Groupoids and Wreath Products of Musical Transformations: A Categorical Approach from poly-Klumpenhouwer Networks

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Abstract. Klumpenhouwer networks (K-nets) and their recent categorical generalization, poly-Klumpenhouwer networks (PK-nets), are network structures allowing both the analysis of musical objects through the study of the transformations between their constituents, and the comparison of these objects between them. In this work, we propose a groupoid-based approach to transformational music theory, in which transformations of PK-nets are considered rather than ordinary sets of musical objects. We show how groupoids of musical transformations can be constructed, and provide an application of their use in post-tonal music analysis with Berg's *Four pieces for clarinet and piano, Op. 5/2*. In a second part, we show how these groupoids are linked to wreath products through the notion of groupoid bisections.

Keywords: Klumpenhouwer network · Transformational music theory · Category theory · Groupoid · Wreath product

1 Groupoids of Musical Transformations

The recent field of transformational music theory, pioneered by the work of Lewin [7,8], shifts the music-theoretical and analytical focus from the "object-oriented" musical content to an operational musical process, wherein transformations between musical elements are emphasized. Within this framework, Klumpenhouwer networks (K-nets) [5,6,9] are network structures allowing both the analysis of musical objects through the study of the transformations between their constituents, and the comparison of these objects between them. A K-net can be informally defined as a labelled graph, wherein the labels of the vertices

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belong to the set of pitch classes, and each arrow is labelled with a transformation that maps the pitch class at the source vertex to the pitch class at the target vertex. The K-net concept, anchored both in group theory and graph theory [12], has been later formalized in a more categorical setting, first as limits of diagrams within the framework of denotators [11], and later as a special case of a categorical construction called poly-Klumpenhouwer networks (PK-nets) [16–18]. K-nets are usually compared in terms of isographies, which correspond to isomorphisms of PK-nets in the categorical approach.

This paper proposes a groupoid-based approach to transformational music theory, in which transformations of PK-nets are considered rather than ordinary sets of musical objects. The first section shows how groupoids of musical transformations can be constructed, with an application to post-tonal music analysis. The second section shows how groupoids are linked to wreath products, which feature prominently in transformational music theory [4], through the notion of groupoid bisections, thus bridging the groupoid approach to the more traditional group-based approach of transformational music theory.

1.1 Introduction to PK-Nets

The groupoid-based approach to transformational music theory presented in this paper stems from the constitutive elements of poly-Klumpenhouwer networks which have been introduced previously [16,17]. We recall the categorical definition of a PK-net, which generalizes the original notion of K-nets in various ways. We assume that the basic notions of transformational music analysis are known, in particular with regards to the so-called T/I group and its action on the set of the twelve pitch classes (see [3] for additional information).

Definition 1. *Let* $\mathbf{C}$ *be a category, and* S *a functor from* $\mathbf{C}$ *to the category* **Sets** *of (small) sets. Let* Δ *be a small category and* R *a functor from* Δ *to* **Sets** *with non-empty values. A* PK-net of form R and of support S *is a 4-tuple* (R, S, F, ϕ)*, in which* F *is a functor from* Δ *to* $\mathbf{C}$*, and* ϕ *is a natural transformation from* R *to* SF*.*

A PK-net can be represented by the diagram of Fig. 1(a). The category $\mathbf{C}$ and the functor $S\colon \mathbf{C} \to \mathbf{Sets}$ represent the musical context of analysis.

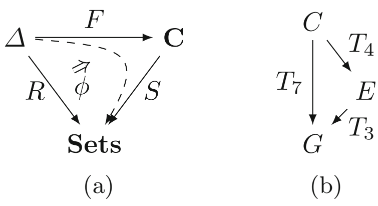


Fig. 1. (a) Diagrammatic representation of a PK-net (R, S, F, ϕ). (b) A K-net describing a major triad. The arrows are labelled with specific transformations in the T/I group indicating the transformations between pitch classes.

The morphisms of the category $\mathbf{C}$ are the musical transformations of interest. Any category $\mathbf{C}$ along with a functor $S\colon \mathbf{C} \to \mathbf{Sets}$ may be considered: transformational music theory often relies on a group acting on a given set of objects, the T/I group acting on the set of the twelve pitch classes being one of the most well-known examples. The category Δ serves as the abstract skeleton of the PK-net: as such, its objects and morphisms are abstract entities, which are labelled by the functor F from Δ to the category $\mathbf{C}$. The objects of Δ do not represent the actual musical elements of a PK-net: these are introduced by the functor R from Δ to $\mathbf{Sets}$. This functor sends each object of Δ to an actual set, which may contain more than a single element, and whose elements abstractly represent the musical objects of study. In the same way the morphisms of Δ represent abstract relationships which are given a concrete meaning by the functor F, the elements in the images of R are given a label in the images of S through the natural transformation ϕ, which ensures the transformational coherence of the whole diagram.

