Non-Contextual JQZ Transformations

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Abstract. Initiated by David Lewin, the contextual PLR-transformations are well known from neo-Riemannian theory. As it has been noted, these transformations are only used for major and minor triads. In this paper, we introduce non-contextual bijections called JQZ transformations that could be used for any kind of chord. These transformations are pointwise, and the JQZ group that they generate acts on any type of n-chord. The properties of these groups are very similar, and the JQZ-group could extend the PLR-group in many situations. Moreover, the hexatonic and octatonic subgroups of JQZ and PLR groups are subdual.

Keywords: Neo-Riemannian group · PLR-group · JQZ-group · Generalized interval systems · Lewin · Parsimonious voice leading

In the neo-Riemannian theory, the use of algebraic structures provides a deep insight into the concept of musical structures and transformational processes. The contextual transformations P (*Parallel*), L (*Leading Tone*) and R (*Relative)* rediscovered by Lewin [15], Hyer [12,13] and Cohn [3–5] from the works of musicologist Hugo Riemann in the late 19th century [16], laid the foundations of the neo-Riemannian theory.

The present paper is organized as follows. After a reminder of some properties of the neo-Riemannian transformations P, L, and R we choose, as we did in 2005 (see [14]), three suitable generators J, Q, and Z for the T/I-group (formed by translations ($T_n(x) = x + n \mod 12$) and inversions ($I_n(x) = -x + n \mod 12$). The T/I-group is known to be isomorphic to the dihedral group D_{12} of 24 elements, the symmetry group of a 12-sided regular polygon [2,11]. In this paper we use the term *JQZ-group* synonymously to the term *T/I-group*. This is in analogy to the usage of the term *PLR-group* synonymously to the term *S/W-group* (Schritt/Wechsel group). This particular system of generators J, Q, and Z has not been systematiclly studied before, but its has very similar properties to the generators P, L and R of the Schritt-Wechsel group. They are not contextually defined and can be applied pointwise. Their definition is unique up to conjugation. Our concrete choice depends upon the choice of the C-major triad $\{0, 4, 7\}$ as a distinctive chord of reference. The choice of any other consonant triad $f(\{0, 4, 7\})$ in this role yields a conjugated triple $fJf^{-1}, fQf^{-1}, fZf^{-1}$ of generators. As in the case of the PLR-group, two subgroups of the JQZ-group are important: the *hexatonic group* generated by transformations J and Q, and the *octatonic group* generated by transformations J and Z.

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1 PLR Transformations

As usual, we encode pitch classes using the standard model of $\mathbb{Z}_{12}$, where $c = 0$, $c\sharp = 1$, $d = 2$, and so on up to $b = 11$. Through this bijection, a major chord is a set of pitches consisting of three notes that has a root x, major third $x + 4$, and perfect fifth $x + 7$; a minor chord is a set of pitches consisting of three notes that has a root x, minor third $x + 3$, and perfect fifth $x + 7$, where the root of the chord ranges through $\mathbb{Z}_{12}$. Major and minor triads are identified with 3-element subsets of $\mathbb{Z}_{12}$ of the form $\{x, x + 4, x + 7\}$ and $\{x, x + 3, x + 7\}$, respectively. The set of the 24 major and minor chords is also called the set of *consonant triads*. Their interval structure can be expressed in terms of third chains $(4, 3)$ and $(3, 4)$, respectively. Rooted interval chains $((a, b), x)$ can be mapped to the associated pitch class sets via: $((a, b), x) \mapsto \{x, x + a, x + a + b\}$. Some neo-Riemannian approaches use *pitch-class segments* denoted by angular brackets. The set Σ of consonant triads then consists of the major segments $\langle x, x + 4, x + 7\rangle$ and minor segments $\langle x, x - 4, x - 7\rangle$ where the (dualistic) root x ranges over $\mathbb{Z}_{12}$. Properties on PLR groups have been established by Thomas Fiore et al. [1,7–10]. The PLR transformations on (dualistic) root position triads within Σ are defined by[1]

$$\begin{aligned} P\langle x, y, z\rangle &= I_{x+z}\langle x, y, z\rangle \\ R\langle x, y, z\rangle &= I_{x+y}\langle x, y, z\rangle \\ L\langle x, y, z\rangle &= I_{y+z}\langle x, y, z\rangle \end{aligned} \tag{1}$$

The transformation P (*Parallel*) exchanges a major triad with the associated parallel minor triad. For instance, $P\langle c, e, g\rangle = P\langle 0, 4, 7\rangle = \langle 7, 3, 0\rangle = \langle g, e\flat, c\rangle$. The transformation R (*Relative*) exchanges a major triad with its relative minor triad (the real root of the *relative minor triad* is a minor third below the root of the major triad) $R\langle c, e, g\rangle = R\langle 0, 4, 7\rangle = \langle 4, 0, 9\rangle = \langle e, c, a\rangle$. And the transformation L (*Leittonwechsel, Leading tone exchange*) exchanges a major triad with a minor triad with its real root a major third above $L\langle c, e, g\rangle = L\langle 0, 4, 7\rangle = \langle 11, 7, 4\rangle = \langle b, g, e\rangle$. In other words, P relates triads that share a common fifth; L relates triads that share a common minor third; and R relates triads that share a common major third.

