Chapter 4
Engineering Magnetostatics and Boundary-Value Problems

4.1 Constitutive Law of Magnetic Material

In a domain Ω, having boundary Γ, containing permanent magnets, i.e. aggregates of magnetic dipoles or, from now on, steady electric current distributed with density $\overline{J}$ (A m^{-2}), a magnetostatic field is set up; it is defined by field strength $\overline{H}$ (A m^{-1}) as well as flux density $\overline{B}$ (Wb m^{-2} = T). In general, the link between $\overline{H}$ and $\overline{B}$, i.e. the constitutive law of the medium, is complicated. Neglecting hysteresis, the law is single-valued and can be expressed, for an isotropic medium in the absence of permanent magnetization, by

$$\overline{B} = \mu\overline{H} \tag{4.1}$$

where μ is called permeability (H m^{-1}) and, in the most general case, is a function of $\overline{|H|}$; the inverse of μ is called reluctivity ν. The observer is supposed to be at rest with respect to the field [4].

4.2 Maxwell's Equations of Magnetostatic Field

The equations governing the magnetic field are in Ω

$$\overline{\nabla} \cdot \overline{B} = 0 \tag{4.2}$$

$$\overline{\nabla} \times \overline{H} = \overline{J} \tag{4.3}$$

and along Γ

$$\overline{n} \cdot \overline{B} = 0 \tag{4.4}$$

P. Di Barba and S. Wiak, *MEMS: Field Models and Optimal Design*, Lecture Notes in Electrical Engineering 573, https://doi.org/10.1007/978-3-030-21496-8_4

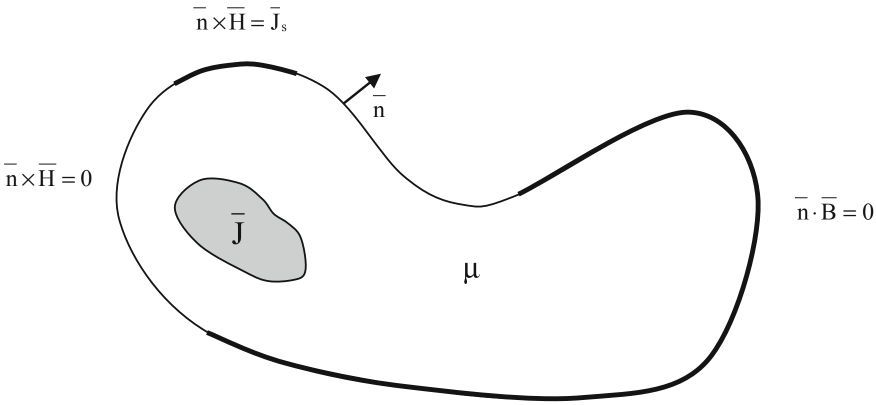


Fig. 4.1 Magnetic field domain with field-based boundary conditions

if Γ is a flux line (flux lines parallel to Γ), or

$$\overline{n} \times \overline{B} = \mu \overline{J}_S \tag{4.5}$$

if current of surface density $\overline{J}_S$ (A m^{-1}) is present, or

$$\overline{n} \times \overline{H} = 0 \tag{4.6}$$

if flux lines are perpendicular to Γ.

For an isotropic and linear medium, in terms of $\overline{B}$, the equations become in Ω

$$\overline{\nabla} \cdot \overline{B} = 0; \quad \overline{\nabla} \times \overline{B} = \mu \overline{J} \tag{4.7}$$

with

$$\overline{n} \cdot \overline{B} = 0 \tag{4.8}$$

or

$$\overline{n} \times \overline{B} = \mu \overline{J}_S \tag{4.9}$$

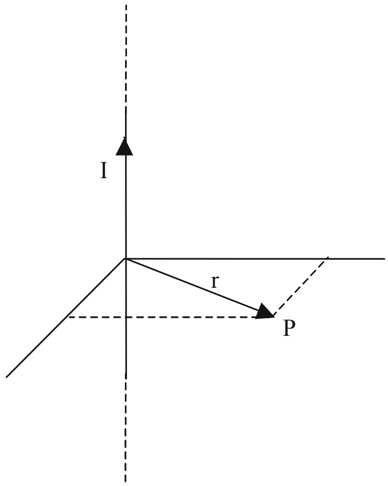
or

$$\overline{n} \times \overline{H} = 0 \quad \text{along } \Gamma \tag{4.10}$$

The equations written above unambiguously define the magnetostatic field which, because of (4.5), is solenoidal [4].

