Constraint-Based Systems of Triads and Seventh Chords, and Parsimonious Voice-Leading

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Abstract. This paper presents a generalization of the neo-Riemannian PLR group to the set of triads with inversions (major, minor, diminished and augmented). A second generalization is proposed, using an extended system of seventh chords with inversions. Both the sets of triads and seventh chords are defined with constraints on semitone separation of voices. In the case of triads, the set of parsimonious transformations is shown to have the structure of a semi-direct product of groups of the form $S_n \ltimes \mathbb{Z}_{12}^{n-1}$, where n is the number of chord types in the set.

Keywords: Constraint-based · Seventh · Triad · Parsimonious · Neo-Riemannian

1 Introduction

1.1 Constraint-Based Systems

In this paper we consider voicings of triads and seventh chords from the viewpoint of semitone separation constraints.

Constraint-based definitions of chords are meant to yield a collection of chords which are close together in the sense of parsimonious voice-leading. In particular, such chords should have spacing between voices which are somewhat similar. We choose to specify such spacing by focusing on the total spread between the highest and lowest pitches, and also the vector of spreads between adjacent pitches. This method also seems to capture some of the well-known and useful chord collections in the case of triads and seventh chords.

The cost of this approach is that we consider systems of chord-types in an absolute sense, not as pitch-class sets. Each chord type is a root position or chord inversion which can be described by the semitone separation type. The benefit of this approach is that we can include chords which have different numbers of inversion types into one system for the purpose of parsimonious voice-leading.

Supported by DigiPen Institute of Technology.

M. Montiel et al. (Eds.): MCM 2019, LNAI 11502, pp. 185–198, 2019.
https://doi.org/10.1007/978-3-030-21392-3_15

For example, the augmented triad has only one separation type, unlike the other triads. Similarly, the dominant seventh chord with flat fifth has only two separation types, unlike the other seventh chords.

The two systems of constraints can be described simply as follows. For the system of major, minor, diminished, and augmented triads, we constrain the separation between pairs of consecutive notes to be from 3 to 6 semitones, and the separation between the upper and lower notes to be from 6 to 9 semitones. All of these triads and their chord inversions are recovered precisely in this way. A similar system of sevenths chords can be defined with separation between pairs of consecutive notes to be from 1 to 4 semitones, and the separation between the upper and lower notes to be from 8 to 11 semitones. All of the standard seventh chords and their chord inversions are recovered in this way, as well as two additional chords obtained from the dominant seventh by lowering or raising the fifth by one semitone.

In order to explore the transformations between these chord types, we consider chords to be ordered tuples of integers such as (a, b, c) or (a, b, c, d), with $a < b < c < d$. This distinguishes a root position chord from its inversions, as separate chord types. In this context, the concept of pitch class set can still be invoked on these chord collections as an equivalence relation. It is also convenient to work with equivalence classes of chord types modulo twelve, but still preserving the types. For example, the set of triad types modulo 12, with C as 0, includes the B major triad in root position, represented as $(11, 15, 18)$ or as $(-1, 3, 6)$, but not as $(3, 6, 11)$. The latter is of course the first inversion, which in this context is not equivalent as a chord type to the previous two.

What is gained from this point of view is a simple approach to parsimonious voice-leading, and the induced groups of transformations. We recover the PLR group for triads as a subgroup of the larger parsimony group, which we show to be isomorphic to $S_{10} \ltimes \mathbb{Z}_{12}^{9}$.

These methods can be put into a wider context, where we start with a system of chord types and consider the group generated by basic parsimonious transformations which swap chord types by changing only one voice by one semitone. If there are n chord types which are defined with a system of constraints similar to the two cases we describe, then it is interesting to investigate whether it is possible to describe the parsimony group as $S_n \ltimes \mathbb{Z}_{12}^{n-1}$.

1.2 Background

In his foundational work, David Lewin [7] explored music theory and composition from the perspective of transformational theory. In this context, algebraic structures, such as groups, play an important role in defining and elucidating musical content. In this branch of transformational music theory, known as *neo-Riemannian theory*, voice-leading between chords plays an important role. The canonical example of this is the PLR group, originally introduced by the 19th-century music theorist Hugo Riemann [8]. The transformations P, L, and R each represent the chord change between a major and minor triad by moving one voice of each chord by one or two semitones. Moreover, these transformations relate the important pairs of such triad relationships such as *Parallel*, *Relative*, or *Leading-Tone Exchange*. These types of voice-leading involving small steps between voices are often called *parsimonious*.