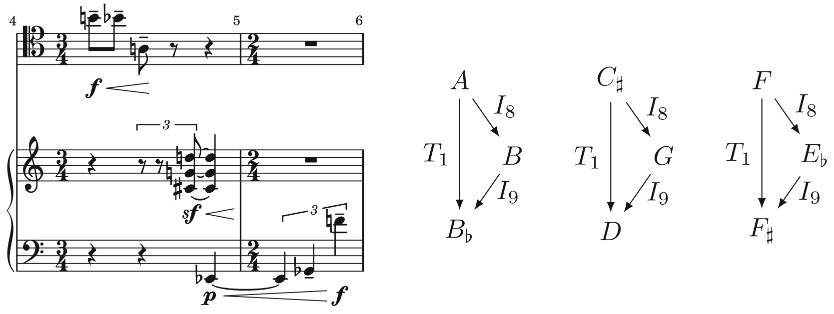
1.2 Reinterpreting the Constitutive Elements of a PK-Net

A basic Klumpenhouwer network describing a C major triad is shown in Fig. 1(b). In the framework of PK-Nets, this network corresponds to the data of

1. a category Δ_3 with three objects X, Y, and Z, and three non-trivial morphisms $f\colon X \to Y$, $g\colon Y \to Z$, and $g \circ f\colon X \to Z$ between them, and
2. a category $\mathbf{C}$ taken here to be the T/I group, with its usual action $S\colon T/I \to$ **Sets** on the set of the twelve pitch-classes, and
3. a functor $F\colon \Delta_3 \to \mathbf{C}$ such that $F(f) = T_4$, $F(g) = T_3$, and $F(g \circ f) = T_7$, and
4. a functor $R\colon \Delta_3 \to \mathbf{Sets}$ sending each object of Δ_3 to a singleton, and a natural transformation ϕ sending these singletons to the appropriate pitch-classes in the image of $\mathbf{C}$ by S.

Upon examination of these constitutive elements, it readily appears that the "major triad" nature of this chord is entirely determined by the category Δ_3 and the functor $F\colon \Delta_3 \to \mathbf{C}$. The images of the morphisms of Δ_3 under F reflect the fact that a major triad is made up of a major third with a minor third stacked above it, resulting in a fifth. This observation is not specific to major triads. Consider for example the chord $\{D, E, G\}$, which is a representative of the set class $[0, 2, 5]$. This chord may be described by a PK-Net in which Δ_3 is the same as above, and in which we consider a different functor $F'\colon \Delta_3 \to \mathbf{C}$ such that $F(f) = T_2$, $F(g) = T_3$, and $F(g \circ f) = T_5$.

One may go further by considering pitch-class sets which are not necessarily transpositionnally related. Figure 2 shows an excerpt of Webern's *Three Little Pieces for Cello and Piano, Op. 11/2*, at bars 4–5, along with a PK-net interpretation of each three-note segment. These three-note segments are clearly not related by transposition, yet the corresponding represented networks share the same functor $\Delta_3 \to \mathbf{C}$. By abstracting this observation, one can consider that



(a) (b)

Fig. 2. (a) Webern, Op. 11/2, bars 4–5. (b) PK-nets corresponding to each of three-note segment of (a).

these three-note pitch-class sets belong to the same *generalized musical class for the PK-net*, which is defined by this particular functor $\Delta_3 \to \mathbf{C}$. The main point of this paper is to generalize further these observations by considering functors $F\colon \Delta \to \mathbf{C}$ as *generalized musical classes for a PK-net.*

Definition 2. *Let* $\mathbf{C}$ *be a category, and* Δ *be a small category. A* generalized musical class for a PK-net of diagram Δ *is a functor* $F\colon \Delta \to \mathbf{C}$.

As is well-known in category theory, functors $F\colon \Delta \to \mathbf{C}$ form a category, known as *the category of functors* $\mathbf{C}^\Delta$.

