6

Derivative Model Applications

6.1 Spot Price Models

The Black–Scholes GBM (geometric Brownian motion) model can be generalized to other models that are more realistic for particular markets. The various simple extensions to the Black–Scholes model assume constant parameters for ease of calculation. In reality, the properties of time series such as volatility, mean reversion, long-term levels and jump behaviour will at the very least vary through time with reasonably predictable patterns. These characteristics can be included in spot models.

6.1.1 Geometric Brownian Motion

The GBM assumption defined in Eq. (5.10) as a process that describes the dynamics of the prices of financial instruments is an approximation of the behaviour observed in real markets. GBM models are frequently used for security prices, interest rates, commodities and other economic and financial variables, and follow what has been defined as a random walk. The Weiner process is the continuous limit of a discrete time random walk. A generalized Weiner process introduces the concept of an expected drift rate. The drift rate is the average increase in a stochastic variable for each unit of time. In models for financial variables, the expected drift rate is replaced with a constant drift rate. Another issue in GBM models is that the uncertainty associated with the price path is greater the longer the time horizon. As the variance of the Weiner process increases linearly as the time horizon increases, the standard deviation

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grows as the square root of the time horizon. This is the equivalent to the definition of volatility, where scaling the standard deviation by the square root of T annualizes the volatility σ.

The GBM process represented in Eq. (5.10) was discretized in Eqs. (5.20) and (5.21) for the simulation of a spot price. Figure 6.1 illustrates a GBM process simulated 100 times with the parameters S = 100, $r - \delta$ = 0.05, σ = 0.30, and Δt = 1/250. In this example, $r - \delta$ is the drift, and $\sigma\varepsilon\sqrt{\Delta t}$ is the stochastic component. One observation is that the sample paths in Fig. 6.1 tend to wander from the initial starting point of σ = 100. While this may be realistic for some variables, and can be verified in tests for random walks, it may, however, not be suitable for other financial and economic time series.

6.1.2 Mean Reversion

The usual assumption made for asset price evolution in many markets is the GBM model assumption. This model, however, allows prices to wander off to unrealistic levels when applied to markets such as energy and commodities. Mean reversion was first described by Vasicek (1977) for modelling interest rate dynamics and has subsequently been widely adapted. Mean reversion can

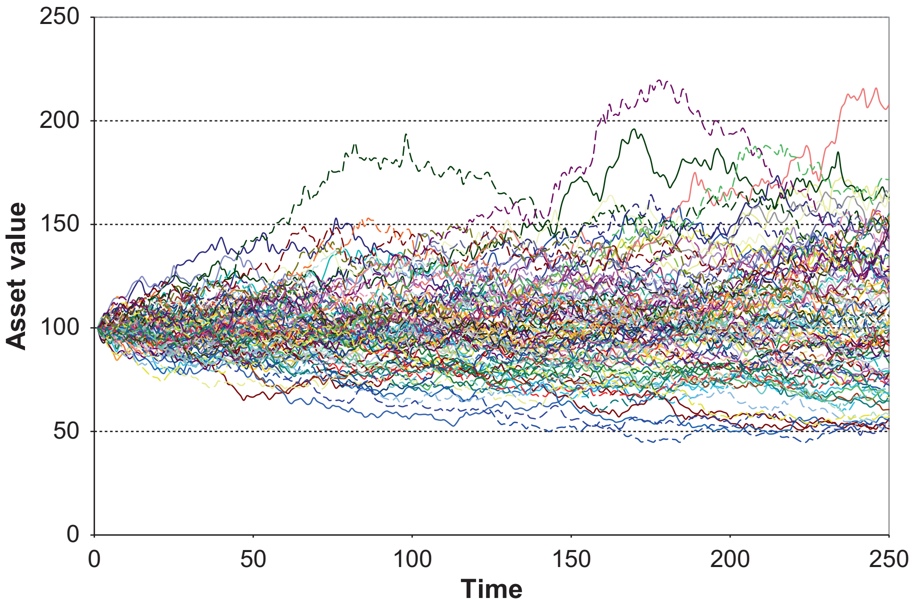

Fig. 6.1 Illustration of 100 simulated GBM paths

be understood by looking at a simple model of a mean reverting spot price (Schwartz 1997), represented by the following equation:

$$dS = \alpha\left(\mu - \ln S\right)Sdt + \sigma S dz \tag{6.1}$$

Figure 6.2 illustrates the log form of a mean reverting process simulated 100 times with the parameters S = 100, α = 3, $\bar{S} = 100$, σ = 0.30, and Δt = 1/250. In this model, the spot price mean reverts to the long-term level $\bar{S} = e^{\mu}$ at a speed given by the mean reversion rate, α, that is taken to be strictly positive. If the spot price is above the long-term level $\bar{S}$, then the drift of the spot price will be negative and the price will tend to revert back towards the long-term level. Similarly, if the spot price is below the long-term level, then the drift will be positive and the price will tend to move back towards $\bar{S}$. Note that, at any point in time, the spot price will not necessarily move back towards the long-term level as the random change in the spot price may be of the opposite sign and greater in magnitude than the drift component. This formulation of the mean reversion process represents one of a number of possible equations that capture the same type of market evolution of prices over time. In reality, the spot price does not mean revert to a constant long-term