Transformations between voicings of triads are also considered in [4], where the authors consider extensions of P, L and R to linear functions defined on all of $\mathbb{Z}_{12}^3$.

In addition to the algebraic action of the PLR group on the set of major and minor triads, a geometric model called the Tonnetz is central to the study of neo-Riemannian transformations. For a full description of the Tonnetz and operations in the PLR group as Dihedral group, we refer the reader to [1] and [3].

In Sect. 2 we recall some facts about the PLR group, in particular its structure as a semi-direct product of groups. In Sect. 3 we identify the structure of the parsimony group G for the constraint-based system of triads with inversions. In Sect. 4 we describe the constraint-based system of seventh chords, and in Sect. 5 we propose some future work.

2 The *PLR* Group as a Semi-direct Product of Groups

The well-known PLR group is a group of transformations on the set of 24 consonant (major and minor) triads. Here we consider triads as pitch-class sets, each consisting of three elements: root, third, and fifth. We will label the sets M and m of major and minor triads as:

$$M = \{M_0, M_1, \ldots, M_{11}\}, \quad m = \{m_0, m_1, \ldots, m_{11}\}$$

where $M_0 = C$ major, $M_1 = C\sharp$ major ... $M_{11} = B$ major, and $m_0 = C$ minor, $m_1 = C\sharp$ minor ... and $m_{11} = B$ minor.

Next recall the three neo-Riemannian transformations:

- P (parallel) swaps major and minor triads by lowering the third (of major triads) or raising the third (of minor triads) by one semitone
- L (leading tone) swaps major and minor triads by moving the root (of major triads) down a semitone, or the fifth (of minor triads) up a semitone
- R (relative) swaps major and minor triads by moving the fifth (of major triads) up a whole tone, or moving the root (of minor triads) down a whole tone

The set of all transformations on the set of major and minor triads which are generated from these is called the PLR group, which we label here as:

$$G_{PLR} = \langle P, L, R \rangle.$$

We can also represent these transformations with indices as follows:

P: $M_i \mapsto m_i$, $m_i \mapsto M_i$ L: $M_i \mapsto m_{i+4}$, $m_i \mapsto M_{i-4}$ R: $M_i \mapsto m_{i+9}$, $m_i \mapsto M_{i-9}$

These transformations can be described by an ordered pair $(s, \mathbf{t})$, where s takes the value σ if the transformation swaps M and m, and 1 (the identity permutation) if the transformation does not swap M and m. The value $\mathbf{t}$ is an integer vector $(t, -t)$ which indicates the shift (or translation) t (modulo twelve) on the index i of a major triad, and the shift $-t$ (modulo 12) on the index j of a minor triad.

For instance, we describe the three transformations as:

$$P : (\sigma, (0, 0)), \quad L : (\sigma, (4, -4)), \quad R : (\sigma, (9, -9))$$

Note: We differ slightly from Hook's notation in [5] where the symbols $+$ and $-$ are used instead of 1 and σ to describe the mode of his Uniform Triadic Transformations, or UTT's.

If $S_2 = \{1, \sigma\}$ is the symmetric group consisting of permutations of the two symbols M and m, and $\mathbb{Z}_{12}$ is the group of integers modulo 12, then we can represent any element of G_{PLR} as an ordered pair $(s, \mathbf{t})$ in the set product:

$$S_2 \times \mathbb{Z}_{12} \times \mathbb{Z}_{12}.$$

Finally, we quote here the well-known structure theorem for G_{PLR} (see [5] for a proof).

Theorem 1. *The neo-Riemannian group G_{PLR} is isomorphic to a semi-direct product $S_2 \ltimes \mathbb{Z}_{12}$.*

Note: Since the semi-direct product $S_2 \ltimes \mathbb{Z}_{12}$ is isomorphic to the dihedral group of order 24 (see for instance [3]), we also get the standard representation of the PLR group as a dihedral group of order 24.

