Chapter 1
Description of Signals

Abstract Signals are everywhere. Literally. The universe is bathed in the background radiation, the remnant of the original Big Bang and, as your eyes scan this page, a signal is being transmitted to your brain where different sets of neurons analyze it and process it. All human activities are based on processing and analysis of sensory signals but the goal of this book is somewhat narrower. The signals we will be mainly interested in can be described as *data* resulting from quantitative measurements of some physical phenomena and our emphasis will be on data that display *randomness* that may be due to different causes, be it errors of measurements, the algorithmic complexity, or the chaotic behavior of the underlying physical system itself.

1.1 Types of Random Signals

For the purpose of this book, signals will be functions of real variable t interpreted as time. To describe and analyze signals we will adopt the functional notation: $x(t)$ will denote the value of a nonrandom signal at time t. The values themselves can be real or complex numbers, in which case we will symbolically write $x(t) \in \mathbf{R}$, or, respectively, $x(t) \in \mathbf{C}$. In certain situations it is necessary to consider vector-valued signals with $x(t) \in \mathbf{R}^d$, where d stands for the dimension of the vector $x(t)$ with d real components.

Signals can be classified into different categories depending on their features. For example:

- *Analog signals* are functions of continuous time and their values form a continuum. *Digital signals* are functions of discrete time dictated by the computer's clock and their values are also discrete and dictated by the resolution of the

W. A. Woyczyński, *A First Course in Statistics for Signal Analysis*, Statistics for Industry, Technology, and Engineering, https://doi.org/10.1007/978-3-030-20908-7_1

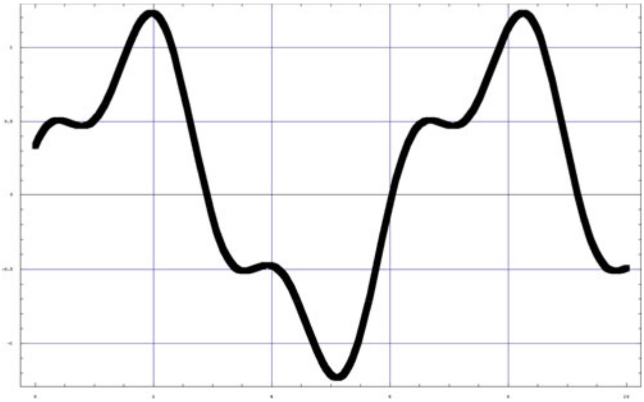

Fig. 1.1 Signal $x(t) = \sin(t) + \frac{1}{3}\cos(3t)$ [V] is analog and periodic with period $P = 2\pi$ [s]. It is also deterministic

system. Of course, one can also encounter mixed type signals which are sampled at discrete times but whose values are not restricted to any discrete set of numbers.

- *Periodic signals* are functions whose values are periodically repeated. In other words, for a certain number $P > 0$, we have $x(t + P) = x(t)$, for any t. Number P is called the *period of the signal*. *Aperiodic signals* are signals that are not periodic.
- *Deterministic signals* are signals not affected by random noise; there is no uncertainty about their values. *Random signals*, often also called *stochastic processes*, include an element of uncertainty; their analysis requires use of statistical tools and providing such tools is the principal goal of this book.

For example, signal $x(t) = \sin(t) + \frac{1}{3}\cos(3t)$ [V] shown in Fig. 1.1 is deterministic, analog, and periodic with period $P = 2\pi$ [s]. The same signal, digitally sampled during the first 5 s at time intervals equal to 0.5 s, with resolution 0.01 V, gives tabulated values:

t	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
$x(t)$	0.50	0.51	0.93	1.23	0.71	−0.16	0.51	−0.48	−0.78	−1.21

This sampling process is called the *analog-to-digital conversion*: given the *sampling period* T and the *resolution* R, the digitized signal $x_d(t)$ is of the form

$$x_d(t) = R\left\lfloor \frac{x(t)}{R} \right\rfloor, \quad \text{for} \quad t = T, 2T, \ldots, \tag{1.1.1}$$

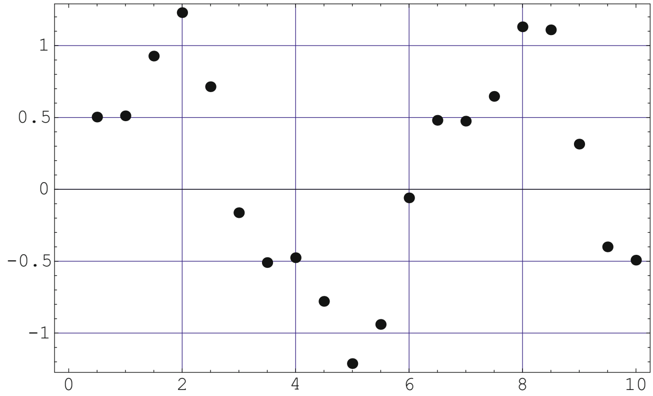

Fig. 1.2 Signal $x(t) = \sin(t) + \frac{1}{3}\cos(3t)$ [V] digitally sampled at time intervals equal to 0.5 s with resolution 0.01 V

where the, convenient to introduce here, "floor" function $\lfloor a \rfloor$ is defined as the largest integer not exceeding real number a. For example, $\lfloor 5.7 \rfloor = 5$, but $\lfloor 5.0 \rfloor = 5$, as well.

Note the role the resolution R plays in the above formula. Take, for example, $R = 0.01$. If the signal $x(t)$ takes all the continuous values between $m = \min_t x(t)$ and $M = \max_t x(t)$, then $x(t)/0.01$ takes all the continuous values between $100 \cdot m$ and $100 \cdot M$, but $\lfloor x(t)/0.01 \rfloor$ takes only integer values between $100 \cdot m$ and $100 \cdot M$. Finally, $0.01 \lfloor x(t)/0.01 \rfloor$ takes as its values only all the discrete numbers between m and M that are 0.01 apart (Fig. 1.2).

Randomness of signals can have different origin, be it quantum *uncertainty principle*, *computational complexity* of algorithms, *chaotic behavior* in dynamical systems, or random fluctuations and errors in measurement of outcomes of independently repeated experiments.[1] The usual way to study them is via their aggregated statistical properties. The main purpose of this book is to introduce some of the basic mathematical and statistical tools useful in analysis of random signals that are produced under *stationary conditions*, that is, in situations where the measured signal may be stochastic and contain random fluctuations, but the basic underlying random mechanism producing it does not change over time; think here about outcomes of independently repeated experiments, each consisting of tossing a single coin (Fig. 1.3).

At this point, to help the reader visualize the great variety of random signals appearing in the physical sciences and engineering, it is worthwhile to review a gallery of pictures of random signals, both experimental and simulated, presented in Figs. 1.4, 1.5, 1.6, 1.7, and 1.8. The captions explain the context in each case.