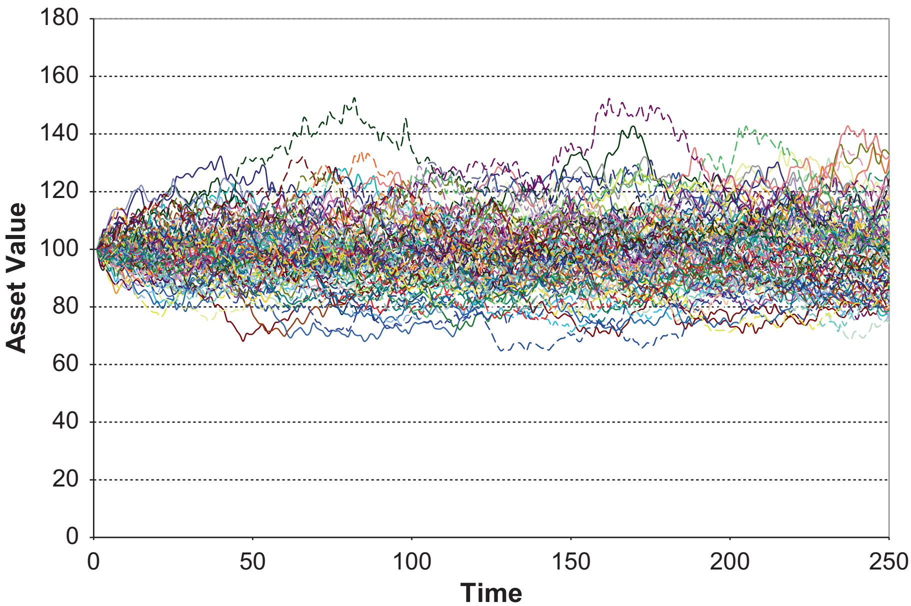

Fig. 6.2 Illustration of 100 simulated mean reversion paths

level. Information on the level to which the spot price mean reverts is contained in the forward curve prices and volatilities.

6.1.3 Jumps and Seasonal Patterns

Jumps can be a significant component of the behaviour of spot prices. This type of behaviour, where the price exhibits sudden, large changes, can be modelled by using jump processes. A simple and realistic model for a spot price, which is identical to the Black–Scholes model except for the addition of a jump process, is the jump-diffusion model introduced by Merton (1976). This model is described by the following stochastic differential equation (SDE):

$$dS = \mu S dt + \sigma S dz + \kappa S dq \tag{6.2}$$

where the lognormal jumps are driven by a Poisson process, and the annualized frequency of jumps is given by ϕ, the average number of jumps per year (ϕ) is defined by prob(dq = 1) = ϕdt). The proportional jump size is κ, which is random and determined by the natural logarithm of the proportional jumps being normally distributed:

$$\ln(1+\kappa) \sim N\left(\ln(1+\bar{\kappa}) - \frac{1}{2}\gamma^2, \gamma^2\right) \tag{6.3}$$

where $\bar{\kappa}$ is the mean jump size and γ is the standard deviation of the proportional jump size. The jump process (dq) is a discrete time process, that is, jumps do not occur continuously, but at specific instants of time. Therefore, for typical jump frequencies, most of the time dq = 0 and only takes the value 1 when a randomly timed jump occurs. When no jump occurs, the spot price behaviour is identical to GBM and only differs when a jump occurs. The proportional jumps (or equivalently jump returns) in Eq. (6.2) are normally distributed and therefore symmetrical, that is, the number of positive and negative jumps and the range of sizes of the proportional jumps will be equal on average.

Season patterns can be taken into account by including seasonality as a deterministic process in the stochastic process for the underlying price path. Discrete methods such as Fourier Transforms that include Sine, Cosine and Fast Fourier Transforms can be specified as continuous processes and included in the specification of the underlying price path.

6.2 Stochastic Volatility

The assumption in the Black–Scholes model that volatility is constant does not always hold. The GARCH (generalized autoregressive conditional hetero-skedasticity) process is one representation of a stochastic volatility model. Many other models have been proposed for the behaviour of volatility. The Heston (1993) form of the stochastic volatility model is described by the following processes for the spot price and the spot price return variance $V = \sigma^2$;

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{\upsilon_t} dW_t \tag{6.4}$$

$$dv_t = \kappa\left(\theta - \upsilon_t\right)dt + \sigma_v \sqrt{v_t}\, dZ_t \tag{6.5}$$

Equation (6.4) is the GBM model with volatility ν_t, which is not constant and changes randomly. The behaviour of the volatility is determined by Eq. (6.5), which specifies the process followed by the variance, the square of the volatility. The variance mean reverts to a long-term level θ at a rate given by κ. The absolute volatility of the variance is $\sigma_v\sqrt{\nu}$, which is proportional to the square root of the variance, that is, the volatility of the spot price. The source of randomness in the variance, dZ_t, is different from the dW_t driving the spot price, although it may be correlated with correlation coefficient ρ.

The following illustrates the estimation of the parameters for the Heston stochastic volatility model.

Two sources of uncertainty reflected in FX options are the stochastic FX rate and stochastic volatility. The Black–Scholes model addresses the first, while stochastic volatility models address both the first and second. The Heston model (1993) is a common method applied to capture stochastic diffusion volatility:

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{\upsilon_t} dW_t \tag{6.6}$$

$$dv_t = \kappa\left(\theta - \upsilon_t\right)dt + \sigma_v \sqrt{v_t}\, dZ_t \tag{6.7}$$

$$dW_t dZ_t = \rho dt \tag{6.8}$$

Stochastic volatility induces smiles and skews that decrease as the option maturity increases. The positive volatility of volatility (σ_v) generates a smile, or fatter tails in the distribution, while a non-zero correlation (ρ) generates skew of the same sign, that is, shifts the probability weight to either one of the tails of the distribution.

Calibrating the Heston model ensures the model matches the market and avoids arbitrage. The Heston model requires the estimation of five parameters:

- kappa (κ), the rate of mean reversion in the volatility
- theta (θ), the long run mean
- the asset volatility (v_t)
- volatility of volatility (σ_v), which influences the kurtosis of the distribution, and
- the correlation (ρ)

This is achieved by finding the set of parameters that produce Heston model prices that match vanilla Black–Scholes option market prices.

The Black–Scholes model has a number of applications in the OTC FX options market:

- Market prices are quoted as Black–Scholes implied volatilities instead of option prices, and are provided at a Black–Scholes delta (δ) instead of the strike.
- Liquidity is typically at five delta levels, 10 δ put, 25 δ put, 0 δ straddle, 25 δ call and 10 δ call.