3 Constraint-Based System of Triads

Triads can be obtained in root position by stacking major or minor thirds. This produces the four triad types: major, minor, diminished and augmented. In music theory it is often preferred to think in terms of pitch class sets, so the chord inversions of these four triads are taken to be equivalent to their root position versions. In this paper, we consider each inversion as a separate entity. In order to distinguish them, we identify each chord by its "successive-interval array". This notion is first defined by Chrisman in [2], in a more general context. For our purpose, we include only the semitone gaps between successive notes, leaving out Chrisman's inclusion of the semitone gap between the highest pitch and one octave above the lowest pitch. In this paper we refer to this simplified array of semitone gaps as *si-type*, for "successive interval type".

We will refer to chords (in equal temperament) by integer tuples which indicate pitches relative to some fixed starting value. For example, if middle C is represented as 0, then a piano with 88 keys is represented by the values -39 (A_0) to 48 (C_8). The triple $(0, 4, 7)$ then represents a C Major triad in root position, with root middle C. We will always assume that such a triple (a, b, c) of integers satisfies $a < b < c$, or equivalently that the pitch values are increasing from left to right.

The si-type $[x, y]$ describes a triad (a, b, c) where $x = b - a$ and $y = c - b$. Thus the si-type of the C Major triad above, or any other Major triad in root position, is $[4, 3]$. The first inversion of this triad then is represented by $(4, 7, 12)$ and has si-type $[3, 5]$. The four types of triad, together with their chord inversions, yield 10 different si-types, which are listed in the following Table 1:

Table 1. si-types of triads

chord name (and symbol)	Root	1^{st} Inv	2^{nd} Inv
Major triad (M)	[4, 3]	[3, 5]	[5, 4]
minor triad (m)	[3, 4]	[4, 5]	[5, 3]
diminished triad (o)	[3, 3]	[3, 6]	[6, 3]
augmented triad ($+$)	[4, 4]	[4, 4]	[4, 4]

Constraint-based definition of triad (based on si-type): We define a triad (a, b, c), given with integers $a < b < c$, to be one with si-type $[x, y] = [b - a, c - b]$ satisfying the following constraints:

$$3 \leq x, y \leq 6 \quad \text{and} \quad 6 \leq x + y \leq 9.$$

It is easy to check that the above 10 si-types in the table are the only ones which satisfy these constraints.

Now consider parsimonious voice-leading transformations from one triad to another which are of the simplest type: changing one of a, b, or c by only one semitone. (Note: we refer to the three voices of the chord based on their position, not their function as root, third or fifth.)

The following table lists all such transformations which yield another chord in this collection. Here we indicate the transformation with the notation a_+ to mean that the note value a is replaced with $a + 1$:

$$a_+ \ : \ (a, b, c) \rightarrow (a + 1, b, c)$$

and a_- to mean that a is replaced with $a - 1$:

$$a_- \ : \ (a, b, c) \rightarrow (a - 1, b, c),$$

and so on for b and c (Table 2).

Triads modulo 12

We define the set T of triads (a, b, c) mod 12, according to their si-type and lowest pitch value a. We take the value $a = 0$ to be the pitch C, etc. Since there are 10 si-types and 12 possible values for a, we have 120 elements in T. It is important to note that these elements are not pitch class sets, since we are still distinguishing between inversions as separate triads. We will use the common notations but add superscripts and subscripts to indicate chord types

Table 2. Parsimonious Transformations on si-types

symbol	si-type	a_-	a_+	b_-	b_+	c_-	c_+
M	[4, 3]	[5, 3]	[3, 3]	[3, 4]			[4, 4]
M_1	[3, 5]	[4, 5]			[4, 4]	[3, 4]	[3, 6]
M_2	[5, 4]		[4, 4]	[4, 5]	[6, 3]	[5, 3]	
m	[3, 4]	[4, 4]			[4, 3]	[3, 3]	[3, 5]
m_1	[4, 5]		[3, 5]	[3, 6]	[5, 4]	[4, 4]	
m_2	[5, 3]	[6, 3]	[4, 3]	[4, 4]			[5, 4]
o	[3, 3]	[4, 3]					[3, 4]
o_1	[3, 6]				[4, 5]	[3, 5]	
o_2	[6, 3]		[5, 3]	[5, 4]			
$+$	[4, 4]	[5, 4]	[3, 4]	[3, 5]	[5, 3]	[4, 3]	[4, 5]

and inversions. (Since lower case letters are already being used for voices, we write Cm for a C minor triad instead of c.) For example, one could list:

$$C = (0,4,7), A\flat_1 = (0,3,8), F_2 = (0,5,9), Cm = (0,3,7), Am_1 = (0,4,9),$$

$$Fm_2 = (0,5,8), C^o = (0,3,6), A_1^o = (0,3,9), F\sharp_2^o = (0,6,9), C^+ = (0,4,8)$$

for chords occuring in the ten different si-types having starting pitch C, or $a = 0$.

