Chapter 9
From the Bottom Up—Reinventing Realistic Mathematics Education in Southern Argentina

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Abstract This chapter focuses on the Grupo Patagónico de Didáctica de la Matemática (GPDM), a collective of about twenty teachers and teacher educators in Southern Argentina who, united by a shared interest in making mathematics meaningful, relevant, and accessible to all students, have been learning about, adapting, implementing, and contributing to Realistic Mathematics Education (RME). The chapter is organised as follows. First, we outline the state of mathematics education reform in Argentina in the 1990s. Next, we describe how the GPDM was formed, how participants learned about and implemented RME in their classrooms, and how the group's sphere of influence in Grades K–12 and in pre-service and in-service mathematics teacher education expanded from the local to the regional, the national, and the international level. We close with a reflection on what we have learned throughout this creative appropriation process. Throughout the chapter, a selection of annotated vignettes on the work of GPDM teachers and their students illustrate the manner in which the legacy of Hans Freudenthal materialised and continues to materialise in Argentinean classrooms.

Keywords Realistic mathematics education · K–16 study group · Professional development · Instructional design · (Re)contextualising · Modelling

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9.1 Introduction

The focus of this chapter is the Grupo Patagónico de Didáctica de la Matemática (GPDM),[1] a collective of about twenty teachers and teacher educators in Southern Argentina. For the past fifteen years, this group, united by a shared interest in making mathematics meaningful, relevant, and accessible to all students, has been learning about, adapting/adopting, implementing, contributing to, and disseminating Realistic Mathematics Education (RME) (Freudenthal, 1973, 1983, 1991).

Freudenthal (1991) proposes to view mathematics as the human activity of mathematising that is, organising or structuring subject matter by mathematical means. As described by Freudenthal (1991), Gravemeijer and Terwel (2000), Van den Heuvel-Panhuizen (1996, 2005a, 2005b), Treffers (1991), and Streefland (1991a), the central principles of RME are: (1) realistic, in the sense of realisable or imaginable, contexts and situations as points of departure for *horizontal* mathematising; (2) the central place of students' productions and constructions in the teaching-and-learning process; (3) teacher-guided emergence and development of models that support *vertical* mathematising; (4) intertwining of curriculum strands and connections across school subjects; and (5) interaction aimed at comparing, contrasting, and reflecting upon different ways/levels of schematising, diagramming, modelling, symbolising, and formalising the problematic situations at hand.

The chapter is organised as follows. First, we outline the state of mathematics education reform in Argentina in the 1990s. Next, in a narrative organised chronologically in three phases, we describe how the GPDM was formed, how its participants learned about and implemented RME in their classrooms, and how the group's sphere was of influence in pre- and in-service teacher education, as well as how instructional design increased from the local to the regional, national, and international levels. A collection of vignettes on the work of GPDM teacher participants and their students illustrates the manner in which the legacy of Freudenthal and his colleagues and disciples materialised and continues to materialise in Argentinean classrooms. The chapter ends with a reflection on what we have learned throughout this creative appropriation process.

9.1.1 Curricular Innovation in Mathematics Education

In Argentina, curricular innovation began in the 1990s with the newly released Common Basic Contents (CBC) (Consejo Federal de Cultura y Educación, 1995), which effected radical changes in the content and methods of mathematics education. The CBC signalled a move away from the structuralist approach of the Modern Mathematics or New Math (dominant in Argentina since the late 1960s). The above-mentioned reform documents emphasised not only conceptual development and procedural skills, but also the attitudes and dispositions associated with the practice of

[1] Patagonian Group of Mathematics Didactics.

doing mathematics. A second level of curriculum reform materialised in support documents prepared by the National Department of Education and distributed to the 23 provinces and the Autonomous City of Buenos Aires.

The 1990s standards and guidelines were influenced by the French Didactique, an approach that had entered Argentina in the 1980s with the translation of the work of Guy Brousseau and his colleagues at the Bordeaux IREM.[2] Professors of didactics of mathematics travelled to Paris, Strasbourg, and Bordeaux to pursue doctoral research and brought back materials related to their teaching experiments that were then disseminated and adopted nationwide. The degree of comprehension of the French Didactique among mathematics teacher educators and teachers varies significantly, thus resulting in transpositions of not always the same quality as that of the original experiments.

