5

Derivatives

5.1 Futures, Forwards and Options

A derivative is a financial instrument whose payoff depends on the values of other more basic variables. The variables underlying derivatives are often the prices of traded securities. Derivatives separate market and credit risks from the underlying assets and liabilities, and offer the ability to reduce a risk exposure through its transfer to a party that is prepared to take on and manage those risks. Derivative securities are also known as contingent claims, and can be contingent on almost any variable—from the price of a commodity to weather outcomes. There are two basic types of derivatives—futures/forwards and options.

Forward and futures contracts are agreements to buy or sell an underlying asset at a predetermined time in the future for a specified price. Futures are exchange standardized contracts, whereas forward contracts are direct agreements between two parties. The cash flows of the two contracts also occur at different times. Futures are marked-to-market daily, with cash flows passing between the long and the short position to reflect the daily futures price change, whereas forwards are settled once at maturity. If future interest rates are known with certainty, then futures and forwards can be treated as the same for pricing purposes.

There are two sides to every forward contract. The party who agrees to buy the asset holds a long forward position, while the seller holds a short forward position. At the maturity of the contract (the 'forward date'), the short position delivers the asset to the long position in return for the cash amount

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agreed in the contract, often called the delivery price. Figure 5.1 shows the profit and loss profile to the long forward position at the maturity of the contract. If T represents the contract maturity date, a long forward payoff is expressed as $S_T - K$, where S_T represents the asset price at time T, and K represents the agreed delivery price. The payoff can be positive or negative, depending on the relative values of S_T and K. The short position has the opposite payoff to the long position, that is, $K - S_T$, as every time the long position makes a profit, the short incurs a loss and vice versa. As the holder of a long forward contract is guaranteed to pay a known fixed price for the spot asset, futures and forwards can be seen as insurance contracts providing protection against the price uncertainty in the spot markets.

For an arbitrage relationship to exist, the forward price has to equal the cost of financing the purchase of the spot asset today and holding it until the forward maturity date. Let F represent a forward contract price on a spot asset that is currently trading at S, T the maturity date of the contract, c the cost of holding the spot asset (which includes the borrowing costs for the initial purchase and any storage costs) and d the continuous dividend yield paid out by the underlying asset. The price of a forward contract at time t and the spot instrument on which it is written are related via the 'cost of carry' formula:

$$F = S\mathrm{e}^{(c-d)(T-t)} \tag{5.1}$$

where $T - t$ represents time in years. The continuous dividend yield, for example, can be interpreted as the yield on index futures, the foreign interest

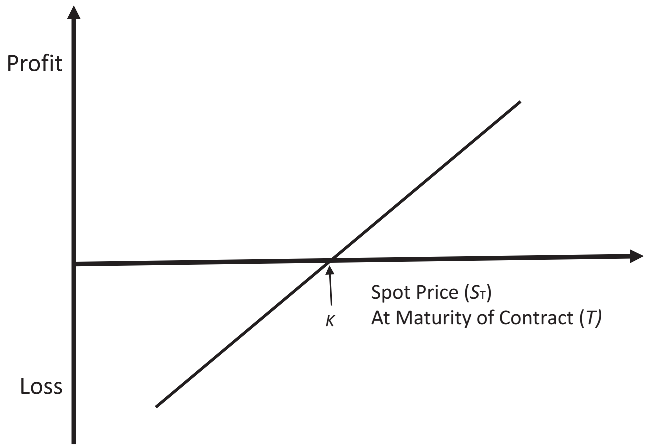

Fig. 5.1 Payoff to a long forward position

rate in foreign exchange futures contracts and the convenience yield for various energy contracts.

Options contracts are the second foundation to the derivatives markets. Options are asymmetrical relationships where the option holder has a right, but not an obligation, to transact at a contracted price called the exercise price. There are two basic types of options. A call option gives the holder the right but not the obligation to buy the spot asset on or before a predetermined date (the maturity date) at a certain price (the strike price), which is agreed today. A put option is the right to sell at the exercise price. Option sellers, or writers, are obliged to commit to the purchaser's decision. Figure 5.2 shows the payoff to the holder of a call option.

Options differ from forward and futures contracts in that a payment, or the option price or premium, must be made by the buyer, usually at the time when entering the contract. If the spot asset price is below the agreed strike or exercise price K at the maturity or expiration date, the holder lets the option expire worthless, forfeits the premium and buys the asset in the spot market. For asset prices greater than K, the holder exercises the option, buying the asset at K and has the ability to immediately make a profit equal to the difference between the two prices less the initial premium. The holder of the call option therefore essentially has the same positive payoff as the long forward contract without the downside risk.

