Tropical Generalized Interval Systems

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Abstract. This paper aims to refine the formalization of David Lewin's Generalized Interval System (GIS) by the means of tropical semirings. Such a new framework allows to broaden the GIS model introducing a new operation and consequently new musical and conceptual insights and applications, formalizing consistent relations between musical elements in an original unified structure. Some distinctive examples of extensions of well-known infinite GIS for lattices are then offered and the impossibility to build tropical GIS in the finite case is finally proven and discussed.

Keywords: Tropical semiring · Ordering · Generalized Interval System

1 Introduction

One of the most significant results achieved by the introduction of the Generalized Interval System (GIS) is to offer a consistent formal framework for musical elements and transformations between them. Although David Lewin's construction is very general and have been exploited for formalizing several different musical elements and transformations, one of its most straightforward use is for offering a formal framework for lattices, as the name itself suggests. Lattices have served as the basis of the ideas of this paper, thus, all the example that will be offered will be taken in their context. However, there may certainly be other possibilities and applications. In fact, the GIS notion in Lewin's own words "generalizes certain intuitions we have concerning traditional sorts of intervals that are directed from one pitch (or pitch class) to another. Generalized intervals are similarly directed, from one object of a GIS to another. These objects need not be pitches or pitch classes; they may have rhythmic, timbral, or other sort of character" [15].

According to [14], a Generalized Interval System can be defined as follows.

Definition 1. *A* ***Generalized Interval System (GIS)*** *is an ordered triple* (M, G, φ)*, where* M *is a set of musical objects,* G *is a group and* φ *is an action of* G *on* M *which is free and transitive.*

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Thus, for any $m \in M$, $\varphi(g, m) = m$ if and only if g is the identity element of G (free group action), and for any pair of elements $m_1, m_2 \in M$ there is one (and only one) $g \in G$ such that $\varphi(g, m_1) = m_2$ (transitive group action).[1]

It is important to underline the fundamental ontological and theoretical distinction that a GIS makes between the undertaken musical elements and the transformations among them: the former constituting a set of musical objects that per se are not ordered, and the latter being a set of transformations that forms a group and that acts on the aforesaid elements, defining and making explicit the structure of their set. This distinction is not obvious at all and was born as a generalization of Milton Babbitt and Allen Forte's ideas [7,12,13]. Furthermore, such an approach seems to meet neat structuralist criteria, in fact a GIS - misquoting Babbitt referring to a twelve tone-system - "like any formal system whose abstract model is satisfactorily formulable, can be characterized completely by stating its elements, the stipulated relation [...] among these elements, and the defined operations upon the so-related elements" [3].

In this framework, this paper aims to answer the following question: is it possible to refine the notion of the GIS so to find an even more detailed formalization that could represent in further detail musical elements and transformations among them?

There are certainly different ways to generalize GIS constructions, for instance by relaxing the simple-transitivity condition [18] or via the concept of groupoids and partial actions [16,17]. We have particularly investigated algebraic structures that might substitute the group one in the conventional definition of the GIS mainly in the context of lattices. The first apparently natural step has been to try to replace the group with a ring, keeping the group binary operation as the addition and introducing a multiplication that is distributive in respect to addition and under which the structure is a monoid.[2] As a result, in the specific case of lattices, this formalization leads to musical nonsense. If addition ends up representing the meaningful previously defined operation between transformation - that can be consistently seen as an ordered application of both the transformations - it is difficult to find a musical meaning for the multiplication, or to define it so to have one. For instance, let us consider the traditional case of intervals seen as a counting of semitones, hence constituting an algebraic structure that is isomorphic to the additive group $\mathbb{Z}$ in the case of pitches and to $\mathbb{Z}/12\mathbb{Z}$ in the case of pitch classes. Multiplying 3 by 2 ends in repeating the action of the interval 3 two times just for a mathematical contingency given by the numerical representation of intervals. In fact, in the aforesaid systems 2 is not a quantity but an interval, and multiplying two intervals has no acknowledged musical meaning. Therefore, due to the purpose of refinement of this paper, the

[1] According to Lewin's exact definition, as given in [14], the group action on the set is required to be "simply transitive". That is equivalent to the requirement of a free and transitive group action.

[2] The definition of a monoid requires only associativity and the existence of the identity element for the binary operation.

ring structure seems to be a dead end with respect to lattices.[3] Fortunately, if we stay close to the ring structure - dealing with semigroups and semirings - and we put the group binary operation of the GIS as the multiplication, we can find GIS refinements that fit our need and such as both operations are musically meaningful and could offer a more detailed insight of the undertaken musical elements.