Each of the si-types determines a subset of T, so we use the symbol or the si-type for subsets as well. For example:

$$M = [4,3] = \{C = (0,4,7), C\# = (1,5,8), D = (2,6,9), \ldots, B = (11,15,18)\}$$

is the subset of 12 major triads, and

$$o_1 = [3,6] = \{A_1^o = (0,3,9), B\flat_1^o = (1,4,10), B_1^o = (2,5,11), \ldots, A\flat_1^o = (11,14,20)\}$$

is the subset of 12 first inversion diminished triads, etc. (Note that we maintain the order $a < b < c$, and that we consider (a,b,c) as representative of an equivalence class modulo 12. So we could use $A\flat_1^o = (-1,2,8)$ but not $(11,2,8)$ since it is an ordered triple but not a pitch-class set.)

Next, we define transformations between si-types, in particular the swaps of order two between si-types which are induced by raising or lowering one voice a, b, or c, by one semitone. As permutations on the set T, these are *involutions* since they swap all triads of one si-type with triads of another si-type, and performing this swap twice results in the identity permutation. (Note the usual P and L transformations are now factored into three swaps on si-types.) We indicate each transformation with its corresponding pair of adjustments to one voice (Table 3).

Table 3. Parsimonious transformation labels

$P_0: M \longleftrightarrow m, \quad b_-, b_+$	$P_1: M_1 \longleftrightarrow m_1, \quad a_-, a_+$	$P_2: M_2 \longleftrightarrow m_2, \quad c_-, c_+$
$L_0: M \longleftrightarrow m_2, \quad a_-, a_+$	$L_1: M_1 \longleftrightarrow m, \quad c_-, c_+$	$L_2: M_2 \longleftrightarrow m_1, \quad b_-, b_+$
$f_0: M \longleftrightarrow o, \quad a_+, a_-$	$f_1: M_1 \longleftrightarrow o_1, \quad c_+, c_-$	$f_2: M_2 \longleftrightarrow o_2, \quad b_+, b_-$
$g_0: m \longleftrightarrow o, \quad c_-, c_+$	$g_1: m_1 \longleftrightarrow o_1, \quad b_-, b_+$	$g_2: m_2 \longleftrightarrow o_2, \quad a_-, a_+$
$\alpha_0: M \longleftrightarrow +, \quad c_+, c_-$	$\alpha_1: M_1 \longleftrightarrow +, \quad b_+, b_-$	$\alpha_2: M_2 \longleftrightarrow +, \quad a_+, a_-$
$\beta_0: m \longleftrightarrow +, \quad a_-, a_+$	$\beta_1: m_1 \longleftrightarrow +, \quad c_-, c_+$	$\beta_2: m_2 \longleftrightarrow +, \quad b_-, b_+$

These transformations are pictured (as swaps of chord types) in the following diagram:

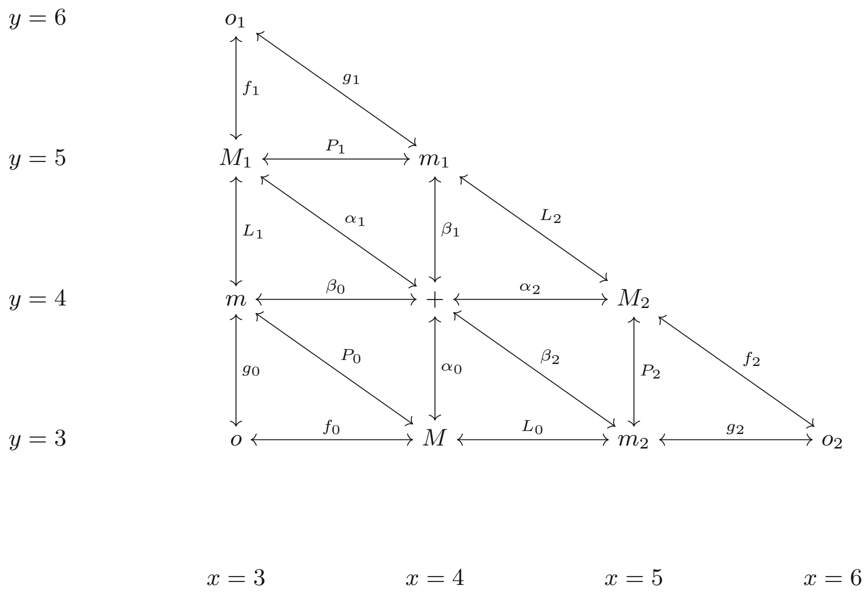
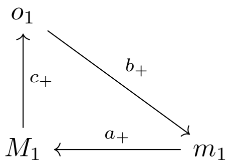