The EUR-USD is used as an illustration. Calibrating the Heston model consists of:

1. converting the EUR-USD option delta quotes into strikes
2. deriving the Black–Scholes option prices using the implied volatilities, the derived strikes, EUR-USD forwards and interest rates for the two currencies, and
3. calibrating the Heston model parameters to the Black–Scholes prices across the volatility surface

The Black–Scholes option model is equivalent to the Black model when the risk-free interest rate is zero, which reflects the forward price, and then discounted to derive the present value.

Figure 6.3 shows the EUR-USD implied volatility surface.

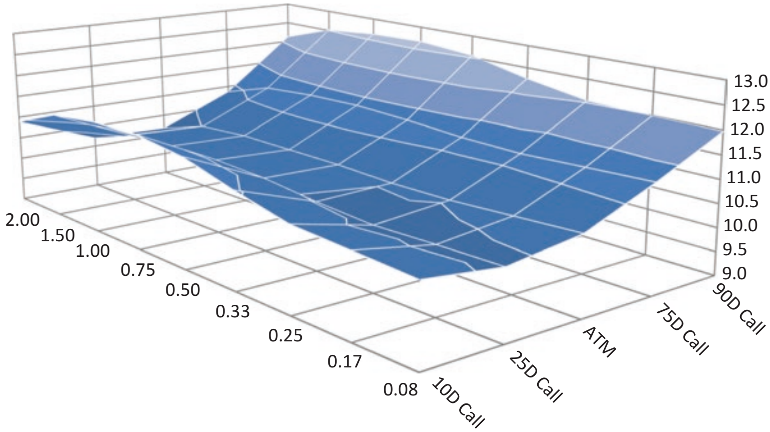

Fig. 6.3 EUR-USD implied volatility surface. *As at 5/15/08*

The five FX option quotes for each maturity are:

- *A delta-neutral straddle (ATMV)* implied volatility. A straddle equals the sum of a call and a put with the same strike. Delta neutral implies that δ(c) plus δ(p) is equal to zero, with $N(d_+)$ equal to 0.5, and d_+ equal to zero. Therefore, ATMV ≡ IV(50 δ c) (= IV(−50 δ p) by put call parity). IV is the implied volatility, c equals call, p equals put, d is the delta and N() is a standard normal function.
- *25-delta Risk Reversal (RR25)*. The RR25 describes the slope of a smile, which represents the skew in the risk neutral distribution of the return. RR25 ≡ IV (25δc) − IV (25δp).
- *25-delta Strangle Margin (SM25)*, or a butterfly spread. A strangle equals the sum of a call and a put with two different strikes, and captures the smile curvature, or the distribution's kurtosis. SM25 ≡ (IV (25δc) + IV (25δp)) /2 − ATMV.
- *10-delta Risk Reversal (10RR)*, and
- *10-delta Strangle Margin (10SM)*

The implied volatility quotes for the five deltas have the following relationships:

- IV (0δs) = ATMV
- IV (25δc) = ATMV + RR25/2 + SM25
- IV (25δp) = ATMV − RR25/2 + SM25
- IV (10δc) = ATMV + RR10/2 + SM10, and
- IV (10δp) = ATMV − RR10/2 + SM10

Table 6.1 The Heston parameters

Kappa (κ)	2.044
Theta (θ)	1.16%
Asset volatility (V)	1.57%
Vol of Vol (σ)	35.62%
Correlation (ρ)	0.173

The EUR-USD option deltas are converted into the strike surface as:

$$K = F\exp\left[\pm IV(\delta,\tau)\sqrt{\tau}N^{-1}\left(\pm e^{r_f\tau}\delta\right)+\frac{1}{2}IV(\delta,\tau)^2\tau\right] \tag{6.9}$$

where for each maturity:

K = the option strike,
F = FX Forward,
IV = implied volatility,
δ = strike delta,
τ = time to expiry,
$N^{-1}()$ = the inverse of the standard normal cumulative distribution, and
r_f = the USD interest rate.

Out-of-the-money Black prices are then calculated across the surface using the implied volatility quotes and the derived strikes. Finally, the stochastic volatility parameters are derived by minimizing the squared error, scaled by a weight derived as the inverse of the delta implied volatility bid/offer spread quote, between the Heston and Black prices across the whole surface.

Table 6.1 illustrates the Heston model parameter estimates for the EUR-USD FX option.

6.3 Forward Curve Models

Forwards and futures markets are often used by risk managers to hedge risk, with liquid forward prices providing a price discovery mechanism to determine the fair value for future delivery. Forward curves contain information about the prices an investor can lock into today to buy or sell at a certain time in the future. Forward curves are well known and understood in the debt markets. Forward rate agreements and exchange traded futures contracts are heavily traded and allow users to lock in borrowing and lending rates for future time periods.

In contrast to futures and forwards, price forecasts are predictions on the likely spot price for periods in the future, and can differ widely between market participants. Forward prices, however, depend on the relationship between traded instruments. Tradable prices today for future spot transactions can be locked in using forward prices, and as such, capture the market reality. Therefore, prices from forwards and futures markets are key inputs to many derivative pricing models, and are as essential in the pricing of derivatives as spot prices.

In the past, the majority of work on modelling prices has focused on stochastic processes for the spot price and other key variables, such as the dividend yields, convenience yields and interest rates. This approach, however, can have some fundamental disadvantages. The first is that key state variables, such as the convenience yield, are unobservable, and second, the forward price curve is an endogenous function of the model parameters, and therefore, will not necessarily be consistent with the market observable forward prices. As a result, many industry practitioners require the forward curve to be an input into the derivative pricing model, rather than an *output* from it.