The payoff to a call option is defined as:

$$\max\left(S-K,0\right) \tag{5.2}$$

The second basic type of option, a put option, gives the holder the right, but not the obligation, to sell the asset on or before the maturity date at the strike price.

The payoff for a put option is defined as:

$$\max\left(K-S,0\right) \tag{5.3}$$

Figure 5.3 shows the payoff to the holder of a put option.

As with forwards, there are two sides to every option contract. One party buys the option and has the long position, while the other party writes or sells the option and takes a short position. Figure 5.4 shows the four possible combinations of payoffs for long and short positions in European call and put options at the maturity date T. Options are also classified with respect to their exercise conventions. European options can only be exercised on the maturity

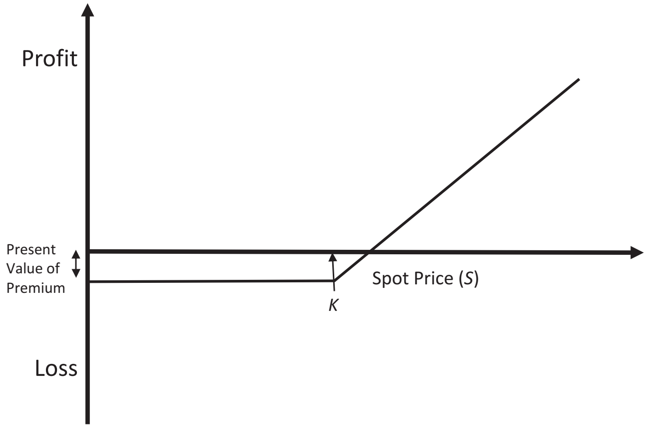

Fig. 5.2 Payoff for a call option

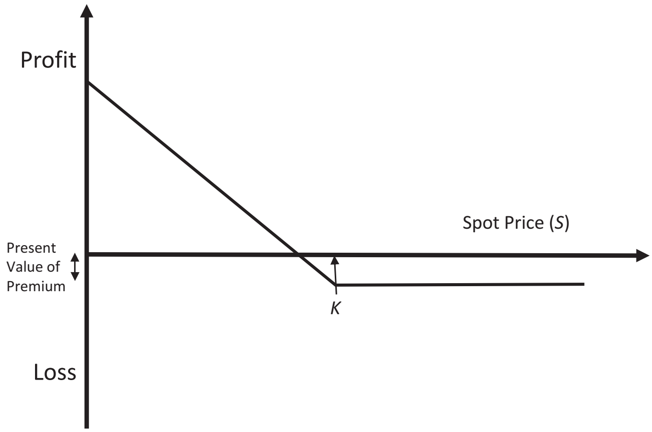

Fig. 5.3 Payoff for a put option

date itself, whereas American-style options can be exercised at any time, up to and including the expiration date. While early exercise of an American option is generally not optimal, there are exceptions to the rule. One example is where the underlying asset pays dividends, reducing the value of the asset and any call options on that asset, in which case, the call option may be exercised before maturity.

Forwards and options are also the key building blocks of more complex derivatives, and these building blocks are themselves interdependent. The decomposition of derivatives into their components assists in identifying a derivative's risk characteristics, which promotes more accurate pricing and