Definition 3. *The* category of functors $\mathbf{C}^\Delta$ *has*

1. *functors* $F\colon \Delta \to \mathbf{C}$ *as objects, and*
2. *natural transformations* $\eta\colon F \to F'$ *between functors* $F\colon \Delta \to \mathbf{C}$ *and* $F'\colon \Delta \to \mathbf{C}$ *as morphisms.*

These natural transformations can be seen as generalized musical transformations between the corresponding generalized musical classes. The additional data of functors R and S, and of a natural transformation $\phi\colon R \to SF$, leads to individual musical sets derived from the generalized musical class $F\colon \Delta \to \mathbf{C}$ for a PK-net of diagram Δ. The purpose of this paper is to investigate the structure of $\mathbf{C}^\Delta$ and of specific functors from this category to **Sets**, and to relate these constructions with known group-theoretical results in transformational music analysis.

One may notice that different categories Δ and functors $F\colon \Delta \to \mathbf{C}$ may describe the same musical sets. For example, the major triad of Fig. 1(b) may also be described using a category Γ with three objects X, Y, and Z, and only two non-trivial morphisms $f\colon X \to Y$, $g\colon X \to Z$, between them, and a functor $F''\colon \Gamma \to \mathbf{C} = T/I$ sending f to T_4 and g to T_7. This alternative description focuses on the major third and the fifth without explicitly referencing the minor third. In a broader setting, one can even consider both transposition and inversion operations. For example, the functor F'' could instead send f to I_4 and g to I_7, emphasizing the inversions in the C major triad. Instead of being a limitation, this possibility allows for various transformations between chord types to be examined, as will be seen in the rest of the paper.

As stated previously, the category $\mathbf{C}$ is often a group in musical applications. The following proposition establishes the structure of the category of functors $\mathbf{C}^{\Delta}$ when Δ is a poset with a bottom element O and $\mathbf{C}$ is a group $\mathbf{G}$ considered as a single-object category.

Proposition 1. *Let Δ be a poset with a bottom element O and $\mathbf{G}$ be a group considered as a category. Then*

1. *the category of functors $\mathbf{G}^{\Delta}$ is a groupoid, and*
2. *for any two objects F and F' of $\mathbf{G}^{\Delta}$ the hom-set $Hom(F, F')$ can be bijectively identified with the set of elements of $\mathbf{G}$.*

Proof. Given two objects F and F' of $\mathbf{G}^{\Delta}$, i.e. two functors $F\colon \Delta \to \mathbf{G}$ and $F'\colon \Delta \to \mathbf{G}$, any natural transformation $\eta\colon F \to F'$ between them is invertible, since the components of η are invertible morphisms of $\mathbf{G}$. Thus, the category of functors $\mathbf{G}^{\Delta}$ is a groupoid. Since Δ is a poset with a bottom element O, the natural transformation η is entirely determined by the component η_O, which can be freely chosen in $\mathbf{G}$. Thus the hom-set $\mathrm{Hom}(F, F')$ can be bijectively identified with the set of elements of $\mathbf{G}$. □

Since in such a case the elements of the hom-set $\mathrm{Hom}(F, F')$ can be uniquely identified with the elements of $\mathbf{G}$, we will use the notation $g^{FF'}$, with $g \in \mathbf{G}$, to designate an element of $\mathrm{Hom}(F, F')$ in $\mathbf{G}^{\Delta}$.

We now consider a functor $R\colon \Delta \to \mathbf{Sets}$ and a functor $S\colon \mathbf{G} \to \mathbf{Sets}$. For any object F of $\mathbf{G}^{\Delta}$, let $\mathrm{Nat}(R, SF)$ be the set of all natural transformations $\phi\colon R \to SF$.

Proposition 2. *There exists a canonical functor $P\colon \mathbf{G}^{\Delta} \to \mathbf{Sets}$ such that for any object F of $\mathbf{G}^{\Delta}$, $P(F) = Nat(R, SF)$.*

Proof. We consider the map $P\colon \mathbf{G}^{\Delta} \to \mathbf{Sets}$, which sends each object F of $\mathbf{G}^{\Delta}$ to the set $\mathrm{Nat}(R, SF)$. Then, given two objects F and F' of $\mathbf{G}^{\Delta}$ and any morphism $\eta\colon F \to F'$ between them, we can construct the image of η by P as the map $P(\eta)\colon \mathrm{Nat}(R, SF) \to \mathrm{Nat}(R, SF')$ sending a natural transformation ϕ of $\mathrm{Nat}(R, SF)$ to the natural transformation $S\eta \circ \phi$ of $\mathrm{Nat}(R, SF')$. It is easily verified that P is a functor. □