Term structure consistent models model the dynamics of the entire term structure in a manner that is consistent with the initial observed market data. These models can be further classified into those that fit the term structure of prices such as interest rates, and those that fit the term structure of prices and price volatilities. There are models in the interest rate world and developments in the energy and commodity markets that use term structure approaches. An approach based on modelling the entire forward price curve with multiple sources of uncertainty uses all the information contained in the term structure of futures prices in addition to the historical volatilities of futures returns for different maturities.

6.3.1 A Single Factor Model for the Forward Curve

Forward curve models are defined as models that explicitly model all the forward prices simultaneously instead of just the spot price. A simple single factor model of the forward curve can be represented by the following stochastic differential equation:

$$\frac{dF(t,T)}{F(t,T)} = \sigma \mathrm{e}^{-\alpha(T-t)} dz(t) \tag{6.10}$$

The inputs to the model are the observed forward curve $F(t, T)$, which denotes the forward price at time t for maturity date T, and $\sigma e^{-\alpha(T-t)}$, which is the single 'factor' or volatility function associated with the source of risk $dz(t)$. Equation (6.10) also has no drift term. As futures and forward contracts have zero initial investment, their expected return in a risk-neutral world must be zero, implying that the process describing their evolution has zero drift. The volatility function of Eq. (6.10) has a very simple negative exponential form illustrated in Fig. 6.4.

For this volatility function, short-dated forward returns are more volatile than long-dated forwards. Information occurring in the market today has little effect on, say, the 5-year forward price, but can have a significant effect on the 1-month forward price. The parameter values used for Fig. 6.4 are $\alpha = 1.0$ and $\sigma = 0.40$. Here, σ represents the 'overall' volatility of the forward curve, while α explains how fast the forward volatility curve attenuates with increasing maturity. With an α of 100%, the 1-month forward has a volatility of about 37%, decreasing to approximately 2% for the 3-year forward.

The volatility function is not restricted to have the parameterized form of Eq. (6.10). The function can be generalized as:

$$\frac{dF(t,T)}{F(t,T)} = \sigma(t,T)\,dz(t) \tag{6.11}$$

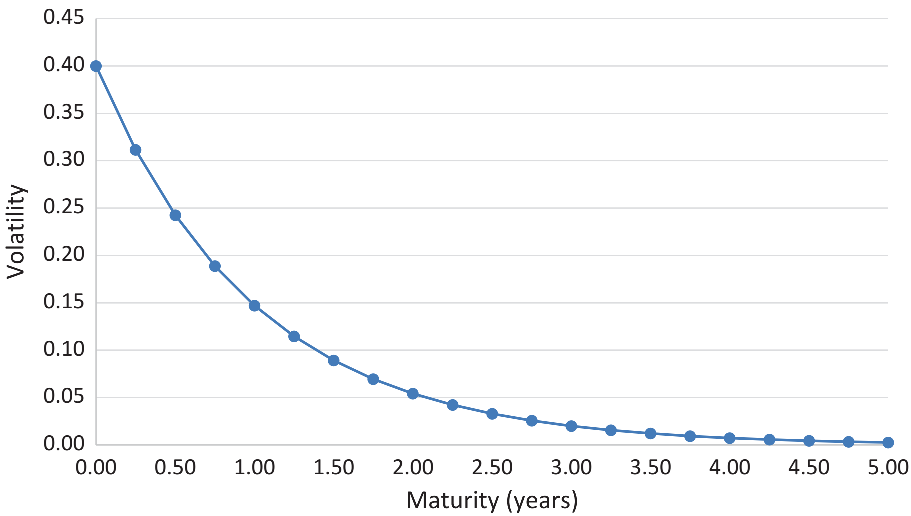

Fig. 6.4 A negative exponential volatility function for forward prices

where $\sigma(t,T)$ is the time t volatility of the T maturity forward price return. The form of $\sigma(t,T)$ can be determined from market data.

6.3.2 The Dynamics of the Forward Curve

An important observation is that forward prices of different maturities are not perfectly correlated. The curves generally move up and down together, but they also change shape in quite complex ways. One method that can be used to determine the set of common factors that drive the dynamics of the forward curve is principal components analysis (PCA), or eigenvector decomposition of the covariance matrix. This procedure can be utilized to simultaneously identify the number of significant factors and estimate the volatility functions. The technique involves calculating the covariances between every pair of forward price returns in a historical time series to form a covariance matrix. The eigenvectors of the covariance matrix yield estimates of the factors driving the evolution of the forward curve.

The implication is that to effectively describe the evolution of the energy forward curve, more than a single factor is required. The model described by Eq. (6.11) can be modified through the addition of sources of risk and volatility functions. For a general multifactor model, the behaviour of the forward curve can be represented by the following equation:

$$\frac{dF(t,T)}{F(t,T)} = \sum_{i=1}^{n} \sigma_i(t,T)\,dz_i(t) \tag{6.12}$$

In this formulation, there are n independent sources of uncertainty, which drive the evolution of the forward curve. Each source of uncertainty has associated with it a volatility function, which determines by how much, and in which direction, that random shock moves each point of the forward curve. Therefore, $\sigma_i(t,T)$ are the n volatility functions associated with the independent sources of risk $dz_i(t)$. In practice, n is usually set to $n = 1, 2$, or 3.