9.1.2 Initial Attempts at Bringing Realistic Mathematics Education to Argentina

In 1984, Diana Rosenberg received a fellowship from the Dutch government to specialise in the didactics of mathematics at OW&OC[3] at Utrecht University. There she participated in two research projects: the HEWET[4] project, aimed at improving mathematics instruction at the secondary level, and the project De Baas over de Computer,[5] which focused on introducing the computer in the early grades of secondary school. In 1986, Jan de Lange and George Schoemaker facilitated workshops for professors at the Universities of Buenos Aires and Tucumán in which they introduced RME and, particularly, ways of using computers to teach various mathematics topics. The next step was to explore the possibility of a collaborative project involving the universities of Buenos Aires and Utrecht. In 1987 a series of seminars was offered at several locations in the Buenos Aires province organised by the Ministry of Education of that province and conducted by Martin Kindt. Among the most successful of these seminars was the one offered in the city of La Plata, in that most participants expressed interest in developing RME-inspired materials adapted to their students' needs. Yet the impossibility of providing one day off per month for each teacher participant to work on that project prevented this initiative from getting off the ground.

In a last attempt at collaborating with the Freudenthal Institute, in 1998 and with the support of the Argentinean National Department of Education, Jan de Lange met with specialists at various public and private institutions and gave a presentation enti-

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[2] Institut de Recherche sur l'Enseignement des Mathématiques (Research Institute for Mathematics Education).

[3] Onderzoek Wiskundeonderwijs en Onderwijs Computercentrum (Mathematics Education Research and Educational Computer Centre).

[4] Herverkaveling Wiskunde I en II (Re-allotment Mathematics I and II).

[5] Master of the Computer.

tled “Mathematics in Reality” at the Centro de Altos Estudios en Ciencias Exactas. De Lange’s work with secondary school teachers spread excitement about RME and its potential for improving mathematics instruction. A series of workshops in the city and province of Buenos Aires as well as in the northern city of Tucumán expanded interest in the RME approach. However, lack of long-term institutional support made it difficult to materialise RME ideas in teacher participants’ classrooms.

9.1.3 San Carlos de Bariloche, Birthplace of the Grupo Patagónico de Didáctica de la Matemática

In December of 1998, Ana Bressan[6] and two colleagues from the Curriculum Office of the National Department of Education met Zolkower, who at the time was working at the City College of New York in the project Mathematics in the City, an NSF-funded,[7] RME-inspired in-service teacher education project directed by Catherine T. Fosnot in collaboration with Willem Uittenbogaard and Maarten Dolk (from the Freudenthal Institute). On that occasion, Zolkower shared informally her experience from this project and that immediately sparked Bressan’s interest in bringing RME to her hometown, San Carlos de Bariloche.

Located in the Patagonian region, San Carlos de Bariloche (population: circa 140,000 habitants) is an international ski tourism destination and an important centre for research and development in science and technology. The city houses the Balseiro Institute, which offers masters and doctoral degrees in physics and nuclear engineering, and the INVAP, a high-tech centre for the design of nuclear reactors, radars, and satellites. Also located in Bariloche are branches of several public and private universities and the Instituto de Formación Docente Continua (IFDC) which is attended by local students and students from nearby cities and towns. All of the above makes of Bariloche a hub strongly tied to a variety of academic, scientific, and cultural centres notwithstanding its distance (1700 km) from Buenos Aires. Perhaps it is that very distance, coupled by its relatively small size, which makes of Bariloche a fertile ground for innovation in science, technology, and education.

Bariloche is a highly class-stratified town. Middle and upper classes (shop-owners, business and administration workers, professionals, and hotel owners and managers) reside in the centre, east, and west sections of town. More than half of the city’s inhabitants live in the southern part; this includes lower and lower middle classes, subsidised workers, immigrants, maids, waiters, and aboriginal people. The educational needs of such a diverse population are served by more than 28 secondary schools, 30 public elementary schools (most with kindergarten annexes), about 30 independent private kindergartens, and 18 private schools (spanning Grades 1 through 12). All

[6] Bressan has been working in mathematics curriculum design and teacher education since 1975 and was a central person in preparing the Río Negro Mathematics Curriculum that later became the foundation for the National Curriculum Standards in Mathematics in Argentina.