The PLR-transformations are involutions $P^2 = L^2 = R^2 = Id$, they are their own inverses. As noted, the group generated by P, L and R is called the PLR-group, the *Schritt/Wechsel* group or the neo-Riemannian group after the late 19th-century music theorist Hugo Riemann. Since $P = R(LR)^3$, the PLR-group is generated by L and R. It has been shown that the PLR-group is the dihedral group of 24 elements. And by corollary, the PLR-group acts simply transitively on the set of consonant triads. In the following, major and minor chords are

[1] Observe, however, that applying the formulae (1) to chord inversions leads to: $L\langle 0, 4, 7\rangle = \langle 11, 7, 4\rangle$ (Em), $L\langle 7, 4, 0\rangle = \langle 9, 0, 4\rangle$ (Am) and $L\langle 4, 7, 0\rangle = \langle 3, 0, 7\rangle$ (Cm). In this case, one can not use the equivalence of chord inversions. But these formulae can alternatively be interpreted as voicing transformations [10].

indicated by compact pitch class notations, with $A = 10$ and $B = 11$: 904 is the minor chord $\langle 9, 0, 4\rangle = \langle e, c, a\rangle$ and 48B is the major chord $\langle 4, 8, 11\rangle = \langle e, g\sharp, b\rangle$. The planar representation of the PLR group is a torus.

It is well known, that the transformations P, L, R are not defined on the 12 pitch classes themselves. P sends $e = 4$ in $< 0, 4, 7 >$ to 3 and in $< 9, 1, 4 >$ to 9. The transformations are contextually defined on the basis of the specific interval structure of the major and minor triads. There is no canonic way to define them in strict analogy on other single chord classes, and there are serious obstacles to extend them to all pitch class sets at once. Therefore it is the goal of the next section is to find non-contextual inversions as partners for P, L and R on $\mathbb{Z}_{12}$, which are then automatically valid for all k-chords.

2 JQZ Transformations

The purpose of this section is to select congenial inversions for the generators P, L and R among the 12 inversions I_k on $\mathbb{Z}_{12}$.

In a first step, we look for an inversion J that behaves like the Wechsel P for the C-major triad, i.e. which transforms the major chord $\{0, 4, 7\}$ into the minor chord $\{0, 3, 7\}$. The transformation $J = I_7$ fulfills this and is uniquely determined. For all $x \in \mathbb{Z}_{12}$, we get $J(\{x, x+4, x+7\}) = \{-x, -x+3, -x+7\}$. In other words, J transforms the major chord rooted at x into the minor chord rooted at $-x$ (mod 12). It can be interpreted as the permutation (in cyclic notation): $J = (0, 7)(1, 6)(2, 5)(3, 4)(8, 11)(9, 10)$.

In the second step, we look for an inversion Q that behaves like the Leitonwechsel L on the C-major triad, i.e. transforming the major chord $\{0, 4, 7\}$ into the Leading-tone-exchange chord (Leittonwechselklang) $\{4, 7, 11\}$. The transformation $Q = I_{11}$ fulfills this and is uniquely determined. For all $x \in \mathbb{Z}_{12}$, we get $Q(\{x, x+4, x+7\}) = \{-x+4, -x+7, -x+11\}$. In other words, Q transforms the major chord rooted at x into the minor chord rooted at $4-x$ (mod 12). The transformation Q is the permutation $Q = (0, 11)(1, 10)(2, 9)(3, 8)(4, 7)(5, 6)$.

In the third and last step, we look for an inversion Z which behaves like the Wechsel R on the C-major triad, i.e. which transforms the major chord $\{0, 4, 7\}$ into the relative minor chord $\{9, 0, 4\}$. The transformation $Z = I_4$ fulfills this and is uniquely determined. For all $x \in \mathbb{Z}_{12}$, we get $Z(\{x, x+4, x+7\}) = \{-x+9, -x, -x+4\}$. In other words, Z transforms the major chord rooted at x into the minor chord rooted at $9-x$ (mod 12). The transformation Z is the permutation: $Z = (0, 4)(1, 3)(5, 11)(6, 10)(7, 9)$.

Since JQZ transformations act pointwise, the order of the pitch classes does not matter.