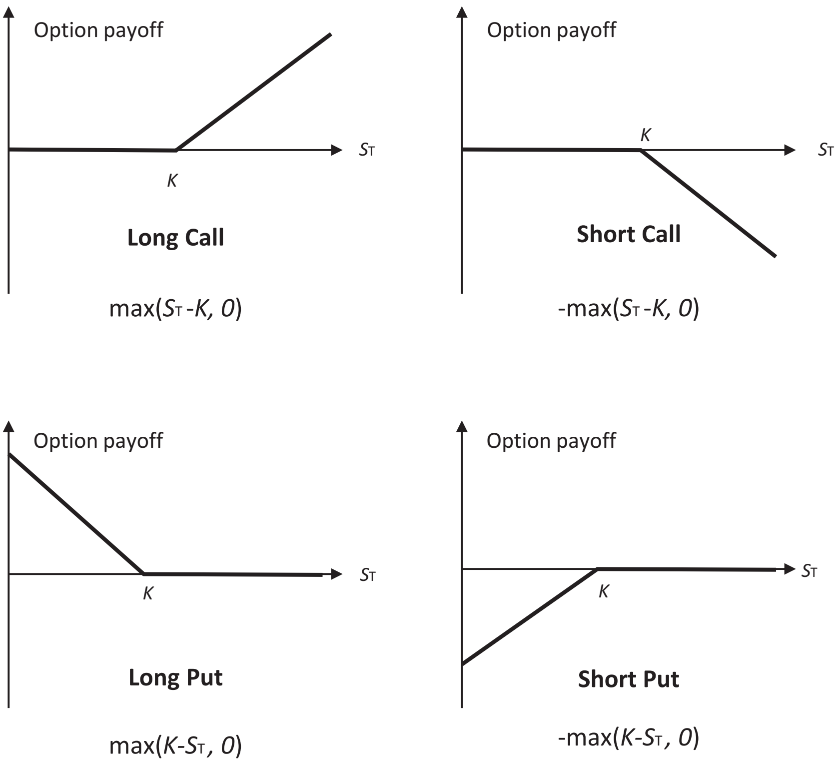

Fig. 5.4 Payoffs for European options

better risk management strategies. The basic futures and options described are the building blocks of all derivative securities, and the principles are consistent across all underlying markets. In some markets, however, derivative structures exhibit a number of important differences from other underlying markets. These differences arise due to the complex contract types that exist in these industries, as well as the complex characteristics of the relevant underlying prices. Both the type of derivative and the associated modelling need to capture the evolution of the underlying prices to reflect these differences.

5.2 The Replicating Portfolio and Risk-Neutral Valuation

The modern theory of option pricing is possibly one of the most important contributions to financial economics. The breakthrough came in the early 1970s, with work by Fisher Black, Myron Scholes and Robert Merton (Black and Scholes 1973; Merton 1973). The Black–Scholes–Merton (BSM) modelling approach not only proved to be important in providing a computationally efficient and relatively easy way of pricing an option, but also demonstrated the principal of no-arbitrage risk-neutral valuation. Their analysis showed that the payoff to an option could be perfectly replicated with a continuously adjusted holding in an underlying asset and a risk-free bond. As the risk of writing an option can be completely eliminated, the risk preferences of market participants are irrelevant to the valuation problem, and it can be assumed that they are risk-neutral. In this construct, all assets earn the risk-free rate of interest, and therefore, the actual expected return on the asset does not appear in the Black–Scholes formula.

Options can be valued by deriving the cost of creating the replicating portfolio such that both the option and the portfolio provide the same future returns, and therefore must sell at the same price to avoid arbitrage opportunities. The portfolio consists of Δ units of an underlying asset S and an amount B borrowed against Δ units at the risk-free rate r. This combination of the borrowing and the underlying asset creates the same cash flows or returns as an option. A binomial model can be used to illustrate the replicating portfolio. The binomial model assumes that the underlying asset price follows a binomial process, where at any time, the asset price S at t_0 can only change to one of two possible values over the time period Δt, either up to uS or down to dS at time t_1, where $u = e^{\sigma\sqrt{\Delta t}}$ and $d = 1/u$. Figure 5.5 is a binomial model for a one-period process, in which a risk-free portfolio consisting of the underlying asset and the call option is illustrated.

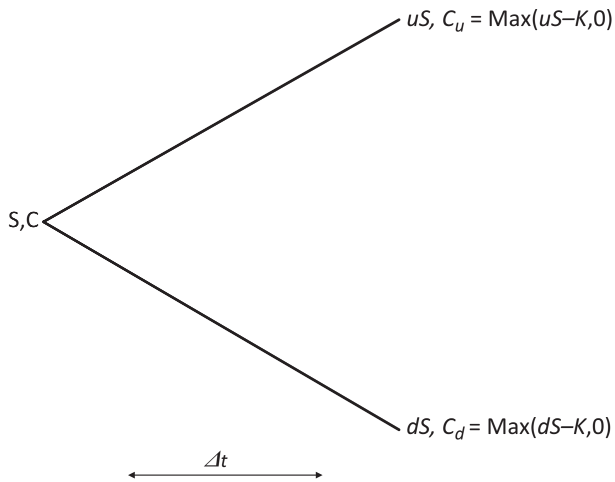

Fig. 5.5 Binomial model of an asset price and call option

The call option is defined as:

$$C \approx (\Delta S - B)$$

The value of the portfolio is the same, regardless of whether the asset price moves up or down over the period Δt:

$$C_u = \Delta u S - (1 + r) B$$

and

$$C_d = \Delta d S - (1 + r) B$$

which after rearranging becomes:

$$-C_u + \Delta u S = -C_d + \Delta d S \tag{5.4}$$

This is the equivalent to:

$$\Delta = \frac{C_u - C_d}{(u - d) S} \tag{5.5}$$

The portfolio must earn the continuously compounded risk-free rate of interest as it is risk-free:

$$(-C + \Delta u S) = \mathrm{e}^{r\Delta t} (-C + \Delta S) \tag{5.6}$$

Substituting into Eq. (5.6) for ΔS, using Eq. (5.5) and rearranging for the call price at t_0 obtains:

$$C = \mathrm{e}^{-r\Delta t} \left(\frac{\mathrm{e}^{r\Delta t} - d}{u - d} C_u + \frac{u - \mathrm{e}^{r\Delta t}}{u - d} C_d \right) \tag{5.7}$$

The actual probabilities of the asset moving up or down have not been used in deriving the option price, and therefore, the option price is independent of the risk preferences of investors. Equation (5.7) can be interpreted as taking discounted expectations of future payoffs under the risk-neutral probabilities.