6.3.3 The Relationships Between Forward Curve and Spot Price Models

Intuitively, a model that describes the evolution of the whole forward curve is implicitly describing the front end of the curve, which is simply the spot energy price, and so, the forward curve models must be related to spot price models. The stochastic differential Eq. (6.12) can be integrated to obtain the following solution:

$$F(t,T) = F(0,T)\exp\left[\sum_{i=1}^{n}\left\{-\frac{1}{2}\int_0^t \sigma_i(u,T)^2\,du + \int_0^t \sigma_i(u,T)\,dz_i(u)\right\}\right] \tag{6.13}$$

This equation expresses the forward curve at time t in terms of its initially observed state (time 0) and integrals of the volatility functions. The spot price is just the forward contract for immediate delivery, and so, the process for the spot price can be obtained by setting $T = t$, that is:

$$S(t) = F(t,t) = F(0,t)\exp\left[\sum_{i=1}^{n}\left\{-\frac{1}{2}\int_0^t \sigma_i(u,t)^2\,du + \int_0^t \sigma_i(u,t)\,dz_i(u)\right\}\right] \tag{6.14}$$

Equation (6.14) can then be differentiated to yield the stochastic differential equation for the spot price:

$$\frac{dS(t)}{S(t)} = \left[\frac{\partial \ln F(0,t)}{\partial t} - \sum_{i=1}^{n}\left\{\int_0^t \sigma_i(u,t)\frac{\partial \sigma_i(u,t)}{\partial t}\,du + \int_0^t \frac{\partial \sigma_i(u,t)}{\partial t}\,dz_i(u)\right\}\right]dt + \sum_{i=1}^{n}\sigma_i(t,t)\,dz_i(t) \tag{6.15}$$

The term in square parentheses in the drift can be interpreted as being equivalent to the sum of the deterministic riskless rate of interest $r(t)$ and a convenience yield $d(t)$, which, in general, will be stochastic. Since the last component of the drift term involves the integration over the Brownian motion, the spot price process will, in general, be non-Markovian—that is, the evolution of the spot price will depend upon its past evolution.

One special case of the general model is the simple single factor model described by Eq. (6.10). For this model, $n = 1$ and $\sigma_1(t,T) = \sigma e^{-\alpha(T-t)}$. Clewlow and Strickland (2000) evaluate Eq. (6.15) with this volatility function and show that the resulting spot price process is given by:

$$\frac{dS(t)}{S(t)} = \left[\frac{\partial \ln F(0,t)}{\partial t} + \alpha\left(\ln F(0,t) - \ln S(t)\right) + \frac{\sigma^2}{4}\left(1 - e^{-2\alpha t}\right)\right]dt + \sigma\,dz(t) \tag{6.16}$$

This implies:

$$\frac{dS(t)}{S(t)} = \left[\mu(t) - \alpha \ln S(t)\right]dt + \sigma dz(t) \tag{6.17}$$

where,

$$\mu(t) = \frac{\partial \ln F(0,t)}{\partial t} + \alpha \ln F(0,t) + \frac{\sigma^2}{4}\left(1 - \mathrm{e}^{-2\alpha t}\right)$$

This single factor forward curve model is therefore just the single factor Schwartz (1997) model with a time-dependent drift term. It is this term in the drift that allows the model to now fit the observed forward prices. Note also that this particular form of the forward curve volatility function results in a 'Markovian' spot price process, as the dependence in the drift on the path of the Brownian motion disappears.

The relationship between the forward curve model and the spot return model also shows that the mean reverting behaviour of the spot price is directly related to the attenuation of volatility of the forward curve. By setting $\alpha = 0$, the Black (1976) model is obtained. This is, therefore, a special case of the general model in Eq. (6.12) with $\sigma(t, T) = \sigma$ and $n = 1$. The main advantage of the forward curve modelling approach is the flexibility that the user has in choosing both the number and form of the volatility functions. These can be chosen in one of two general ways—historically from time series analysis or implied from the market prices of options.

6.4 Convertible Bonds

Convertibles are hybrid derivative securities that combine the characteristics of both bonds and stocks, and include options on the issuer's common stock and debt. A convertible gives the bond holder the right to exchange or convert the bond's par amount for the issuer's common shares at a fixed rate during a specified time period. Convertible bonds can be callable by the issuer on specified dates over the life of the convertible, with the call option decreasing the value of the convertible.

Convertibles can also contain puts where the buyer can put the bond to the issuer, with the put option increasing the value of the convertible. When the stock price is relatively low when compared to the conversion price, the convertible is unlikely to be converted into stock, and is therefore effectively a straight bond. When the stock price is relatively high when compared to the conversion price, the convertible is more likely to be converted into stock, and the convertible price is therefore the conversion parity, or the stock price multiplied by the conversion ratio.

The motives for financing through the issuance of convertibles include delaying equity financing until growth has been realized, and financing when the debt markets is not accessible, while for investors, convertibles can offer a higher yield than common stock dividend yields and the potential upside in the firm's growth and stock conversion.

Figure 6.5 illustrates a convertible bond price as a function of the stock price. The stock price is on the horizontal axis, the conversion ratio on the vertical axis, and the horizontal line represents the bond floor. The bond floor is the equivalent of the market value of a fixed income bond, where the coupons and redemption value are discounted at an interest rate that reflects the credit quality of the issuer. The conversion ratio is the number of ordinary shares at which the bond's notional value is converted, is established at issue and typically stays constant over the life of the convertible bond. The conversion price is the bond's nominal value divided by the conversion ratio. The diagonal line represents parity, derived as the conversion ratio multiplied by the share price, and represents the market value of the shares received at the conversion of the bond.

The option to convert into stock or retain the bond implies that the convertible's value should be the minimum of that of the stock or the bond. The stock and bond's minimum values therefore function as lower bounds on the

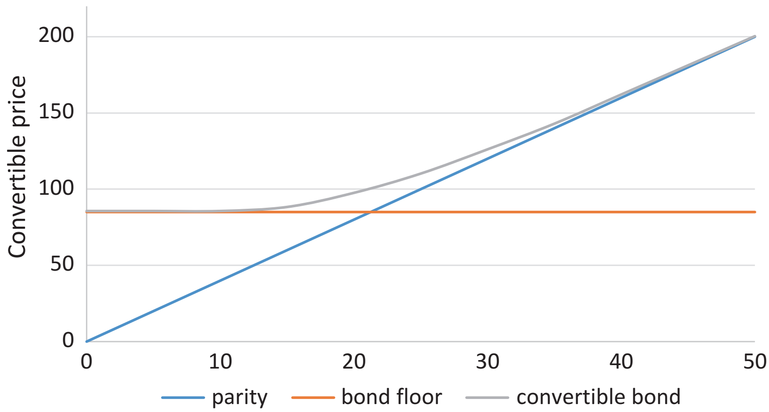

Fig. 6.5 A convertible bond price as a function of the stock price

convertible bond price. The conversion value is the equivalent to a call option on the stock, with the market value of the option to convert reflected in the difference between the bond floor and the convertible value. The conversion becomes more valuable, and the price of the convertible bond will increase as the stock volatility increases, with the conversion line in Fig. 6.5 becoming less convex.