[7] Funded by the U.S. National Science Foundation.

of the above places, demands on teacher preparation that are hardly met by basic, one-size-fits-all content and methods coursework in that many prospective teachers arrive to those courses with weak foundations in basic literacy and mathematics.

In the winter of 1999, invited by Bressan, Zolkower taught in Bariloche a mini-course for teachers entitled Closing the Gap between School Math and Common Sense: Freudenthal's Realistic Mathematics Education and gave a presentation at the Regional Centre of the University of Comahue. These events led to the formation, in February of 2000, of a study group of more than twenty teachers working in early childhood, elementary, and secondary classrooms of public as well as private schools. The collective, which named itself Grupo Patagónico de Didáctica de la Matemática (GPDM),[8] set off to "improve our practice by approaching the problems of learning and teaching mathematics taking RME as an object of study", as expressed by one of the participants. Another participant reflected: "From the start, what intrigued us the most about RME is how it opens up the classroom doors to common sense, imagination, desire to learn, and the mathematising potential of our students."

Since then, the degree and sphere of influence of the GPDM has been increasing steadily through classroom teaching experiments, workshops, courses, specialisation post-degrees, online seminars, conference presentations, and its webpage (gpdmatematica.org.ar). Group participants have produced more than twenty publications in *Novedades Educativas* (a magazine for teachers widely read throughout Latin America), *Yupana* (a journal of the University of Litoral, Argentina), *Paradigma* (a journal from Venezuela), *Premisa* (a journal of the Argentinean Society of Mathematics Education), *Didáctica* (a journal from Uruguay), and in the GPDM's webpage; two book chapters (Bressan, Zolkower, & Gallego, 2004; Zolkower & Bressan, 2012); two books (Bressan & Bressan, 2008; Brinnitzer et al., 2015); dozens of conference presentations (e.g., Zolkower's presentation at the FIUS in Colorado; Zolkower, 2009); and a myriad of translations of seminal work by RME specialists. Roughly once a year, Zolkower travels to Bariloche to offer thematic workshops on topics such as unpacking the teacher's role in conducting whole-class interaction, and the function of diagrams and diagramming in non-routine problem-based lessons, and to co-present at regional conferences. Meanwhile, from the distance, she shares resources with the group; co-designs teaching experiments; co-authors papers that narrate those experiences; and collaborates on a research study of teacher conduction of whole-class interaction (Shreyar, Zolkower, & Pérez, 2010; Zolkower, Shreyar, & Pérez, 2015).

[8] Is part of the Grupo de Educación Bariloche (Fundación GEB), a non-profit organisation devoted to in-service teacher education.

9.2 First Phase (2000–2004): Contexts, Situations, Models, and Strategies

9.2.1 Fractions, Decimals, and Percentages

In the summer of 2000, Zolkower facilitated a four-day workshop in Bariloche on the realistic approach to ratio and proportion, fractions, decimals, and percentages. The decision to start with these topics was motivated by the fact that rational number tends to present difficulties for teachers and students. Typically, fractions, decimals, ratio, proportion, and percentages are taught as separate topics. Students are presented with a set of rules to perform with little emphasis on why those rules work. In particular, with regard to fractions, they are expected to transit rather quickly from fraction as part/whole to fraction as bare number with little attention to fraction as ratio or to the use of the ratio table and open number line as tools for mathematising genuinely problematic situations. This approach often results in the 'multiplication makes bigger, division makes smaller' generalisation leading to errors when operating with rational numbers. Streefland's (1991b) research documents the positive effect of teaching fractions within realistic contexts as antidote to the above misconception.