---

[1] See, e.g., M. Denker and W.A. Woyczyński, *Introductory Statistics and Random Phenomena: Uncertainty, Complexity, and Chaotic Behavior in Engineering and Science,* Birkhäuser-Boston, 1998.

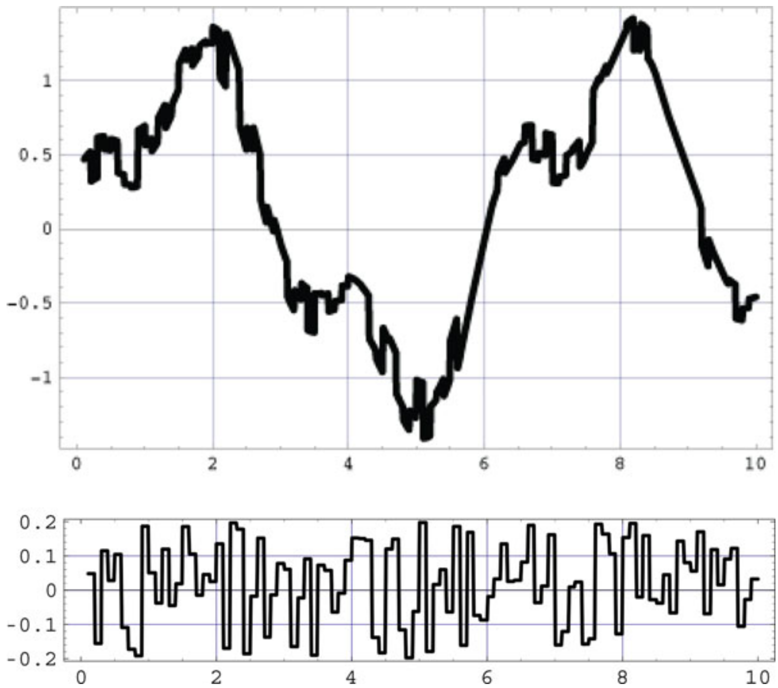

Fig. 1.3 Signal $x(t) = \sin(t) + \frac{1}{3}\cos(3t)$ [V] in presence of additive random noise with average amplitude of 0.2 V. The magnified noise component itself is pictured underneath the graph of the signal

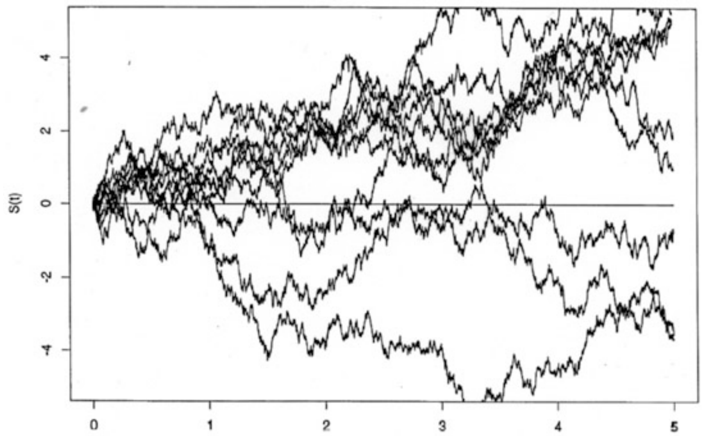

Fig. 1.4 Several, computer-generated trajectories (sample paths) of a random signal called the *Brownian motion* stochastic process or the *Wiener stochastic process*. Its trajectories, although very rough, are continuous. It is often used as a simple model of *diffusion*. The random mechanism that created different trajectories was the same. Its importance for our subject matter will become clear in Chap. 9

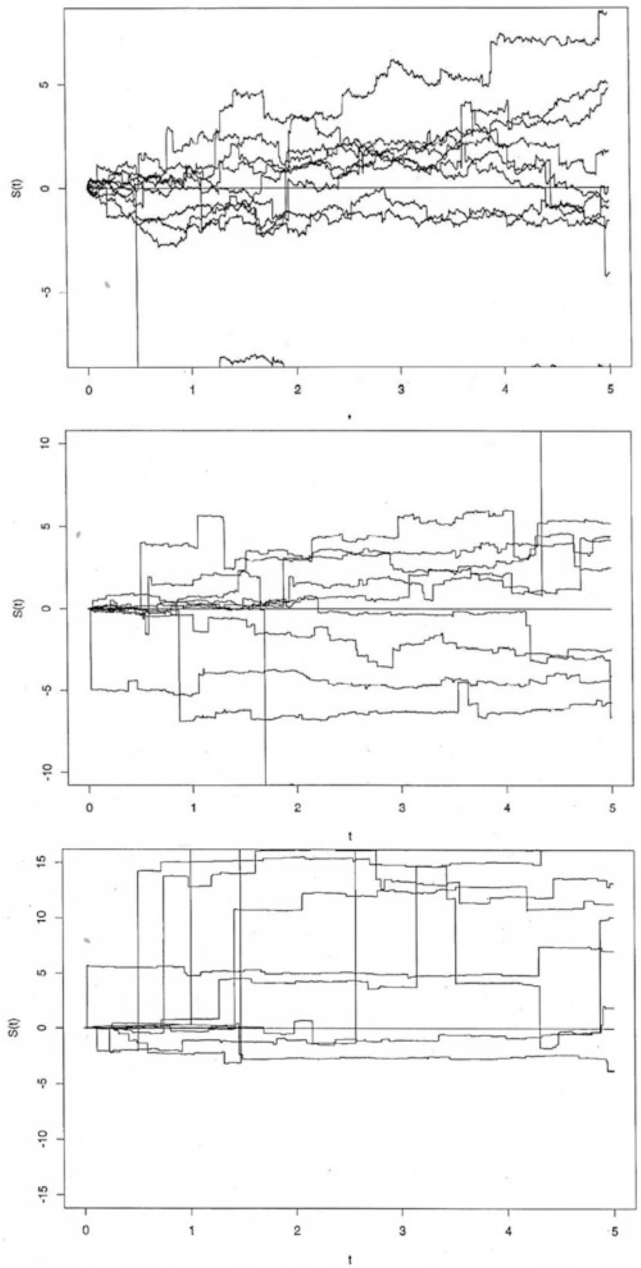

Fig. 1.5 Several, computer-generated trajectories (sample paths) of random signals called *Lévy stochastic processes* with parameter $\alpha = 1.5, 1$, and 0.75, respectively (from top to bottom). They are often used to model anomalous diffusion processes wherein diffusing particles are also permitted to change their position by jumping. Parameter α indicates intensity of jumps of different sizes. Parameter value $\alpha = 2$ corresponds to the Wiener process (shown in Fig. 1.4) which has trajectories that have no jumps. In each figure, the random mechanism that created different trajectories was the same. However, different random mechanisms led to trajectories presented in different figures