Convertible bonds are typically priced with binomial lattice trees that include the bond's embedded option at each tree node. The model assumes that the convertible bond's value is a function of the underlying stock price volatility. The factors that can influence a convertible bond's value include the market parameters, the terms in the prospectus and their behaviour. A binomial one factor model was used for a simple illustration of a convertible bond's theoretical value. The Cox, Ross and Rubinstein (1979) approach was used for the binomial tree. See Sect. 5.5.2 for background on the binomial model.

The following variables are used to illustrate the convertible bond:

- the bond notional value is 100,
- the bond coupon is 10%
- the convertible bond maturity (T) is 5 years
- the conversion ratio (m) is 4
- calls of \$107.5 in year two, which decline by \$2.5 every year to maturity
- the current stock price is \$25
- the volatility is 20%
- the risk-free rate is 5%

A yield to maturity or flat term structure is assumed for the risk-free interest rate. The bond coupon and interest rate are compounded annually, while the volatility can be estimated using historical volatility.

Figure 6.6 illustrates the convertible binomial lattice tree. The convertible bond payoff at each node at maturity is derived as the maximum of the bond redemption value and coupon, and the binomial tree's stock price $S_{T,j}$ multiplied by the conversion ratio.

$$P_{T,j} = \max\left(mS_{T,j}, 100 + coupon\right) \tag{6.18}$$

The convertible price at each node is then derived recursively to the valuation date (t_0) as:

$$P_{i,j} = \max\left[mS_{i,j}, \min\left(e^{-r\Delta t}\left(pP_{i+1,j+1} + (1-p)P_{i+1.j-1}\right), C\right)\right] \tag{6.19}$$

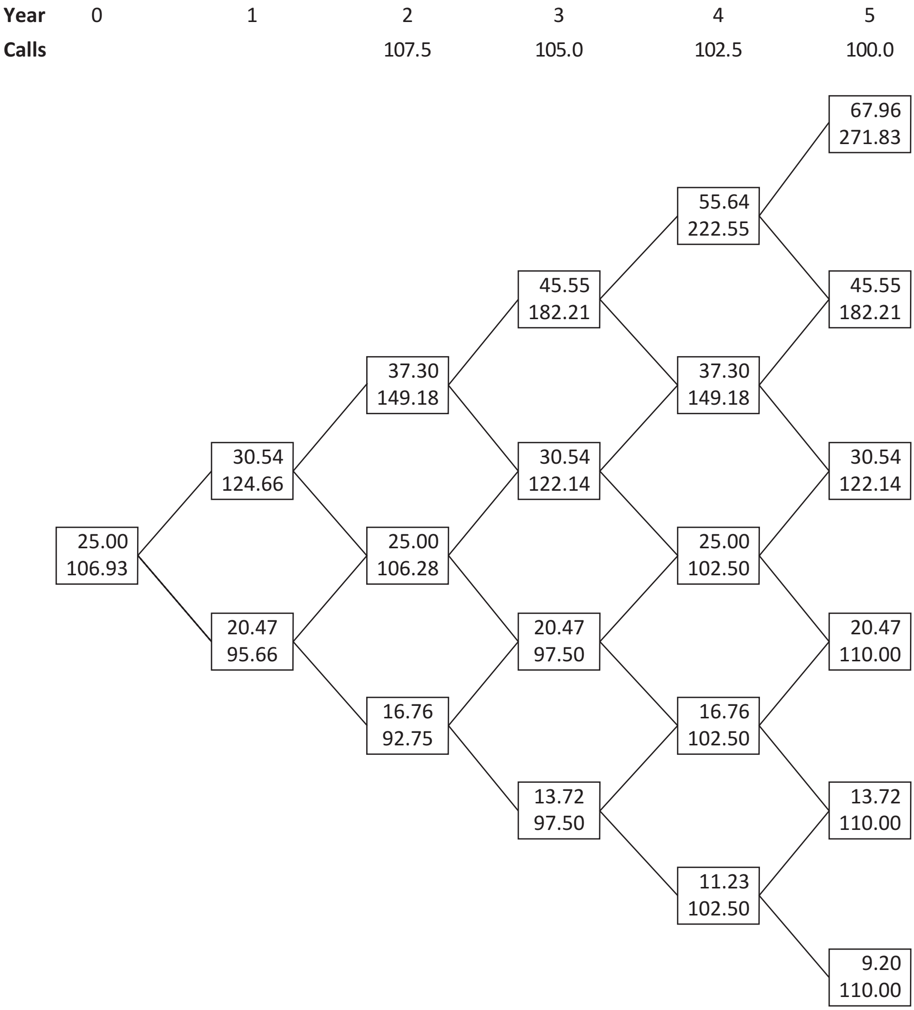

Fig. 6.6 The convertible binomial lattice tree

with the process continuing up the valuation date. The value of the convertible in the example is calculated as \$106.93.

This is a relatively simple example of the valuation of a convertible bond. The valuation can be extended to include stochastic interest rates and the probability of default.

Market factors influence the behaviour of the theoretical value of convertibles in a number of ways. A convertible's value rises along with the parity of the underlying stock, as conversion is more probable. The value of a convertible also increases along with the volatility of the underlying stock, as the option value to convert the bond to stock is larger and near-the-money.