A general field domain is shown in Fig. 4.1.

Fig. 4.2 Line current



If both $\overline{J}_S$ and $\overline{J}$ are given, then it must be

$$\int_{\Gamma} |\overline{J}_s| d\Gamma = \int_{\Omega} |\overline{J}| d\Omega \tag{4.11}$$

i.e. the total current sums up to zero: therefore, densities J_S and J cannot be independent.

In a non-homogeneous domain at the interface between two materials of permeability μ_1 and μ_2, from (4.2) it holds

$$\overline{n} \cdot \left(\overline{B}_2 - \overline{B}_1\right) = 0 \tag{4.12}$$

so that the normal component of $\overline{B}$ is always continuous (Fig. 4.2).

If there is a current of density $\overline{J}_S$ (A m^{-1}), then from (4.3) it follows

$$\overline{n} \times \left(\overline{H}_2 - \overline{H}_1\right) = \overline{J}_s \tag{4.13}$$

If $\overline{J}_s = 0$ the tangential component of $\overline{H}$ is continuous. Equations (4.12) and (4.13) are called transmission conditions.

In the case of a non-homogeneous medium, the following remark can be put forward. After (4.1) and (4.2), considering vector identity (A.14), one has

$$\overline{\nabla} \cdot \mu\overline{H} = \mu\overline{\nabla} \cdot \overline{H} + \overline{\nabla}\mu \cdot \overline{H} = 0 \tag{4.14}$$

In the case of a non-homogeneous medium, field $\overline{H}$ is solenoidal if $\overline{\nabla}\mu$ and $\overline{H}$ are orthogonal vectors; this means that lines separating layers of different μ are parallel to field lines of $\overline{H}$.

Conversely, after (4.1) and (4.3), considering vector identity (A.16), it turns out to be

$$\overline{\nabla} \times \mu^{-1}\overline{B} = \mu^{-1}\overline{\nabla} \times \overline{B} + \overline{\nabla}\mu^{-1} \times \overline{B} = \overline{J} \tag{4.15}$$

It appears that, in a current-free medium (i.e. J = 0), field $\overline{B}$ is irrotational if $\overline{\nabla}\mu^{-1}$ and $\overline{B}$ are parallel vectors; this means that lines separating layers of different μ are orthogonal to field lines of $\overline{B}$. If $\overline{\nabla}\mu^{-1} = 0$ and J = 0 (homogeneous current-free medium), then $\overline{B}$ is always irrotational.

Finally, an extension of constitutive law (4.1) is considered.

In the presence of a permanent magnetization $\overline{B}_0$ in the magnetic material (permanent magnet) the constitutive law is

$$\overline{B} = \mu\overline{H} + \overline{B}_0 \tag{4.16}$$

In this case the field equations are

$$\overline{\nabla} \cdot \overline{B} = 0 \tag{4.17}$$

$$\overline{\nabla} \times \overline{B} = \mu\overline{J} + \overline{\nabla} \times \overline{B}_0 \tag{4.18}$$

In particular, the field inside a permanent magnet is described by (4.18) with $\overline{J} = 0$; it follows that the magnet can be modelled by an equivalent distribution of current given by $\overline{J}_{eq} = \mu^{-1}\overline{\nabla} \times \overline{B}_0$.