Fig. 1. Parsimonious transformation diagram

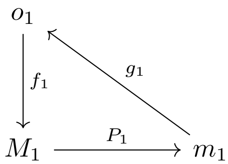
Note that in any small triangle of the following shape, if we start with any chord, then follow the arrows *clockwise*, this results in an increase by one semitone to each of the voices, resulting in a transformation which affects this one chord type as:

$$(a, b, c) \longrightarrow (a+1, b+1, c+1).$$



Similarly, if we follow the arrows *counterclockwise* this results in:

$$(a, b, c) \longrightarrow (a - 1, b - 1, c - 1).$$



A few more observations on this diagram are in order before we state the theorem. First, the only transformations which change the lower voice (a) of a triad are those which appear as horizontal arrows, such as f_0, L_0, etc. The remaining transpositions leave a fixed and hence can be interpreted as *pure swaps* of subsets. By this we mean that if the sets M and m are swapped by P_0 and we index the entries of each of these sets then the major and minor triads (in root position) are swapped at the same index value, with no translation inside the two sets. Another type of transformation is the *pure translation* on subsets, such as $P_0 f_0 g_0 f_0$, which can be seen to shift m up by one semitone, and shift M down by one semitone. If we specify an order to the si-types, or subsets, as:

$$(M, M_1, M_2, m, m_1, m_2, o, o_1, o_2, +)$$

then we can indicate these pure translations with the vector notation. For instance, $P_0 f_0 g_0 f_0$ would be represented as $(-1, 0, 0, 1, 0, 0, 0, 0, 0, 0)$.

With the above notation, it is straight-forward to generate the following such pure translations:

$$(-1, 1, 0, 0, 0, 0, 0, 0, 0, 0), (-1, 0, 1, 0, 0, 0, 0, 0, 0, 0), (-1, 0, 0, 1, 0, 0, 0, 0, 0, 0),$$

$$(-1, 0, 0, 0, 1, 0, 0, 0, 0, 0), (-1, 0, 0, 0, 0, 1, 0, 0, 0, 0), (-1, 0, 0, 0, 0, 0, 1, 0, 0, 0),$$

$$(-1, 0, 0, 0, 0, 0, 0, 1, 0, 0), (-1, 0, 0, 0, 0, 0, 0, 0, 1, 0), (-1, 0, 0, 0, 0, 0, 0, 0, 0, 1)$$

This can be achieved by conjugation, or simply preceeding and following $P_0 f_0 g_0 f_0$ by a series of transpositions which are in reversed orders. For example, we obtain

$$(-1,1,0,0,0,0,0,0,0,0) = L_1(P_0 f_0 g_0 f_0)L_1$$

and

$$(-1,0,1,0,0,0,0,0,0,0) = L_2 g_1 f_1 L_1(P_0 f_0 g_0 f_0)L_1 f_1 g_1 L_2.$$

Next, we define the *parsimony group* G to be the group of transformations generated by all of the above defined transformations, acting as involutions on the set T of 120 triads modulo 12.

$$G = \langle P_0, P_1, P_2, L_0, L_1, L_2, f_0, f_1, f_2, g_0, g_1, g_2, \alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2 \rangle.$$

We will show that G is isomorphic to a semi-direct product of two groups:

$$G \cong S_{10} \ltimes \mathbb{Z}_{12}^9.$$

The first of these is the full permutation group S_{10} on 10 symbols, in this case on the chord symbols or si-types. The second factor in the semidirect product is $\mathbb{Z}_{12}^9$, obtained from the action of the group $\mathbb{Z}_{12}$ on subsets of chords in one si-type. This factor can be seen as the subgroup Z of $\mathbb{Z}_{12}^{10}$ consisting of all vectors $\mathbf{t} = (t_1, \ldots, t_{10})$ satisfying $\sum_{i=1}^{10} t_i = 0$.