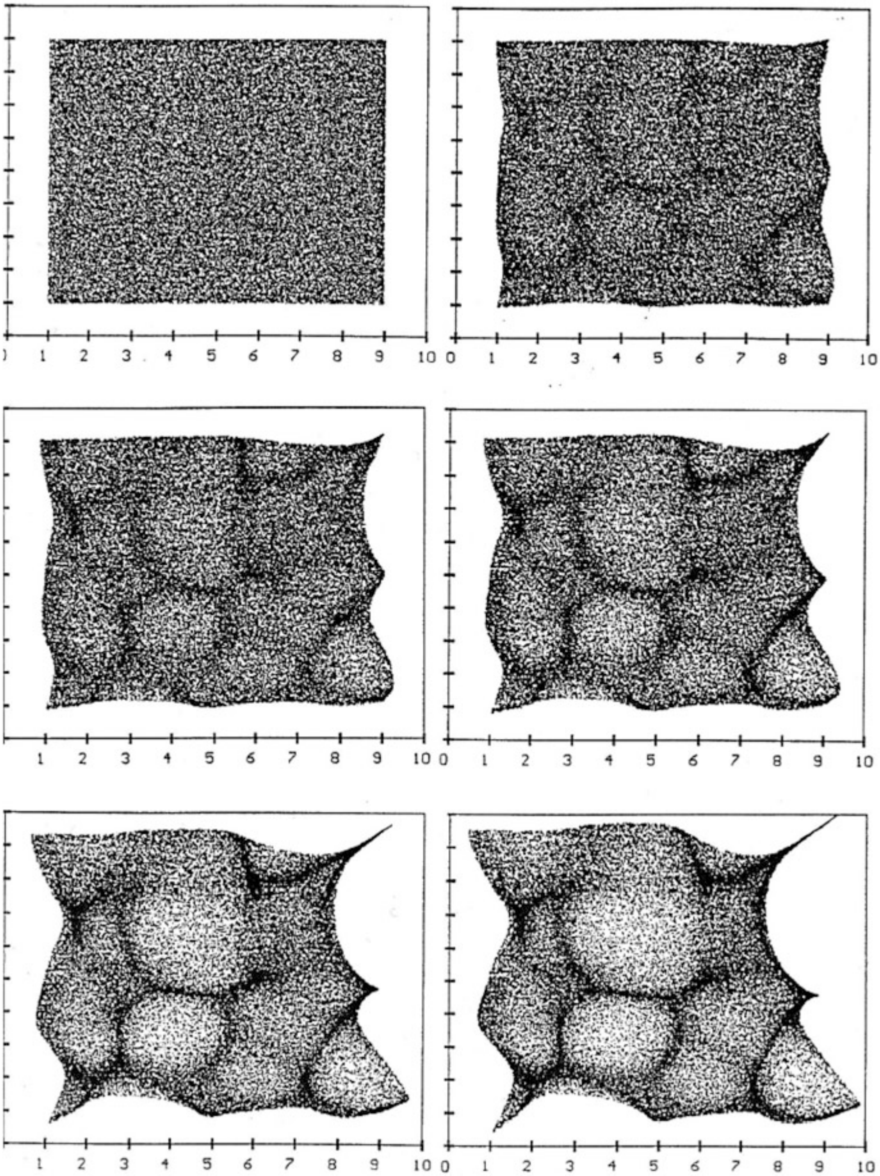

Fig. 1.6 Computer simulation of the evolution of passive tracer density in a turbulent velocity field with random initial distribution and random "shot-noise" initial velocity data. The simulation was performed for 100,000 particles. The consecutive frames show the location of passive tracer particles at times $t = 0.0, 0.3, 0.6, 1.0, 2.0, 3.0\,\mathrm{s}$