1.3 Transformations of Generalized Musical Classes for a PK-net

We now consider the specific case where the category $\mathbf{G}$ is the T/I group considered as a single-object category. We wish here to give examples of transformations of generalized musical classes for a PK-net, in the particular case where

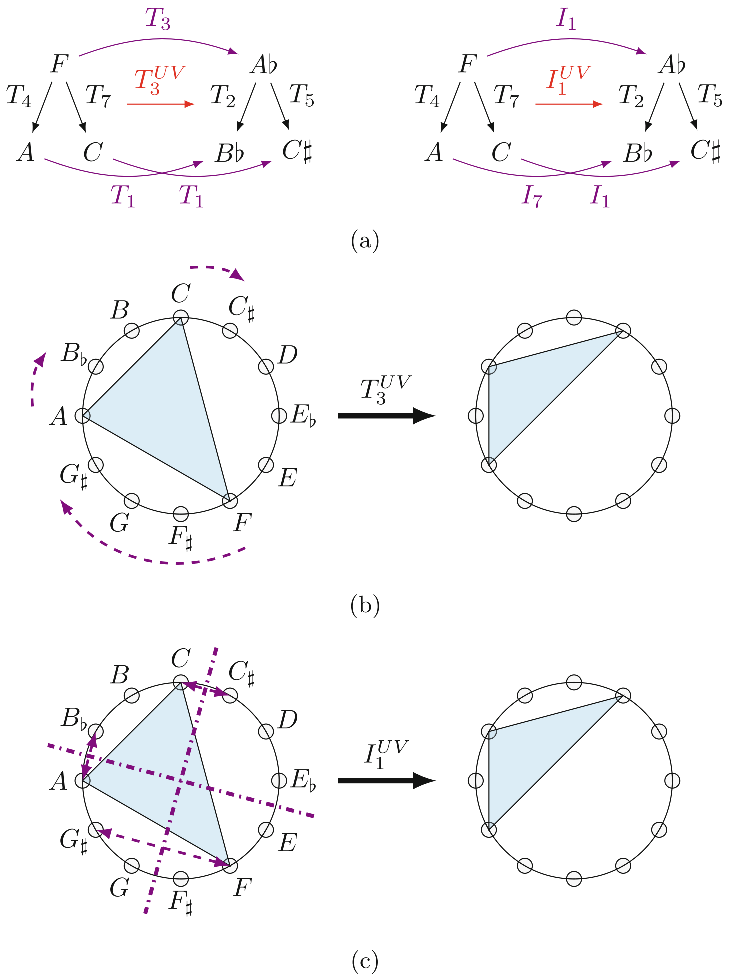


Fig. 3. (a) Action of the morphisms T_3^{UV} and I_1^{UV} of $(T/I)^{\Gamma}$ on the PK-net representing the F major chord, resulting in the $\{A\flat, B\flat, C\sharp\}$ chord. The constitutive elements of the PK-nets (the functor R, S, and the natural transformation ϕ) have been omitted here for clarity). (b) and (c) Graphical representation of the transposition and inversion components of the above morphisms, and their action on the individual pitch classes of the F major chord (dashed lines).

Δ is the category Γ introduced above, and with specific functors $U\colon \Gamma \rightarrow \mathbf{G}$ and $V\colon \Gamma \rightarrow \mathbf{G}$. Our goal is to detail the structure of the hom-set $\mathrm{Hom}(U, V)$. We consider the following objects of $(T/I)^{\Gamma}$:

- the functor $U\colon \Gamma \rightarrow T/I$ sending f to T_4 and g to T_7, and
- the functor $V\colon \Gamma \rightarrow T/I$ sending f to T_2 and g to T_5.

These functors model the set classes of prime form $[0, 4, 7]$ (major triad) and $[0, 2, 5]$. Let $\eta\colon U \to V$ be a natural transformation: it is uniquely determined by the component η_X, which is an element of T/I, from which we can derive the components η_Y and η_Z as follows.

