Exploring the Syntonic Side of Major-Minor Tonality

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Abstract. The description of the Major and Minor modes as fillings of a triadic division of the octave offers the possibility to study them as Pairwise Well-Formed Modes. As a consequence one obtains two projections: the diatonic projection yields the well-known Ionian and Aeolian modes and provides a link between the triadic modes and the pseudo-classical modes. The syntonic projection looks unfamiliar at first sight, but closer inspection shows that these modes provide a common ground for the natural, harmonic, and melodic manifestations of both the Major and the Minor modes.

Keywords: Triadic mode · Diatonic and syntonic mode · Tonal and modal step intervals · Sturmian morphism · Algebraic combinatorics on words · Major/minor tonality

1 Introduction

In scale theory it is common to consider a generic layer of scale degrees together with a specific layer of notes or pitches. The specific layer is sometimes associated with the pitch parameter of actual musical tones in a psycho-acoustical sense. While this option is not always thematized, authors may still consider themselves to be safe in the light of this possibility. A challenging case in this regard is therefore the concept of a *Pairwise Well-Formed Scale*, in particular because our central idea is to conceive the major and minor modes as instances of pairwise well-formed modes. In our recent study [5] we emphasize the abstract context of the investigation, but the concept of pairwise well-formedness has radical consequences in itself, which should be further pursued. In fact, the definition of a *pairwise well-formed scale* involves a specific layer and the concrete realization of the major scale in just intonation with the step-interval pattern (a, c, b, a, c, a, b) and the specific interval sizes $a = log_2(9/8)$, $c = log_2(10/9)$, $b = log_2(16/15)$ turns out to be an instance of this concept. But the verification of the property of pairwise well-formedness involves acts of identification of pairs of these step intervals, i.e. $c = b$, $a = c$ and $a = b$. These identifications are hence situated half-way between the specific level where all three letters are different

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and the generic level, where $a = b = c$. In one of these identifications the two instances of c are replaced by the letter a, which results in the step-interval pattern (a, a, b, a, a, a, b). As the definition requires, it turns out that this is the abstract step-interval pattern of a (non-degenerate) well-formed scale. It corresponds to the Ionian mode. The definition does not require one to single out specific values for c and b. But, although it is sufficient that there exist infinitely many possibilities, one may still consider oneself more safe by singling out either the step sizes $c = log_2(9/8)$ and $b = log_2(256/243)$ of Pythagorean tuning or $c = 2/12$ and $b = 1/12$ of 12-tone-equal temperament. Furthermore, the music-theoretical significance of an abstract Ionian mode seems beyond question. In other words, the music-theoretical postulate of an abstract diatonic layer, halfway between the generic scale degrees $\hat{1}, \ldots, \hat{7}$ and the specific layer of the triadic major mode, seems quite reasonable.

The truly challenging case is the verification of the step-interval pattern (a, b, b, a, b, a, b), which results from the identification of the letters b and c. There are infinitely many possible solutions for a and b.[1] But even though it is by definition not necessary to single out specific values for a and b, the lack of a prominent instance for such a scale might seem disconcerting.

In search of a possibly hidden musical manifestation of the syntonic step-interval pattern (a, b, b, a, b, a, b), it is the goal of the present paper to think through the music-theoretical consequences of our approach in [5] more rigurously: if it is not the specific pitch-height differences which distinguish the step intervals of type a from those of type b, it must be some other musically relevant property. We see this difference in the syntactic behavior of these steps. The step intervals of type a could be called the *tonal* or *fixed* step intervals. Over a fixed tonic they remain unchanged in the typical inflections within major or minor modes and the processes of modal mixture (with respect to C-major and C-minor, step intervals $C-D$ and $F-G$, in major also $A-B$ and in natural minor also $A\flat - B\flat$). The step intervals of type b could be called the *modal* or *moveable* step intervals. They are the locations in the step-interval patterns where these processes take place. While through the identification of the letters b and c these alterations become unnoticeable, the step pattern *abbabab* still remembers the locations where they occur. In Sect. 3 we explore the typical alterations of the major and minor modes, such as their harmonic and melodic forms and show that the syntonic step-interval pattern *abbabab* remains unchanged under these alterations.