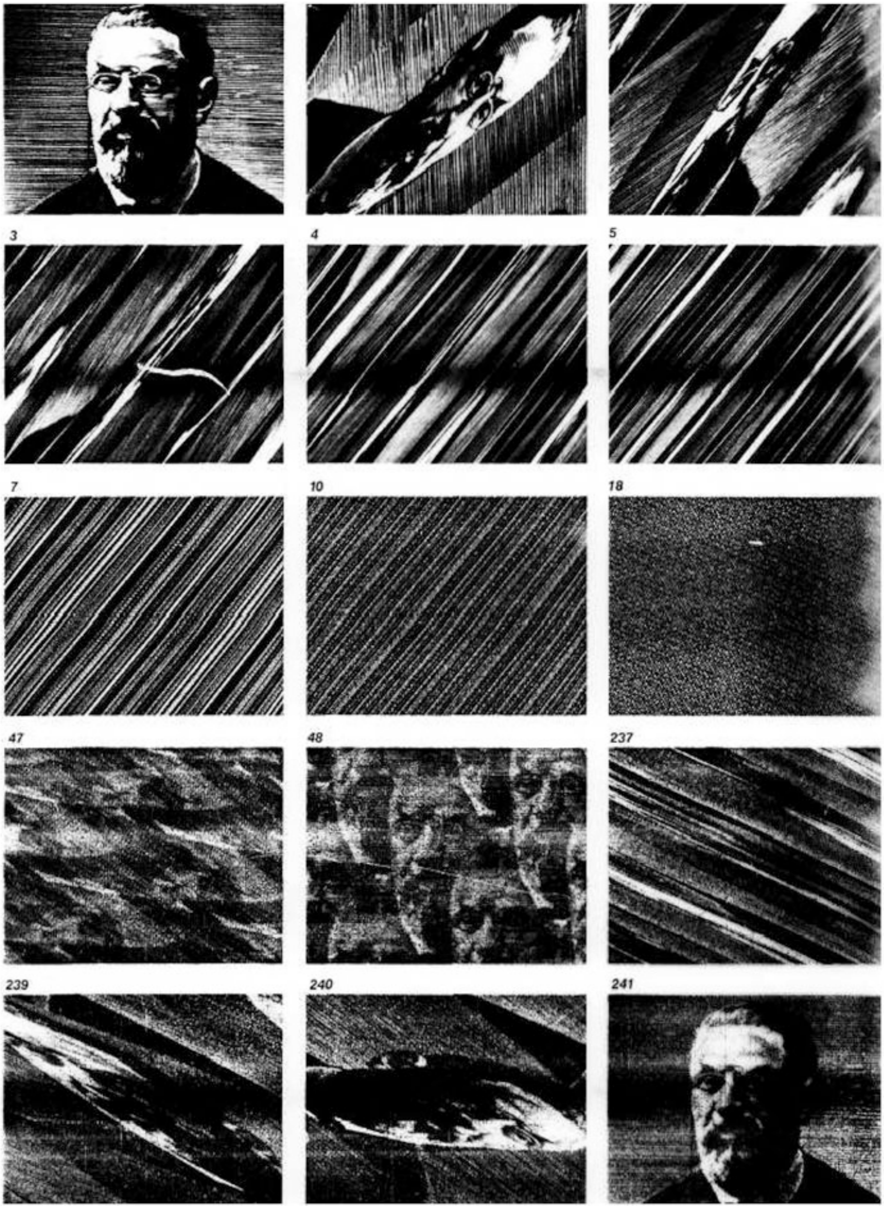

Fig. 1.7 Some deterministic signals (in this case, the images) transformed by deterministic systems can appear random. The above picture shows a series of iterated transformations of the original image via a fixed linear 2D mapping (matrix). The number of iterations applied is indicated in the top left corner of each image. The curious behavior of iterations, the original image first dissolving into seeming randomness only to return later to an almost original condition, is related to the so-called *ergodic* behavior. Thus irreverently transformed is Professor Henri Poincaré (1854–1912) of the University of Paris, the pioneer of ergodic theory of stationary phenomena (From *Scientific American*, reproduced with permission. Copyright 1986, James P. Crutchfield)

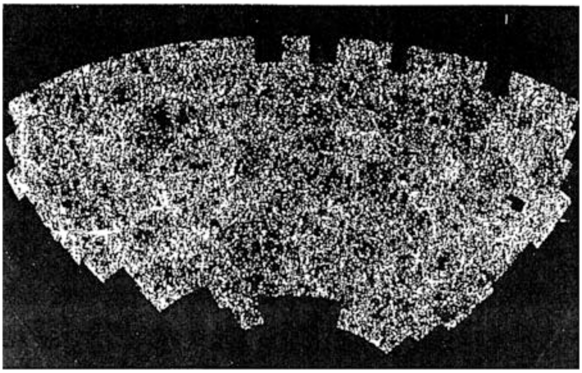

Fig. 1.8 A signal (again, an image) representing the large-scale and apparently random distribution of mass in the universe. The data come from the APM galaxy survey and shows more than two million galaxies in a section of sky centered on the South Galactic pole. The so-called *adhesion model* of the large scale mass distribution in the Universe uses Burgers equation to model the relevant velocity fields

The signals shown in Figs. 1.4 and 1.5 are, obviously, not stationary and have a diffusive character. However, their increments (differentials) are stationary and, in Chap. 9, they will play an important role in construction of the spectral representation of stationary signals themselves. The signal shown in Fig. 1.4 can be interpreted as a *trajectory*, or *sample path*, of a *random walker* moving, in discrete time steps, up or down a certain distance with equal probabilities 1/2 and 1/2. However, in the picture these trajectories are viewed from far away, and in accelerated time, so that both time and space appear continuous.

In certain situations the randomness of the signal is due to uncertainty about initial conditions of the underlying phenomenon which otherwise can be described by perfectly deterministic models such as partial differential equations. A sequence of pictures in Fig. 1.6 shows evolution of the system of particles with an initially random (and homogeneous in space) spatial distribution. The particles are then driven by the velocity field $\vec{v}(t,\vec{x}) \in \mathbf{R}^2$ governed by the so-called *2D Burgers equation*[2]

$$\frac{\partial \vec{v}(t,\vec{x})}{\partial t} + \Big(\nabla \cdot \vec{v}(t,\vec{x})\Big)\vec{v}(t,\vec{x}) = D\left(\frac{\partial^2 \vec{v}(t,\vec{x})}{\partial x_1^2} + \frac{\partial^2 \vec{v}(t,\vec{x})}{\partial x_2^2}\right), \qquad (1.1.2)$$

where $\vec{x} = (x_1, x_2)$, the *nabla* operator $\nabla = \partial/\partial x_1 + \partial/\partial x_2$, and the positive constant D is the coefficient of diffusivity. The initial velocity field is also assumed to be random.

[2]See, e.g., W.A. Woyczyński, *Burgers-KPZ Turbulence–Göttingen Lectures*, Springer-Verlag 1998.

1.2 Characteristics of Signals

Several physical characteristics of signals are of primary interest.

- *The time average of the signal:* For analog, continuous-time signals the time average is defined by the formula

$$\mathbf{AV}_x = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\,dt, \tag{1.2.1}$$

and for digital, discrete-time signals which are defined only for the time instants $t = nT,\ n = 0, 1, 2, \ldots, N-1$, it is defined by the formula

$$\mathbf{AV}_x = \frac{1}{N}\sum_{n=0}^{N-1} x(nT). \tag{1.2.2}$$

For periodic signals, it follows from (1.2.1) that

$$\mathbf{AV}_x = \frac{1}{P}\int_0^P x(t)\,dt, \tag{1.2.3}$$

so that, for the signal $x(t) = \sin t + (1/3)\cos(3t)$ pictured in Fig. 1.1, the time average is 0 as both $\sin t$ and $\cos(3t)$ integrate out to zero over the period $P = 2\pi$.

- *Energy of the signal:* For an analog signal $x(t)$, the total energy

$$\mathbf{EN}_x = \int_0^\infty |x(t)|^2\,dt, \tag{1.2.4}$$

and for digital signals

$$\mathbf{EN}_x = \sum_{n=0}^{\infty} |x(nT)|^2 \cdot T. \tag{1.2.5}$$

Observe that the energy of a periodic signal, such as the one from Fig. 1.1, is necessarily infinite if considered over the whole positive time line. Also note that, since in what follows it will be convenient to consider complex-valued signals, the above formulas include notation for the square of the modulus of a complex number: $|z|^2 = (\mathrm{Re}\,z)^2 + (\mathrm{Im}\,z)^2 = z \cdot z^*$; more about it in the next section.