4.3 From Field to Potentials

(i) From (4.2), since, for any vector $\overline{A}$, $\overline{\nabla} \cdot \left(\overline{\nabla} \times \overline{A}\right) = 0$ holds (see A.8), it is possible to define a vector function $\overline{A}$ (Wb m^{-1}) called vector potential by means of

$$\overline{\nabla} \times \overline{A} = \overline{B} \tag{4.19}$$

and

$$\overline{\nabla} \cdot \overline{A} = 0 \quad \text{(gauge condition)} \tag{4.20}$$

This way (4.2) is fulfilled, while (4.3) becomes

$$\overline{\nabla} \times \mu^{-1}\left(\overline{\nabla} \times \overline{A}\right) = \overline{J} \tag{4.21}$$

For a homogeneous domain, after (A.12) and (4.20) it turns out to be

$$\overline{\nabla}^2 \overline{A} = -\mu \overline{J} \tag{4.22}$$

This is the (Poisson's) vector equation governing $\overline{A}$. In a system of rectangular coordinates it corresponds to the following three scalar equations

$$\begin{aligned}
\left(\overline{\nabla}^2 \overline{A}\right)_x &= \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} = -\mu J_x \\
\left(\overline{\nabla}^2 \overline{A}\right)_y &= -\mu J_y \\
\left(\overline{\nabla}^2 \overline{A}\right)_z &= -\mu J_z
\end{aligned} \tag{4.23}$$

In general, the gradient of an harmonic function η may be added to $\overline{A}$, having all the equations fulfilled. Suitable boundary conditions on Γ must be added in order to define the field in a unique way.

In particular, after (4.18) and (4.22), the potential inside a permanent magnet is given by $\overline{\nabla}^2 \overline{A} = -\overline{\nabla} \times \overline{B}_0$.

(ii) In a two-dimensional domain, vectors $\overline{J}$ and so $\overline{A}$ have only one non-zero component; hence, vector potential can be treated as a scalar quantity.

Boundary conditions (4.8) and (4.10), in terms of $\overline{B} = \left(B_x, B_y\right)$ along the boundary Γ with normal unit vector $\overline{n} = \left(n_x, n_y\right)$ and tangential unit vector $\overline{t} = \left(t_x, t_y\right) = \left(n_y, -n_x\right)$, become, in terms of A,

$$\begin{aligned}
\bar{n} \cdot \overline{B} &= n_x B_x + n_y B_y = n_x \frac{\partial A}{\partial y} - n_y \frac{\partial A}{\partial x} \\
&= -t_y \frac{\partial A}{\partial y} - t_x \frac{\partial A}{\partial x} = -\bar{t} \cdot \overline{\nabla} A = -\frac{\partial A}{\partial t} = 0
\end{aligned} \tag{4.24}$$

i.e. $A = \text{const}$ along Γ and

$$\begin{aligned}
\bar{n} \times \overline{B} &= \left(n_x B_y - n_y B_x\right)\bar{i}_z = \left(-n_x \frac{\partial A}{\partial x} - n_y \frac{\partial A}{\partial y}\right)\bar{i}_z \\
&= -\left(\bar{n} \cdot \overline{\nabla} A\right)\bar{i}_z = -\frac{\partial A}{\partial n}\bar{i}_z = 0
\end{aligned} \tag{4.25}$$

i.e. $\frac{\partial A}{\partial n} = 0$ along Γ, respectively.

(iii) If $\overline{J} = 0$ in Ω and Ω is simply connected, then, along with $\overline{A}$, the field $\overline{H}$ can be described by a scalar function φ (total scalar potential, A) defined as

$$\overline{H} = -\overline{\nabla}\varphi \tag{4.26}$$

In fact, (4.3) is automatically satisfied, while from (4.2) we obtain

$$\overline{\nabla} \cdot \mu\overline{\nabla}\varphi = 0 \quad \text{in } \Omega \tag{4.27}$$

The latter is the Laplace's equation governing total scalar potential φ with suitable boundary conditions.

The condition of simply connected domain can be obtained by suitable cuts, if necessary. If this condition is not fulfilled, nevertheless φ can be still defined, apart from multiples of a constant.

(iv) In a three-dimensional domain, following the $\overline{T} - \Omega$ method, in regions free of impressed current ($J_0 = 0$) an electric vector potential $\overline{T}$ (A m^{-1}) can be defined as

$$\overline{\nabla} \times \overline{T} = \overline{J} \tag{4.28}$$

Comparing (4.28) and (4.3) it turns out that $\overline{H}$ and $\overline{T}$, which have the same curl, must differ by the gradient of a function Ω (dual scalar potential, A)

$$\overline{H} = \overline{T} - \overline{\nabla}\Omega \tag{4.29}$$

This way, the electric and magnetic vectors, $\overline{J}$ and $\overline{H}$, have been reformulated in terms of two potentials.