This provides a means to derive the risk-neutral probabilities directly from the asset price:

$$uSp + dS(1-p) = Se^{r\Delta t} \tag{5.8}$$

for which the return can now be assumed as being the risk-free rate. Rearranging gives:

$$p = \frac{e^{r\Delta t} - d}{u-d} \tag{5.9}$$

for the risk neutral probability for uS, and $1 - p$ for dS. Equation (5.7) can now be written as:

$$C = e^{-r\Delta t}\left(pC_u + (1-p)C_d\right)$$

This is the price of the call option with one period to maturity.

5.3 A Model for Asset Prices

The evolution of uncertainty over time can be conceptualized and modelled as a mathematical expression, known as a stochastic process, which describes the evolution of a random variable over time. Models of asset price behaviour for pricing derivatives are formulated in a continuous time framework by assuming a stochastic differential equation (SDE) describes the stochastic process followed by the asset price. The most well-known assumption made about asset price behaviour, which was made by Black and Scholes (1973), is geometric Brownian motion (GBM).

The GBM assumption in the Black–Scholes model is the mathematical description of how asset prices evolve through time. In the GBM assumption, proportional changes in the asset price, denoted by S, are assumed to have constant instantaneous drift, μ, and volatility, σ. A non-dividend paying asset S following a GBM is represented by the following stochastic differential equation (SDE):

$$dS = \mu S dt + \sigma S dz \tag{5.10}$$

where dS represents the increment in the asset price process during a (infinitesimally) small interval of time dt, and dz is the underlying uncertainty driving the model, representing an increment in a Weiner process during dt. The risk-neutral assumption implies that the drift can be replaced by the risk-free rate of interest (i.e., $\mu = r$). Any process describing the stochastic behaviour of the asset price will lead to a characterization of the distribution of future asset values. An assumption in Eq. (5.10) is that future asset prices are lognormally distributed, or that the returns to the asset are normally distributed. Dividing through by S gives:

$$\frac{dS}{S} = \mu dt + \sigma dz \tag{5.11}$$

In Eq. (5.11), the percentage change or return in the asset price dS/S has two components. The first is that during the small interval of time dt, the average return on the asset is μdt, which is deterministic. The parameter μ is known as the drift. Added to this drift is the random component made up of the change dz, in a random variable z, and a parameter σ, which is generally referred to as the volatility of the asset. The random variable z, or equivalently, the change dz is called a Weiner process. A Weiner process is defined by two key properties. The first is that dz is normally distributed with mean zero and variance dt or the standard deviation of the square root of dt. The second is that the values of dz over two different non-overlapping increments of time are independent. Equations (5.10) and (5.11) are examples of an Itô process, as the drift and volatility only depend on the current value of the variable (the asset price) and time. In general, the stochastic differential equation for a variable S following an Itô process is:

$$dS = \mu(S,t)dt + \sigma(S,t)dz \tag{5.12}$$

where the functions $\mu(S,t)$ and $\sigma(S,t)$ are general functions for the drift and volatility. Many models for the behaviour of asset prices assume that the future evolution of the asset price depends only on its present level and not on the path taken to reach that level. A stochastic process possessing this property is known as Markovian.

The stochastic process followed by any derivative can be inferred from the assumption of the behaviour of the asset price on which the derivative's payoff is dependent. It follows that, using the Black–Scholes concept of constructing a riskless portfolio, a partial differential equation can be derived that governs the price of the derivative security.

5.4 The Black–Scholes Formula

The stochastic differential equation for the asset price S is the starting point for any derivative model. As the process for the asset and the process for the derivative have the same source of uncertainty, it is possible to combine the two securities in a portfolio in such a way as to eliminate that uncertainty. A portfolio consisting of a short position in an option and a long position in an underlying asset can be constructed such that the change in its value over an infinitesimal increment of time is independent of the source of randomness, and is therefore risk-free. This relationship leads to the Black–Scholes partial differential equation. The Black–Scholes formulae for standard European call and put options are the result of solving this partial differential equation.

As the expected return on the underlying asset does not appear in the Black–Scholes partial differential equation, the value of the derivative is independent of the risk preferences of investors. The implication of this risk-neutral pricing is that the present value of any future random cash flow—for example, the payoff for an option—is given by the expected value of the random future value discounted at the riskless rate. Replacing the expectation with the integral and solving obtains the Black–Scholes equation.