Consequently, we shall introduce the mathematics of semirings, tropical semirings, min-plus and max-plus algebras, discussing then their implementation in the proposal for tropical Generalized Interval Systems both in the case of finite and infinite sets of musical elements.

2 Semirings and Tropical Algebras

Let us now introduce the algebraic structures that are going to be employed, in the order: semigroups, semirings, tropical semirings, min-plus and max-plus algebras. From a historical perspective, they are quite recent notions. Semirings have been introduced in 1934, in a short paper by Harry Schultz Vandiver [19], who gave them that name because of their ring-like structure. However, the same concept, although with a different name, has appeared in an earlier work by Richard Dedekind in 1884 [2,11]. The ideas that led to tropical algebra and tropical geometry can be traced back to the end of the fifties, as reported in 1979 in [5].

> In the past 20 years a number of different authors, often apparently unaware of one another's work, have discovered that a very attractive formulation language is provided for a surprisingly wide class of problems by setting up an algebra of real numbers (perhaps extended by symbols such as $-\infty$, etc.) in which, however, the usual operations of multiplication and addition of two numbers are replaced by the operations: (i) arithmetical addition, and (ii) selection of the greater (or dually, the less) of the two numbers, respectively.

Finally, "the adjective *tropical* was coined by French mathematicians, including Jean-Eric Pin, in honor of their Brazilian colleague Imre Simon, who was one of the pioneers in what could also be called min-plus algebra. There is no deeper meaning in the adjective tropical. It simply stands for the French view of Brazil" [20].

Let us now offer some formal definitions.

[3] However, outside the framework of lattices the ring structure can be successfully used to refine a GIS. Lewin gave in [15] an example of a GIS that calls out to be extended to a ring, although he did not carry out the extension himself. In fact, with respect to the GIS of Babbitt's lists investigated in the aforesaid paper, the transformation group can be easily and meaningfully extended to a ring. See Example 3 in Sect. 5 of this paper.

Definition 2. *A **semigroup** is an ordered pair* $(\mathbb{S}, \bullet)$ *such that* $\mathbb{S}$ *is a non empty set and* $\bullet$ *is an associative binary operation; thus for any* a*,* b*, and* c *in* $\mathbb{S}$*:*

$$(a \bullet b) \bullet c = a \bullet (b \bullet c). \tag{1}$$

Notice that no other restrictions are placed on a semigroup: it does not need an identity element and its elements do not need to have inverses within the semigroup. Only closure and associativity are preserved.

Definition 3. *A **semiring** is an ordered triple* $(\mathbb{S}, \oplus, \otimes)$ *such that* $\mathbb{S}$ *is a non empty set,* $\oplus$ *and* $\otimes$ *are respectively called **addition** and **multiplication**, and* $(\mathbb{S}, \oplus)$ *and* $(\mathbb{S}, \otimes)$ *are semigroups such that multiplication left and right distributes over addition; thus for any* a*,* b*, and* c *in* $\mathbb{S}$*:*

$$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c) \quad \textit{and} \quad (a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c). \tag{2}$$

It should be emphasized that: (1) to avoid trivial examples, a semiring is hereby supposed to have at least two elements; (2) a semiring may or may not have identity elements for addition and/or multiplication, which may or may not coincide; (3) a semiring may or may not be additively and/or multiplicatively commutative, however, in case both the semigroups are abelian the semiring is said to be commutative [1]. For instance, $(\mathbb{N}, +, \times)$, with $\mathbb{N}$ the set of all the non-negative integers and $+$ and $\times$ the usual addition and multiplication of integers, is a commutative semiring with identity elements 0 and 1 for addition and multiplication respectively.

According to [11], let us now offer a general definition for tropical semirings.

Definition 4. *A **tropical semiring** is a semiring with idempotent addition; thus for any* a *in* $\mathbb{S}$*:*

$$a \oplus a = a. \tag{3}$$

We can now introduce two of the most investigated tropical semirings.