- If $\eta_X = T_p$, with p in $\{0 \dots 11\}$, then we must have $\eta_Y T_4 = T_2 T_p$, and $\eta_Z T_7 = T_5 T_p$. This leads to $\eta_Y = T_{p+10}$, and $\eta_Z = T_{p+10}$.
- If $\eta_X = I_p$, with p in $\{0 \dots 11\}$, then we must have $\eta_Y T_4 = T_2 I_p$, and $\eta_Z T_7 = T_5 I_p$. This leads to $\eta_Y = I_{p+6}$, and $\eta_Z = I_p$.

Unlike the known action on triads of the transpositions and inversions of the T/I group, wherein the same group element operates on every pitch class of the chord, the two types of morphisms of $\mathrm{Hom}(U, V)$ have different components for each object of Γ. The morphisms T_p^{UV} can thus be considered as "generalized" transpositions, and the morphisms I_p^{UV} as "generalized" inversions between objects U and V. Figure 3 illustrates the action of these morphisms on the PK-net representing the F major chord, resulting in the $\{A\flat,\ B\flat, C\sharp\}$ chord.

1.4 An Application to Berg's Op. 5/2

To illustrate the above concepts, we will focus on a small atonal example from Berg's *Four pieces for clarinet and piano, Op. 5/2*. Figure 4 shows a reduction of the piano right hand part at bars 5–6. To analyse this progression, we consider the group $\mathbf{G} = T/I$, the category Γ described above, and the corresponding groupoid of functors $(T/I)^{\Gamma}$. In particular we consider the following objects of $(T/I)^{\Gamma}$:

- the functor $U\colon \Gamma \to T/I$ sending f to I_3 and g to I_{10},
- the functor $U'\colon \Gamma \to T/I$ sending f to I_7 and g to I_3,

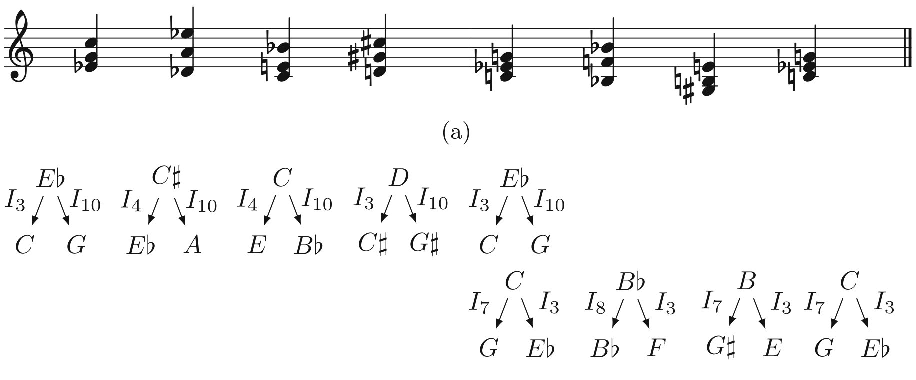


Fig. 4. (a) Berg, Op. 5/2, reduction of the piano right hand part at bars 5–6. (b) A PK-net interpretation of the first five chords (top row) and of the last four chords (bottom row) of the progression of (a).

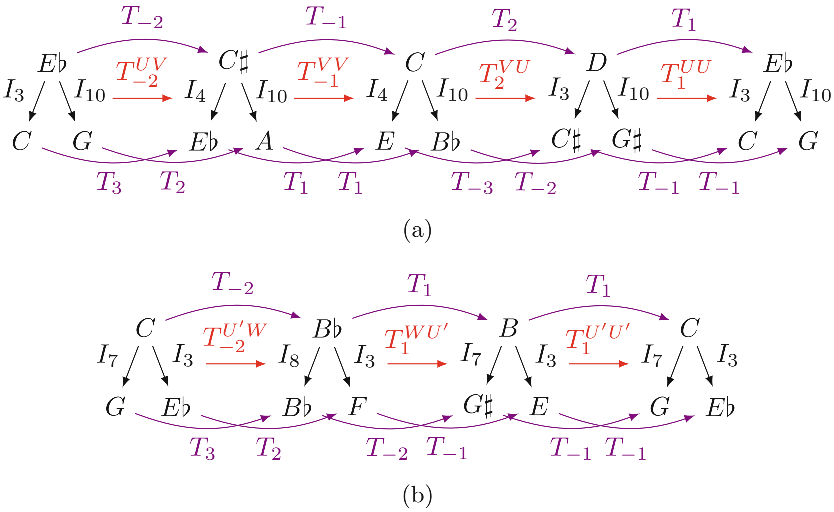


Fig. 5. Analysis of the progression of the (a) first five chords and of the (b) last four chords of Fig. 4(a) using morphisms of $(T/I)^{\Gamma}$.