As noted, the transformations J, Q, Z depend upon the choice of the C-major triad. But analogous transformations can be chosen with a different chord of reference, such that they behave like the transformations P, L, R on this chord. Introducing the transformations $J(x, +) = I_{2x+7}$, $Q(x, +) = I_{2x-1}$, $Z(x, +) = I_{2x+4}$ for a major chord, and the transformations $J(x, -) = I_{2x+7}$, $Q(x, -) = I_{2x+3}$, $Z(x, -) = I_{2x-2}$ for a major chord, we have the following result.

Theorem 1. *For the major chord $X = \{x, x+4, x+7\}$ we define the inversions:*

$$J(x,+) = I_{2x+7}, \quad Q(x,+) = I_{2x-1}, \quad Z(x,+) = I_{2x+4}$$

For the minor chord $X = \{x, x+3, x+7\}$ we define the inversions

$$J(x,-) = I_{2x+7}, \quad Q(x,-) = I_{2x+3}, \quad R = Z(x,-) = I_{2x-2}$$

In both cases $s = \pm$ we obtain $J(x,s)(X) = P(X), Q(x,s)(X) = L(X), Z(x,s)(X) = R(X)$. Moreover, the transformations $J(x,s), Q(x,s), Z(x,s)$ are conjugates of J, Q and Z, respectively:

$$\begin{aligned} &J(x,+) = T_x J T_x^{-1}, \, Q(x,+) = T_x Q T_x^{-1}, \, Z(x,+) = T_x Z T_x^{-1}, \\ &J(x,-) = I_x J I_x, \quad Q(x,-) = I_x Q I_x, \quad Z(x,-) = I_x Z I_x. \end{aligned}$$

Proof. The proof is straightforward using Eq. (1). For instance, one has:

$$P \langle x, x+4, x+7 \rangle = I_{2x+7} \langle x, x+4, x+7 \rangle = J(x,+)$$

Using the properties (see Eq. (2) below), we have

$$T_x J T_x^{-1} = T_x I_7 T_{-x} = T_x I_{7+x} = I_{2x+7} = J(x,+) = J(x,-)$$

Other relationships are shown in the same way.

These conjugation relations allow us to choose a reference point (here $x = 0$), but it can be adapted for musical purpose. The Cayley-graph of the action of the neo-Riemannian group on the consonant triads with respect to the generators P, L, and R is known under the nickname "chickenwire torus". Analogously, the Cayley graph of the JQZ transformations is represented on Fig. 1. The torus is visualized by gluing the right border with the left side, and the top side with the bottom, according to the chords.

3 The JQZ and PLR Groups

The JQZ-transformations are involutions $J^2 = Q^2 = Z^2 = Id$. They generate the JQZ-group of order 24, with presentation:

$$\langle J, Q, Z \rangle = \left\langle J, Q, Z \mid J^2 = Q^2 = Z^2 = (JQ)^3 = (JZQ)^2 = (JZ)^4 = 1 \right\rangle$$

The toroidal representation was made by several authors including Richard Cohn in [3–5]. Derek Waller was one of the first authors to introduce the torus in musical representations [17]. The donut that is made by gluing the edges of the tonnetz was called the *chicken-wire torus* by Douthett and Steinbach [6]. Whether we start from the PLR network or from the JQZ network, we find the same topological figure (see Fig. 2).

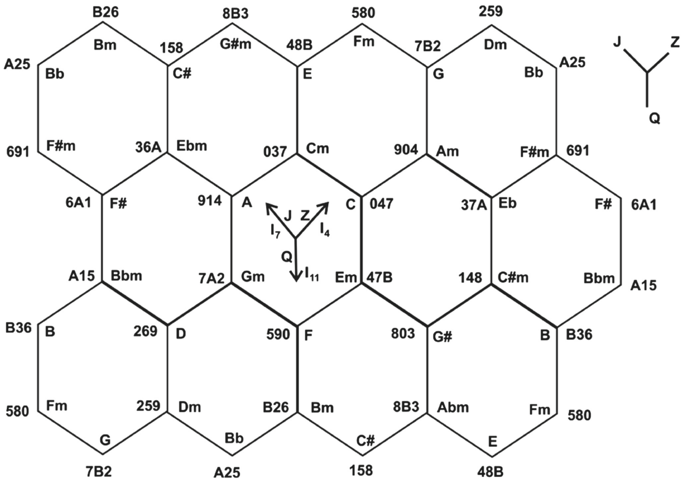

Fig. 1. Tonnetz of the JQZ-transformations

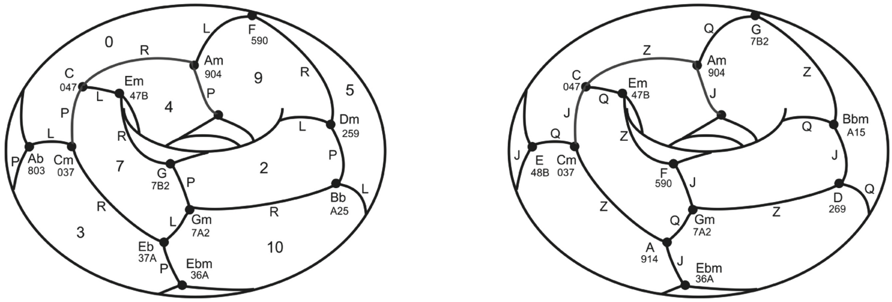

Fig. 2. Chicken-Wire Torus (left PLR, right JQZ)