We may to some degree align ourselves in the history of theory with Hauptmann. Hauptmann's axioms are the "directly intelligible" intervals of the perfect octave, perfect fifth, and major third ([8], p. 5), which leads to the just major scale expressed by the word *acbacab*, and to the just (natural) minor scale, expressed as the word *abcabac*. The latter is not a conjugate of the major form, but a conjugate of its reversal. To wit, if *acbacab* represents the just C-major scale, then the circular reversal beginning from the fourth letter may represent

[1] For example the generator $g = (2 - \sqrt{2})/2$ yields a seven-note scale with the step intervals of sizes $a = 3 - 2\sqrt{2}$ and $b = 3/\sqrt{2} - 2$.

the just C-minor scale: take the prefix, *acba*, reverse it to form *abca*, and follow it by the reversal of the suffix *cab*, yielding *bac*. (The authentic division into perfect fifth and perfect fourth is in play here.) We will see below that transformationally the minor form is Twisted Triadic Aeolian. In this concrete expression of the scales we understand greater major steps ($a = log_2(9/8)$), lesser major steps ($c = log_2(10/9)$), and minor steps or diatonic semitones ($b = log_2(16/15)$). The chromatic alterations are the lesser and greater augmented primes, respectively $log_2(25/24)$ and $log_2(135/128)$. We will also call them the *modal* and *tonal* augmented primes, respectively. As between C-major and C-minor scales, we have alterations of modal step intervals by the lesser augmented prime at the locations represented in major and in minor by letters c and b. In fact, the most instructive way to describe the transformation from Major to Minor is the letter exchange $E_{(bc)}$, replacing each instance of b by c and vice versa: $E_{(bc)}((a, c, b, a, c, a, b)) = (a, b, c, a, b, a, c)$.[2] Hauptmann justifies these alterations in some situations via fixed mediating notes. f (see Sect. 3).

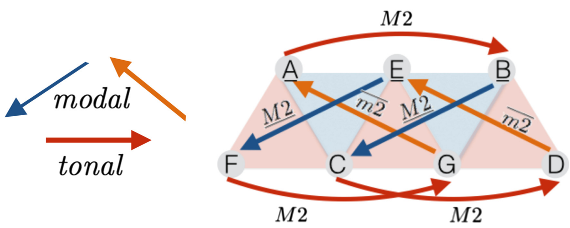

Fig. 1. Illustration of the tonal steps (horizontal) and the modal steps (diagonal either way) in a tone net representation of the Major scale.

2 The Two Sides of the Major and Minor Modes

Here we spare the reader negotiations about the quite abstract content of [5]. Instead we will focus on the main music-theoretical result in its concrete form, leaving generalities aside. The transformational orientation of that paper shall be only briefly recapitulated: the acts of filling the three triadic intervals with step intervals are studied as transformations on three-letter words. This ultimately motivates the coincidence of the number 3 of triadic intervals (major third, minor third and perfect fourth) with the number 3 of different step intervals

[2] In consideration of the fact that Dahlhaus ([7], e.g. p. 46), in his critical reflection on Hauptmann's dialectical concept of the Major and Minor keys, localizes traces of Dualism, it is worthwhile to highlight that these are not implied by the scalar structure. The flattening of scale degrees $\hat{3}, \hat{6}$ and $\hat{7}$ (by a lesser augmented prime) in a major scale leads to the corresponding minor scale. No triad needs to be turned upside down. This transformation corresponds to Hindemith's idea of *Trübung* (disturbance, turbidation), who emphatically rejects a dualistic concept. ([9], p. 78 and also [10], p. 147).

(traditionally called: greater major step, lesser major step, minor step). In several ways the transformations on three letters can be merged into transformations on two-letter words. Four of these mergings turn out to be Sturmian morphisms, as discussed in [5]. This finding constitutes the transformational formulation of the Pairwise Well-formedness Property. There are two main types of Sturmian mergings: diatonic morphisms and syntonic morphisms. They are defined on two-letter words and describe the acts of filling divisions of the octave into fifths and fourths (diatonic case), or into thirds and sixths (syntonic case). Here again the transformational setup motivates the coincidence of the number 2 of division intervals with the number 2 of step intervals.

For our purposes it suffices to describe the major and minor modes merely as triples of three-letter words and their mergings as pairs of two-letter words. The Table below shows the complete list of all 14 triadic modes. Its order appears as an interlocking of the separate Tables 3 and 5 in [5] and exemplifies a general property of (non-singular) pairwise well-formed scales: the existence of a Q-cycle (see [3]), analogous to Cohn's maximally smooth cycles [6].