During the workshop, participants worked on selected activities from the textbook series *Mathematics in Context* (MiC) (Wisconsin Center for Education Research & Freudenthal Institute, 1997–1998), in particular the units *Some of the Parts* (Van Galen, Wijers, Burrill, & Spence, 1997–1998) and *Fraction Times* (Keijzer et al., 1997–1998) and, in so doing, became familiar with Streefland's (1991b) classroom experiments on introducing fraction as ratio via fair sharing situations while guiding the development of level-raising tools (e.g., fraction strips, number line, ratio table, pie chart, and bar model). The collaborative study of MiC activities benefited from the GPDM's heterogeneous composition. Whereas lower grades teachers became aware of weaknesses in their understanding of rational number, those teaching mathematics in upper grades appreciated the pivotal function of contexts and models in supporting students' understanding of and fluency with fractions. Furthermore, comparing and contrasting productions at different levels allowed participants to make sense, through their own experiencing and reflective discussions, of two central notions in RME, namely guided reinvention and progressive mathematising (Freudenthal, 1991).

> Students should "reinvent mathematising rather than mathematics; abstracting rather than abstractions; schematising rather than schemes; formalising rather than formulas; algorithmising rather than algorithms; verbalising rather than language…". (Freudenthal, 1991, p. 49)

The above-described work proved so productive and intriguing that the idea of classroom try-outs emerged almost immediately. Processes and results from those experiences were analysed and discussed in group meetings giving participants the opportunity to consider student-generated strategies, the function of contexts and models, the advantages of heterogeneous grouping, and the paramount role of the teacher in guiding whole-class interaction.

The following is an example of what came out of such a discussion. It is an excerpt from a journal entry in which Silvia Pérez, co-author and GPDM participant, narrates an event in her 5th grade classroom.

> For the whole-class share I posted on the board strips with fractions and we worked together on how to complete the whole starting off with each of the fractions. Next the students had to think about different ways to make a whole by combining 1/2, 1/4, 1/3, 1/8, 1/6, and/or 1/5 (Fig. 9.1). All kinds of calculations were proposed using addition, subtraction, multiplication, and division. And this led to more examples, surprising strategies, unexpected questions, and new discoveries.

The above event and other similar experiments in GPDM classrooms called for revisiting the forms and functions of assessment. From the perspective of RME, assessments are viewed as serving foremost a didactical purpose, namely to gather information about each student's learning which teachers can use, before, during, and after each instructional sequence to guide individual as well as collective learning processes (Van den Heuvel-Panhuizen, 1996, 2005b). For example, the end-of-the-unit assessment Pérez (2004) designed for her 5th graders included bare number and context problems involving fraction as ratio, operator (measuring), part/whole, division, and bare number. In Fig. 9.2 there are four items from that assessment.

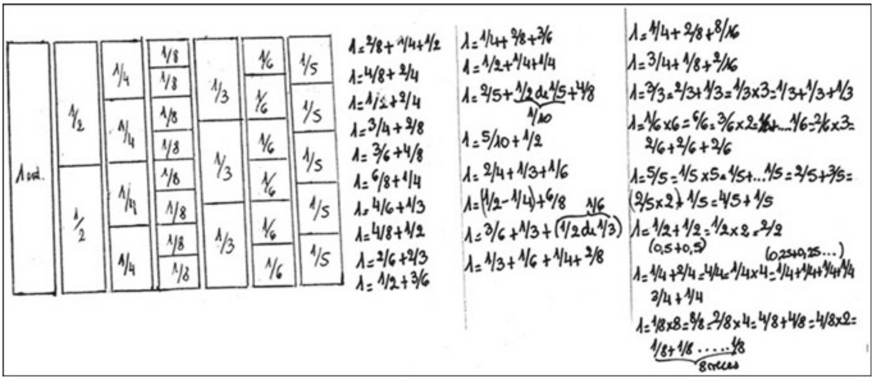


Fig. 9.1 Fraction strips posted on the board by Silvia

- 8 children share 5 pancakes. How much pancake does each child get?
- Can we fit 3/5 of a can and 5/10 of a can in one can? Explain!
- Solve each of the following calculations […]. Then choose one of them and invent a problem such that that calculation could be used to solve it. 1/2 + 3/5 = 2/3 + 3/4 =
  Compare 9/12 and 5/8 using three different strategies.