Definition 5. *A **min-plus algebra** and a **max-plus algebra**, are the two tropical semirings* $(\mathbb{R} \cup \{\infty\}, \oplus, \otimes)$ *and* $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$*, with the operations as follows:*

$$x \oplus y = min\{x, y\} \quad \textit{or, respectively,} \quad x \oplus y = max\{x, y\} \quad \textit{and} \tag{4}$$

$$x \otimes y = x + y. \tag{5}$$

For them, the identity elements for the addition are ∞ and $-\infty$, respectively, and the multiplication is the usual addition of real numbers with 0 as the identity element. Commonly, min-plus and max-plus algebras are defined as such, but notice that in general in a tropical semiring the existence of an identity element is not a requirement, both for addition and multiplication. In order to extend the concept of a GIS, in which the elements of the algebras are - from a broader point of view - intervals and transformations between musical elements, the definition of an infinite one is thereby without any use, sense and practical application.

3 Extended and Tropical GIS

Considering the purpose of extending the notion of a Generalized Interval System, we shall then replace the group structure with a semiring one.

Definition 6. *An* ***Extended Generalized Interval System (eGIS)*** *is an ordered triple* $(M, \mathbb{S}, \varphi)$*, where* M *is a set of musical objects,* $\mathbb{S}$ *is a semiring* $(\mathbb{S}, \oplus, \otimes)$ *such that with multiplication* $(\mathbb{S}, \otimes)$ *constitutes a group, and* φ *is an action of* $(\mathbb{S}, \otimes)$ *on* M *which is free and transitive.*

Notice that to preserve the feature of the binary operation between the intervals/transformations of a conventional GIS we have imposed the semiring multiplication to be a group. This way, it is possible to consider a standard GIS and to expand it in an extended one, keeping the group operation as the multiplication of the semiring and introducing a consistent addition that would satisfy its axioms. In this respect, idempotent operations - and in particular kind of minimum and maximum ones - ensure the definition of an order on the transformations, that to some extent one could read between the lines of the original idea of Lewin's GIS. In fact, in [14] Lewin refers to the transformations as "a family of directed measurements, distances, or motions of some sort". Therefore, the ordering of the elements of the transformation by the means of an idempotent binary operation lets explicit an otherwise not formally obvious metric of some musical meaning.

In fact, as is known, there is a strict correlation between the ordering of a set and some kinds of binary operations, as shown in the following two lemmas. Their proofs are simple and thus only sketched.[4]

Lemma 1. *To define a linear order in a set is equivalent to defining a binary operation that is associative, commutative and such that its outcome is always one of the two operands.*

Proof. Let a linear order be given. Define $a \oplus b = min\{a, b\}$. This is a binary operation. The required properties are obvious. Conversely, let be given a binary operation with those properties. Define $a \leq b$ if and only if $a \oplus b = a$. This is a binary relation. Since the outcome is always one of the two operands, the relation is reflexive and, if it is an order, it is a linear one. Antisymmetry is obvious. Since the operation is associative, the relation is transitive.

Lemma 2. *To define in a set a binary operation that is associative, commutative, and idempotent is equivalent to defining a (partial) order in which, for any two elements, there is a greatest lower bound, and also to defining a (partial) order in which, for any two elements, there is a least upper bound.*

Proof. If an order is given, define $a \oplus b = inf\{a, b\}$. The required properties are obvious. If, conversely, a binary operation is given, define $a \leq b$ if and only if $a \oplus b = a$. This relation is reflexive, owing to idempotency. It is obviously an order

[4] See also [8], Proposition 2.1.

relation, not necessarily linear. Anyway, it is easy to prove, using associativity and idempotency, that $(a \oplus b) \leq a$ and $(a \oplus b) \leq b$; moreover, if $c \leq a$ and $c \leq b$, then - again by associativity - $c \leq (a \oplus b)$, so $a \oplus b = inf\{a, b\}$, as required.

Notice that the hypotheses of Lemma 1 are a particular case of the ones of Lemma 2. Let us then narrow down the concept of the eGIS to one that considers only semirings with idempotent addition.

Definition 7. *A* ***Tropical Generalized Interval System (tGIS)*** *is an eGIS* $(M, \mathbb{S}, \varphi)$ *such that* $\mathbb{S}$ *is a tropical semiring.*

The advantage of introducing extended GIS and tropical GIS is not only of theoretical and explanatory nature in seeking new and more refined conceptual insights of a musical space, but it can offer also benefits on the application side. In a tropical GIS a kind of minimum or maximum binary operator works as the addition, as well as the composition between transformations works as the multiplication, and the two operations can be dealt together in long mathematical expressions of musical meaning that can be reduced and solved just as in traditional arithmetic. This is much different than simply requiring a linear order on the intervals of a GIS. In fact, it could be possible to define a linear order that implies a binary operation that could not satisfy the semiring features. The musical meaning of such an impasse will be discussed in Sect. 5.