– the functor $V\colon \Gamma \to T/I$ sending f to I_4 and g to I_{10}, and
– the functor $W\colon \Gamma \to T/I$ sending f to I_8 and g to I_3.

We also consider the functor $R\colon \Gamma \to$ **Sets** sending each object of Γ to a singleton, and the functor $S\colon T/I \to$ **Sets** given by the action of the T/I group on the set of the twelve pitch-classes. It can then easily be checked that the first five chords of the progression of Fig. 4a are instances of PK-nets using R and S, and whose functor from Γ to T/I is either U or V, as shown in Fig. 4b. Similarly, the last four chords of the progression of Fig. 4a are instances of PK-nets whose functor from Γ to T/I is either U' or W.

The structure of the hom-sets $\mathrm{Hom}(U,U)$, $\mathrm{Hom}(V,V)$, $\mathrm{Hom}(U,V)$, etc. can be determined as indicated previously. Regarding the first five PK-nets of Fig. 4b, it can readily be seen that this progression can be analyzed through the successive application of T_{-2}^{UV}, T_{-1}^{VV}, T_2^{VU}, and T_1^{UU}, as shown in Fig. 5a. Similarly the progression of the chords represented by the last four PK-nets of Fig. 4b can be analyzed through the successive application of $T_{-2}^{U'W}$, $T_1^{WU'}$, and $T_1^{U'U'}$, as shown in Fig. 5b. One should observe in particular that $T_2^{VU} \circ T_{-1}^{VV} = T_1^{VU}$, which has the same components as $T_1^{WU'}$, evidencing the similar logic at work behind these two progressions.

1.5 Construction of Sub-groupoids of $\mathbf{G}^{\Delta}$ and their application in music

In the examples considered in Sects. 1.3 and 1.4, for any object U of the groupoid $(T/I)^{\Gamma}$, the hom-set $\mathrm{Hom}(U,U)$ can be bijectively identified with elements of the

T/I group, and thus contains "generalized" transpositions and inversions. For transpositionally-related chords however, it may be useful to consider only a sub-category of $(T/I)^{\Gamma}$ wherein the hom-set $\mathrm{Hom}(U, U)$ only contains transposition-like morphisms. We show here how such a sub-category can be constructed by exploiting the extension structure of the T/I group.

We consider the general case where $\mathbf{G}$ is an extension $1 \to \mathbf{Z} \to \mathbf{G} \to \mathbf{H} \to 1$. This is the case for the T/I group for example, which is an extension of the form $1 \to \mathbb{Z}_{12} \to T/I \to \mathbb{Z}_2 \to 1$. Since $\mathbf{G}$ is an extension, the elements of $\mathbf{G}$ can be written as $g = (z, h)$ with $z \in \mathbf{Z}$, and $h \in \mathbf{H}$. Given a poset Δ, we define a functor $\Pi\colon \mathbf{G}^{\Delta} \to \mathbf{H}^{\Delta}$ induced by the homomorphism $\pi\colon \mathbf{G} \to \mathbf{H}$ as follows.

Definition 4. *For a given poset Δ with a bottom element, the* functor *$\Pi\colon \mathbf{G}^{\Delta} \to \mathbf{H}^{\Delta}$ induced by the homomorphism $\pi\colon \mathbf{G} \to \mathbf{H}$, is the functor which*

– *is the identity on objects, and*
– *sends any morphism $g^{FF'} = (z, h)^{FF'}$ in $\mathbf{G}^{\Delta}$ to $\pi(g)^{FF'} = h^{FF'}$ in $\mathbf{H}^{\Delta}$.*

By Proposition 1, we deduce immediately that the functor Π is full. We now consider a sub-category $\widetilde{\mathbf{H}^{\Delta}}$ of $\mathbf{H}^{\Delta}$ such that for any object U of $\widetilde{\mathbf{H}^{\Delta}}$, the group $\mathrm{End}(U)$ of endomorphisms of U is trivial, and such that the inclusion functor $\iota\colon \widetilde{\mathbf{H}^{\Delta}} \to \mathbf{H}^{\Delta}$ is the identity on objects. It is obvious to see that for any objects U and V of $\widetilde{\mathbf{H}^{\Delta}}$, the hom-set $\mathrm{Hom}(U, V)$ is reduced to a singleton which can be identified with one element of H. The choice of hom-sets $\mathrm{Hom}(U, V)$ is not unique and determines the sub-category $\widetilde{\mathbf{H}^{\Delta}}$. We now arrive to the definition of the desired category $\widetilde{\mathbf{G}^{\Delta}}$.