A convertible's value also increases with the put option held by the owner of the convertible. A higher put level has a larger value due to the protection provided by the put in declining markets, and is greater at lower parity levels. The increase in value as a result of puts is also more evident with higher interest rates, as the convertible's fixed-income value is lower.

A convertible's value will decrease as interest rates rise, with the rate of decrease larger as the maturity of the convertible increases. The rate of decrease, however, is less than an equivalent conventional bond due to the offsetting influence of the conversion option, which rises in value as interest rates increase. A convertible's value also decreases as the credit spread increases, with the sensitivity to the credit spread increasing with longer maturities.

Call options also decrease the convertible value, with lower call levels providing the issuer a larger probability for early conversion and reducing the conversion option's time value. The decrease in value with calls is higher at higher parity levels. The decrease in a convertible's value with calls is also more evident with lower interest rates, as the convertible's fixed-income value is larger.

6.5 Compound Options

Compound options, or options on options, where the payoff is another option, allow the holder to buy or sell another option for a fixed price. There are four compound option types—a call on a call, a put on a put, a call on a put, and a put on a call. Projects and investments that are staged as a sequence are compound options, where the initial investment cost is the exercise price for the subsequent option on the next stage of the investment. Plant development, product development and research and development are examples of sequential compound options. Compound options are useful for analysing the impact of an investment on a firm. Many projects and investments are not independent, as assumed in a discounted cash flow (DCF) analysis, but are a series of interrelated cash flows where the initial investment is a prerequisite for the following outlays.

Geske (1979) developed the original closed form solution for a compound option as a call option on a firm's equity, which itself is a European call option on the total value of the firm. The compounding in this specification occurs simultaneously, as the firm's equity, a call option on the leveraged value of the firm, and the call option on the equity appear at the same time. Both simultaneous and sequential compound options can be solved in trees, although the valuation progressively more complicated as more options are added.

The Geske compound option model is specified as a call option on a stock which itself is an option on the firm's assets. The functional representation of this relationship is:

$$C = f(S,t) = f\left(g(V,t),t\right) \tag{6.20}$$

where C is the value of a call option, S is the firm's stock and V is the value of the firm. Transformations in the call option value are therefore defined as a function of transformations in firm value and time. The Geske model transforms the option's underlying state variable from the firm's stock to the firm's market value (V), or the total market value of the firm's equity and debt. The Geske specification therefore provides a measure of firm value when applied to listed options.

The Geske model is specified as:

$$\begin{aligned} C &= VN_2\left(h_1 + \sigma_v\sqrt{T_1 - t}, h_2 + \sigma_v\sqrt{T_2 - t}; p\right) \\ &\quad - Me^{-r_{F_2}(T_2 - t)}N_2\left(h_1, h_2; p\right) - Ke^{-r_{F_1}(T_1 - t)}N_1\left(h_1\right) \end{aligned} \tag{6.21}$$

where,

$$h_1 = \frac{\ln(V / V*) + \left(r_{F_1} - \frac{1}{2}\sigma_v^2\right)(T_1 - t)}{\sigma_v\sqrt{T_1 - t}}$$

$$h_2 = \frac{\ln(V / M) + \left(r_{F_2} - \frac{1}{2}\sigma_v^2\right)(T_2 - t)}{\sigma_v\sqrt{T_2 - t}}$$

$$p = \sqrt{\frac{T_1 - t}{T_2 - t}}$$

$V*$ represents the firm's critical total market value, where the firm's stock level S_{T1} is equal to the option strike K. S_{T1} is derived using Merton's definition of the Black–Scholes model, where a firm's stock is the equivalent to an option:

$$S = VN_1\left(h_2 + \sigma_V\sqrt{T_2 - t}\right) - Me^{-rF_2(T_2 - t)}N_1\left(h_2\right) \tag{6.22}$$

Therefore, at $t = T_1$ when $S_{T1} = K$

$$S_{T1} = V_{T1}^{*} N_1\left(h_2 + \sigma_V \sqrt{T_2 - T_1}\right) - M e^{-rF_2(T_2 - t)} N_1\left(h_2\right) = K \quad (6.23)$$

where h_2 is defined as given earlier. The variable M is the face value of a firm's debt, while T_2 represents the debt's duration. The addition of the M term to the Black–Scholes model reflects the effects of leverage, where leverage changes the firm's equity volatility. The Black–Scholes model assumes that a firm's equity volatility is not a function of the level of equity. The Geske model, however, considers that a firm's equity volatility has an inverse relationship with a firm's stock level. As a firm's stock level increases, the firm's leverage and stock volatility will fall, and the inverse of this relationship also holds. The Geske model also implies the volatility of a firm's total market value, conforms to Miller and Modigliani, and is the equivalent to the Black–Scholes model when the firm has no debt.

A summary of the Geske model variables follows:

C = current market value of a firm's stock call option
S = current market value of the firm's stock
V = current market value of the firm's securities (debt + equity)
V^* = the critical total firm market value where $V \geq V^*$ which implies $S \geq K$
M = face value of market debt (debt outstanding for the firm)
K = strike price of the option
r_{Ft} = the risk-free rate of interest to date t
σ_V = the instantaneous volatility of the firm market value return
σ_s = the instantaneous volatility of the equity return
t = current time
T_1 = expiration date of the option
T_2 = duration of the market debt
N_1 (.) = univariate cumulative normal distribution function
N_2 (…) = bivariate cumulative normal distribution function
ρ = the correlation between the two option exercise opportunities at T_1 and T_2

Refer to Chap. 10 for an example of an application of the Geske compound option model.

6.6 Model Risk

Over the last 50 years, there has been a huge growth in the use of theoretical models for valuation and pricing in financial markets. A large body of the theory relates to derivatives, financial instruments where value is derived from underlying assets. These theories have been extended into real options, where models have been developed for options on real assets. Relying on models to analyse and quantify value and risk, however, carries its own risks. The term *model risk* has many connotations and is used in many different contexts. The following is based on Rebonato (2001) definition. Model risk is the risk, at some point in time, of a significant difference between the modelled value of a complex and/or illiquid asset and the realized value of that same asset.