In order to define $\overline{T}$ uniquely, a gauge must be introduced.

The equations governing electric and magnetic field can be now expressed in terms of $\overline{T}$ and Ω. In fact, from (4.3) taking the curl of both members and taking into account (4.2) and (4.29), one has

$$\begin{aligned} \overline{\nabla} \times \left(\sigma^{-1}\overline{\nabla} \times \overline{T}\right) &= \overline{\nabla} \times \sigma^{-1}\overline{J}_0 - \frac{\partial}{\partial t}\mu\left(\overline{T} - \overline{\nabla}\Omega\right) \\ \overline{\nabla} \times \left(\sigma^{-1}\overline{\nabla} \times \overline{T}\right) &= \overline{\nabla} \times \sigma^{-1}\overline{J}_0 \end{aligned} \tag{4.30}$$

and from (4.2)

$$\overline{\nabla} \cdot \mu\left(\overline{T} - \overline{\nabla}\Omega\right) = 0 \tag{4.31}$$

In regions where $\sigma = 0$ one has $\overline{J} = 0$ and therefore, from (4.2), $\overline{\nabla} \times \overline{T} = 0$.

Moreover, imposing the gauge $\overline{\nabla} \cdot \overline{T} = \mu\sigma \frac{\partial \Omega}{\partial t}$ $\overline{\nabla} \cdot \overline{T} = 0$, from (4.30) and (4.31) one obtains two independent equations for T and Ω, namely

$$\overline{\nabla}^2 \overline{T} - \mu\sigma \frac{\partial \overline{T}}{\partial t} = -\overline{\nabla} \times \overline{J}_0$$
$$\overline{\nabla}^2 \overline{T} = -\overline{\nabla} \times \overline{J}_0 \tag{4.32}$$

and

$$\nabla^2 \Omega - \mu\sigma \frac{\partial \Omega}{\partial t} = 0$$
$$\nabla^2 \Omega = 0 \tag{4.33}$$

subject to appropriate boundary conditions. They are

$$\overline{n} \times \overline{T} = 0, \quad \Omega = 0 \tag{4.34}$$

or

$$\overline{n} \cdot \overline{T} = 0, \quad \frac{\partial \Omega}{\partial n} = 0 \tag{4.35}$$

if the boundary is normal to a flux line (i.e. $\overline{n} \times \overline{B} = 0$) or it is parallel to a flux line (i.e. $\overline{n} \cdot \overline{B} = 0$), respectively.

After determining $\overline{T}$, Ω is given by

$$\Omega(t) = \Omega_0 + (\mu\sigma)^{-1} \int_0^t \overline{\nabla} \cdot \overline{T}(t') dt' \tag{4.36}$$

with Ω_0 to be determined.

The following remark can be put forward.

In the two-dimensional case, the magnetic vector potential has only one non-zero component, and this makes the computational cost low. In contrast, if a formulation based on magnetic vector potential is used for a three-dimensional problem, all three vector components are unknown; therefore, the computational cost is high. The $\overline{T}$-Ω formulation is a good compromise: in fact, the use of vector potential $\overline{T}$ in current-carrying conductors makes it possible an accurate modeling of current distribution, while the use of scalar potential Ω elsewhere leads to economy in computation. Suitable conditions are needed at the boundary between conducting and non-conducting materials.

(v) When in (4.1) permeability μ depends on $|\overline{H}|$, one has $|\overline{B}| = \mu(|\overline{H}|)|\overline{H}|$ and for the solution of (4.22) one should resort to an iterative procedure. According e.g. to the Newton-Raphson method, the residual *r(A)* of the governing Eq. (4.22) is developed in Taylor's series, truncating the development at the first order

$$r(A_k) = r(A_{k-1}) + \left(\frac{dr}{dA}\bigg|_{A=A_{k-1}} \right)(A_k - A_{k-1}) + o(A_k) \tag{4.37}$$

If an estimate of solution A_{k-1} at the (k − 1)-th iteration is available, the subsequent prediction A_k at the k-th iteration is given by (4.28) after imposing $r(A_k) = 0$. It turns out to be

$$A_k = A_{k-1} - \left[\frac{dr}{dA}\bigg|_{A=A_{k-1}} \right]^{-1} r(A_{k-1}) \tag{4.38}$$

Then, μ and so $\left|\overline{H}\right|$ are updated by means of the new estimate of A, and the problem is solved again. The procedure stops when the error between two successive solutions is less than the prescribed threshold. It is necessary to know an initial prediction A_0 and the value of the derivative $\frac{dr}{dA}$ at each iteration.