The Black–Scholes formula for a European call option on a non-dividend paying stock is:

$$c = S_0 N(d_1) - K\mathrm{e}^{-r(T-t)} N(d_2) \tag{5.13}$$

where,

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

while the corresponding equation for the European put is:

$$p = K\mathrm{e}^{-r(T-t)} N(-d_2) - S_0 N(-d_1) \tag{5.14}$$

and the parameters are:

S_0 = the value of S at time zero,
K = the strike price of the option,
r = the risk-free interest rate,
t = a point in time,
T = Time at maturity of a derivative.

One of the qualities that has led to the enduring success of the Black–Scholes model is its simplicity. The inputs of the model are defined by the contract being priced or are directly observable from the market. The only exception to this is the volatility parameter, and there is now a vast amount of published material in the finance literature for deriving estimates of this figure, either from historical data or as implied by the market prices of options.

One widely used relaxation of the original formula takes into account assets that pay a constant proportional dividend. Assets of this kind are handled by reducing the expected growth rate of the asset by the amount of the dividend yield. If the asset pays a constant proportional dividend at a rate d over the life of the option, then the original Black–Scholes call formula (5.13) can be used with the adjustment where the parameter S is replaced by the term $Se^{-d(T-t)}$. This adjustment has been applied to value options on broad-based equity indices, options on foreign exchange rates, and real options that allow for competition, where the fall in value due to competition is equivalent to the dividend yield.

The intuition of the replicating portfolio concept can be illustrated with the Black–Scholes formula. The Black–Scholes formula can be defined as a combination of two binary options—a cash-or-nothing call and an asset-or-nothing call:

Asset-or-nothing call:

$$Se^{-\delta(T-t)}N(d_1) \tag{5.15}$$

Cash-or-nothing call:

$$Ke^{-r(T-t)}N(d_2) \tag{5.16}$$

A European call option represents a long position in an asset-or-nothing call and a short position in a cash-or-nothing call, where the cash payoff on the cash-or-nothing call is equivalent to the strike price. A European put is a long position in a cash-or-nothing put and a short position in an asset-or-nothing put, where the strike price represents the cash payoff on the cash-or-nothing put. $N(d_1)$, the option delta, is the number of units of the underlying asset required to form the portfolio, and the cash-or-nothing term is the number of bonds, each paying $1 at expiration.

Although it is possible to obtain closed-form solutions such as Eq. (5.13) for certain derivative pricing problems, there are many situations when analytical solutions are not obtainable, and therefore, numerical techniques need to be applied. Examples include American options and other options where there are early exercise opportunities, 'path-dependent' options with discrete observation frequencies, models that incorporate jumps and models dependent on multiple random factors. The description of two of these techniques is the subject of the next section.

5.5 Numerical Techniques

Two numerical techniques that are most commonly used by practitioners to value derivatives in the absence of closed-form solutions are binomial and trinomial trees and Monte Carlo simulation. Practitioners also use other techniques such as finite difference schemes, numerical integration, finite element methods and others. It is possible to price not only derivatives with complicated payoff functions dependent on the final price using trees and Monte Carlo simulation techniques, but also derivatives whose payoff is determined also by the path the underlying price follows during its life.

5.5.1 Monte Carlo Simulation

Monte Carlo simulation provides a simple and flexible method for valuing complex derivatives for which analytical formulae are not possible. The method can easily deal with multiple random factors, can be used to value complex path-dependent options, and allows the inclusion of price processes such as price jumps. In general, the present value of an option is the expectation of its discounted payoff. Monte Carlo simulation derives an estimate of this expectation by simulating a large number of possible paths for the asset price from time zero to the option maturity, and computing the average of the discounted payoffs.

GBM for non-dividend spot prices with constant expected return μ and volatility σ is represented by the SDE in Eq. (5.10). The Black–Scholes perfect replication argument leads to the risk-neutral process in which the actual drift of the spot price μ is replaced by the interest rate r:

$$dS = rSdt + \sigma Sdz \tag{5.17}$$

If the asset pays a constant continuous dividend yield δ, then the risk-neutral process becomes:

$$dS = (r - \delta)Sdt + \sigma Sdz \tag{5.18}$$

Transforming the spot price to the natural log of the spot price $x = \ln(S)$ gives the following process for x:

$$dx = vdt + \sigma dz \tag{5.19}$$

where $v = r - \delta - \frac{1}{2}\sigma^2$. The transformed GBM process represented in Eq. (5.10) can be discretized as:

$$x_{t+\Delta t} = x_t + \left(v\Delta t + \sigma\left(z_{t+\Delta t} - z_t\right)\right) \tag{5.20}$$

In terms of the original asset price, the discrete form is:

$$S_{t+\Delta t} = S_t \exp\left(v\Delta t + \sigma\left(z_{t+\Delta t} - zt\right)\right) \tag{5.21}$$

