On the Solution of the Fredholm Equation with the Use of Quadratic Integro-Differential Splines

I. G. Burova(✉) and N. S. Domnin

St. Petersburg State University, 7/9 Universitetskaya nab., St. Petersburg 199034, Russia
i.g.burova@spbu.ru, burovaig@mail.ru

Abstract. Currently there are a number of papers in which certain types of splines are used to solve the Fredholm equation. Now much attention is paid to the application of a new type of spline, the so-called integro-differential spline to the solution of various problems. In this paper we consider the solution of the Fredholm equation using polynomial integro-differential splines of the third order approximation. To calculate the integral in the formula of a quadratic integro-differential spline, we propose the corresponding quadrature formula. The results of numerical experiments are given.

Keywords: Polynomial splines · Integro-differential splines · Fredholm equation

1 Introduction

At present, the theory of approximation by local interpolation splines continues to evolve. Approximation with local splines of the Lagrange or the Hermite types can be used in many applications. Approximation with the use of these splines is constructed on each mesh interval separately as a linear combination of the products of the values of the function and/or its derivatives at the grid nodes and basic functions. We obtain the basic functions as a solution of a system of linear algebraic equations (approximation relations). The approximation relations are formed from the conditions of accuracy of approximation on the functions forming the Chebyshev system. The constructed basic splines provide an approximation of the prescribed order. Using basic splines, one can construct continuous or continuously differentiable predetermined types of approximation. There are new types of splines that we call integro-differential splines (see [2–9]), which compete with existing polynomial and nonpolynomial splines of the Lagrange type. The main features of integro-differential splines are the following: the approximation is constructed separately for each grid interval (or elementary rectangular); the approximation constructed as the sum of products of the basic splines and the values of function in nodes and/or the values of integrals of

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this function over subintervals. Basic splines are determined by using a solving system of equations which are provided by the set of functions. It is known that when integrals of the function over the intervals are equal to the integrals of the approximation of the function over the intervals then the approximation has some physical parallel. The splines which are constructed here satisfy the property of the third order approximation. Here, the one-dimensional polynomial basic splines of the third order approximation are constructed when the values of the function are known in each point of interpolation. For the construction of the spline, we use quadrature with the appropriate order of approximation. These basic splines can be used to solve various problems, including the approximation of a function of one and several variables; the construction of quadrature and cubature formulas; the solution of boundary value problems; the solution of the Fredholm equation, and the Cauchy problem. Currently there are papers in which certain types of splines are used to solve the Fredholm equation (see [1,10–12,14–16]), boundary value problems (see [13,17–19]).

In this paper we consider the solution of the Fredholm equation using polynomial integro-differential splines of the third order approximation. To calculate the integral in the formula of a quadratic integro-differential spline, we propose the corresponding quadrature formula. The results of numerical experiments are given.

2 Construction of a Solution of the Fredholm Equation with the Use of Quadratic Polynomial Splines

Suppose that a, b are real numbers. Consider the Fredholm equation

$$\varphi(x) - \int_a^b K(x,s)\varphi(s)ds = f(x). \tag{1}$$

Suppose that n is a natural number. We construct on the interval $[a,b]$ a uniform grid $\{x_j\}_{j=0}^n$ with step h: $h = \frac{b-a}{n}$.

We construct an approximate solution of the integral equation by applying quadratic polynomial splines as follows. First we represent the integral in (1) in the following form:

$$\int_a^b K(x,s)\varphi(s)ds = \int_a^{b-h} K(x,s)\varphi(s)ds + \int_{b-h}^b K(x,s)\varphi(s)ds. \tag{2}$$

In the first integral of (2) we apply the following transformation using integro-differential splines. We replace the function $\varphi(s)$, $s \in [x_j, x_{j+1}]$, by $\widetilde{\varphi}(s)$:

$$\widetilde{\varphi}(s) = \varphi(x_j)\omega_j(s) + \varphi(x_{j+1})\omega_{j+1}(s) + \int_{x_{j+1}}^{x_{j+2}} \varphi(\tau)d\tau \cdot \omega_j^{<1>}(s). \tag{3}$$

Here $\omega_j(s), \omega_{j+1}(s), \omega_j^{<1>}(s)$ are the continuous integro-differential splines which will be defined later.