Table 1. The table shows the step-interval patterns of all triadic modes as a Q-cycle of length 14. Consecutive modes arise out of each other by a flip of adjacent letters (under- and over-lined, respectively). The corresponding diatonic projections change only every other time. The syntonic projections traverse two cycles of length 7. The major and minor modes correspond to opposite positions in this long cycle (Plain Triadic Ionian vs. Twisted Triadic Aeolian; they are seven moves apart from each other in either direction along the cycle) and share the same syntonic projection $(ab, babab)$.

Triadic mode	Step intervals	Diatonic steps	Syntonic steps
Plain Triadic Ionian	$(\underline{ac}, \overline{b}\overline{a}, cab)$	$(aaba, aab)$	$(ab, babab)$
Twisted Triadic Ionian	$(\overline{ca}, ba, c\underline{ab})$	$(aaba, aab)$	$(ba, babab)$
Plain Triadic Mixolydian	$(ca, b\underline{a}, \underline{c}\overline{b}\overline{a})$	$(aaba, aba)$	$(ba, babba)$
Twisted Triadic Mixolydian	$(c\underline{a}, \underline{b}\overline{c}, \overline{a}ba)$	$(aaba, aba)$	$(ba, bbaba)$
Plain Triadic Dorian	$(\underline{c}\overline{b}, \overline{a}c, ab\underline{a})$	$(abaa, aba)$	$(bb, ababa)$
Twisted Triadic Dorian	$(\overline{a}b, ac, \underline{ab}\overline{c})$	$(abaa, aba)$	$(ab, ababb)$
Plain Triadic Aeolian	$(ab, \underline{ac}, \overline{b}\overline{a}c)$	$(abaa, baa)$	$(ab, abbab)$
Twisted Triadic Aeolian	$(\underline{ab}, \overline{ca}, bac)$	$(abaa, baa)$	$(ab, babab)$
Plain Triadic Phrygian	$(\overline{b}\overline{a}, ca, b\underline{ac})$	$(baaa, baa)$	$(ba, babab)$
Twisted Triadic Phrygian	$(ba, c\underline{a}, \underline{b}\overline{ca})$	$(baaa, baa)$	$(ba, babba)$
Plain Triadic Locrian	$(b\underline{a}, \underline{c}\overline{b}, \overline{a}ca)$	$(baab, aaa)$	$(ba, bbaba)$
Twisted Triadic Locrian	$(\underline{b}\overline{a}, \overline{c}b, ac\underline{a})$	$(baab, aaa)$	$(bb, ababa)$
Plain Triadic Lydian	$(\overline{a}c, ab, \underline{ac}\overline{b})$	$(aaab, aab)$	$(ab, ababb)$
Twisted Triadic Lydian	$(ac, \underline{ab}, \overline{ca}b)$	$(aaab, aab)$	$(ab, abbab)$

The Major mode appears in the form of the *Plain Triadic Ionian* mode. The triple (ac, ba, cab) encodes the step-interval pattern $(M2\underline{M2}, \overline{m2}M2, \underline{M2}M2\overline{m2})$.

Diatonically, this reduces to the divided step-interval pattern $(aaba, aab)$, encoding the authentically divided Ionian mode in terms of major seconds and minor seconds $(M2M2m2M2, M2M2m2)$. Syntonically, this reduces to the step-interval pattern $(ab, babab)$, whose musical meaning shall be uncovered. The Minor mode appears in the form of the *Twisted Triadic Aeolian* mode. The triple (ab, ca, bac) encodes the step-interval pattern $(M2\overline{m2}, \underline{M2}M2, \overline{m2}M2\underline{M2})$. Diatonically, this reduces to the divided step-interval pattern $(abaa, baa)$, encoding the authentically divided Aeolian mode in terms of major seconds and minor seconds $(M2m2M2M2, m2M2M2)$. Syntonically, this reduces to the same step-interval pattern $(ab, babab)$, as in the case of the Major mode. The opposite positions of the major and minor modes on the long cycle of length 14 corresponds to the fact, that the letter exchange $E_{(bc)}$ represents the unique element of order 2 within the cyclic group of order 14. This is an interesting result and has been established in a quite general form in [5]. From a music-theoretical point of view the coincidence of the syntonic step-interval pattern of the Major and the Minor modes provides a common ground for the modes of triadic tonality. But at this point a potentially crucial objection must be raised: the "pure" major and minor modes can hardly be accepted as the only representatives of major and minor tonality. In particular the typical alterations of the 6th and 7th degrees must be taken into account as well.