Equations (5.20) or (5.21) can be used to simulate the evolution of the spot price through time. The change in the random Brownian motion, $z_{t+\Delta t} - z_t$, has a mean of zero and a variance of Δt. It can therefore be simulated using random samples from a standard normal distribution multiplied by $\sqrt{\Delta t}$, that is, $\sqrt{\Delta t}\varepsilon$ where $\varepsilon \sim N(0,1)$. In order to simulate the spot price, the time period $[0, T]$ is divided into N intervals such that $\Delta t = T/N$, $t_i = i\Delta t$, $i = 1, \ldots, N$. Using, for example, Eq. (5.21) gives:

$$S_{t_i} = S_{t_{i-1}} \exp\left(v\Delta t + \sigma\sqrt{\Delta t}\varepsilon_i\right) \tag{5.22}$$

As the drift and volatility terms do not depend on the variables S and t, the discretization is correct for any chosen time step. Therefore, the option can be simulated to the maturity date in a single time step if the payoff is a function of the terminal asset value and does not depend on the asset's path during the life of the option. Repeating this process N times, and choosing ε_i randomly each time, leads to one possible path for the spot price for each simulation.

At the end of each simulated path, the terminal value of the option C_T is evaluated. Let $C_{T,j}$ represent the payoff to the contingent claim under the jth simulation. For example, a standard European call option terminal value is given by:

$$C_{T,j} = \max\left(S_{T,j} - K, 0\right) \tag{5.23}$$

Each payoff is discounted using the simulated short-term interest rate sequence:

$$C_{0,j} = \exp\left(-\int_0^T r_u du\right) C_{T,j} \tag{5.24}$$

In the case of constant or deterministic interest rates, Eq. (5.24) simplifies to:

$$C_{0,j} = P\left(0,T\right) C_{T,j} \tag{5.25}$$

This value represents the value of the option along one possible asset price path. The simulations are repeated M times and the average of all the outcomes is taken to compute the expectation, and hence, the option price:

$$\hat{C}_0 = \frac{1}{M} \sum_{j=1}^{M} C_{0,j} \tag{5.26}$$

Therefore, $\hat{C}_0$ is an estimate of the true value of the option, C_0, but with an error due to the fact that it is an average of randomly generated samples, and so, is itself random. In order to obtain a measure of the error, the standard error SE(.) is estimated as the sample standard deviation, SD(.), of $C_{t,j}$ divided by the square root of the number of samples:

$$SE\left(\hat{C}_0\right) = \frac{SD\left(C_{0,j}\right)}{\sqrt{M}} \tag{5.27}$$

where $SD(C_{0,j})$ is the standard deviation of C_0:

$$SD\left(C_{0,j}\right) = \sqrt{\frac{1}{M-1} \sum_{j=1}^{M} \left(C_{0,j} - \hat{C}_0\right)^2} \tag{5.28}$$

For many American-style options, early exercise can be optimal, depending on the level of the underlying price. It is rare to find closed-form solutions for prices and risk parameters of these options, so numerical procedures must be applied. Using Monte Carlo simulation for pricing American-style options, however, can be difficult. The problem arises because simulation methods generate trajectories of state variables forward in time, whereas a backward dynamic programming approach is required to efficiently determine optimal exercise decisions for pricing American options. Therefore, practitioners usually use binomial and trinomial trees for the pricing of American options.

5.5.2 The Binomial and Trinomial Method

The binomial model of Cox et al. (1979) is a well-known alternative discrete time representation of the behaviour of asset prices following GBM. This model is important in several ways. First, the continuous time limit of the proportional binomial process is exactly the GBM process. Second, and perhaps most importantly, the binomial model is the basis of the dynamic programming solution to the valuation of American options. Section 5.2 discussed a one-step binomial tree as part of the overview of the replicating portfolio. To price options with more than one period to maturity, the binomial tree is extended outwards for the required number of periods to the maturity date of the option. Figure 5.6 illustrates a binomial tree for an option that expires in four periods of time.