- *Power of the signal:* Again, for an analog signal, the (average) power

$$\mathbf{PW}_x = \lim_{T\to\infty} \frac{1}{T}\int_0^T |x(t)|^2\,dt \tag{1.2.6}$$

and for a digital signal

$$\mathbf{PW}_x = \lim_{N\to\infty} \frac{1}{NT} \sum_{n=0}^{N-1} |x(nT)|^2 \cdot T = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |x(nT)|^2. \tag{1.2.7}$$

As a consequence, for an analog periodic signal with period P,

$$\mathbf{PW}_x = \frac{1}{P} \int_0^P |x(t)|^2 \, dt. \tag{1.2.8}$$

For example, for the signal in Fig. 1.1,

$$\mathbf{PW}_x = \frac{1}{2\pi} \int_0^{2\pi} \Big(\sin t + (1/3)\cos(3t) \Big)^2 dt \tag{1.2.9}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(\sin^2 t + \frac{2}{3} \sin t \cos(3t) + \frac{1}{9} \cos^2(3t) \right) dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{2}(1 - \cos(2t)) + \frac{2}{3}\frac{1}{2}(\sin(4t) - \sin(2t)) + \frac{1}{9}\frac{1}{2}(1 + \cos(6t)) \right) dt$$

$$= \frac{1}{2\pi} \left(\frac{1}{2} 2\pi + \frac{1}{9}\frac{1}{2} 2\pi \right) = \frac{5}{9}.$$

The above routine calculation, deliberately carried out here in detail, was somewhat tedious because of the need for various trigonometric identities. To simplify such manipulations and make the whole theory more elegant, we will introduce in the next section a complex number representation of the trigonometric functions via the so-called de Moivre formulas.

Remark 1.2.1 (Timeline Infinite in Both Direction) Sometimes it is convenient to consider signals defined for all time instants t, $-\infty < t < +\infty$, rather than just for positive t. In such cases all of the above definitions have to be adjusted in obvious ways, replacing the one-sided integrals and sums by two-sides integrals and sums, and adjusting the averaging constants correspondingly.

1.3 Time Domain and Frequency Domain Descriptions of Periodic Signals

The Time Domain Description The trigonometric functions

$$x(t) = \cos(2\pi f_0 t), \qquad \text{and} \qquad y(t) = \sin(2\pi f_0 t),$$

represent a harmonically oscillating signal with period $P = 1/f_0$ (measured, say, in seconds [s]), and the frequency f_0 (measured, say, in cycles per second, or Hertz [Hz]), and so do the trigonometric functions

$$x(t) = \cos(2\pi f_0(t+\theta)), \qquad \text{and} \qquad y(t) = \sin(2\pi f_0(t+\theta))$$

shifted by the phase-shift θ. The powers

$$\mathbf{PW}_x = \frac{1}{P}\int_0^P \cos^2(2\pi f_0 t)\,dt = \frac{1}{P}\int_0^P \frac{1}{2}(1+\cos(4\pi f_0 t))\,dt = \frac{1}{2}, \tag{1.3.1}$$

$$\mathbf{PW}_y = \frac{1}{P}\int_0^P \sin^2(2\pi f_0 t)\,dt = \frac{1}{P}\int_0^P \frac{1}{2}(1-\cos(4\pi f_0 t))\,dt = \frac{1}{2}, \tag{1.3.2}$$

using the trigonometric formulas from Tables 1.1 and 1.2. The phase shifts, obviously do not change the power of the above harmonic signals.

Table 1.1 Trigonometric formulas

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha;$
$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta;$
$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2};$
$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2};$
$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2};$
$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2};$
$\sin^2\alpha - \sin^2\beta = \cos^2\beta - \cos^2\alpha = \sin(\alpha+\beta)\sin(\alpha-\beta);$
$\cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha = \cos(\alpha+\beta)\cos(\alpha-\beta);$
$\sin\alpha\cos\beta = \frac{1}{2}\Big[\sin(\alpha+\beta) + \sin(\alpha-\beta)\Big];$
$\cos\alpha\cos\beta = \frac{1}{2}\Big[\cos(\alpha+\beta) + \cos(\alpha-\beta)\Big];$
$\sin\alpha\sin\beta = \frac{1}{2}\Big[\cos(\alpha-\beta) - \cos(\alpha+\beta)\Big];$

Table 1.2 Complex numbers and De Moivre formulas

(i) By definition,

$$j = \sqrt{-1}.$$

(ii) Hence, for any integer m,

$$j^{4m} = 1, \quad j^{4m+1} = j, \quad j^{4m+2} = -1, \quad j^{4m+3} = -j.$$

(iii) Cartesian representation of the complex number:

$$z = a + jb, \quad a = \operatorname{Re} z, \quad b = \operatorname{Im} z,$$

where both a and b are real numbers and are called, respectively, the real and imaginary components of z. The complex number,

$$z^* = a - jb,$$

is called the complex conjugate of z.

(iv) The polar representation of the complex number (it is a good idea to think about complex numbers as representing points, or vectors, in the two-dimensional plane spanned by the two basic unit vectors, 1 and j):

$$z = |z|(\cos\theta + j\sin\theta) = |z| \cdot e^{j\theta},$$

and

$$z^* = |z|(\cos\theta - j\sin\theta) = |z| \cdot e^{-j\theta},$$

where

$$|z| = \sqrt{a^2 + b^2} = \sqrt{z \cdot z^*}, \quad \text{and} \quad \theta = \operatorname{Arg} z = \arctan\frac{\operatorname{Im} z}{\operatorname{Re} z},$$

is called, respectively, the *modulus* of z, and the *argument* of z. Alternatively,

$$\operatorname{Re} z = \frac{z + z*}{2} = |z|\cos\theta, \quad \operatorname{Im} z = \frac{z - z*}{2j} = |z|\sin\theta.$$

(v) For any complex number $w = \beta + j\alpha$,

$$e^w = e^{\beta + j\alpha} = e^{\beta}(\cos\alpha + j\sin\alpha).$$

(vi) For any complex number $z = a + jb = |z|e^{j\theta}$, and any integer n,

$$z^n = |z|^n e^{jn\theta} = (a^2 + b^2)^{n/2}(\cos n\theta + j\sin n\theta).$$

Taking their linear combination (like the one in Fig. 1.1), with amplitudes A and B, respectively,

$$z(t) = Ax(t) + By(t) = A\cos(2\pi f_0(t+\theta)) + B\sin(2\pi f_0(t+\theta)), \tag{1.3.3}$$

also yields a periodic signal with frequency f_0. For a signal written in this form we no longer need to include the phase shift explicitly since

$$\cos(2\pi f_0(t+\theta)) = \cos(2\pi f_0 t)\cos(2\pi f_0\theta) - \sin(2\pi f_0 t)\sin(2\pi f_0\theta),$$

and

$$\sin(2\pi f_0(t+\theta)) = \sin(2\pi f_0 t)\cos(2\pi f_0\theta) + \cos(2\pi f_0 t)\sin(2\pi f_0\theta),$$

so that

$$z(t) = a\cos(2\pi f_0 t) + b\sin(2\pi f_0 t), \tag{1.3.4}$$

with the new amplitudes

$$a = A\cos(2\pi f_0\theta) + B\sin(2\pi f_0\theta), \qquad \text{and} \qquad b = B\cos(2\pi f_0\theta) - A\sin(2\pi f_0\theta).$$

