpha_{21}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{23}\alpha_{33}
\end{aligned}
\tag{1.17}
$$

The following has been assumed:

(a) $X_1, Y_1, Z_1$ is a fixed and stationary coordinate system.
(b) $X, Y, Z$ is a body fixed coordinate system where the body rotates around some fixed point $O$.
(c) The direction cosines between the $X$ coordinate and coordinates $X_1, Y_1, Z_1$ are $\alpha_{11}, \alpha_{12}, \alpha_{13}$.
(d) Similarly, for the direction cosines between the $Y$ coordinate and coordinates $X_1, Y_1, Z_1$, we have: $\alpha_{21}, \alpha_{22}, \alpha_{23}$, etc.

Hence, the transformation of a vector from the stationary $X_1, Y_1, Z_1$ coordinate system to the rotating $X, Y, Z$ coordinate system can be written in matrix form as follows:

$$
\begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} x_{ea} \\ y_{ea} \\ z_{ea} \end{bmatrix} \tag{1.18}
$$

where the vector $\begin{bmatrix} x_{ea} \\ y_{ea} \\ z_{ea} \end{bmatrix}$ is in the $X_1, Y_1, Z_1$ coordinate frame and the vector $\begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix}$ is in the $X, Y, Z$ coordinate frame. The rotation of the vector $X_1, Y_1, Z_1$ coordinates into the vector in $X, Y, Z$ coordinates can be described by Euler angular transformations in matrix form as follows:

$$
\begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} x_{ea} \\ y_{ea} \\ z_{ea} \end{bmatrix} \tag{1.19}
$$

Equating the two transformations we note that:

$$\begin{aligned}
&\alpha_{11} = \cos\theta\cos\psi;\;\; \alpha_{12} = \cos\theta\sin\psi;\;\; \alpha_{13} = -\sin\theta \\
&\alpha_{21} = \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi;\;\; \alpha_{22} = \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi \\
&\alpha_{23} = \sin\phi\cos\theta;\;\; \alpha_{31} = \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\
&\alpha_{32} = \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi;\;\; \alpha_{33} = \cos\phi\cos\theta
\end{aligned} \tag{1.20}$$

Taking the product of $\alpha_{22} * \alpha_{33}$

$$\begin{aligned}
\alpha_{22} * \alpha_{33} &= (\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi) * (\cos\phi\cos\theta) \\
&= \cos\phi\cos\theta\sin\phi\sin\theta\sin\psi + \cos\theta\cos^2\phi\cos\psi
\end{aligned} \tag{1.21}$$

Similarly the product of $-\alpha_{23} * \alpha_{32}$

$$\begin{aligned}
-\alpha_{23} * \alpha_{32} &= -(\sin\phi\cos\theta) * (\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi) \\
&= -(\cos\phi\cos\theta\sin\phi\sin\theta\sin\psi) + \cos\theta\sin^2\phi\cos\psi
\end{aligned} \tag{1.22}$$

Calculating $\alpha_{22} * \alpha_{33} - \alpha_{23} * \alpha_{32}$, we have:

$$\begin{aligned}
\alpha_{22} * \alpha_{33} - \alpha_{23} * \alpha_{32} &= (\cos\phi\cos\theta\sin\phi\sin\theta\sin\psi) \\
&\quad - (\cos\phi\cos\theta\sin\phi\sin\theta\sin\psi) \\
&\quad + \cos\theta\cos^2\phi\cos\psi + \cos\theta\sin^2\phi\cos\psi \\
&= \cos\theta\cos\psi = \alpha_{11}
\end{aligned} \tag{1.23}$$

Similarly for $\alpha_{12}$, we have: $\alpha_{12} = \alpha_{23} * \alpha_{31} - \alpha_{33} * \alpha_{21} = \cos\theta\sin\psi$, which is expanded in the following equation:

$$\begin{aligned}
\alpha_{23} * \alpha_{31} &= (\sin\phi\cos\theta) * (\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi) \\
&= \sin\phi\cos\theta\cos\phi\sin\theta\cos\psi + \sin^2\phi\cos\theta\sin\psi \\
&\quad - \alpha_{33} * \alpha_{21} = -(\cos\phi\cos\theta)(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi) \\
&= \cos^2\phi\cos\theta\sin\psi - \cos\phi\cos\theta\sin\phi\sin\theta\cos\psi \\
&\Rightarrow \alpha_{23} * \alpha_{31} - \alpha_{33} * \alpha_{21} = \cos\theta\sin\psi
\end{aligned} \tag{1.24}$$

And finally for $\alpha_{21} = \alpha_{32} * \alpha_{13} - \alpha_{12} * \alpha_{33}$, the result is:

$$
\begin{aligned}
\alpha_{32} * \alpha_{13} &= (\sin\phi\cos\psi - \cos\phi\sin\theta\sin\psi)(\sin\theta) \\
&= \sin\theta\sin\phi\cos\psi - \sin^2\theta\cos\phi\sin\psi \\
&\quad - \alpha_{12} * \alpha_{33} = -(\cos\theta\sin\psi)(\cos\phi\cos\theta) = -\cos^2\theta\sin\psi\cos\phi \\
&\quad \Rightarrow \alpha_{21} = \sin\theta\sin\phi\cos\psi - \sin^2\theta\cos\phi\sin\psi - \cos^2\theta\sin\psi\cos\phi \\
&= \sin\theta\sin\phi\cos\psi - \cos\phi\sin\psi
\end{aligned}
\tag{1.25}
$$

Following a similar procedure, all of the identities between the Euler angles and the direction cosines are as follows:

1. $\alpha_{11} = \alpha_{22} * \alpha_{33} - \alpha_{23} * \alpha_{32} = \cos\theta\cos\psi$
2. $\alpha_{12} = \alpha_{23} * \alpha_{31} - \alpha_{33} * \alpha_{21} = \cos\theta\sin\psi$
3. $\alpha_{13} = \alpha_{21} * \alpha_{32} - \alpha_{31} * \alpha_{22} = -\sin\theta$
4. $\alpha_{21} = \alpha_{32} * \alpha_{13} - \alpha_{12} * \alpha_{33} = \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi$
5. $\alpha_{22} = \alpha_{33} * \alpha_{11} - \alpha_{13} * \alpha_{31} = \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi$
6. $\alpha_{23} = \alpha_{31} * \alpha_{12} - \alpha_{11} * \alpha_{32} = \sin\phi\cos\theta$
7. $\alpha_{31} = \alpha_{12} * \alpha_{23} - \alpha_{22} * \alpha_{13} = \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi$
8. $\alpha_{32} = \alpha_{13} * \alpha_{21} - \alpha_{11} * \alpha_{23} = \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi$
9. $\alpha_{33} = \alpha_{11} * \alpha_{22} - \alpha_{12} * \alpha_{21} = \cos\phi\cos\theta$