Lemma 1. *Let function $u(x)$ be such that $u \in C^3[x_{j-1}, x_{j+1}]$. The following formula is valid:*

$$\int_{x_j}^{x_{j+1}} u(x)dx \approx \frac{h}{12}(5u(x_{j+1}) + 8u(x_j) - u(x_{j-1})). \tag{4}$$

Proof. We put $\int_{x_j}^{x_{j+1}} u(x)dx \approx \int_{x_j}^{x_{j+1}} \widetilde{u}(x)dx$, where $\widetilde{u}(x) = u(x_{j-1})w_{j-1}(x) + u(x_j)w_j(x) + u(x_{j+1})w_{j+1}(x), x \in [x_j, x_{j+1}]$,

$$w_{j-1}(x) = \frac{(x-x_j)(x-x_{j+1})}{(x_{j-1}-x_j)(x_{j-1}-x_{j+1})}, \quad w_j(x) = \frac{(x-x_{j-1})(x-x_{j+1})}{(x_j-x_{j-1})(x_j-x_{j+1})},$$

$$w_{j+1}(x) = \frac{(x-x_{j-1})(x-x_j)}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)}.$$

We obtain formula (4) after integration. The proof is complete.

Remark 1. It is not difficult to obtain the following relation:

$$|u(x) - \widetilde{u}(x)| \le K_0 h^3 \|u'''\|_{[x_{j-1}, x_{j+1}]}, x \in [x_j, x_{j+1}], K_0 > 0.$$

It can be shown that

$$|\int_{x_j}^{x_{j+1}} u(x)dx - \frac{h}{12}(5u(x_{j+1}) + 8u(x_j) - u(x_{j-1}))| \le K_1 h^4 \|u'''\|.$$

Now (3) for $s \in [x_j, x_{j+1}]$ has the form:

$$\widetilde{\varphi}(s) = \varphi(x_j)\omega_j(s) + \varphi(x_{j+1})\omega_{j+1}(s) + \frac{h}{12}(5\varphi(x_{j+2}) + 8\varphi(x_{j+1}) - \varphi(x_j))\omega_j^{<1>}(s). \tag{5}$$

Lemma 2. *Suppose φ be such that $\varphi \in C^3[x_j, x_{j+2}]$ and $\widetilde{\varphi}(s)$ is given by (3). The following formula is valid $\widetilde{\varphi}(s) = \varphi(s)$, $\varphi(s) = 1, s, s^2$ where*

$$\omega_j(s) = (s - h - jh)(3s - 5h - 3jh)/(5h^2), \tag{6}$$

$$\omega_{j+1}(s) = -(s - jh)(9s - 14h - 9jh)/(5h^2), \tag{7}$$

$$\omega_j^{<1>}(s) = 6(s - h - jh)(s - jh)/(5h^3). \tag{8}$$

Proof. Using (3), (4) and the Taylor expansion, it is not difficult to obtain the relations (6), (7), (8). The proof is complete.

Remark 2. If $s \in [x_j, x_{j+1}]$, $t \in [0,1]$, $s = x_j + th$, then the basic splines can be written in the form:
$\omega_j(x_j + th) = (t-1)(3t-5)/5, \quad \omega_{j+1}(x_j + th) = -t(9t-14)/5,$
$\omega_j^{<1>}(x_j + th) = 6t(t-1)/(5h).$
It is not difficult to obtain the following relation:

$$|\varphi(x) - \widetilde{\varphi}(x)| \le K_2 h^3 \|u'''\|_{[x_j, x_{j+2}]}, x \in [x_j, x_{j+1}], K_2 > 0.$$

In the second integral of (2) we apply the following transformation using integro-differential splines. We replace the function $\varphi(s)$, $s \in [x_j, x_{j+1}]$, by $\widetilde{\varphi}(s)$:

$$\widetilde{\varphi}(s) = \varphi(x_j)\widetilde{\omega}_j(s) + \varphi(x_{j+1})\widetilde{\omega}_{j+1}(s) + \int\limits_{x_{j-1}}^{x_j} \varphi(\tau)d\tau \cdot \widetilde{\omega}_j^{<-1>}(s). \tag{9}$$

Here $\widetilde{\omega}_j(s)$, $\widetilde{\omega}_{j+1}(s)$, $\widetilde{\omega}_j^{<-1>}(s)$ are the continuous integro-differential splines which will be defined later.