A state in the tree is referred to as a node, and is labelled as node (i,j), where i indicates the number of time steps from time zero and j indicates the number of upward movements the asset price has made since time zero. Therefore, the level of the asset price at node (i,j) is $S_{i,j} = Su^j d^{i-j}$ and the option price will be $C_{i,j}$. At the lowest node at every time step $j = 0$, and j will remain the same when moving from one node to another via a downward branch, as the number of upward moves that have occurred would not have changed. It is generally assumed that there are N time steps in total, where the Nth time step corresponds to the maturity date of the option. As is the case with the one period example, the value of a call option at the maturity date is the payoff:

$$C_{N,j} = \max\left(S_{N,j} - K, 0\right) \tag{5.29}$$

As the value of the option at any node in the tree is its discounted expected value, at any node in the tree before maturity:

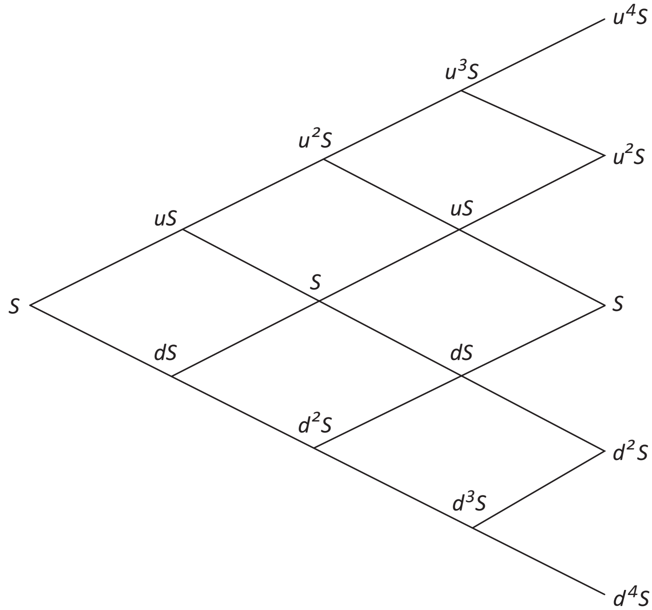

Fig. 5.6 A four-step binomial tree for an asset

$$C_{i,j} = \mathrm{e}^{-r\Delta t}\left(pC_{i+1,j+1} + \left(1-p\right)C_{i+1,j}\right) \tag{5.30}$$

where the binomial risk neutral probabilities p and $(1 - p)$ are derived as:

$$p = \frac{\mathrm{e}^{r\Delta t} - d}{u - d}$$

and r is the risk free rate.

Using Eqs. (5.29) and (5.30), the value of the option can be computed at every node for time step $N-1$. Equation (5.30) can then be reapplied at every node for every time step, working backwards through the tree to compute the value of the option at every node in the tree. The value of a European option can be derived using this procedure. To derive the value of an American option, the value of the option, if it is exercised, is compared at every node to the option value if it is not exercised, and the value at that node set to the greater of the two.

Although binomial trees are used by many practitioners for pricing American-style options, trinomial trees offer a number of advantages over the binomial tree. As there are three possible future movements over each time period, rather than the two of the binomial approach, the trinomial tree provides a better approximation to a continuous price process than the binomial tree for the same number of time steps. The trinomial tree is also easier to work with because of its more regular grid and is more flexible, allowing it to be fitted more easily to market prices of forwards and standard options, an important practical consideration. A discussion of trinomial trees follows.

In the following, it is more convenient to work in terms of the natural logarithm of the spot price as defined in Eq. (5.19). Consider a trinomial model of the asset price in which, over a small time interval Δt, the asset price can increase by Δx (the space step), stay the same or decrease by Δx, with probabilities p_u, p_m and p_d, respectively. This is depicted in terms of x in Fig. 5.7.

The drift and volatility parameters of the asset price are now captured in this simplified discrete process by Δx, p_u, p_m and p_d. It can be shown that the space step cannot be chosen independently of the time step, and that a good choice is $\Delta x = \sigma\sqrt{3\Delta t}$. The relationship between the parameters of the continuous time process and the trinomial process is obtained by equating the mean and variance over the time interval Δt and requiring that the probabilities sum to one, that is:

$$E\left[\Delta x\right] = p_u\left(\Delta x\right) + p_m\left(0\right) + p_d\left(-\Delta x\right) = \nu\Delta t \tag{5.31}$$

$$E\left[\Delta x^2\right] = p_u\left(\Delta x^2\right) + p_m\left(0\right) + p_d\left(\Delta x^2\right) = \sigma^2\Delta t + \nu^2\Delta t^2 \tag{5.32}$$

$$p_u + p_m + p_d = 1 \tag{5.33}$$

Solving Eqs. (5.31–5.33) yields the following explicit expressions for the transitional probabilities:

$$p_u = \frac{1}{2}\left(\frac{\sigma^2\Delta t + \nu^2\Delta t^2}{\Delta x^2} + \frac{\nu\Delta t}{\Delta x}\right) \tag{5.34}$$

$$p_m = 1 - \frac{\sigma^2\Delta t + \nu^2\Delta t^2}{\Delta x^2} \tag{5.35}$$

$$p_d = \frac{1}{2}\left(\frac{\sigma^2\Delta t + \nu^2\Delta t^2}{\Delta x^2} - \frac{\nu\Delta t}{\Delta x}\right) \tag{5.36}$$

The single period trinomial process in Fig. 5.7 can be extended to form a trinomial tree. Figure 5.8 depicts such a tree.