Before proceeding with this task it is interesting to understand the musical meaning of the cyclic order in the table. It is evident from the third column, with the fifth-generated diatonic projections, that every other mode in the cycle appears to progress one step further with respect to circle-of-fifths order: Ionian - Mixolydian - Dorian - etc. If one keeps a fixed tonic, then the signature changes in the same way as in a modulation of the same mode to a new tonic at (downward) fifth-distance (as one moves downward on the table, from the C-major collection to F-major collection, etc.). The syntonic modes (in the fourth column), however, change with every new triadic mode along the cycle. Hauptmann describes these smaller changes as a *stretching out of the key-system to dominant or subdominant*. The dominant key shares the key signature with that of the Lydian mode, and the tonic of the dominant key corresponds to the fifth-divider of the Lydian-mode, i.e. with the framing note of the Hypo-Lydian mode. This is true for both forms of the triadic Lydian: the plain and the twisted. But the interesting observation to be made here is this: the step-interval pattern of Hauptmann's intermediate key between C-Major and G-Major coincides with the twisted Lydian mode (ac, ab, cab). The note $F\sharp$ replaces the F of the C-Plain-Triadic-Ionian, but the species of the fifth *cabc* on D contains three modal steps and one tonal step. So it cannot serve as a dominant. But the signature and step-interval pattern of the twisted Lydian mode on C coincide with the twisted Aeolian mode on E (beginning on E in twisted C-Lydian yields the twisted Aeolian pattern *abcabac*). And E is the third-divider of the syntonic projection of the Plain Triadic Ionian mode. In other words, the Minor modes are half way between the Major modes and vice versa. Note, however that this does not correspond to the order in Heinichen's circle of keys.

This implies a two-fold concept of modulation, involving a diatonic and a syntonic component. Modulations between relative major and minor are purely syntonic (diatonically unnoticeable): when we move from plain triadic C-Ionian to twisted triadic C-Ionian, we have no change of signature, and to achieve the twisted A-Aeolian form it suffices to start on scale degree 6. We see no change in the diatonic steps column in Table 1, but we do see a change in the syntonic steps column. Conversely, modulations between parallel major and minor keys are purely diatonic (syntonically unnoticeable): when we move from plain triadic C-Ionian to twisted triadic C-Aeolian we change the signature, registering a change in the diatonic steps column, but no change in the syntonic steps column.

3 Alteration as Conjugation

Let $\mathcal{A}^*$ denote the free monoid generated by the letters of the finite alphabet $\mathcal{A}$. Within the free group $F_{\mathcal{A}}$, generated by the letters of $\mathcal{A}$ we consider all elements of the form $xz^{-1}y$, where x, y, z each run through the letters from $\mathcal{A}$. In addition we assume we have further letters at our disposal in order to denote all these elements with unique letters. In other words, we consider the subset $\tilde{\mathcal{A}} = \{xz^{-1}y \,|\, x, y, z \in \mathcal{A}\} \subset F_{\mathcal{A}}$ together with a bijection $\mathfrak{l} : \tilde{\mathcal{A}} \xrightarrow{\sim} \overline{\mathcal{A}}$ into a larger alphabet $\overline{\mathcal{A}}$ containing $\mathcal{A}$ and satisfying $\mathfrak{l}(x) = x$ iff $x \in \mathcal{A}$.

Definition 3.1. *Consider three letters $x, y, z \in \mathcal{A}$ together with the letter $t = \mathfrak{l}(xz^{-1}y)$. The two-letter words $\overset{\curvearrowright z}{xy} = tz$ and $\overset{z \curvearrowleft}{xy} = zt$ are called the* right *and the* left z-alteration *of the two-letter word xy, respectively.*

The idea behind this definition is that in an alteration one has a two-letter word xy, whose commutative image remains unchanged, while one of the two letters is superceded by a third letter z. In the act conjugation of the replaced letter by either z or z^{-1} the letter z sneaks in while the concatenation of the replaced letter with z^{-1}, (the commutative image of which is their difference) are joined with the passive letter. The simplest and trivial case is an alteration which does not change the two-letter word xy.