Lemma 3. *Let function $u(x)$ be such that $u \in C^3[x_j, x_{j+2}]$. The following formula is valid:*

$$\int\limits_{x_j}^{x_{j+1}} u(x)dx \approx \frac{h}{12}(5u(x_j) + 8u(x_{j+1}) - u(x_{j+2})). \tag{10}$$

Proof. We put $\int\limits_{x_j}^{x_{j+1}} u(x)dx \approx \int\limits_{x_j}^{x_{j+1}} \widetilde{u}(x)dx$, where

$$\widetilde{u}(x) = u(x_j)\widetilde{w}_j(x) + u(x_{j+1})\widetilde{w}_{j+1}(x) + u(x_{j+2})\widetilde{w}_{j+2}(x), x \in [x_j, x_{j+1}],$$

where

$$\widetilde{w}_j(x) = \frac{(x - x_{j+1})(x - x_{j+2})}{(x_j - x_{j+1})(x_j - x_{j+2})}, \quad \widetilde{w}_{j+1}(x) = \frac{(x - x_j)(x - x_{j+2})}{(x_{j+1} - x_j)(x_{j+1} - x_{j+2})},$$

$$\widetilde{w}_{j+2}(x) = \frac{(x - x_j)(x - x_{j+1})}{(x_{j+2} - x_j)(x_{j+2} - x_{j+1})},$$

after integration we obtain formula (10). The proof is complete.

Remark 3. It is not difficult to obtain the following relation

$$|u(x) - \widetilde{u}(x)| \le K_3 h^3 \|u'''\|_{[x_j, x_{j+2}]}, x \in [x_j, x_{j+1}], K_3 > 0.$$

It can be shown that $|\int\limits_{x_j}^{x_{j+1}} u(x)dx - \frac{h}{12}(5u(x_j) + 8u(x_{j+1}) - u(x_{j+2}))| \le K_4 h^4 \|u'''\|$, $K_4 > 0$.

Now (9), $s \in [x_j, x_{j+1}]$, has the form:

$$\widetilde{\varphi}(s) = \varphi(x_j)\widetilde{\omega}_j(s) + \varphi(x_{j+1})\widetilde{\omega}_{j+1}(s) + \frac{h}{12}(5\varphi(x_{j-1}) + 8\varphi(x_j) - \varphi(x_{j+1}))\widetilde{\omega}_j^{<-1>}(s). \tag{11}$$

Lemma 4. *Suppose $\widetilde{\varphi}$ be such that $\widetilde{\varphi} \in C^3[x_j, x_{j+2}]$ and $\widetilde{\varphi}(s)$ is given by (9). The following formula is valid: $\widetilde{\varphi}(s) = \varphi(s)$, $\varphi(s) = 1, s, s^2$ where $s \in [x_j, x_{j+1}]$*

$$\widetilde{\omega}_j(s) = -(9s + 5h - 9jh)(s - h - jh)/(5h^2), \tag{12}$$

$$\widetilde{\omega}_{j+1}(s) = (3s + 2h - 3jh)(s - jh)/(5h^2), \tag{13}$$

$$\widetilde{\omega}_j^{<-1>}(s) = 6(s - h - jh)(s - jh)/(5h^3). \tag{14}$$

Proof. Using (9), (10) and the Taylor expansion, it is not difficult to obtain the relations (12), (13), (14). The proof is complete.