In the *Cube Dance*, another figure introduced by Douthett and Steinbach in [6], each vertex represents a consonant triad or an augmented triad, and each solid edge is labeled by either P or L. A slightly differing figure can be obtained with the transformations J and Q: Observe, that four cubes dance in two pairs here (see Fig. 3).

Although there is an isomorphism between PLR and JQZ groups, this isomorphism is not obvious. In the T/I group, the transpositions commute $T_nT_m = T_mT_n = T_{m+n}$ but the inversions do not commmute. They satisfy the relations:

$$I_nI_m = T_{n-m} \quad T_nI_m = I_{n+m} \quad I_mT_n = I_{m-n} \tag{2}$$

PLR -Transformations commute with transpositions ($PT_n = T_nP$, $LT_n = T_nL$, $RT_n = T_nR$), while J, Q and Z anticommute ($JT_n = T_{-n}J$, $QT_n = T_{-n}Q$, $ZT_n = T_{-n}Z$).

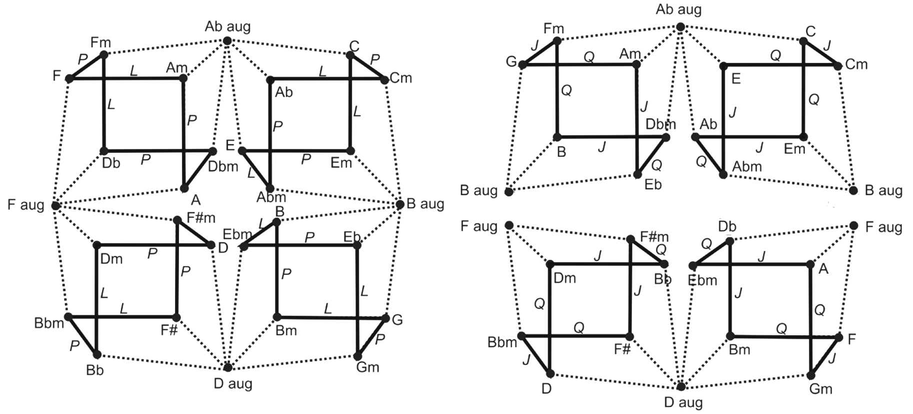

Fig. 3. Douthett's Cube Dance (left PLR, right JQZ)

Theorem 2. *The transformations J,Q,Z satisfy the relations*

$$JI_n = I_{2-n}J, \quad ZI_n = I_{8-n}Z, \quad QI_n = I_{10-n}Q$$

Commutation relations are obtained for $JI_1 = I_1J$, $ZI_{10} = I_{10}Z$, $QI_5 = I_5Q$.

Proof. Substituting J, Q, Z by respectively I_7, I_{11}, I_4 leads to the relations. For instance,

$$JI_n = I_7I_n = T_{7-n} = T_{2-n+5} = T_{2-n-7} = I_{2-n}I_7 = I_{2-n}J$$

Commutation relations are obtained only if $I_mI_n = T_6$. Since $I_mI_n = T_{m-n}$, n must be equal to $n = m + 6 \bmod 12$. Then for $m = 7$, $n = 1$, for $m = 10$, $n = 4$ and for $m = 4$, $n = 10$.

Theorem 3. *The transformations P, L, R on the set of consonant triads* Σ *satisfy the relations:*

$$T_{x-y}PL = T_{y-x}LP, \quad T_{z-y}PR = T_{y-z}RP, \quad T_{z-x}LR = T_{x-z}RL$$

Proof. We have $PL\langle x, y, z\rangle = P\langle y + z - x, z, y\rangle = \langle y, 2y - x, y + z - x\rangle$. On the other hand, the computation: $T_{2y-2x}LP\langle x, y, z\rangle = T_{2y-2x}L\langle z, x + z - y, x\rangle = T_{2y-2x}\langle 2x - y, x, x + z - y\rangle = \langle y, 2y - x, y + z - x\rangle = PL\langle x, y, z\rangle$ leads to the formula. The proof of the two other relations is similar.

In the following table, we give a dictionary between the two groups. already remarked, the JQZ transformations do not depend on element The transpositions T_n can be expressed as a combination of JQZ transformations. The correspondence of an element of the JQZ-group with an element in the PLR-group is given by their concordant behavior on a chosen triad. If this element $\langle x, y, z\rangle$ is a minor chord, the correspondance is given by $J \leftrightarrow P$, $Q \leftrightarrow L$, $Z \leftrightarrow R$.