Let us now study tropical Generalized Interval Systems both in the infinite and finite cases.

4 Infinite Tropical Generalized Interval Systems

Perhaps, the simplest example of a tropical GIS in the infinite case is the one obtained extending the conventional GIS $(P, \mathbb{Z}, \varphi)$ such as P is the infinite set of equal tempered pitches and its group of intervals is isomorphic to $\mathbb{Z}$. Let us define the two operations as follows:

$$x \oplus y = min\{x, y\} \quad \text{and} \quad x \otimes y = x + y \tag{6}$$

where $+$ is the usual addition in $\mathbb{Z}$. The tropical semiring structure of $(\mathbb{Z}, \oplus, \otimes)$ can be inferred from the min-plus algebra $(\mathbb{R} \cup \{\infty\}, \oplus, \otimes)$, of which $(\mathbb{Z}, \oplus, \otimes)$ is a subsemiring. Our tGIS is then $(P, (\mathbb{Z}, \oplus, \otimes), \varphi)$, for which there is no identity element for $\oplus$.

Another example can be given extending the GIS $(F, \mathbb{Q}, \varphi)$, where F are all the frequencies represented as positive rational numbers and the group is the multiplicative one $\mathbb{Q}$ of frequency ratios, that are positive rationals written as irreducible fractions as well. Its simplest tropical extension can be achieved defining the two operations such as:

$$x \oplus y = min\{x, y\} \quad \text{and} \quad x \otimes y = x \times y \tag{7}$$

where $\times$ is the usual multiplication in $\mathbb{Q}$. In this case, $\oplus$ simply outputs the shortest interval between the two operating ones.

However, different solutions embodying musical meaning can be defined for $\oplus$, keeping the multiplication from the group structure of $\mathbb{Q}$. For instance, reminded that we are dealing with irreducible fractions, it is possible to consider the following operation:

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{GCD(a,c)}{LCM(b,d)}, \tag{8}$$

such as GCD stands for the greatest common divisor and LCM for the least common multiple of two integers. Consequently, such an operation outputs one of the two elements only if the multiplication of it by a natural number gives the other operand; in case it does not, it outputs the biggest element in $\mathbb{Q}$ such as the two operands can be obtained multiplicating it by natural numbers. It is an idempotent operation and multiplication left and right distributes over it. From the point of view of music, it outputs the biggest interval expressed as a ratio in respect of which the two operands can be seen as natural harmonics.

Let us now consider the GIS $(F(\sqrt[12]{2}), \mathbb{Q}^*(\sqrt[12]{2}), \varphi)$. $F(\sqrt[12]{2})$ is the set of all the frequencies that can be obtained as ratios combined with the ones in equal temperament. Thus, it is the set of all the frequencies that can be written as follows:

$$a_{11}(\sqrt[12]{2})^{11} \times a_{10}(\sqrt[12]{2})^{10} \times \cdots \times a_1 \sqrt[12]{2} \times a_0, \tag{9}$$

with $a_0, a_1, \ldots, a_{11} \in \mathbb{Q}$. Here $\mathbb{Q}^*(\sqrt[12]{2})$ is a subgroup of the multiplicative group of the algebraic number field $\mathbb{Q}(\sqrt[12]{2})$, generated by $\mathbb{Q}$ and $\sqrt[12]{2}$. Such a GIS can be extended with the semiring addition introduced in Eq. 7. On the contrary, the one offered in Eq. 8 does not work, because not all the elements of $F(\sqrt[12]{2})$ can be reduced to fractions, and, moreover, because a common submultiple between two elements in $\mathbb{Q}^*(\sqrt[12]{2})$ does not necessarily exist; hence not every couple of intervals can be seen as natural harmonics in a series.