Definition 5. *The* category *$\widetilde{\mathbf{G}^{\Delta}}$ is defined as the pull-back of the following diagram.*

$$\begin{array}{ccc} \widetilde{\mathbf{G}^{\Delta}} & \dashrightarrow & \widetilde{\mathbf{H}^{\Delta}} \\ \downarrow & & \downarrow \iota \\ \mathbf{G}^{\Delta} & \xrightarrow{\Pi} & \mathbf{H}^{\Delta} \end{array}$$

The following propositions are immediate from the definition.

Proposition 3. *For any object U of $\widetilde{\mathbf{G}^{\Delta}}$, the endomorphism group $End(U)$ is isomorphic to $\mathbf{Z}$.*

Proposition 4. *For any objects U and V of $\widetilde{\mathbf{G}^{\Delta}}$, the hom-set $Hom(U, V)$ is in bijection with a coset of $\mathbf{Z}$ in $\mathbf{G}$.*

In the specific case where $\mathbf{G}$ is the T/I group, there exists a projection functor $\Pi\colon T/I^{\Delta} \to \mathbb{Z}_2{}^{\Delta}$ induced by the homomorphism $\pi\colon T/I \to \mathbb{Z}_2$, and one can select an appropriate subcategory $\widetilde{\mathbb{Z}_2}^{\Delta}$. The subcategory $\widetilde{T/I}^{\Delta}$ obtained by the construction described above is then such that

– for any object U of $\widetilde{T/I}^{\Delta}$, the endomorphism group $\mathrm{End}(U)$ is isomorphic to $\mathbb{Z}_{12}$ and its elements correspond to generalized transpositions as exposed in Sect. 1.3, and
– for any objects U and V of $\widetilde{T/I}^{\Delta}$, the elements of hom-set $\mathrm{Hom}(U,V)$ correspond either to generalized inversions or to generalized transpositions (but not both). Their nature depends on the choice of the subcategory $\widetilde{\mathbb{Z}_2}^{\Delta}$.

2 Groupoid Bisections and Wreath Products

Wreath products have found many applications in transformational music theory [13,14], most notably following the initial work of Hook on Uniform Triadic Transformations (UTT) [4]. In this section, we show how groupoids are related to wreath products through *groupoid bisections*, thus generalizing the work of Hook.

2.1 Bisections of a Groupoid

Let $\mathbf{C}$ be a connected groupoid with a finite number of objects. By convention, we will index the objects of $\mathbf{C}$ by $i \in \{1, \ldots, n\}$, where n is the number of objects in $\mathbf{C}$. We denote by G the group of endomorphisms of any object i of $\mathbf{C}$. We first give the definition of a *bisection of a groupoid*. This notion, which has been studied in the theory of Lie groupoids, is a particular case of the notion of a local section of a topological category as introduced by Ehresmann [1], who later studied the category of such local sections [2]. The word *bisection* is due to Mackenzie [10].

Definition 6. *A* bisection of $\mathbf{C}$ *is the data of a permutation* $\sigma \in S_n$ *and a collection of morphisms* $a_{i\sigma(i)} \colon i \to \sigma(i)$ *of* $\mathbf{C}$ *for* $i \in \{1, \ldots, n\}$. *A bisection will be notated as* $(\ldots, a_{i\sigma(i)}, \ldots)$.

Bisections can be composed according to:

$$(\ldots, b_{i\tau(i)}, \ldots) \circ (\ldots, a_{i\sigma(i)}, \ldots) = (\ldots, b_{\sigma(i)\tau\sigma(i)} a_{i\sigma(i)}, \ldots),$$

and form a group $\mathrm{Bis}(\mathbf{C})$. The main result of this section is the following theorem (which can be easily extended to all small connected groupoids), which establishes the structure of $\mathrm{Bis}(\mathbf{C})$.

Theorem 1. *The group* $Bis(\mathbf{C})$ *is isomorphic to the wreath product* $G \wr S_n$.

Proof. We construct an explicit isomorphism from $\mathrm{Bis}(\mathbf{C})$ to $G \wr S_n$. By an abuse of notation, we will denote by g_{ii} both an endomorphism of an object i of $\mathbf{C}$ and the corresponding element of G.