But if the element is a minor chord, we have to inverse the element given by the minor chord. For instance, for the transposition T_3 up to a minor chord $T_3 = I_7 I_4 = JZ$, the corresponding element in PLR is PR for a minor chord and $(PR)^{-1} = RP$ for a major chord. The verification is straightforward:

$$\begin{aligned} PR\,\langle x, x+8, x+5\rangle &= P\,\langle x+8, x, x+3\rangle = \langle x+3, x+11, x+8\rangle \\ &= T_3\,\langle x, x+8, x+5\rangle \end{aligned}$$

For the inversions, the dependence is not only related to the major or minor chord but also to the root of the chord. For instance, we get

$$\begin{aligned} PRL\,\langle x, x+4, x+7\rangle &= PR\,\langle x+11, x+7, x+4\rangle = P\,\langle x+7, x+11, x+2\rangle \\ &= \langle x+2, x+10, x+7\rangle = I_{2x+2}\,\langle x, x+4, x+7\rangle \end{aligned}$$

In the following table, the index n in $2x+n$ must be computed first in order to use the PLR column. For instance, the passage from $A\flat$ (803) to Fm (085) is I_8. The chord $A\flat$ is a major chord rooted at $x = 8$. It follows that $2x+n = 8$ implies $n = 4$. In the column *Major,* at line I_{2x+4} the corresponding PLR transformation is R. Thus the passage from $A\flat$ to Fm is the R transform. Moreover, since inversions are involution, the passage from Fm (085) to $A\flat$ (803) is also I_8. The chord root of the minor chord Fm is $x = 0$ since the minor chord are of the form $\langle x, x+8, x+5\rangle$. It follows from $2x+n = 8$ and $x = 0$ that $n = -4$. In the column *Minor* at line I_{2x-4}, the corresponding PLR transformation is R. Thus the passage from Fm to $A\flat$ is the R transform.

T/I	JQZ	$Major$	$Minor$
T_0	1	1	1
T_1	$QJZJ$	$(LPRP)^{-1}$	$LPRP$
T_{11}	$(QJZJ)^{-1}$	$(PRPL)^{-1}$	$PRPL$
T_2	$(QZ)^2$	$(LR)^{-2}$	$(LR)^2$
T_{10}	$(QZ)^{-2}$	$(LR)^2$	$(LR)^{-2}$
T_3	JZ	$(PR)^{-1}$	PR
T_9	$(JZ)^{-1}$	PR	$(PR)^{-1}$
T_4	QJ	$(LP)^{-1}$	LP
T_8	$(QJ)^{-1}$	LP	$(LP)^{-1}$
T_5	ZQ	$(RL)^{-1}$	RL
T_7	$(ZQ)^{-1}$	RL	$(RL)^{-1}$
T_6	$(ZJ)^2$	$(RP)^{-2}$	$(RP)^2$

T/I	JQZ	$Major$	$Minor$	PLR
I_0	JQZ	I_{2x}	I_{2x}	PLR
I_1	ZJZ	I_{2x+1}	I_{2x-1}	RPR
I_2	JZQ	I_{2x+2}	I_{2x-2}	PRL
I_3	JQJ	I_{2x+3}	I_{2x-3}	PLP
I_4	Z	I_{2x+4}	I_{2x-4}	R
I_5	$ZQZQJ$	I_{2x+5}	I_{2x-5}	$RLRLP$
I_6	QZQ	I_{2x+6}	I_{2x-6}	LRL
I_7	J	I_{2x+7}	I_{2x-7}	P
I_8	QJZ	I_{2x+8}	I_{2x-8}	LPR
I_9	ZQZ	I_{2x+9}	I_{2x-9}	RLR
I_{10}	JZJ	I_{2x+10}	I_{2x-10}	PRP
I_{11}	Q	I_{2x+11}	I_{2x-11}	L

A dictionnary between T/I, JQZ and PLR elements

With the presentation of the PLR-group and the JQZ-group, finding an isomorphism between the two groups which has a musical meaning is not obvious. However, there exists some isomorphisms between the PLR group and the JQZ group through the permutations of the symmetric group S_{24}. The calculation was done with the GAP software. It shows that if the PLR group is built with the generators:

$$
\begin{aligned}
P &= (0,19)(1,15)(2,14)(3,11)(4,0)(5,18)(6,7)(8,22)(9,23)(12,21)(13,20)(16,17)\\
R &= (0,16)(1,20)(2,21)(3,23)(4,22)(5,17)(6,19)(7,18)(8,13)(9,12)(10,15)(11,14)\\
L &= (0,4)(1,2)(3,5)(6,8)(7,9)(10,12)(11,13)(14,16)(15,17)(18,20)(19,21)(22,23)
\end{aligned}
$$

and the group JQZ is built with the generators:

$$
\begin{aligned}
J &= (0,7)(1,6)(2,5)(3,4)(8,11)(9,10)\\
Q &= (0,11)(1,10)(2,9)(3,8)(4,7)(5,6)\\
Z &= (0,4)(1,3)(5,11)(6,10)(7,9)
\end{aligned}
$$

then the group isomorphism is defined by $P \to J$, $L \to Q$ and $R \to Z$.