Fig. 9.2 Sample items from an end-of-the-unit assessment on fractions

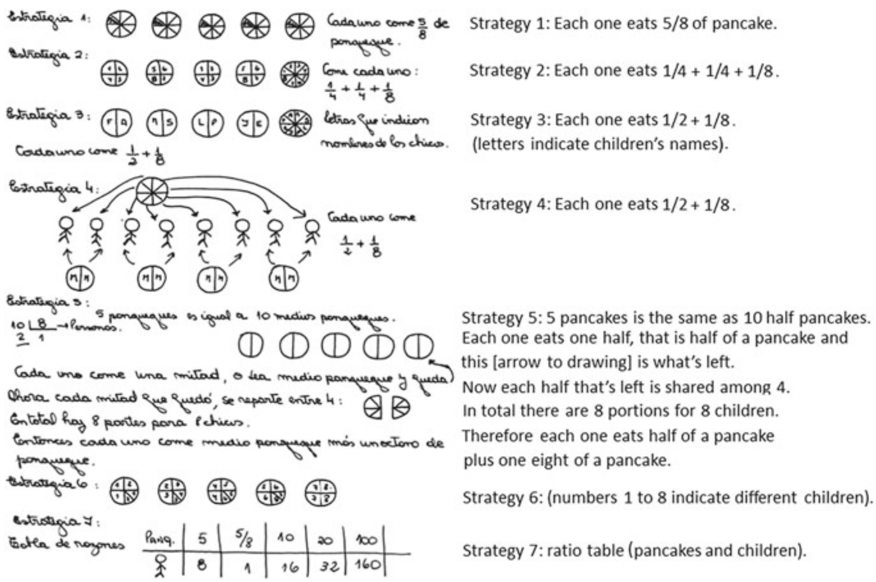


Fig. 9.3 Sample solutions for the problem about 5 pancakes for 8 children

Figure 9.3 shows the strategies students used to solve the pancakes problem. Worth highlighting is the use of circles as a model of the pancakes; different ways of cutting the pancakes leading to different, yet equivalent, expressions for each child's share; and the use of the ratio table with doubling and 'ten times' strategies.

The students' work in Fig. 9.4, concerning the comparison two fractions (9/12 and 5/8), offers further evidence of these 5th graders' appropriation of models for representing, working with, and thinking about fractions (fraction bar, circular model: pie chart and clock, and number line) as well as flexible strategies (e.g., using 3/4 as a benchmark and generating equivalent fractions). This assessment yielded invaluable information about students' strengths and weaknesses that the teacher took into account as she planned subsequent lessons.

The following year, these same students worked with Graciela Méndez, also a GPDM member. At a bi-weekly meeting, Graciela shared her 6th graders' work on problems involving rational numbers that, to everyone's amazement, evidenced their strong number sense, including the flexible use of a variety of tools and strategies. Figure 9.5 presents nine responses to: "7/12 is smaller than 70%. Is that true or false? How do you know?"

Worth underlining in the students' work is their ability to use different representations of rational number, the use of the double number line (G) and strategies such as completing the whole (H), benchmarks (A, C, D, I), equivalent fractions (B, C, E), and approximation and estimation (A, D, H).