Proposition 3.2. *$\overset{\curvearrowright z}{xy} = xy$ iff $z = y$ and $\overset{z \curvearrowleft}{xy} = xy$ iff $z = x$.*

Proof. $\overset{\curvearrowright z}{xy} = xy$ iff $xz^{-1}y = x$ iff $z = y$. $\overset{z \curvearrowleft}{xy} = xy$ iff $xz^{-1}y = y$ iff $z = x$.

The following examples are based on a three-letter alphabet $\mathcal{A} = \{a, b, c\}$ as well as on its two-letter subalphabet $\{a, b\}$. As the set $\tilde{\mathcal{A}}$ also contains those two elements $\{ab^{-1}a, ba^{-1}b\}$ which are built from this sub-alphabet, we may use the same letter-providing map $\mathfrak{l}$ on $\tilde{\mathcal{A}}$ in both cases without risking conflicts.

The first case is an exchange of the two letters a and b, namely $ba = \overset{\curvearrowright a}{ab} = \overset{b \curvearrowleft}{ab}$.

Example 3.3. Consider the alphabet $\mathcal{A} = \{a, b\}$. With the interpretation $a = M2, b = m2$ the word $abaabaa$ denotes the step-interval pattern of the Aeolian species of the octave. Applying the right b-alteration to the second occurrence of the factor ba yields $abaa\ \overset{\frown b}{ba}\ a = abaaaba$, which is the Dorian species of the octave. Applying the right b-alteration again to the suffix ba in this pattern yields $abaaa\ \overset{\frown b}{ba} = abaaaab$, which can be recognized as the step interval pattern of the melodic minor mode (in terms of the intervals $a = M2$ and $b = m2$). Anticipating Example 3.5 we note, that we could go on to perform the unnoticeable alteration $\overset{\frown a}{aa}$ on the rightmost factor of type aa, without changing the pattern $abaaaab$.

The next interesting case is the (right- or left-) alteration of aa by another letter b. A third letter $d = \mathfrak{l}(ab^{-1}a)$ comes into play. The relevant musical example is the augmented second.

Example 3.4. Again starting from $abaabaa$ we apply the right b-alteration to the suffix aa of the Aeolian species of the octave: $abaab\ \overset{\frown b}{aa} = abaabdb$. This can be recognized as the step-interval pattern of the harmonic minor mode in terms of the intervals $a = M2, b = m2$ and $d = A2$.

Now we turn to three letters. With the refined interpretation $a = M2, b = \overline{m2}, c = \underline{M2}$ we again go through the alterations from Examples 3.3 and 3.4.

Example 3.5. The starting point is the word $abcabac$ denoting the step-interval pattern of the minor mode in the shape of the twisted triadic Aeolian species of the octave. Applying the right b-alteration to the second occurrence of the factor ba yields $abca\ \overset{\frown b}{ba}\ c = abcaabc$. Applying the right b-alteration to its suffix bc yields $abcaa\ \overset{\frown b}{bc} = abcaacb$. The analogue to the redundant third alteration in Example 3.3 is not redundant here: we finally apply a left c-alteration to the factor ac and obtain $abca\ \overset{c\frown}{ac}\ b = abcacab$, which we regard to be the step-interval pattern of the melodic minor mode in terms of the intervals $a = M2$, $c = \underline{M2}$ and $b = \overline{m2}$).

Example 3.6. Now consider the letter $\underline{d} = \mathfrak{l}(ab^{-1}c)$ and apply the right b-alteration to the suffix ac of the Minor species of the octave: $abcab\ \overset{\frown b}{ac} = abcab\underline{d}b$. This can be recognized as the step interval pattern of the harmonic minor mode in terms of the intervals $a = M2, b = \overline{m2}, c = \underline{M2}$ and $\underline{d} = \underline{A2}$.