From the musical point of view, the JQZ and PLR paths in the tonnetz are very similar. The examples studied by Alissa S. Crans, Thomas M. Fiore, and Ramon Satyendra in [2] are reinterpreted here by JQZ relations. For instance, in the "Grail" theme of the *Prelude of Parsifal* (1882), the chord progression written under the compact form:

$$
\begin{matrix} A\flat \\ 803 \end{matrix} \to \begin{matrix} Fm \\ 580 \end{matrix} \to \begin{matrix} D\flat \\ 158 \end{matrix} \to \begin{matrix} B\flat m \\ A15 \end{matrix} \to \begin{matrix} A\flat \\ 803 \end{matrix}
$$

is interpreted in two ways. The PLR interpretation shows the importance of relative chords while the JQZ interpretation highlights the action of the octatonic subgroup, namely the inversion ZJZ between chords. The vertical arrows are not the same in the PLR transformations (one is R, and the other RLR) while they are perfectly symmetrical ($ZJZ = J^Z$) in the case of the group JQZ, representing the inversion I_1. Taking the chord $A\flat$ as a reference will lead to simpler relationships.

PLR Interpretation: JQZ Interpretation:

$$
\begin{tikzcd}
A\flat \arrow[r, dash, "R"] \arrow[d, dash, "RLR"'] & Fm \arrow[d, dash, "L"] \\
B\flat m \arrow[r, dash, "R"'] & D\flat
\end{tikzcd}
\qquad
\begin{tikzcd}
A\flat \arrow[r, dash, "QJZ"] \arrow[d, dash, "ZJZ"'] & Fm \arrow[d, dash, "ZJZ"] \\
B\flat m \arrow[r, dash, "QZQ"'] & Db
\end{tikzcd}
$$

In the *Lento occulto* of Feruccio Busoni's *Sonatina seconda*, the similarity between PLR and JQZ group is clear in the chord progression,

$$
\begin{tikzcd}
E\flat \arrow[r, "ZJZQ", "RPRL"'] & C\sharp \arrow[r, "ZJQJ", "PRPL"'] & D \arrow[r, "ZJQJ", "PRPL"'] & E\flat \arrow[r, "ZJQJ", "PRPL"'] & E \arrow[d, "RL"', "QZ"] \\
E\flat \arrow[u] & E \arrow[l, "JQJZ"', "PLPR"] & C \arrow[l, "QJ"', "PL"] & F \arrow[l, "QZ"', "RL"] & B \arrow[l, "JZJZ"', "PRPR"]
\end{tikzcd}
$$

as well as in Debussy's *Danseuses de Delphes* (*Preludes* vol. 1):

$$
\begin{tikzcd}
B\flat \arrow[r, dash, "J", "L"'] & Dm \arrow[r, "ZQ", "RL"'] & Gm \arrow[r, dash, "JQZ", "PRL"'] & C \arrow[r, dash, "ZQZ", "RLR"'] & Dm \arrow[r, dash, "J", "L"'] & B\flat \arrow[r, "ZQZQ", "RLRL"'] & C \arrow[r, "ZQ", "LR"'] & F
\end{tikzcd}
$$

4 Dual and Subdual Groups

In this section, we study two particularly interesting subgroups: the hexatonic group and the octatonic group. The hexatonic group is the group generated by transformations J and Q. It has the presentation

$$\langle J, Q \rangle = \left\langle J, Q \mid J^2 = Q^2 = 1,\ JQJ = QJQ \right\rangle$$

The group has six elements: $\langle J, Q \rangle = \{T_0, I_7, T_4, I_3, T_8, I_{11}\} = \{1, J, QJ, JQJ, JQ, Q\}$. Each element of the group is compute using the relations $J = I_7$ and $Q = I_{11}$. For instance,

$$JQ(x) = J(-x+11) = -(-x+11)+7 = x-4 = x+8 \mod 12 = T_8(x)$$

The group acts on the major and minor chords, but likewise it acts on any other set class. The 3-chord $\{0, 6, 11\}$ has the 6-element orbit

$$\{\{0,6,11\},\{1,7,8\},\{3,4,10\},\{3,4,9\},\{2,7,8\},\{0,5,11\}\}$$

within the set class 3–5 (Forte's nomenclature).