In fact, regrettably, to extend an infinite GIS is not always as straightforward as it could appear. For instance, we have tried without success to consider a third proposal for $\oplus$ in order to extend $(F, \mathbb{Q}, \varphi)$ involving Euler's consonance degree value, *gradus suavitatis*, as it was described in [6] and investigated, amongst many others, in [4,10]. First of all, given an irreducible fraction $\frac{a}{b}$ and following Euler's principles, let us define the function $C_e : \mathbb{Q} \rightarrow \mathbb{N}^*$ that associates a fraction with its degree of consonance as follows:

$$C_e\left(\frac{1}{2^n}\right) = n + 1 \tag{10}$$

otherwise, for $a, b \in \mathbb{N}^*$ such as $\frac{a}{b} \neq \frac{1}{2^n}$,

$$\begin{aligned} C_e\left(\frac{a}{b}\right) &= C_e\left(\frac{1}{LCM\left(\frac{a}{GCD(a,b)}, \frac{b}{GCD(a,b)}\right)}\right) \\ &= C_e\left(\frac{1}{p_1^{k_1} \times p_2^{k_2} \times \cdots \times p_m^{k_m}}\right) = \sum_{i=1}^{m}(p_i^{k_i} - k_i) + 1 \end{aligned} \tag{11}$$

Thus, we have tried to define $\oplus$ accordingly:

$$\frac{a}{b} \oplus \frac{c}{d} = \begin{cases} \frac{a}{b}, & \text{if } C_e\left(\frac{a}{b}\right) < C_e\left(\frac{c}{d}\right); \\ \frac{c}{d}, & \text{if } C_e\left(\frac{c}{d}\right) < C_e\left(\frac{a}{b}\right); \\ min\{\left(\frac{a}{b}\right), \left(\frac{c}{d}\right)\} & \text{otherwise.} \end{cases} \tag{12}$$

In this case, $\oplus$ outputs the most consonant interval between the two (the one with the lowest degree of consonance); otherwise, if the two ratios share the degree, it outputs the smallest of them. Unfortunately, the distributive law does not work. In fact,

$$5 \otimes \left(\frac{1}{5} \oplus \frac{1}{8}\right) = 5 \otimes \frac{1}{8} = \frac{5}{8}, \tag{13}$$

but, at the same time,

$$\left(5 \otimes \frac{1}{5}\right) \oplus \left(5 \otimes \frac{1}{8}\right) = 1 \oplus \frac{5}{8} = 1. \tag{14}$$

A last successful example can be given extending the GIS $(F_J, \mathbb{Q}_J, \varphi)$, where F_J are all the pitches in 5-limit just intonation represented as frequencies and $\mathbb{Q}_J$ is the group of the just intonation ratios, i.e. the multiplicative subgroup of $\mathbb{Q}$ made up of all the numbers that can be expressed in the form $2^i3^j5^k$ such that $i, j, k \in \mathbb{Z}$. An idempotent addition can be defined as follows:

$$2^i3^j5^k \oplus 2^{i^*}3^{j^*}5^{k^*} = \begin{cases} 2^i3^j5^k, & \text{if } k > k^*, \\ & \text{if } k = k^* \text{ and } j > j^*, \\ & \text{if } k = k^*, j = j^* \text{ and } i > i^*, \\ 2^{i^*}3^{j^*}5^{k^*} & \text{otherwise.} \end{cases} \tag{15}$$

Therefore, for instance $\frac{8}{27} \oplus \frac{1}{5} = 2^3 3^{-3} 5^0 \oplus 2^0 3^0 5^{-1} = \frac{8}{27}$. Such an operation might be easily adjusted to be used to extend different GIS with n-limit extended just tuning intervals in the form $2^i3^j \ldots n^k$ with n prime.

Nevertheless, these are just a few examples of the multitude of potential extensions of infinite GIS that can be built for better or alternative insights and applications.

5 Finite Tropical Generalized Interval Systems

Considering our musical aims, it would be natural to try to apply the same structures to modular arithmetics, due to the importance of groups of intervals isomorphic to $\mathbb{Z}/12\mathbb{Z}$ and $\mathbb{Z}/7\mathbb{Z}$ in music theory. But, unfortunately, the following theorem (which is, perhaps, the main result of this paper) makes it impossible to build finite not trivial tropical GIS.

Theorem 1. *Given a finite set of at least two elements, it is not possible to define two binary operations such as one is associative, commutative and idempotent and the other is a group and is distributive over the first.*

Proof. Let us consider a finite set for which $\otimes$ is a group and $\oplus$ is associative, commutative and idempotent. Therefore, because of Lemma 2, there is an order in the set in which, for any two elements, there is a greatest lower bound, i.e., $a \oplus b$ is the greatest lower bound of a and b. Let 0 be the identity element for $\otimes$, $s \neq 0$, and $t = 0 \oplus s$. If $t \neq 0$, then $t \oplus 0 = t$ and, applying the distributive law, we get:

$$t \otimes t = t \otimes (t \oplus 0) = (t \otimes t) \oplus (t \otimes 0) = (t \otimes t) \oplus t. \tag{16}$$