4 Alteration and Letter Projection

Definition 4.1. *For any pair x and y of distinct letters of an alphabet $\mathcal{A}$ one has the associated* Letter Projection *with x in the role of the* abandoned letter *and y in the role of the* receiving letter*:*

$$\pi_{x\to y} : \mathcal{A} \to \mathcal{A}\backslash\{x\} \text{ with } \pi_{x\to y}(z) := \begin{cases} z \text{ if } z \neq x, \\ y \text{ if } z = x. \end{cases} .$$

Letter projections $\pi_{x \to y} : \mathcal{A}^* \to \mathcal{A}\backslash\{x\}^*$ can be naturally extended to the larger letter domain and the words formed with them: $\overline{\pi}_{x \to y} : \overline{\mathcal{A}}^* \to \overline{\mathcal{A}\backslash\{x\}}^*$, where $\overline{\pi}_{x \to y}(rt^{-1}s) = \mathfrak{l}(\pi_{x \to y}(r)(\pi_{x \to y}(t))^{-1}\pi_{x \to y}(s))$. Thereby letter projections can be applied to alterations: If $\overset{\curvearrowright z}{xy} = tz$ with $t = \mathfrak{l}(xy^{-1}x)$ one obtains $\overline{\pi}_{x \to y}(\overset{\curvearrowright z}{xy}) = \overline{\pi}_{x \to y}(tz)$, and analogously $\overline{\pi}_{x \to y}(\overset{z \curvearrowleft}{xy}) = \overline{\pi}_{x \to y}(zt)$. This extension is natural in the sense that alteration and letter projection commute:

Proposition 4.2. *For any five letters $x, y, r, s, t \in \mathcal{A}$ one finds:*

$$\overset{\curvearrowright \pi_{x \to y}(t)}{\pi_{x \to y}(r)\pi_{x \to y}(s)} = \overline{\pi}_{x \to y}(\overset{\curvearrowright t}{rs}) \;\; \textit{and} \;\; \overset{\pi_{x \to y}(t) \curvearrowleft}{\pi_{x \to y}(r)\pi_{x \to y}(s)} = \overline{\pi}_{x \to y}(\overset{t \curvearrowleft}{rs}).$$

The proof of this is straightforward. With this preparation we may now return to the Examples from Sect. 3 and inspect them with respect to letter projections.

4.1 Melodic Minor

Examples 3.3 and 3.5 provide two separate derivations of the step interval pattern of the melodic minor mode over the alphabets $\{a, b, c\}$ and $\{a, b\}$, respectively. In the upper and middle rows of Fig. 2 we can see, that they are connected under the letter projection $\overline{\pi}_{c \to a}$. Considering the aim of this paper, the most interesting insight can be drawn from the bottom row of Fig. 2, which shows the syntonic projections of these derivations. It turns out that the left- and the right-most interval patterns coincide *abbabab*. It is the unaltered step-interval pattern of the Standard syntonic mode. In other words, the melodic alteration of the Minor mode is diatonically noticeable and syntonically unnoticeable. The syntonic side offers itself as a common ground for the unaltered and the melodically altered Minor mode.

Hauptmann does not explicitly distinguish the two different kinds of alterations. But pursuing his principle that step intervals are mediated through two harmonic intervals he makes implicit decisions which we may interpret as choices. Tonal steps a have to be altered by tonal augmented primes and are always mediated through a fifth and a fourth. Modal steps have to be altered by modal augmented primes and are always mediated by a fourth and a third. The altered step interval in a tonal alteration can therefore not be mediated by the same tone, as the latter forms either a diminished fifth or an augmented fourth with one of the two tones of the altered step interval. But the altered step interval in a modal alteration can very well be mediated by the same tone, namely if the unaltered and the altered tone both form a third with that tone. Otherwise the mediation tone needs to be altered or replaced. The possibility to maintain the mediation tone may—theoretically—serve as a defining property for modal alterations. Hauptmann seems choose mediations tones by a criterium of key membership, as we will see below. In the key of C-major the tonal step from C to D as well as the modal step from D to E are both mediated through the tone G in the just described manner ([8], p. 37). The notes A and B on scale degrees $\hat{6}$ and $\hat{7}$ are mediated through the note E and form a tonal step interval a.