The power of the signal $z(t)$, in view of (1.3.1) and (1.3.2), is given by the Pythagorean-like formula

$$\mathbf{PW}_z = \frac{1}{P}\int_0^P z^2(t)\,dt = \frac{1}{P}\int_0^P (a\cos(2\pi f_0 t) + b\sin(2\pi f_0 t))^2\,dt$$

$$= a^2 \cdot \mathbf{PW}_x + b^2 \cdot \mathbf{PW}_y + 2ab\frac{1}{P}\int_0^P \cos(2\pi f_0 t)\sin(2\pi f_0 t)\,dt = \frac{1}{2}(a^2+b^2), \tag{1.3.5}$$

because (see Tables 1.1 and 1.2, again)

$$\frac{1}{P}\int_0^P \cos(2\pi f_0 t)\sin(2\pi f_0 t)\,dt = \frac{1}{P}\int_0^P \frac{1}{2}\sin(4\pi f_0 t)\,dt = 0. \tag{1.3.6}$$

The above property (1.3.6), called *orthogonality* of the sine and cosine signals, will play a fundamental role in this book.

The next observation is that signals

$$z(t) = a\cos(2\pi(mf_0)t) + b\sin(2\pi(mf_0)t), \qquad m = 0, 1, 2, \ldots,$$

have the frequency equal to the multiplicity m of the *fundamental* frequency f_0, and as such have, in particular, period P (but also period P/m). Their power is also equal to $(a^2 + b^2)/2$. So, if we superpose M of them, with possibly different amplitudes a_m and b_m, for different $m = 0, 1, 2, \ldots, M$, the result is a periodic signal

$$
\begin{aligned}
x(t) &= \sum_{m=0}^{M} \Big(a_m \cos(2\pi(mf_0)t) + b_m \sin(2\pi(mf_0)t) \Big) \\
&= a_0 + \sum_{m=1}^{M} \Big(a_m \cos(2\pi(mf_0)t) + b_m \sin(2\pi(mf_0)t) \Big) \qquad (1.3.7)
\end{aligned}
$$

with period P, and the fundamental frequency $f_0 = 1/P$, which has the mean and power

$$
\mathbf{AV}_x = a_0, \qquad \text{and} \qquad \mathbf{PW}_x = a_0^2 + \frac{1}{2} \sum_{m=1}^{M} (a_m^2 + b_m^2). \qquad (1.3.8)
$$

The above result follows from the fact that not only sine and cosine signals (of arbitrary frequencies) are orthogonal to each other (see, (1.3.6)) but also cosines of different frequencies are *orthogonal* to each other, and so are sines. Indeed, if $m \neq n$, that is, $m - n \neq 0$, then

$$
\frac{1}{P} \int_0^P \cos(2\pi m f_0 t) \cos(2\pi n f_0 t)\, dt \qquad (1.3.9)
$$

$$
= \frac{1}{P} \int_0^P \frac{1}{2} \Big(\cos(2\pi(m-n) f_0 t) + \cos(2\pi(m+n) f_0 t) \Big)\, dt = 0,
$$

and

$$
\frac{1}{P} \int_0^P \sin(2\pi m f_0 t) \sin(2\pi n f_0 t)\, dt \qquad (1.3.10)
$$

$$
= \frac{1}{P} \int_0^P \frac{1}{2} \Big(\cos(2\pi(m-n) f_0 t) - \cos(2\pi(m+n) f_0 t) \Big)\, dt = 0.
$$

Example 1.3.1 (Superposition of Simple Cosine Oscillations) Consider the signal

$$
x(t) = \sum_{m=1}^{12} \frac{1}{m^2} \cos(2\pi m t). \qquad (1.3.11)
$$

Its fundamental frequency is $f_0 = 1$, its average $\mathbf{AV}_x = 0$, and its power (see, (1.3.8))

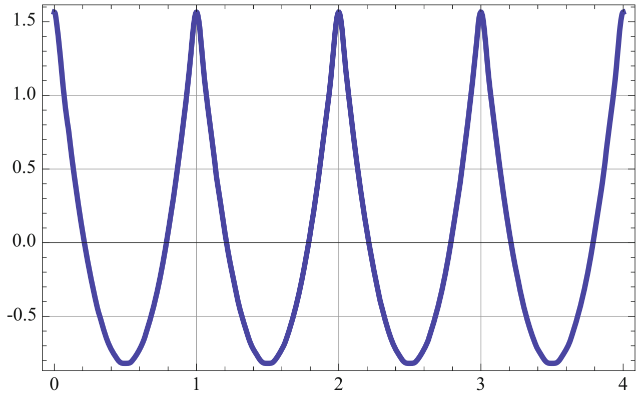

Fig. 1.9 Signal $x(t) = \sum_{m=1}^{12} m^{-2}\cos(2\pi mt)$ in its time-domain representation

$$\mathbf{PW}_x = \frac{1}{2}\sum_{m=1}^{12}\left(\frac{1}{m^2}\right)^2 \approx 0.541.$$

With its sharp cusps, the shape of the above signal is unlike that of any simple harmonic oscillation and one could start wondering what kind of other periodic signals can be well represented (approximated) by superpositions of harmonic oscillations of the form (1.3.7). The answer, discussed at length in Chap. 2, is that almost all of them can, as long as their power is finite (Fig. 1.9).

The Frequency Domain Description The signal $x(t)$ in Example 1.3.1 would be completely specified if, instead of writing the whole formula (1.3.11), we just listed the frequencies present in the signal and the corresponding amplitudes, that is, considered the list

$$(1, 1/1^2), (2, 1/2^2), (3, 1/3^3), \ldots, (12, 1/12^2).$$

Similarly, in the case of the general superposition (1.3.7), it would be sufficient to list the cosine and sine frequencies and associated amplitudes, that is, compile the lists

$$(0, a_0), (1f_0, a_1), (2f_0, a_2), \ldots, (Mf_0, a_M), \tag{1.3.12}$$

and

$$(1f_0, b_1), (2f_0, b_2), \ldots, (Mf_0, b_M). \tag{1.3.13}$$

The lists (sequences) ((1.3.12) and (1.3.13)) are called the *frequency domain (spectral) representation* of the signal (1.3.7).