The octatonic group is generated by the transformations J and Z. It has the presentation: $\langle J, Z \rangle = \left\langle J, Z \mid J^2 = Z^2 = 1,\ (ZJ)^2 = (JZ)^2 \right\rangle$. The group has eight elements: $\langle J, Z \rangle = \{T_0, I_7, T_9, I_{10}, T_6, I_1, T_3, I_4\} = \{1, J, ZJ, JZJ, (ZJ)^2, ZJZ, JZ, Z\}$. For instance, we compute

$$JZ(x) = J(-x+4) = -(-x+4)+7 = x+3 \mod 12 = T_3(x).$$

Since the groups *T/I, PLR* and *JQZ* are isomorphic, they have the same center. This centre is the group of ordre 2 consisting of $\{1, T_6\}$. The group $\mathcal{Q}$ of *quasi uniform triadic transformations* of order 1152 (GAP 1152#32554) introduced by Hook [11] is a refinement of the group $\mathcal{U}$ of *uniform triadic transformations* of order 288 (GAP 288#239). The group $\mathcal{U}$ is isomorphic to the wreath product of the cyclic group C_{12} of order 12 by the cyclic group C_2 of order 2. The group $\mathcal{Q}$ is the wreath product of the T/I group by the symmetric group of 2 elements S_2.

$$\mathcal{U} = C_{12} \wr C_2 \quad \mathcal{Q} = T/I \wr S_2 \simeq D_{12} \wr S_2$$

The notion of dual groups in the sense of Lewin has been introduced in [8].

Definition 1. *Two subgroups G_1, G_2 of a group $\mathfrak{G}$ are dual (in the sense of Lewin) if both act simply and transitively and are each others centralizers.*

$$C_{\mathfrak{G}}(G_1) = G_2 \quad \textit{and} \quad C_{\mathfrak{G}}(G_2) = G_1$$

For instance, the centralizer of the T/I group in $\mathcal{Q}$ is the JQZ-group (or PLR-group) and the centralizer of the JQZ-group (or PLR-group) in $\mathcal{Q}$ is the T/I group. The computation is straightforward with the GAP software.

$$C_{\mathcal{Q}}(T/I) = \langle J, Q, Z \rangle \quad \text{and} \quad C_{\mathcal{Q}}(\langle J, Q, Z \rangle) = T/I$$

The same is true in the symmetric group $Sym(\Sigma) \simeq S_{24}$ of permutations of 24 elements [2] instead of $\mathcal{Q}$: the JQZ-group and the T/I-group are subgroups of $Sym(\Sigma)$. The subduality of the hexatonic group and the octatonic group has been studied by Thomas Fiore and Thomas Noll in the same reference [8]. They define the concept of subduality in the following sense.

Definition 2. *Let G_1, G_2 be dual subgroups of the symmetric group S_{24} of the 24 triads and let H_1, H_2 be two subgroups of G_1, G_2 respectively. H_1 and H_2 are subdual groups if both act simply and transitively on a subset $X \subset S_{24}$ and are each others centralizers within the symmetric group S_X of this subset.*

$$C_{S_X}(H_1) = H_2 \quad \textit{and} \quad C_{S_X}(H_2) = H_1$$

The hexatonic JQ-group $\langle J, Q\rangle$ has 4 orbits, namely two pairs of tritone-related sets of six triads each.

$\{C,\ Cm,\ E,\ Em,\ A\flat,\ A\flat m\}$ and $\{D,\ Dm,\ F\sharp,\ F\sharp m,\ B\flat,\ B\flat\ m\}$ as well as $\{D\flat,\ E\flat m,\ F,\ Gm,\ A,\ Bm\}$ and $\{D\flat m,\ E\flat,\ Fm,\ G,\ Am,\ B\}$. The orbits in the first pair are triads in proper hexatonic collections, i.e. they are also orbits of the hexatonic PL-group $\langle P, L\rangle$. The orbits in the second pair form orbits under a conjugate of $\langle P, L\rangle$, namely: $R\,\langle P, L\rangle\,R$. This leads to the following result.

Theorem 4. *With respect to the first two orbits the group $\langle J, Q\rangle$ is subdual to $\langle P, L\rangle$. With respect to the second two orbits $\langle J, Q\rangle$ is subdual to $R\,\langle P, L\rangle\,R$.*

The octatonic JZ-group $\langle J, Z\rangle$ has 3 orbits of 8 triads each. The first one $\{F\sharp m,\ F\sharp,\ E\flat m,\ A,\ Cm,\ C,\ Am,\ E\flat\}$ is also orbit of the octatonic PR-group. The two others $\{Bm,\ C\sharp,\ G\sharp m,\ E,\ Fm,\ G,\ Dm,\ B\flat\}$ and $\{B\flat m,\ D,\ Gm,\ F,\ Em,\ G\sharp,\ C\sharp m,\ B\}$ are orbits under conjuguation $(RL)\,\langle P, R\rangle\,(RL)^{-1}$. This lead to the following result.