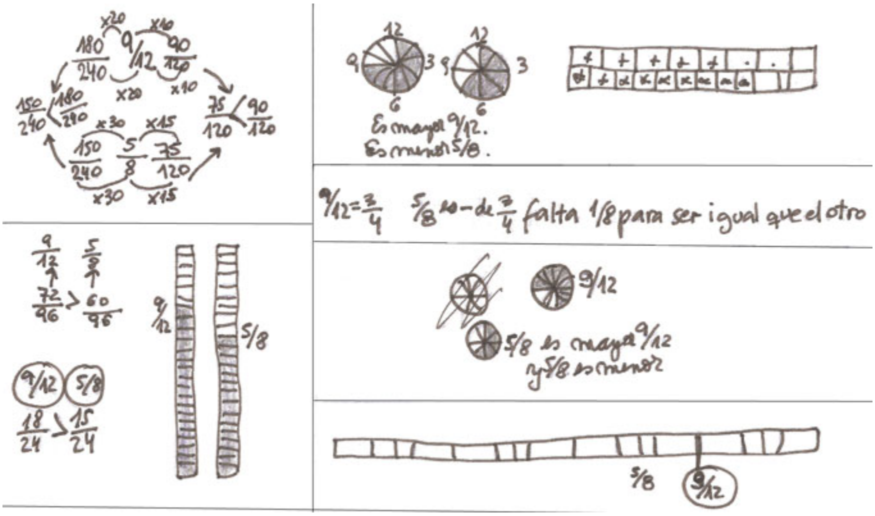


Fig. 9.4 Use of models to compare 9/12 and 5/8

A. It's true because 50% is 6/12 and 6/12+1.5/12=7.5/12= 62.5% and that's a bit less than 70%.	B. 7/12 → 3.5/6 → 1.75/3 I know that 1.75/3 is smaller than 2/3 and 2/3 is smaller than 70%.	C. 2/3 is less than 70% 2/3=8/12 8/12 is more than 7/12 So, 7/12 is less than 70%
D. 6/12=50% and 1/12 is about 8% So, 7/12 is about 58%, that is less than 70%.	E. 70%=7/10 7/12=35/60 7/10=42/60 35<42 So 7/12<70%	F. It is less than 70%. To be equal it would need to be 8/12.
G. 6/12 7/12 8/12 9/12 50% 75% 70% is only 5% away from 75%. That's a bit more than 8/12. So, 70% must be more than 7/12.	H. If you have 7/12, you need 5/12 to get to 1. That means that 7/12 is more than ½. It's about 65%.	I. It is true because 6/12=50% and 1/12 is less than 20%. So, 7/12 cannot be more than 70%.

Fig. 9.5 Sample responses to the 70% and 7/12 comparison

9.2.2 *City Buses*

Experiences that brought RME into the classroom extended to other topics and grade levels. Inspired by the work of Van den Brink (1991), Mary Collado, another GPDM teacher, introduced to her 1st grade students the city bus as a context for early addition and subtraction. The instructional sequence began with play-acting, the bus conductor (paper hat on his head) circulating around the room picking up and dropping off passengers at various stops (desks). This was followed by a whole-class

Teacher:	How do we show that they got ON the bus? And how do we say they got OFF? What can we put here? [Points to the empty rectangle + vertical stick representing a bus stop.]
Carolina:	Put another bus!
Josefina:	Change the number!
Florencia:	Erase that number.
Jeremy:	WE COULD USE A PLUS!
Teacher:	Why plus?
Jeremy:	Because when they get on, there are more people
Teacher:	And here, did people get ON or they got OFF?
Sebastian:	Somebody got off!
Teacher:	How do I put that?
Sebastian:	MINUS, PUT A MINUS!!

Fig. 9.6 Getting on and off the city bus: emergence of the plus and minus signs (capitals indicate emphasis)

conversation about different ways of drawing bus trajectories, as shown in the brief excerpt below (Fig. 9.6).

The city bus served as a springboard for students to invent their own bus stories and, in the process, learn to use arrow language (dynamic) as a precursor of standard (static) expressions with the = sign. All of the above included teacher-guided opportunities for sharing, comparing, contrasting, and reflecting upon students' constructions and productions geared towards level rising. Figures 9.7a–f show how the city bus evolved from context, to model *of*, to model *for* (Streefland, 1985; Van den Heuvel-Panhuizen, 2003) thereby supporting progressive schematisation towards formal addition and subtraction. It is worth highlighting in the samples below the contrast between the bus stories by Fede (Fig. 9.7a) and Nata (Fig. 9.7f). Whereas Fede's presents a school bus' early morning trajectory picking up children and bringing them to school, Nata's depicts the opposite trajectory, the bus delivering children to their homes at the end of the school day (*todos* in Spanish means *all*).