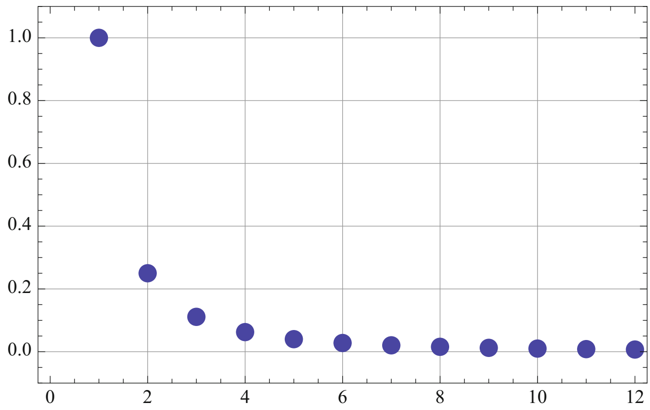

Fig. 1.10 Signal $x(t) = \sum_{m=1}^{12} m^{-2} \cos(2\pi mt)$ in its frequency domain representation. Only the amplitudes of frequencies $m = 1, 2, \ldots, 12$, are shown since all the phase shifts are zero

Remark 1.3.1 (Amplitude-Phase Form of the Spectral Representation) Alternatively, if the signal $x(t)$ in (1.3.7) is rewritten in the amplitude-phase form,

$$x(t) = \sum_{m=0}^{M} c_n \cos(2\pi(mf_0)(t + \theta_m)),$$

then the frequency domain representation must list the frequencies present in the signal, mf_0, $m = 0, 1, \ldots, M$, and the corresponding amplitudes c_m $m = 0, 1, \ldots, M$, and phases θ_m, $m = 0, 1, \ldots, M$.

For the signal from Example 1.3.1, such a representation is graphically pictured in Fig. 1.10. We will see in Chap. 2 that, for any periodic signal, the spectrum is always concentrated on a discrete set of frequencies, namely, the multiplicities of the fundamental frequency.

Finally, the formula (1.3.8) shows how the total power of signal $x(t)$ is distributed over different frequencies. Such a distribution, provided by the list

$$(0, a_0^2), (1f_0, (a_1^2 + b_1^2)/2), (2f_0, (a_2^2 + b_2^2)/2), \ldots, (Mf_0, (a_M^2 + b_M^2)/2), \tag{1.3.14}$$

is called the *power spectrum* of the periodic signal (1.3.7).

Observe that, in general, knowledge of the power spectrum is not sufficient for the reconstruction of the signal $x(t)$ itself, while knowledge of the whole representation in the frequency domain is.

To complete our elementary study of periodic signals note that if an arbitrary signal is studied only in a finite time interval $[0, P]$, then it can always be treated as a periodic signal with period P since one can extend its definition periodically to the whole time line by copying its waveform from the interval $[0, P]$ to intervals $[P, 2P]$, $[2P, 3P]$, and so on.

1.4 Building a Better Mousetrap: Complex Exponentials

Catching the structure of periodic signals via their decomposition into a superposition of basic trigonometric functions leads to some cumbersome calculations employing various trigonometric identities (as we have seen in Sect. 1.3). A greatly simplified and also more elegant approach to the same problem employs a representation of trigonometric functions in terms of exponential functions of the imaginary variable. The cost of moving into the complex domain is not high as we will rely, essentially, on a single relationship

$$e^{j\alpha} = \cos\alpha + j\sin\alpha, \quad \text{where} \quad j = \sqrt{-1}, \tag{1.4.1}$$

which is known as *de Moivre formula*,[3] and which immediately yields two identities

$$\cos\alpha = \frac{1}{2}(e^{j\alpha} + e^{-j\alpha}), \qquad \text{and} \qquad \sin\alpha = \frac{1}{2j}(e^{j\alpha} - e^{-j\alpha}). \tag{1.4.2}$$

In what follows, we are going to routinely utilize the complex number techniques. Thus, for the benefit of the reader, the basic notation and facts about them are summarized in Table 1.2.

Since de Moivre formula is so crucial for us, it is important to understand where it is coming from. The proof is straightforward and relies on the power series expansion of the exponential function,

$$e^{j\alpha} = \sum_{k=0}^{\infty} \frac{j^k \alpha^k}{k!}. \tag{1.4.3}$$

However, the powers of the imaginary unit j can be expressed via a simple formula

$$j^k = \begin{cases} 1, & \text{if} k = 4m; \\ j, & \text{if} k = 4m+1; \\ -1, & \text{if} k = 4m+2; \\ -j, & \text{if} k = 4m+3, \end{cases}$$

so the whole series (1.4.3) splits neatly into the real part, corresponding to even indices of the form $k = 2n$, $n = 0, 1, 2, \ldots,$, and the imaginary part, corresponding to the odd indices of the form $k = 2n+1$, $n = 0, 1, 2, \ldots$:

[3] Throughout this book we denote the imaginary unit $\sqrt{-1}$ by the letter j, which is a standard usage in the electrical engineering signal processing literature as the, usual in the mathematical literature, letter i is reserved for electrical current.

$$\sum_{k=0}^{\infty} \frac{j^k \alpha^k}{k!} = \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n}}{(2n)!} + j \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!}.$$

Now, it suffices to recognize in the above formula the familiar power series expansions for trigonometric functions,

$$\cos\alpha = \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n}}{(2n)!}, \qquad \sin\alpha = \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!}$$

to obtain de Moivre formula.

Given de Moivre formulas which provides a representation of sine and cosine functions via the complex exponentials, we can now rewrite the general superposition of harmonic oscillation

$$x(t) = a_0 + \sum_{m=1}^{M} a_m \cos(2\pi m f_0 t) + \sum_{m=1}^{M} b_m \sin(2\pi m f_0 t), \tag{1.4.4}$$

in terms of the complex exponentials

$$x(t) = \sum_{m=-M}^{M} z_m e^{j2\pi m f_0 t}, \tag{1.4.5}$$

with the real amplitudes, a_m and b_m, in representations (1.4.4), and the complex amplitudes z_m in the representation (1.4.5), connected by the formulas

$$a_0 = z_0, \qquad a_m = z_m + z_{-m}, \qquad b_m = j(z_m - z_{-m}), \qquad m = 1, 2, \ldots,$$

or, equivalently,

$$z_0 = a_0 \qquad z_m = \frac{a_m - jb_m}{2}, \qquad z_{-m} = \frac{a_m + jb_m}{2}, \qquad m = 1, 2, \ldots$$

The above relationships show that for the signal of the form (1.4.5) to represent a real-valued signal $x(t)$ it is necessary and sufficient that the paired amplitudes for symmetric frequencies, mf_0 and $-mf_0$, be complex conjugates of each other:

$$z_{-m} = z_m^*, \qquad m = 1, 2, \ldots \tag{1.4.6}$$

However, in the future it will be convenient to consider general complex-valued signals of the form (1.4.5) without the restriction (1.4.6) on its complex amplitudes.