We need to specify the group operation in G which is given by the homomorphism

$$\phi : S_{10} \longrightarrow Aut(Z)$$

with $\phi(s) = s(\mathbf{t})$, or in other words, the action of $\phi(s)$ on a vector $\mathbf{t}$ in Z is to simply permute the vector components of $\mathbf{t}$. We can represent any element of G as a pair $(s, \mathbf{t})$ and then the product is given as:

$$(s, \mathbf{t}) \cdot (s', \mathbf{t}') = (ss', \mathbf{t} + s(\mathbf{t}')),$$

where

$$s(\mathbf{t}') = (t'_{s(1)}, t'_{s(2)}, \ldots, t'_{s(10)}).$$

To see this, let the vector of si-types be relabeled with superscripts so that

$$(M, M_1, M_2, m, m_1, m_2, o, o_1, o_2, +) = (T^1, T^2, \ldots, T^{10}).$$

Each of the twelve chords inside each si-type can then be indicated with subscripts, so that C major root position is now T_0^1, and B augmented, or B^+, is now T_{11}^{10}. With this notation we can see that

$$(s, \mathbf{t})(T_j^i) = T_{j+t_i}^{s(i)}.$$

This product then satisfies the properties:

Lemma 1. *For any elements* $(s, \mathbf{t})$ *and* $(s', \mathbf{t}')$ *in the parsimony group* G*, we have:*

- $(s, \mathbf{t}) \cdot (s', \mathbf{t}') = (ss', \mathbf{t} + s(\mathbf{t}'))$
- $(s, \mathbf{t})^{-1} = (s^{-1}, s^{-1}(-\mathbf{t}))$
- $(s, \mathbf{t})(1, \mathbf{t}')(s, \mathbf{t})^{-1} = (1, s(\mathbf{t}'))$

Proof. The product (with first factor acting first) is verified by:

$$((s, \mathbf{t}) \cdot (s', \mathbf{t}'))(T^i_j) = (s', \mathbf{t}')(T^{s(i)}_{j+t_i}) = T^{s'(s(i))}_{j+t_i+t'_{s(i)}} = T^{(ss')(i)}_{j+t_i+s(t'_i)},$$

and the form of the inverse follows directly. The last property is verified as:

$$(s, \mathbf{t})(1, \mathbf{t}')(s, \mathbf{t})^{-1} = (s, \mathbf{t})(1, \mathbf{t}')(s^{-1}, s^{-1}(-\mathbf{t})) = (s, \mathbf{t})(s^{-1}, \mathbf{t}' + s^{-1}(-\mathbf{t}))$$
$$= (1, \mathbf{t} + s(\mathbf{t}' + s^{-1}(-\mathbf{t}))) = (1, s(\mathbf{t}')).$$

□

From the last property of the Lemma we obtain:

Corollary 1. *The group Z of pure translations is a normal subgroup of G.*

Next, we recall that a group G is a semi-direct product, written $K \ltimes_\phi H$ if the following hold:

- K and H are subgroups of G
- H is normal in G
- $KH = G$
- $\phi : K \longrightarrow Aut(H)$ is a homomorphism, $k \mapsto \phi_k$
- The product in G is $(k, h)(k', h') = (kk', h\phi_k(h'))$

Theorem 2. *The parsimony group G defined above is isomorphic to $S_{10} \ltimes \mathbb{Z}^9_{12}$.*

Proof. The proof follows the outline of the proof of Theorem 1 in [1]. There are two steps: (1) We show that the permutation part of this group contains all transpositions on the sets of triad types, and (2) We show that the vectors of integers modulo 12 contain all elements of the type

$$(t_1, t_2, \ldots, t_{10})$$

satisfying $\sum_{i=1}^{10} t_i = 0$, which shows that the subgroup of pure translations Z is isomorphic to $\mathbb{Z}^9_{12}$. The first part follows from Fig. 1 where we can identify a sequence of transpositions which generate all of S_{10}. In fact, we can generate all transpositions, or swaps of two si-types, where we avoid any translations. This is done simply by following arrows in the diagram from one si-type to another but avoiding the horizontal arrows. For example, the swap between type o (diminished triad in root position) and type m_2 (minor triad second inversion) can be obtained as:

$$g_0 L_1 \alpha_1 \beta_2 \alpha_1 L_1 g_0.$$

Since the transpositions generate the full symmetric group, the first part is done. It is evident that the generators of G satisfy the property that the translation vector has sum of its components equal to zero. The second part follows by noting that we can express any element of the specified type as a sum of the elements generated above, in particular:

$$(t_1, t_2, \ldots, t_{10}) = t_2(-1, 1, 0, 0, 0, 0, 0, 0, 0, 0) + \cdots + t_{10}(-1, 0, 0, 0, 0, 0, 0, 0, 0, 1)$$

where the first coordinate is automatically correct since $t_1 = -(t_2 + \cdots + t_{10})$. □