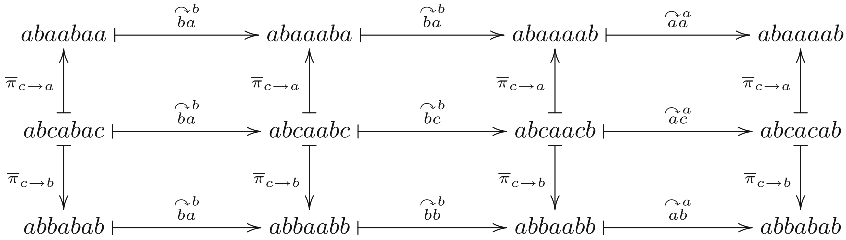

Fig. 2. Alteration chains for the derivation of the step-interval pattern of the melodic minor mode. The middle row starts with the octave species of the twisted triadic Aeolian mode *abcabac* and reaches the desired pattern *abcacab* on the right. The upper row shows the corresponding diatonic projection, starting from the octave species of the authentic Aeolian mode *abaabaa* and reaches *abaaaab*. The middle and upper rows are connected along the letter projection $\overline{\pi}_{c\to a}$. The bottom row shows the corresponding syntonic projection, starting from the octave species of the syntonic Standard mode *abbabab* and ends with the same mode. The middle and bottom rows are connected along the letter projection $\overline{\pi}_{c\to b}$

With regard to the scale of the C-Minor key, we get into an interesting tension with Hauptmann's position. His starting point is not the natural minor scale, but the three-triad system constituted by the C-Minor, F-Minor and G-Major Triads. The implied scale is the harmonic minor scale, which we consider in the next subsection. But with regard to the *scale of the minor key* Hauptmann sees the need to close the gap between $A\flat$ and B though an alteration $A\flat \to A$, resulting in a melodic minor scale. But precisely there we get into conflict with Hauptmann. The mediation for a tonal step between A and B must go though E. But Hauptmann discards this option, as the tone E does not belong to the key. Instead he chooses D as the mediating tone for a modal step, the same tone would serve to mediate between G and A as a tonal step. In other words, Hauptmann favors the step-interval pattern *abbaabb*, the penultimate node in the diagram of Fig. 2. As a drawback he gets a complicated alteration with a switch from the mediating tone $E\flat$ for the modal minor step $G - A\flat$ to D.

We tend to regard harmonic and melodic minor to be both alterations of the natural minor mode, where we have a tonal step between $A\flat$ and $B\flat$. In this perspective it seems more natural to expect a binding between the two altered notes, keeping the tonal step between them intact.[3]

4.2 Harmonic Minor, Harmonic and Melodic Major

Our position is further strengthened in the case of the harmonic alteration of the natural minor mode, as Fig. 3 shows. Under the diatonic projection $\overline{\pi}_{c\to a}$ the letter $\underline{d} = \mathfrak{l}(ab^{-1}c)$ is mapped to the letter $d = ab^{-1}a$. Under the syntonic

[3] In a separate paper we will have a closer look into this tension, collecting more arguments in favor and against both interpretations of the step pattern of melodic minor.

projection $\overline{\pi}_{c\to b}$ the letter $\underline{d} = \mathfrak{l}(ab^{-1}c)$ is mapped to the letter $ab^{-1}b = a$. The augmented prime $\underline{d}$ is syntonically unnoticeable.

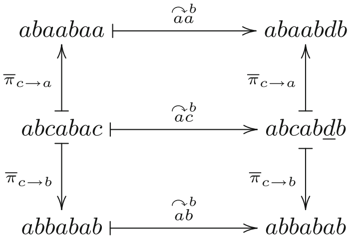

Fig. 3. Harmonic alteration of the minor mode. The middle row starts with the octave species of the twisted triadic Aeolian mode *abcabac* and yields $\underline{d}$ on the right. The upper row shows the corresponding diatonic projection, starting from *abaabaa* and reaches *abaabdb*. The middle and upper rows are connected along the letter projection $\overline{\pi}_{c\to a}$. The bottom row shows the corresponding syntonic projection, starting from the octave species of the syntonic Standard mode *abbabab* and ends with the same mode. The middle and bottom rows are connected along the letter projection $\overline{\pi}_{c\to b}$

Replacing all instances of the letters b and c in the middle rows of the two diagrams in Figs. 2 and 3 yields analogous derivations of the harmonic and melodic alterations of the major mode (c.f. Hauptmann's *Minor Major key*). While this letter-exchange leads to different diatonic projections it turns out that the syntonic projections are precisely the same as in Figs. 2 and 3.

In particular we can conclude that the (anti-standard) syntonic mode $(ab, babab)$ provides a unifying structure for the prominent forms of the major and minor keys.

5 Outlook: Embracing the Complete Syntonic Hierarchy

This outlook is inspired by Eytan Agmon's [2] discussion of the connection between counterpoint and harmony in harmonic tonality.