4.3.1 Field of a Line Current in a Three-Dimensional Domain: Differential Approach

A current $I(A)$, concentrated at $r = 0$ and directed along the z axis in a system of cylindrical coordinates (r, φ, z), is considered (Fig. 4.2) [1].

The symmetry implies $\overline{H} = (0, H, 0)$ and from (4.3) the field equation is

$$\overline{\nabla} \times \overline{H} = \frac{1}{r}\frac{\partial rH}{\partial r} = \frac{\partial H}{\partial r} + \frac{1}{r}H = I\delta(r), \quad r > 0 \tag{4.39}$$

where H vanishes as r approaches infinity. The general solution is

$$H(r) = \frac{1}{r}\left(I \int_0^r \rho\delta(\rho)d\rho + k \right) \tag{4.40}$$

The Dirac's δ in a cylindrical geometry can be approximated by

$$\delta = \lim_{\alpha \to 0} \delta_\alpha, \quad \alpha > 0 \tag{4.41}$$

with $\delta_\alpha = \frac{1}{\pi\alpha^2}, \quad r \leq \alpha$ and $\delta_\alpha = 0$ elsewhere. Consequently, the field H can be approximated as

$$H = \lim_{\alpha \to 0} H_\alpha \tag{4.42}$$

For $r \leq \alpha$ it turns out to be

$$H_\alpha = \frac{1}{r}\left(I \int_0^r \rho \delta_\alpha d\rho + k_\alpha \right) = \frac{1}{r}\left(\frac{I}{\pi\alpha^2}\frac{r^2}{2} + k_\alpha \right)$$
$$= \frac{Ir}{2\pi\alpha^2} + \frac{k_\alpha}{r} \tag{4.43}$$

Since δ_α is a regular function near the origin, also H_n will be regular near zero; therefore $k_\alpha = 0$.

For $r \geq \alpha$ it turns out to be

$$H_\alpha = \frac{1}{r}\left(I \int_0^\alpha \rho \delta_\alpha d\rho + k_\alpha \right) = \frac{1}{r}\left(\frac{I}{\pi\alpha^2} \int_0^\alpha \rho d\rho \right)$$
$$= \frac{1}{r}\left(\frac{I}{\pi\alpha^2}\frac{\alpha^2}{2} \right) = \frac{I}{2\pi r}, \quad r > 0 \tag{4.44}$$

The Biot-Savart's law follows

$$H(r) = \lim_{\alpha \to 0} H_\alpha(r) = \frac{I}{2\pi r}, \quad r > 0 \tag{4.45}$$

Alternatively, the Stokes's theorem can be applied to (4.3), giving $\oint_l \overline{H} \cdot \bar{t} d\ell = I$, if ℓ is a closed line linking the conductor once. Considering the field geometry, ℓ can be taken as a circular line centred at $r = 0$; therefore, (4.41) follows.

From (4.41) and (4.19) the vector potential is

$$\overline{A} = \frac{I}{2\pi\nu} \ln r \bar{i}_z, \quad r > 0 \tag{4.46}$$

4.4 Magnetostatic Energy

Given a magnetostatic field characterized by strength $\overline{H}$ and flux density $\overline{B}$ in a linear medium, the specific energy (J m^{-3}) of the field is defined as $\frac{1}{2}\overline{H} \cdot \overline{B}$; if the medium is isotropic, the energy W(J) stored in an unbounded region Ω is given by

$$W = \frac{1}{2}\int_\Omega HBd\Omega \tag{4.47}$$

If the constitutive relationship of the magnetic material is non-linear, the specific energy is $\int_0^B H\, dB'$ and the total energy is