We can immediately identify the PLR group as a subgroup of the parsimony group G. In particular, since each operation now factors as a product of three transpositions, we have:

$$G_{PLR} \cong \langle P_0P_1P_2, L_0L_1L_2, R_0R_1R_2 \rangle.$$

4 Constraint-Based System of Seventh Chords

Extensions of the PLR group to seventh chords are explored in several recent papers. In [6] Kerkez defines a PS-group, isomorphic to G_{PLR}, which acts on the major and minor seventh chords. In [1] Cannas, Antonini, and Pernazza, define a group called $PLRQ$ which generalizes the PLR group to five types of seventh chords: dominant, minor, half-diminished, major, and diminished, and they show that this group is isomorphic to the semi-direct product $S_5 \ltimes \mathbb{Z}_{12}^4$. Continuing in this vein, we now extend these results to a larger constraint-based system of seventh chords.

A typical definition of seventh chord might be: *"A four-note chord obtained by stacking thirds based on a major or minor scale."* One should also add that such a chord can be inverted in the usual way, giving three other equivalent four-note chords. If we are interested primarily in pitch classes, then of course these all represent the same pitch-class set. In this paper we consider these as individual chords in their own right, and note that their structure gives way to a very simple constraint-based description of seventh chords.

If seventh chords are assumed to come from the major or minor scale construction alluded to above, then we have the following seven types, which we call the classical types of seventh chords.

Dominant seventh (7), minor seventh (m7), half-diminished seventh (∅7), Major seventh (M7), minor-Major seventh (mM7), augmented Major seventh (+M7), diminished seventh (o7).

As we did for triads, recalling Chrisman's "successive-interval array" in [2], we introduce the "successive-interval type", or si-type: $[x, y, z]$ for a seventh chord. In equal temperament, we can represent a four-note chord by an integer vector of values (a, b, c, d). Here we will assume that the note values are listed in increasing order $a < b < c < d$.

We define the successive-interval type, or si-type: $[x, y, z]$ for a seventh chord (a, b, c, d) to be:

$$[x, y, z] = [b - a, c - b, d - c],$$

or simply the numbers of semitones separating the notes of the chord, from left to right.

For example, if we use 0 to represent middle C, then the chord $(0, 4, 7, 10)$ would be C Dominant seventh chord in root position. The si-type for this chord is then $[4, 3, 3]$. The first inversion of this chord is $(4, 7, 10, 12)$, with si-type $[3, 3, 2]$. Note: The si-type describes a chord inversion, but is not an invariant of the pitch class set.

It is easy to see that there are 25 si-types associated to these classical seventh chords with all of their inversions. The only chord whose inversions do not generate new si-types is the full diminished chord. It is also easy to check that the semitone separation variables x, y, z exhibited in these classical seventh chord types always assume values $1, 2, 3$ or 4, and that any such chord (a, b, c, d) with one of these types must have total spread $x + y + z$ to be at least 8 semitones (a minor sixth) and at most 11 semitones (a Major seventh). Next, we take the above description of si-types and turn it into a definition of seventh chord:

Constraint-based definition of seventh chord (based on si-type): We define a seventh chord (a, b, c, d), given with integers $a < b < c < d$, to be one with si-type $[x, y, z] = [b - a, c - b, d - c]$ satisfying the following constraints:

$$1 \le x, y, z \le 4 \quad and \quad 8 \le x + y + z \le 11.$$

Practiclly speaking, we are defining a seventh chord to be one which can be played on the piano by simply choosing four notes in such a way that: (1) any two adjacent notes are separated by a major third, a minor third, a whole step, or a half step, and (2) the spread from the first to the last notes is at least a minor sixth, and at most a major seventh.