In the physical sciences, where quantitative modelling originated, predictions can be made reasonably accurately. Variables in physical science models such as time, position and mass exist, regardless of the existence of humans. The fundamental unknown in financial markets, however, is certainty. Many financial and real assets only trade at certain discrete times, while financial variables also only symbolize human expectations. Risk and return refers to expected risk and return, variables that are unobserved and not realized. In most circumstances, however, models based on financial concepts and theory assume causation and stability between the values of these unobserved variables and asset values.

There a number of ways in which the development of a financial model can go wrong:

- The most fundamental risk is that modelling is just not appropriate. Modelling requires knowledge and context within a discipline. Mathematics is a representation or an abstraction of a discipline, and is a means to an end and is not the end itself.
- All the factors that affect valuation may not have been included in the model.
- Although a model may be theoretically correct, the model variables such as forward prices, interest rates, volatilities, correlations and spreads may be poorly estimated. A model's variables, for example, may be based on historical data from a past regime, and therefore, not provide a good estimate of future value.
- Incorrect assumptions can be made about the properties of the asset values being modelled and the relationships between the variables in a model.

- A model may be inappropriate in the existing market environment, or some of the assumptions such as the distributions of variables may not be valid. Even if a model itself is satisfactory, the world it is predicting may be unstable.

Financial modelling draws on a multitude of disciplines—from business management and financial theory to mathematics and computer science—and is as much art as it is theory and quantitative techniques. An intimate knowledge of markets and how market participants think about valuation and risk are also part of the model practitioner's skill set. Derman (1996) provides some procedures for constructing financial models:

- Identify and isolate the most important variables used by market participants to analyse value and risk, and decide which variables can be used in mathematical modelling.
- Separate the dependent variables and the independent variables.
- Determine which variables are directly measurable and those that are more in the nature of human expectations, and so, are only indirectly measurable.
- Specify which variables can be treated as deterministic and those that must be considered as stochastic. Uncertainty will have little effect on the future values for some variables and these therefore can be approximated. For other variables, however, uncertainty will be a critical issue.
- Build a quantitative picture that characterizes how the dependent variables are influenced by the independent ones.
- Determine how to obtain the market values of independent observable variables, and how to derive the implied values of indirectly measurable ones.
- Create a mathematical picture of the problem, and determine which stochastic process best describes the evolution of the independent stochastic variables. Determine whether an analytical or numerical solution is appropriate.
- Deliberate the issues and difficulties in solving the model, and simplify it if necessary to make the solution as easy as possible. Only give up substance, however, for a relatively easy or elegant analytical solution when it is absolutely necessary.
- Finally, programme the model, test it and apply to the valuation problem.

The application of financial modelling draws from a palette that includes knowledge of the markets, the applicability of the financial model, the relevance of the mathematics used to solve the problem, the systems and software used to implement and present it, and the accurate communication and dissemination of the information and knowledge gained from the analysis. Drawing from these various disciplines can address the issues and reduce the risks associated with the application of financial modelling.

Bibliography

Black, F. The pricing of commodity contracts, The Journal of Financial Economics, 3, (Jan–March): 167–79, 1976.

Carr, P. Wu, L. Stochastic Skew Models for FX Options, 2004.

Clewlow, L. and Strickland, C. Energy Derivatives, Pricing and Risk Management, Lacima, 2000.

Clewlow, L. and Strickland, C. Implementing Derivative Models, Wiley, 1998.

Copeland, T. and Antikarov, V. Real Options, Texere, 2001.

Cox, J.S., Ross, S. and Rubinstein, M. Option pricing: a simplified approach, Journal of Financial Economics, 7 (September): 229–63, 1979.

Derman, E. Quantitative Strategies, Research Notes: Model Risk, Goldman Sachs, 1996.

Dixit, A. and Pindyck, R. Investment Under Uncertainty, Princeton University Press, 1994.

Geske, R. The valuation of compound options, Journal of Financial Economics, 7: 63–81, 1979.

Geske, R. and Zhou, Y. Predicting Risk and Return of the S&P 500: Evidence from Index Options, UCLA Working Paper, 2007.

Geske, R. and Zhou, Y. Capital Structure Effects on Prices of Firm Stock Options: Tests Using Implied Market Values of Corporate Debt, UCLA Working Paper, 2007a.

Goldman Sachs. Valuing convertible bonds as derivatives, Quantitative Strategies Research Notes, November 1994.

Haug, E.G. The Complete Guide to Option Pricing Formulas, McGraw Hill, second edition, 2007.

Heston, S. L. ‘A closed-form solution for options with stochastic volatility with applications to bonds and currency options’, The Review of Financial Studies 6(2), 327–343, 1993.

Hull, J. and White, A. An analysis of the bias in option pricing caused by stochastic volatility, Advances in Futures and Options Research, 3: 29–61, 1988.

Merton, R. Option pricing when underlying stock returns are discontinuous, Journal of Financial Economics, 3: 125–44, 1976.

Moodley, N. The Heston Model: A Practical Approach with Matlab Code, 2005.

Natixis Asset Management. Investing in global convertible bonds – stylized facts, pricing and strategies, research paper number eight, January 2017.

Rebonato, R. Model risk: new challenges, new solutions, Risk, March 2001.
Schwartz, E. The stochastic behavior of commodity prices: implications for valuation and hedging, Journal of Finance, LII(3): 923–73, 1997.
Vasicek, O. An equilibrium characterization of the term structure, Journal of Financial Economics, (5): 177–88, 1977.
Zhou, Y. Pricing Individual Stock Options on Firms with Leverage, Anderson School of Management at UCLA, 2007.