In Mary Collado's own words:

> The bus context gave meaning to addition and subtraction and served afterwards as a model for the children to fall back to in order to make sense of other homologous situations, or when working with bare number problems. This context was also fruitful to generate a wide range of mathematisable situations including: (1) geometry: trajectories, location via points of reference, distance, sketching the inside of the buses, top and side views, symmetries, and so on; and (2) arithmetic: ticket fares, coin combinations for the ticket machine, numbers on the tickets, capacity of buses, school bus trajectories (first adding on and on, then subtracting on and on), short distance versus long distance buses, and so on. (Collado, Bressan, & Gallego, 2003, p. 15)

9.2.3 *From Necklaces to Number Lines*

Another RME-inspired context/object appropriated by GPDM teachers in elementary and secondary grade classes was the beads necklace. Necklaces proved invaluable

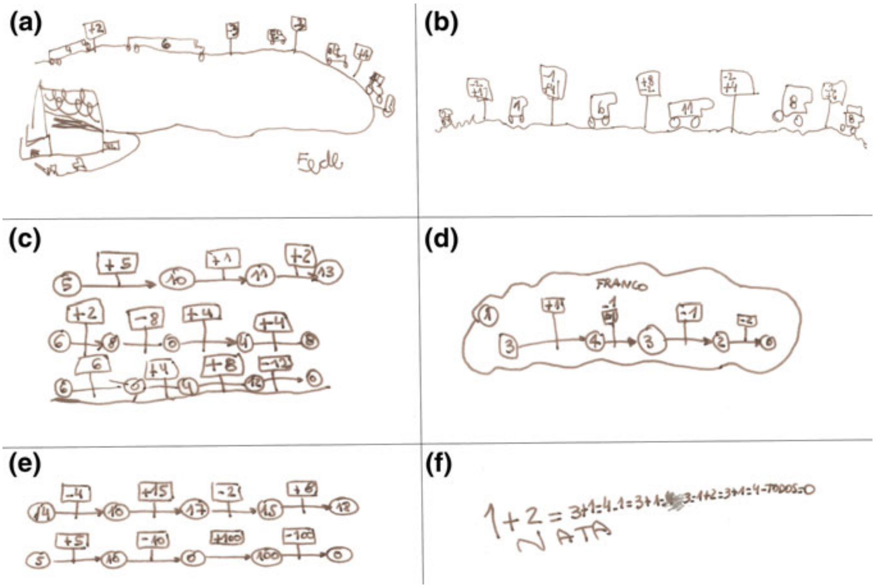


Fig. 9.7 First graders' own productions (bus stories)

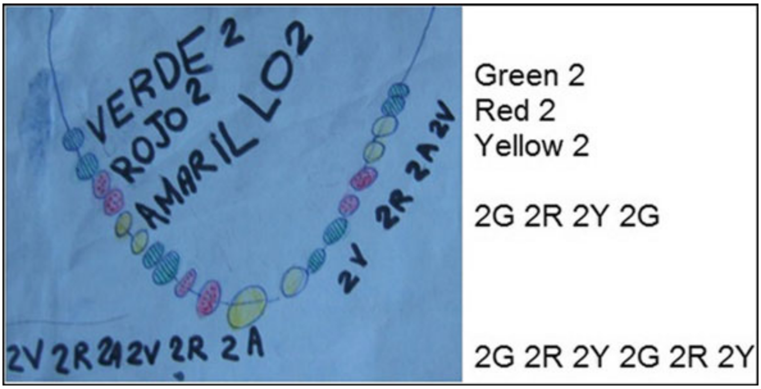
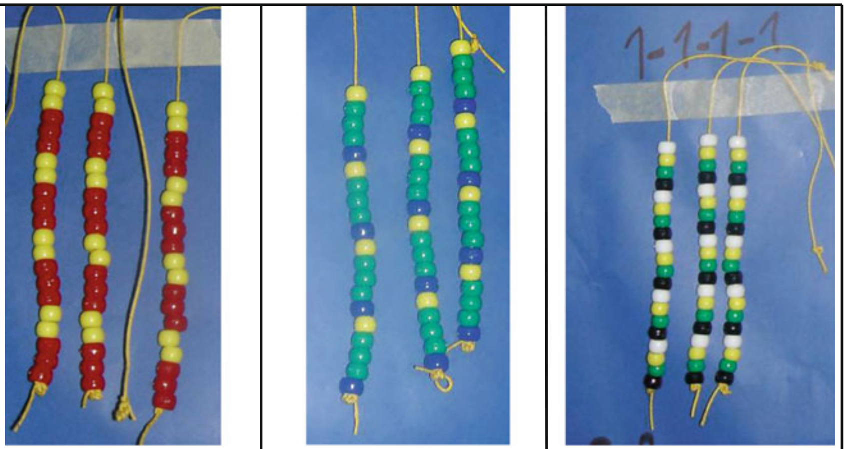