$$W = \int_{\Omega} \left(\int_{0}^{B} H \, dB' \right) d\Omega \tag{4.48}$$

In some cases it is convenient to introduce the specific co-energy $\int_0^H B\,dH'$ and the total co-energy is

$$W' = \int_{\Omega} \left(\int_{0}^{H} B \, dH' \right) d\Omega \tag{4.49}$$

In the case of linear medium $W = W'$ holds.
In the linear case, taking into account the following identity (see A.13)

$$\overline{H} \cdot \overline{B} = \overline{H} \cdot \left(\overline{\nabla} \times \overline{A}\right) = \overline{A} \cdot \left(\overline{\nabla} \times \overline{H}\right) - \overline{\nabla} \cdot \left(\overline{H} \times \overline{A}\right) = \overline{A} \cdot \overline{J} - \overline{\nabla} \cdot \left(\overline{H} \times \overline{A}\right) \tag{4.50}$$

and (4.3), the total energy stored in a region Ω of boundary Γ is

$$W = \frac{1}{2} \int_{\Omega} \overline{H} \cdot \overline{B} \, d\Omega = \frac{1}{2} \int_{\Omega} \overline{A} \cdot \overline{J} \, d\Omega - \frac{1}{2} \int_{\Gamma} \left(\overline{H} \times \overline{A}\right) \cdot \overline{n} \, d\Gamma \tag{4.51}$$

The equation above reduces to $W = \frac{1}{2} \int_{\Omega} \overline{A} \cdot \overline{J} \, d\Omega$ if either $\overline{A} \times \overline{n} = 0$ or $\overline{H} \times \overline{n} = 0$ along Γ [3].

4.5 Forces and Torques in the Magnetostatic Field

4.5.1 Principle of Virtual Work

Given a structure in the field region, on which force $\overline{F}$ is to be computed, a virtual linear displacement ds in the direction of $\overline{F}$, supposing that the magnetic system is supplied by a constant current I creating a linkage flux Φ, the sum of mechanical work Fds and variation of magnetic energy dW is equal to the input energy $Id\Phi$ so that the following balance equation

$$\begin{aligned} F\,ds + dW &= Id\Phi \\ F\,ds &= d(I\Phi - W) \\ F &= \frac{d}{ds}(I\Phi - W) \end{aligned} \tag{4.52}$$

In the case of an angular displacement $d\vartheta$, the torque M with respect to the rotation axis is

$$M = \frac{d}{d\vartheta}(I\Phi - W) \tag{4.53}$$

The quantity $I\Phi$-W, denoted by W', is the complementary energy or co-energy of the system.

On the other hand, if the magnetic system is isolated, mechanical work Fds and variation of magnetic energy dW take place so that

$$F\,ds + dW = 0 \tag{4.54}$$

Therefore, the force can be evaluated as

$$F = -\frac{dW}{ds} \tag{4.55}$$

while the torque is

$$M = -\frac{dW}{d\vartheta} \tag{4.56}$$

If the system is linear, W' and W coincide.

4.5.2 Lorentz's Method

It is based on the definition of flux density; in the empty space, the force $\overline{F}$ exerted on current I carried by a linear conductor of length ℓ is $\overline{F} = I\overline{\ell} \times \overline{B}$ where $\overline{B}$ is the external field, i.e. the flux density in the absence of current. In general, the force $\overline{F}$ exerted on current distributed with density $\overline{J}$ in the region Ω is

$$\overline{F} = \int_{\Omega} \overline{J} \times \overline{B}\, d\Omega \tag{4.57}$$

Direction of force is orthogonal to the plane defined by flux density and current density vectors.

4.5.3 Method of Maxwell's Stress Tensor

Defined a closed surface Γ enclosing the structure, then force $\overline{F}$ is evaluated as

$$\overline{F} = \int_{\Omega} \overline{\nabla} \cdot \overline{\overline{T}}\, d\Omega = \int_{\Gamma} \overline{\overline{T}} \cdot \bar{n}\, d\Gamma \tag{4.58}$$

where $\bar{n}$ is the outward normal unit vector.