The "special structural status" of the perfect octave $P8$, perfect fifth $P5$ and perfect fourth $P4$ for the diatonic system can also be related to their role as periods. $P8$ is the main period of all diatonic modes. $P5$ and $P4$ are the periods of the doubly-periodic Guidonian hexachord *aabaa* (in the word-theoretic sense of periodicity). The corresponding prefixes *aaba* and *aab* are—at the same time—the Ionian species of the fifth and the fourth, whose concatenations yield the Ionian and Hypo-Ionian species of the octave.

When Agmon speaks of the thirds and sixths as cyclic generators, he thinks of the triads and seventh chords as chains of thirds within the generic layer of the diatonic scale, i.e. the cyclic group $\mathbb{Z}_7$ of the seven scale degrees. The third-generated structures have been motivated by the fact they afford smooth diatonic voice leading among themselves [1]. The concept of the *syntonic mode*

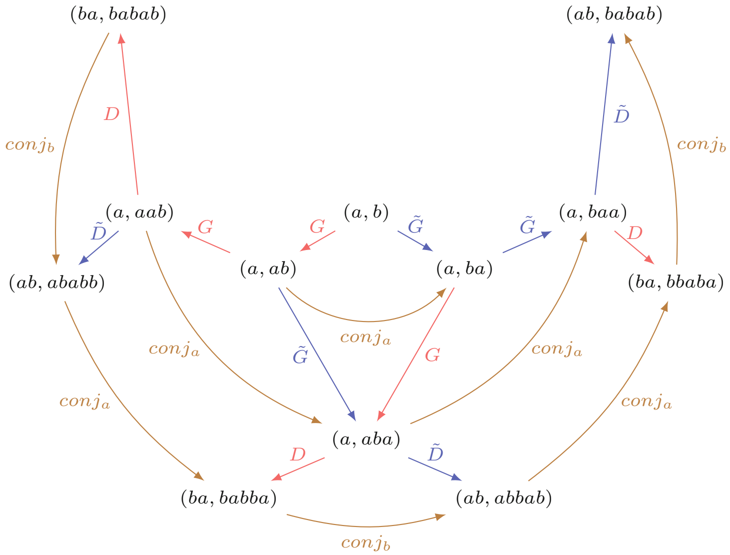

Fig. 4. The nodes along each of the concentric circular arcs form a complete conjugation class of special Sturmian morphisms. Each morphism f is represented by the pair $(f(a), f(b))$ of images of the letters a and b. The node (a, b) in the center represents the Identity map. Then from inside outwards the conjugation classes of G, GG, and DGG are displayed. The graph forms a subgraph of the Cayley graph of the automorphism group $Aut(F_2)$ with respect to the generators $G, \tilde{G}, D, \tilde{D}, conj_a, conj_b$. Each single conjugation class forms a linear graph, whose arrows are all labeled with one of the conjugations $conj_a$ or $conj_b$. The outward reaching arrows, connecting nodes on successive arcs, are labeled with the generators $G, \tilde{G}, D, \tilde{D}$ of the special Sturmian monoid (= monoid of special positive automorphisms).

is a refinement of this approach, in the sense that the syntonic modes are third-generated. We see in Fig. 4 six out of the seven syntonic modes, which form a complete conjugation class of Special Sturmian morphisms. The missing mode is the—so called—bad conjugate $(bb, ababa)$. They are shown along the outer arc of the diagram in Fig. 4 and form the syntonic analogoue of the six authentic Glarean modes. This analogy is—from the outset—a purely mathematical one. But the present paper is a first attempt to interpret them in musical terms. We may understand the interior arcs, moving inward, as the inversions of the seventh chords and of the triads, respectively. The inner node represents the basic division of the octave into the imperfect consonances third and sixth, and the divided words represent the filling-in of the third and the sixth. The missing elements—the bad conjugates—are the 6/4 triadic inversion (b, aa) and the 6/4/2 seventh-chord inversion (b, aaa).

On the one hand it is not surprising, that the innermost levels of the diatonic and the syntonic hierarchies are formed by the perfect and the imperfect consonances, respectively—perfect fifths and fourths as well as thirds and sixths:

pairs of octave complements. After all, they are the inner-triadic intervals. But this "tautological" consequence of our study of triadic modes offers, on the other hand, a new perspective on the concept of consonance, by itself. The consonances are particular simultaneities in polyphony. But at the same time they are spaces to be filled in subsequent transformations. It is a challenging project to review the emergence and development of polyphony in the light of this observation.

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