Let k be an object of $\mathbf{C}$, and let $\{c_{ki}, i \in \{1,\ldots,n\}\}$ be the set obtained by choosing a morphism c_{ki} of $\mathbf{C}$ for every object i of $\mathbf{C}$. This defines a collection of morphisms $\{c_{ij} = c_{kj}c_{ki}^{-1}, i \in \{1,\ldots,n\}, j \in \{1,\ldots,n\}\}$ such that for any objects p, q, and r of $\mathbf{C}$, we have $c_{qr}c_{pq} = c_{pr}$. The morphisms c_{ij} induce automorphisms ϕ_{ij} of G given by $\phi_{ij}(g_{ii}) = c_{ij}g_{ii}c_{ij}^{-1}$, with the added property that for any objects p, q, and r, we have $\phi_{qr} \circ \phi_{pq} = \phi_{pr}$.

Since $\mathbf{C}$ is a groupoid, any bisection $(\ldots, a_{i\sigma(i)}, \ldots)$ of $\mathrm{Bis}(\mathbf{C})$ can then be uniquely written as $(\ldots, c_{i\sigma(i)}g_{ii}, \ldots)$. Let $\chi\colon \mathrm{Bis}(\mathbf{C}) \to G \wr S_n$ be the bijective map which sends an element $(\ldots, a_{i\sigma(i)}, \ldots)$ of $\mathrm{Bis}(\mathbf{C})$ to the element $\langle(\ldots, \phi_{i1}(g_{ii}), \ldots), \sigma\rangle$ of $G \wr S_n$. It can be easily shown that the map χ is indeed an isomorphism (the technical proof is left to the reader). □

2.2 Application to Musical Transformations

The following proposition shows how can one pass from a groupoid action on sets to a corresponding group action.

Proposition 5. *Let* $\mathbf{C}$ *be a connected groupoid with a finite number of objects, with* G *the group of endomorphisms of any object, and let* S *be a functor from* $\mathbf{C}$ *to* **Sets**. *There is a canonical group action of* $G \wr S_n$ *on the disjoint union of the image sets* $S(i)$.

Proof. Let $\bigsqcup S(i) = \bigcup\{(x,i), x \in S(i), i \in \{1,\ldots,n\}\}$ be the disjoint union of the image sets $S(i)$ and let $(\ldots, a_{i\sigma(i)}, \ldots)$ be a bisection of $\mathbf{C}$. The group action of $G \wr S_n$ on $\bigsqcup S(i)$ is directly given by the action defined as

$$(\ldots, a_{i\sigma(i)}, \ldots) \cdot (x,i) = (S(a_{i\sigma(i)})(x), \sigma(i)).$$

□

As a direct application to musical transformations, consider a subgroupoid $\widetilde{T/I^{\Delta}}$ as constructed in Sect. 1.5. Given a functor $R\colon \Delta \to \mathbf{Sets}$, we know from Proposition 2 that there exists a canonical functor $P\colon T/I^{\Delta} \to \mathbf{Sets}$, which extends to a functor $\widetilde{P}\colon \widetilde{T/I^{\Delta}} \to \mathbf{Sets}$. From Proposition 5, we thus deduce that there exists a group action of $\mathbb{Z}_{12} \wr S_n$ on the disjoint union of the image sets $\widetilde{P}(i)$, or in other words the set of all PK-nets (R, S, i, ϕ). In the case R is representable and $n = 2$, it is easy to see that one recovers a wreath product acting on two different types of chords analog to Hook's UTT group.

3 Conclusions

Building on the categorical framework of poly-Klumpenhouwer networks, this paper has introduced the notion of generalized musical class for a PK-net of diagram Δ as a functor $F\colon \Delta \to \mathbf{C}$, where Δ is a small category and $\mathbf{C}$ is a musically relevant category. In the case $\mathbf{C}$ is a group and Δ is a poset with a bottom element, it has been shown that generalized musical transformations emerge as morphisms of the groupoid of functors $\mathbf{C}^{\Delta}$. Through the example of Berg's *Four pieces for clarinet and piano, Op. 5/2*, an application to post-tonal music has been presented, extending the range of possibilities for analysis beyond transpositionally related pitch-class sets. In addition, the notion of groupoid bisection bridges this categorical approach with group-theoretical wreath products which have been introduced by previous authors. This categorical framework may readily be implemented in high-level programming languages [15] and would thus provide an opportunity for computer-aided music analysis and composition.

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