The Maxwell's magnetic stress tensors $\overline{\overline{T}}$, assuming a system of rectangular coordinates, in a three-dimensional domain can be represented in matrix form as

$$\overline{\overline{T}} = \begin{bmatrix} \frac{1}{2}(H_x B_x - H_y B_y - H_z B_z) & H_x B_y & H_x B_z \\ H_y B_x & \frac{1}{2}(H_y B_y - H_x B_x - H_z B_z) & H_y B_z \\ H_z B_x & H_z B_y & \frac{1}{2}(H_z B_z - H_x B_x - H_y B_y) \end{bmatrix} \tag{4.59}$$

In order the tensor be uniquely defined, surface Γ should not be coincident with the interface between materials having different permeability [2].

4.5.4 Link Between Lorentz's and Maxwell's Approach

There is a link between Lorentz's and Maxwell's approach to force calculation. In fact, using (4.1), (4.3) and (4.57), the force density $\overline{f}\left(\mathrm{Nm}^{-3}\right)$ is

$$\overline{f} = \overline{J} \times \overline{B} = \left(\overline{\nabla} \times \nu\overline{B}\right) \times \overline{B} \tag{4.60}$$

In particular, the x-directed component is

$$f_x = \nu B_z \frac{\partial B_x}{\partial z} - \nu B_z \frac{\partial B_z}{\partial x} - \nu B_y \frac{\partial B_y}{\partial x} + \nu B_y \frac{\partial B_x}{\partial y} \tag{4.61}$$

After adding and subtracting the term $\frac{\nu}{2}\frac{\partial B_x^2}{\partial x}$ it follows

$$f_x = \frac{\nu}{2}\frac{\partial B_x^2}{\partial x} + \nu B_z \frac{\partial B_x}{\partial z} + \nu B_y \frac{\partial B_x}{\partial y} + \\ - \frac{\nu}{2}\frac{\partial}{\partial x}\left(B_x^2 + B_y^2 + B_z^2\right) \tag{4.62}$$

It turns out to be

$$f_x = \frac{\nu}{2}\frac{\partial B_x^2}{\partial x} + \nu\frac{\partial(B_xB_z)}{\partial z} - \nu B_x\frac{\partial B_z}{\partial z} + \nu\frac{\partial\left(B_xB_y\right)}{\partial y} + \\ - \nu B_x\frac{\partial B_y}{\partial y} - \nu B_x\frac{\partial B_x}{\partial x} - \frac{\nu}{2}\frac{\partial}{\partial x}\left(B_y^2 + B_z^2\right) \tag{4.63}$$

$$f_x = \nu\left[\frac{\partial}{\partial x}\left(B_x^2 - \frac{1}{2}\left|\overline{B}\right|^2\right) + \frac{\partial\left(B_xB_y\right)}{\partial y} + \frac{\partial(B_xB_z)}{\partial z} - B_x\overline{\nabla}\cdot\overline{B}\right] \tag{4.64}$$

Due to (4.2) the last term of (4.64) is zero; then, if vector

$$\overline{v}_1 = \nu\left(B_x^2 - \frac{1}{2}\left|\overline{B}\right|^2, B_xB_y, B_xB_z\right) \\ = \left(\frac{1}{2}\left(H_xB_x - H_yB_y - H_zB_z\right), H_xB_y, H_xB_z\right) \tag{4.65}$$

is defined, f_x can be viewed as its divergence, apart from a constant k which can be set to zero, namely

$$f_x = \overline{\nabla}\cdot\overline{v}_1 \tag{4.66}$$

A similar result holds for force density components f_y and f_z; it follows

$$\overline{v}_2 = \left(H_yB_x, \frac{1}{2}\left(H_yB_y - H_xB_x - H_zB_z\right), H_yB_z\right) \tag{4.67}$$

such that

$$f_y = \overline{\nabla}\cdot\overline{v}_2 \tag{4.68}$$

and

$$\overline{v}_3 = \left(H_zB_x, H_zB_y, \frac{1}{2}\left(H_zB_z - H_xB_x - H_yB_y\right)\right) \tag{4.69}$$

such that

$$f_z = \overline{\nabla}\cdot\overline{v}_3 \tag{4.70}$$

respectively. Therefore, according to (4.58), the force $\overline{F}(N)$ can be computed as the flux, leaving surface Γ, of tensor $\overline{\overline{T}}$ represented by matrix (4.59), in which the row entries are the components of vectors $\overline{v}_k, \quad k = 1, 3$.