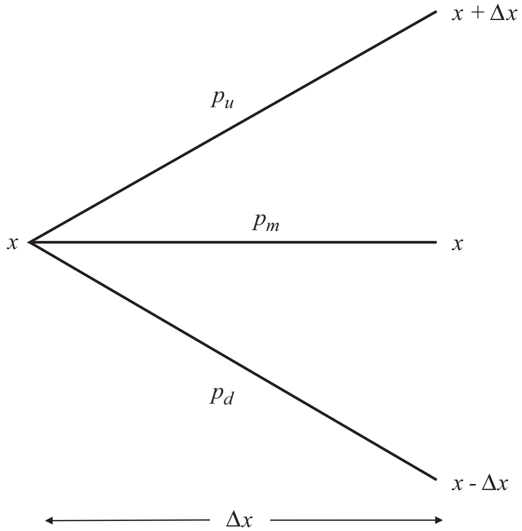

Fig. 5.7 The trinomial model of an asset price

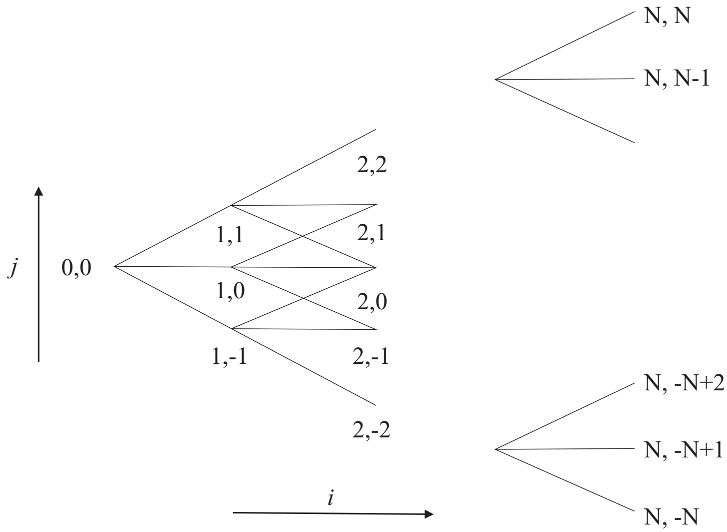

Fig. 5.8 A trinomial tree model of an asset price

Let i denote the number of the time step and j, the level of the asset price relative to the initial asset price in the tree. If $S_{i,j}$ denotes the level of the asset price at node (i,j), then $t = t_i = i\Delta t$, and an asset price level of $S\exp(j\Delta x)$. Once the tree has been constructed, the spot price is known at every time and every state of the world consistent with the original assumptions about its behaviour process, and the tree can be used to derive prices for a wide range of derivatives.

The procedure is illustrated with reference to pricing a European and American call option with a strike price K on the spot price. The value of an option is represented at node (i,j) by $C_{i,j}$. In order to value an option, the tree is constructed as representing the evolution of the spot price from the current date out to the maturity date of the option. Let time step N correspond to the maturity date in terms of the number of time steps in the tree, that is, $T = N\Delta t$. The values of the option at maturity are determined by the values of the spot price in the tree at time step N and the strike price of the option:

$$C_{N,j} = \max\left(S_{N,j} - K, 0\right); j = -N, \ldots, N \tag{5.37}$$

It can be shown that option values can be computed as discounted expectations in a risk-neutral world, and therefore, the values of the option at earlier nodes can be computed as discounted expectations of the values at the following three nodes to which the asset price could jump:

$$C_{i,j} = \mathrm{e}^{-r\Delta t}\left(p_u C_{i+1,j+1} + p_m C_{i+1,j} + p_d C_{i+1,j-1}\right) \tag{5.38}$$

where $\mathrm{e}^{-r\Delta t}$ is the single period discount factor. This procedure is often referred to as 'backwards induction' as it links the option value at time i to known values at time $i + 1$. The attraction of this method is the ease with which American option values can be evaluated. During the inductive stage, the immediate exercise value of the option is compared with the value if not exercised as computed from Eq. (5.38). If the immediate exercise value is greater, then this value is stored at the node, that is:

$$C_{i,j} = \max\left\{\mathrm{e}^{-r\Delta t}\left(p_u C_{i+1,j+1} + p_m C_{i+1,j} + p_d C_{i+1,j-1}\right), S_{i,j} - K\right\} \tag{5.39}$$

This method also provides the optimal exercise strategy for the American option, since for every possible future state of the world, that is, every node in the tree, it can be determined whether to exercise the option or not. The value of the option today is given by the value in the tree at node (0,0), $C_{0,0}$.

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