Remark 4. If $s \in [x_j, x_{j+1}]$, $t \in [0, 1]$, $s = x_j + th$, the basic splines can be written in the form:

$$\widetilde{\omega}_j(x_j + th) = -(9t + 5)(t - 1)/5, \quad \widetilde{\omega}_{j+1}(x_j + th) = t(3t + 2)/5,$$

$$\widetilde{\omega}_j^{<-1>}(x_j + th) = 6t(t - 1)/(5h).$$

It is not difficult to obtain the following relation:

$$|\varphi(x) - \widetilde{\varphi}(x)| \leq K_4 h^3 \|u'''\|_{[x_{j-1}, x_{j+1}]}, x \in [x_j, x_{j+1}], K_4 > 0.$$

Using (5), (6)–(8), (11), (12)–(14) and the following notations:

$$A_j^{<l>}(x) = \int_{x_j}^{x_{j+1}} K(x, s)(\omega_j(s) - \frac{h}{12}\omega_j^{<1>}(s))ds,$$

$$B_j^{<l>}(x) = \int_{x_j}^{x_{j+1}} K(x, s)\left(\omega_{j+1}(s) + \frac{2h}{3}\omega_j^{<1>}(s)\right)ds,$$

$$C_j^{<l>}(x) = \frac{5h}{12}\int_{x_j}^{x_{j+1}} K(x, s)\omega_j^{<1>}(s)ds,$$

$$A_{n-1}^{<r>}(x) = \frac{5h}{12}\int_{x_{n-1}}^{x_n} K(x, s)\widetilde{\omega}_{n-1}^{<-1>}(s)ds,$$

$$B_{n-1}^{<r>}(x) = \int_{x_{n-1}}^{x_n} K(x, s)\left(\widetilde{\omega}_{n-1}(s) + \frac{2h}{3}\widetilde{\omega}_{n-1}^{<-1>}(s)\right)ds,$$

$$C_{n-1}^{<r>}(x) = \int_{x_{n-1}}^{x_n} K(x,s) \left(\widetilde{\omega}_n(s) - \frac{h}{12} \widetilde{\omega}_{n-1}^{<-1>}(s) \right) ds$$

we get the following system of equations for calculating $\varphi(x_i)$, $i = 0, \ldots, n$:

$$\varphi(x_i) - \sum_{j=0}^{n-2} \left(\varphi(x_j) A_j^{<l>}(x_i) + \varphi(x_{j+1}) B_j^{<l>}(x_i) + \varphi(x_{j+2}) C_j^{<l>}(x_i) \right)$$

$$- \left(\varphi(x_{n-2}) A_{n-1}^{<r>}(x_i) + \varphi(x_{n-1}) B_{n-1}^{<r>}(x_i) + \varphi(x_n) C_{n-1}^{<r>}(x_i) \right) = f(x_i).$$

Table 1. Numerical solutions when $n = 10$ and $n = 100$

$K(x,s)$	$\varphi(x)$	$n = 10$	$n = 100$
$K(x,s) = x^2 s^2$	$\varphi(x) = x^{\frac{3}{2}} \sin(x)$	$0.21 \cdot 10^{-5}$	$0.24 \cdot 10^{-7}$
$K(x,s) = e^x \cos(s)$	$\varphi(x) = x^{\frac{3}{2}} \sin(x)$	$0.37 \cdot 10^{-3}$	$0.44 \cdot 10^{-6}$
$K(x,s) = xs$	$\varphi(x) = \frac{1}{1+25x^2}$	$0.60 \cdot 10^{-4}$	$0.61 \cdot 10^{-8}$

3 Numerical Results

Here we present some numerical results. In Table 1 one can see the absolute values of the difference between the exact solution and solutions, obtained with suggested method, when $a = 0$, $b = 1$, with $n = 10$ and $n = 100$, Digits=15. Here $f(x)$ is obtained using $K(x,s)$ and $\varphi(s)$.

4 Conclusion

The quadratic polynomial integro-differential splines proposed in this paper showed the possibility of solving the Fredholm integral equation. In the proposed method, it is necessary to calculate the integrals $A_j^{<l>}(x)$, $B_j^{<l>}(x)$, $C_j^{<l>}(x)$, $A_{n-1}^{<r>}(x)$, $B_{n-1}^{<r>}(x)$, $C_{n-1}^{<r>}(x)$. In future papers, the application of nonpolynomial splines to solve the Fredholm equation will be investigated.

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