Thus,

$$(t \otimes t) \oplus t = t \otimes t \tag{17}$$

and, because of order transitivity,

$$(t \otimes t) \oplus 0 = t \otimes t. \tag{18}$$

Let us consider $t^m = t \otimes t \otimes \cdots \otimes t$, m times, and such as $t^m = 0$ (m surely exists, because the group is finite). Then, by distributivity,

$$(t \otimes t^{m-1}) \oplus (t \otimes 0) = t \otimes t^{m-1}, \tag{19}$$

thus, $0 \oplus t = 0$, a contradiction. If $t = 0$, then $0 = 0 \oplus s$ and the proof is similar.

Let us discuss some related constructions which could be considered erroneously as counterexamples.

Example 1. Two-element Boolean algebra (also called Boolean semiring). This is not a counterexample to Theorem 1, because the two elements, with multiplication, do not form a group: 0 has no multiplicative inverse. All the other properties are verified [9].

Example 2. Finite fields (also called Galois fields). None of them is a counterexample. The addition is not idempotent (the only idempotent element is 0) and the element, with multiplication, do not form a group (the multiplicative group of any field contains the elements different from 0). Note that the field $(\mathbb{Z}/2\mathbb{Z}, +, \times)$ has two elements, as the Boolean algebra of Example 1. The only difference between these two algebraic structures is the sum $1 + 1$ (see next example).

Example 3. Direct product of finite fields. In [15] David Lewin uses the term "Boolean sum" for the sum modulo 2. In the above Example 1, the meaning is different: in the two-element Boolean algebra $1 + 1 = 1$; in the sum modulo 2, on the contrary, $1 + 1 = 0$. In the same paper, the author uses the group $((\mathbb{Z}/2\mathbb{Z})^4, +)$. So, let us consider for a moment also finite products of finite fields. None of them is a counterexample to Theorem 1 above. In fact, we can repeat what we have already shown in Example 2. Moreover, in such products there exist zero divisors, so the multiplicative group cannot even contain all the elements different from 0. Note that, if we consider a finite product of copies of $(\mathbb{Z}/2\mathbb{Z}, +, \times)$, we get a Boolean ring (i.e., the multiplication is idempotent). The group used by Lewin can be extended to such a ring.

The result offered by Theorem 1 is meaningful especially on the musical side. In fact, the sensitive requirement that cannot be satisfied is the distributive law, that de facto connects the two operations. Moreover, we have shown that to define on a set a binary operation that is associative, commutative, and idempotent is equivalent to defining an order in which, for any two elements, there is a greatest lower bound and also to defining an order in which, for any two elements, there is a least upper bound. As a matter of fact, cyclicity and linearity are not compatible concepts: if we try to deal with a finite - thus, in some sense, cyclical[5] - GIS from the point of view of a linear order we break its cyclical nature that made any combination of transformations meaningful. As a consequence, such linear orders could have only a theoretical significance on a taxonomic level[6], before any combination, and cannot offer any general systematic insight. Thus, a finite GIS cannot be extended to a tropical one.

6 Conclusions

We have introduced a refinement of David Lewin's Generalized Interval System by the means of tropical semirings, investigated the links between a tropical semiring addition and an ordering, studied some examples obtained extending well-known infinite GIS mainly in the context of lattices, and proven and discussed the impossibility to build tropical GIS in the finite case. Such new theoretical framework has shown the capability to embody a more detailed knowledge of the formalized musical elements in a unified structure. Moreover, we believe that the several possibilities in the definition of the semiring addition, both in the tropical and in the more general case of extended GIS, could trigger new ideas and conceptual insights in dealing with such elements, as we have shown in some of the examples offered. Therefore, further studies may be conducted in deepening the various extending possibilities. Finally, we have deepened finite and infinite tropical GIS, but the more general case of extended GIS is still open to further studies, especially about the finite case.

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[5] In fact, every element of a finite group generates a cyclical subgroup.

[6] For instance, it would be meaningful to order the elements in $\mathbb{Z}/12\mathbb{Z}$ from 0, the minimum, to 11, the maximum, or, alternatively, in the following sequence: $0, 1, 11, 2, 10, 3, 9, 4, 8, 5, 7, 6$, in which it is taken the distance from 0 on both side, favoring the right one in case of the same value.

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