An obvious question to ask is whether the above constraints are a description of precisely the above collection of 25 si-types, or have we introduced something new? The answer is that indeed there are precisely two new chords in this family: the flat 5 seventh (7♭5) and the sharp 5 seventh (7♯5).

The si-types of the classical seventh chords as well as these two additional chords are listed in the following Table 4:

Table 4. si-types of constraint-based system of seventh chords

chord name (and symbol)	Root	1^{st} Inv	2^{nd} Inv	3^{rd} Inv
Dominant seventh (7)	[4, 3, 3]	[3, 3, 2]	[3, 2, 4]	[2, 4, 3]
minor seventh (m7)	[3, 4, 3]	[4, 3, 2]	[3, 2, 3]	[2, 3, 4]
half-diminished seventh (⌀7)	[3, 3, 4]	[3, 4, 2]	[4, 2, 3]	[2, 3, 3]
Major seventh (M7)	[4, 3, 4]	[3, 4, 1]	[4, 1, 4]	[1, 4, 3]
minor-Major seventh (mM7)	[3, 4, 4]	[4, 4, 1]	[4, 1, 3]	[1, 3, 4]
augmented Major seventh (+M7)	[4, 4, 3]	[4, 3, 1]	[3, 1, 4]	[1, 4, 4]
diminished seventh (o7)	[3, 3, 3]	[3, 3, 3]	[3, 3, 3]	[3, 3, 3]
flat 5 seventh (7♭5)	[4, 2, 4]	[2, 4, 2]	[4, 2, 4]	[2, 4, 2]
sharp 5 seventh (7♯5)	[4, 4, 2]	[4, 2, 2]	[2, 2, 4]	[2, 4, 4]

Since the 7♭5 chord only generates two si-types, while the 7♯5 generates four types, we have a total of 31 si-types for this constraint-based system of seventh chords. We label the set of these 31 si-types S_7:

$$S_7 = \{[x, y, z] : 1 \le x, y, z \le 4, 8 \le x + y + z \le 11\}.$$

Let's call the set of integer vectors (a, b, c, d) representing a seventh chord as above C_7, which is a subset of $\mathbb{Z}^4$:

$$C_7 = \{(a, b, c, d) : a < b < c < d, x = b - a, y = c - b, z = d - c, 1 \leq x, y, z \leq 4, \\ 8 \leq x + y + z \leq 11\}.$$

Finally, we consider C_7 modulo translation by the group $\mathbb{Z}_{12}$. By this we mean that two chords (a, b, c, d) and (a', b', c', d') are considered equivalent if they have the same si-type $[x, y, z]$ and $a \equiv a' \pmod{12}$. We can represent each of these equivalence classes by a chord (a, b, c, d) with $0 \leq a \leq 11$. Denote the equivalence class of a chord (a, b, c, d) by simply $(a, b, c, d)_{12}$. Then we define:

$$X_7 = \{(a, b, c, d)_{12} : (a, b, c, d) \in C_7\}.$$

The size of X_7 is $31 \cdot 12 = 372$.

We define the parsimony group G_7 for this set X_7 of seventh chords to be the group generated by all parsimonious transformations which raise or lower one of the four voices, a, b, c or d, of the seventh chord by one semitone, but only allowing such transformations in the case where the resulting chord is in the same set X_7. Just as with the parsimony group G for triads, each such transformation can be seen as a swap of two s-types, with a possible shift modulo 12.

5 Future Work

We propose to investigate the following question in a continuation of this work:

Is the parsimony group G_7 defined above isomorphic to $S_{31} \ltimes \mathbb{Z}_{12}^{30}$?

We can define an infinite graph on the constraint-based system of seventh chords with edges which exist if there is a parsimonious voice-leading transformation between the two chords. We have developed software to play random walks on this graph, and propose to use this type of system for generative music.

X_7 breaks up naturally into some subsets which can be described as stabilizers of permutation actions on the si-type. Such subsets are:

$$X_1 = \{7, m7, \varnothing 7\},\ X_2 = \{M7, mM7, +M7\},\ X_3 = \{o7\},\ \text{and}\ \ X_4 = \{7\flat 5, 7\sharp 5\}.$$

We propose to study further these subsets, and the corresponding subgroups of the parsimony group G_7, and their significance for voice-leading and generative music.

Acknowledgements. We would like to thank the referee and the organizer Thomas Noll for their very helpful comments and suggested references.

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