Correspondingly, the torque is given by $\overline{M} = \int_\Gamma \bar{r}_{PO} \times \overline{\overline{T}} \cdot \bar{n}\, d\Gamma$ where $\bar{r}_{PO}$ is the position vector of point P on Γ with respect to the rotation axis in O.

It can be remarked that a solenoidal vector $\overline{w}$ may be added to (4.65), (4.67) and (4.69) leaving force density components (4.66), (4.68) and (4.70) fulfilled. This means that stress tensor (4.59) is not uniquely defined.

As far as a comparison of methods is concerned, the following remark can be put forward. In order Lorentz's method to apply, a current density must be defined in Ω; in contrast, virtual work principle (VWP) and Maxwell's stress tensor method (MST) are more general. VWP is computationally more expensive, because the derivative of energy or co-energy is approximated by means of a finite difference, involving two displaced positions of the structure. Therefore, two field analyses are necessary to compute force or torque at a given position. MST require only one field analysis.

4.6 Worked Example

4.6.1 Force on an Electromagnet

Let an electromagnet with a movable plunger be considered (Fig. 4.3), [1].

The iron core is supposed to have infinite permeability. The air gaps in the x direction are supposed to be much smaller than the air gap t in the y direction.

The circulation of the magnetic field H, along a line linking the excitation current NI and crossing the air gap t in the normal direction, reduces to

$$NI = Ht \tag{4.71}$$

Therefore at the air gap

$$H = \frac{NI}{t} \tag{4.72}$$

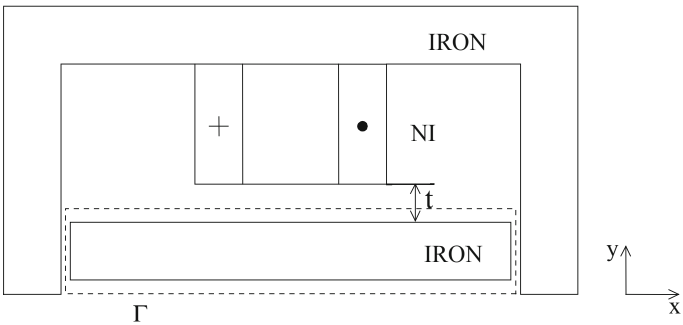


Fig. 4.3 Model of the electromagnet

while in the iron part $H = 0$. Following (4.49), the co-energy stored in the air gap is given by

$$W' = \frac{1}{2}\mu_0 H^2 St = \frac{\mu_0 (NI)^2 S}{2t} \tag{4.73}$$

where S is the cross-section of the central limb and μ_0 is the air permeability.

If NI is constant, according to (4.52), the force acting on the movable part is

$$F_t = \frac{\partial W'}{\partial t} = -\frac{\mu_0 S}{2}\left(\frac{NI}{t}\right)^2 \tag{4.74}$$

The force is negative, i.e. opposite to the direction of increasing t; therefore, it is attractive, regardless of the sign of I.

In order to apply the method of Maxwell's stress tensor, an integration surface Γ enclosing the movable part is considered having $\overline{n}$ as its outward normal unit vector. Taking into account the field distribution, it follows

$$\overline{\overline{T}} = \begin{bmatrix} -\frac{1}{2}H_y B_y & 0 \\ 0 & \frac{1}{2}H_y B_y \end{bmatrix} \tag{4.75}$$

$$\overline{F} = \int_\Gamma \overline{\overline{T}} \cdot \overline{n} d\Gamma = \left(0, \frac{1}{2}H_y B_y S\right) \tag{4.76}$$

Therefore it turns out to be

$$F_y = \frac{1}{2}\mu_0 H_y^2 S = \frac{1}{2}\mu_0 S\left(\frac{NI}{t}\right)^2 \tag{4.77}$$

The force is attractive, because variables t and y are oriented in opposite directions.

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