Theorem 5. *With respect to the first orbit the group $\langle J, Z\rangle$ is subdual to $\langle P, R\rangle$. With respect to the second two orbits $\langle J, Z\rangle$ is subdual to $(RL)\,\langle P, R\rangle\,(RL)^{-1}$.*

5 The Atonal Triad

The study of the interplay of the PLR group and the JQZ group is helpful in atonal analysis. We will illustrate this for the atonal triad $\langle 0, 1, 6\rangle$.

The main advantage of JQZ transformations is to be able to consider all types of chords and not only consonant chords. For instance, if the JQZ group acts on the atonal triad 016 $\{c, c\sharp, f\sharp\}$, the action leads to a new lattice. In the other hand, if we replace the set Σ of consonant triads by the set Γ of atonal triads rooted on x of the form $\langle x, x+1, x+6\rangle$ and $\langle x, x-1, x-6\rangle$ we get new relations

$$P\,\langle x, x+1, x+6\rangle = \langle x+6, x+5, x\rangle\,,\ \ P\,\langle x, x+11, x+6\rangle = \langle x+6, x+7, x\rangle$$
$$R\,\langle x, x+1, x+6\rangle = \langle x+1, x, x+7\rangle\,,\ \ R\,\langle x, x+1, x+6\rangle = \langle x+11, x, x+5\rangle$$
$$L\,\langle x, x+1, x+6\rangle = \langle x+7, x+6, x+1\rangle\,,\ \ L\,\langle x, x+11, x+6\rangle = \langle x+5, x+6, x+11\rangle$$

This allows us to interpret Georges Crumb's *Gargoyles*, a piano piece, extract from *Makrokosmos*, in both ways:

$$238 \xrightarrow[(PL)^3]{SZ} 56B \xrightarrow[PR]{QZ} 016 \xrightarrow[(PL)^4]{SZ} 349 \xrightarrow[LPRP]{QS} 781 \xrightarrow[RL]{(SZ)^2} 127$$

$$107 \xrightarrow[(LP)^3]{SZ} 43A \xrightarrow[RP]{QZ} BA5 \xrightarrow[(LP)^3]{SZ} 218 \xrightarrow[(LP)^4]{QS} 650 \xrightarrow[LR]{(ZS)^2} 0B6$$

The right hand evolves in the same way as the left hand: the atonal triads are linked by the transformations (JZ, QJ). The same transformations are used in both hands, except in the last triads. But if we consider a crossing of the hands: 781 goes to 0B6 by the transformation J, just as 650 goes to 127 by the same transformation.

Another interpretation is possible. As we saw for the PLR group, we can consider two kinds of atonal triads, one of structure (1, 5) and the other of structure (5, 1). From the relations of PLR seen above, we can introduce the transformations for atonal triads $\{x, x+1, x+6\}$,

$$J(x,+) = I_{2x+6}, \quad Q(x,+) = I_{2x+7}, \quad Z(x,+) = I_{2x+1}$$

and for the atonal triads $\{x, x+5, x+6\}$,

$$J(x,-) = I_{2x+6}, \quad Q(x,-) = I_{2x-7}, \quad Z(x,-) = I_{2x-1}$$

For $x = 0$, we have three new transformations $J = I_6, Q = I_7, Z = I_1$. The tritone (6) has replace the fifth (7) in the definition of J. Applying these transformations to Olivier Messiaen's *Regard de l'Onction terrible* (extract of *Vingt Regards de l'Enfant Jesus* #18), leads to the following chords progression, which evolves against the same chord A94, on the left hand. We use the following notation for the conjuguaison: $Z^J = JZJ$.

$$\begin{tikzcd}
650 \arrow[r, "JQ", "PL"'] \arrow[d, "RPL"', "Z^J Q"] & 54B \arrow[r, "JQ", "PL"'] \arrow[d, "JZ"', "RP"] & 43A \arrow[r, "JQ", "PL"'] \arrow[d, "ZQ"', "RL"] & 329 \arrow[r, "JQ", "PL"'] \arrow[d, "ZJ"', "RP^L"] & 218 \arrow[r, "JQ", "PL"'] \arrow[d, "(LP)^4"', "ZQ^J"] & 107 \arrow[r] \arrow[d, "(LP)^3"', "Z^J J^Q"] & 107 \arrow[d, "JQ"', "(RP)^2"] \\
A94 \arrow[r] & A94 \arrow[r] & A94 \arrow[r] & A94 \arrow[r] & A94 \arrow[r] & A94 \arrow[r, "JQ", "LP"'] & 0B6
\end{tikzcd}$$

This demonstrates the interest and power of these non-contextual transformations for the analysis of all kind of music, but especially for atonal music.

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