At the first sight, the above introduction of complex numbers and functions of complex-valued variables may seem as an unnecessary complication in the analysis of signals. But let us calculate the power of the signal $x(t)$ given by (1.4.5). The need for unpleasant trigonometric formulas disappears as now we need to integrate only exponential functions. Indeed, remembering the $|z|^2 = z \cdot z^*$ now stands for the square of the modulus of a complex number, we have

$$\mathbf{PW}_x = \frac{1}{P}\int_{t=0}^{P} |x(t)|^2\,dt = \frac{1}{P}\int_{t=0}^{P} \Big|\sum_{m=-M}^{M} z_m e^{j2\pi m f_0 t}\Big|^2\,dt$$

$$= \frac{1}{P}\int_{t=0}^{P} \Big(\sum_{m=-M}^{M} z_m e^{j2\pi m f_0 t} \cdot \sum_{k=-M}^{M} z_k^* e^{-j2\pi k f_0 t}\Big)\,dt$$

$$= \frac{1}{P}\sum_{m=-M}^{M}\sum_{k=-M}^{M} z_m z_k^* \int_{t=0}^{P} e^{j2\pi(m-k)f_0 t}\,dt = \sum_{m=-M}^{M} |z_m|^2, \tag{1.4.7}$$

because, for $m-k \neq 0$,

$$\frac{1}{P}\int_{t=0}^{P} e^{j2\pi(m-k)f_0 t}\,dt = \frac{1}{j2\pi(m-k)f_0} e^{j2\pi(m-k)f_0 t}\Big|_{t=0}^{P} = 0, \tag{1.4.8}$$

as the function $e^{j2\pi(m-k)f_0 t} = \cos(2\pi(m-k)f_0 t) + j\sin(2\pi(m-k)f_0 t)$ is periodic with period P, and for $m-k=0$,

$$\frac{1}{P}\int_{t=0}^{P} e^{j2\pi(m-k)f_0 t}\,dt = 1. \tag{1.4.9}$$

Thus all the off-diagonal terms in the double sum in (1.4.7) disappear. The formulas (1.4.8) and (1.4.9) express mutual orthogonality and normalization of the complex exponential signals,

$$e^{j2\pi m f_0 t}, \qquad m = 0, \pm 1, \pm 2, \ldots, \pm M.$$

In view of (1.4.7), the distribution of the power of the signal (1.4.5) over different multiplicities of the fundamental frequency f_0 can be written as a list with simple structure,

$$(mf_0, |z_m|^2), \qquad m = 0, \pm 1, \pm 2, \ldots, \pm M. \tag{1.4.10}$$

Remark 1.4.1 (Aperiodic Signals) Nonperiodic signals can also be analyzed in terms of their frequency domains but their spectra are not discrete. We will study them later on.

1.5 Problems and Exercises

1* Find the real and imaginary parts of $(j+3)/(j-3)$; $(1+j\sqrt{2})^3$; $1/(2-j)$; $(2-3j)/(3j+2)$.[4]

2* Find the moduli $|z|$ and arguments θ of complex numbers $z=5$; $z=-2j$; $z=-1+j$; $z=3+4j$.

3* Find the real and imaginary components of complex numbers $z=5\,e^{j\pi/4}$; $z=-2\,e^{j(8\pi+1.27)}$; $z=-1\,e^{j}$; $z=3\,e^{je}$.

4* Show that

$$\frac{5}{(1-j)(2-j)(3-j)}=\frac{j}{2},\quad \text{and}\quad (1-j)^4=-4.$$

5* Sketch sets of points in complex plane (x,y), $z=x+jy$, such that $|z-1+j|=1$; $|z+j|\le 3$; $\mathrm{Re}\,(z^*-j)=2$; $|2z-j|=4$; $z^2+(z^*)^2=2$.

6* Using de Moivre's formulas find $(-2j)^{1/2}$ and $\mathrm{Re}\,(1-j\sqrt{3})^{77}$. Are these complex numbers uniquely defined?

7 Write the signal $x(t)=\sin t+\cos(3t)/3$ from Fig. 1.1 as a sum of phase-shifted cosines.

8 Using de Moivre's formulas write the signal $x(t)=\sin t+\cos(3t)/3$ from Fig. 1.1 as a sum of complex exponentials.

9 Find the time average and power of the signal $x(t)=-2e^{-j2\pi 4t}+3e^{-j2\pi t}+1-2e^{j2\pi 3t}$. What is the fundamental frequency of this signal? Plot the distribution of power of $x(t)$ over different frequencies. Write this (complex) signal in terms of cosines and sines. Find and plot its real and imaginary parts.

10* Using de Moivre's formula derive the complex exponential representation (1.4.5) of the signal $x(t)$ given by the cosine series representation $x(t)=\sum_{m=1}^{M}c_m\cos(2\pi m f_0(t+\theta_m))$.

11 Find the time average and power of the signal $x(t)$ from Fig. 1.9. Use a symbolic manipulation language such as *Mathematica* or *Matlab* if you like.

12* Using a computing platform such as *Mathematica*, *Maple*, or *Matlab* produces plots of the signals

$$x_M(t)=\frac{\pi}{4}+\sum_{m=1}^{M}\left[\frac{(-1)^m-1}{\pi m^2}\cos mt-\frac{(-1)^m}{m}\sin mt\right],$$

[4] Solutions of the problems marked by the asterisk can be found at the end of the book in the chapter *Solutions to Selected Problems and Exercises.*

for $M = 0, 1, 2, 3, \ldots, 9$ and $-2\pi < t < 2\pi$. Then produce their plots in the frequency-domain representation. Calculate their power (again, using *Mathematica*, *Maple*, or *Matlab*, if you wish). Produce plots showing how power is distributed over different frequencies for each of them. Write down your observations. What is likely to happen with the plots of these signals as we take more and more terms of the above series, that is, as $M \to \infty$? Is there a limit signal $x_\infty(t) = \lim_{M\to\infty} x_M(t)$? What could it be?

13* Use the analog-to-digital conversion formula (1.1.1) to digitize signals from Problem 13 for a variety of sampling periods and resolutions. Plot the results.

14* Use your computing platform to produce a discrete-time signal consisting of a string of random numbers uniformly distributed on the interval [0,1]. For example, in *Mathematica*, the command

```
Table[Random[], {20}]
```

will produce the following string of 20 random numbers between 0 and 1:

```
{0.175245, 0.552172, 0.471142, 0.910891, 0.219577,
0.198173, 0.667358, 0.226071, 0.151935, 0.42048,
0.264864, 0.330096, 0.346093, 0.673217, 0.409135,
0.265374, 0.732021, 0.887106, 0.697428, 0.7723}
```

Use the "random numbers" string as additive noise to produce random versions of the digitized signals from Problem 14. Follow the example described in Fig. 1.3. Experiment with different string length and various noise amplitudes. Then center the noise around zero and repeat your experiments.