Fig. 9.8 Initial work of a first grader with bead necklaces

artefacts for developing students' mental arithmetic skills; working on ratio, proportion, and fractions; factors, multiples, divisibility, and remainders; and early algebra activities (e.g., describing and symbolising repeating and patterns and generalising those via building formulas). Figure 9.8 shows the initial work of a first grader with necklaces.

Activities involving patterns in necklaces included introducing them as material artefacts, drawing them, describing their structure (e.g., by relating the number of beads in the repeating pattern unit or 'chunk' with the total number of beads), invent-



Necklace A: the repeating chunk is aabbb	Necklace B: the repeating chunk is abbbc	Necklace C: the repeating chunk is abcd

Fig. 9.9 Three student-generated 20-bead necklaces

ing abbreviated descriptions (by means of icons; letter strings; and letters, numbers, and parentheses); creating necklaces with beads of different colour, size, price, or different number of beads; making necklaces that satisfy given constraints (e.g., the number of beads, the repeating chunk or pattern unit, the ratio between different colour beads, the length necklace, or the price).

Called upon to imagine themselves as workers in a necklace factory, students in Carolina Moreno's 2nd grade classroom were asked to design bicolour 20-bead necklaces with a complete repeating pattern. After completing this task, they arrived at the following conclusions: (1) if a necklace has a complete repeating pattern, the colour of the last bead is the same as the colour of the last bead in the chunk; (2) the pattern is complete when the number of beads in the repeating chunk fits an exact number of times in the total number of beads in the necklace; (3) if you have 20 beads, you cannot make a necklace with a pattern of length 3 or 9; and (4) you can make a 20 bead necklace with patterns of length 2, 4, 5, and 10. Also, beginning with a certain amount of beads in the chunk, they played at making different necklaces (Fig. 9.9).

When asked to make all possible 36-bead necklaces with complete repeating patterns, they used the language of multiplication, 'times' (*veces*) and 'goes into' (*entra en*), to describe, explain, and justify their findings. Figure 9.10 shows, summarised by the teacher, how these second graders expressed different decompositions of 36 as the product of two whole numbers.

In line with RME, a bi-colour 100-bead necklace structured in 10 groups of 10 beads with alternating colours served as a flexible (adaptable to each student's level) material artefact and a precursor to the open number line (Fig. 9.11), a schematic and continuous and, thus, more abstract linear model (Van den Heuvel-Panhuizen,

Number of bead sin the pattern unit	Total number of beads
2	2 x 18
3	3 x 12
4	4 x 9
6	6 x 6
9	9 x 4
12	12 x 3
18	18 x 2

Fig. 9.10 Decomposition of 36 in the context of bead necklaces

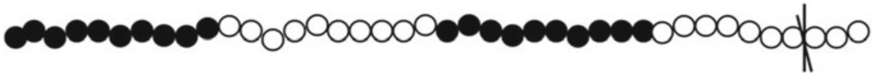


Fig. 9.11 Locating 37 on the 100-bead necklace

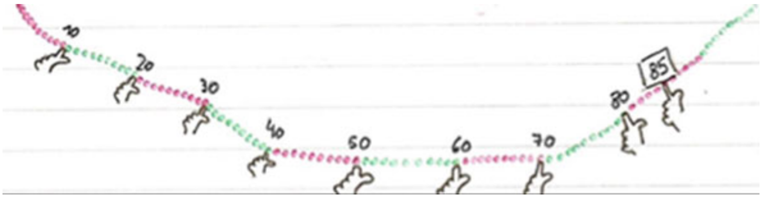


Fig. 9.12 Locating 85 on the 100-bead necklace

2008). This necklace functioned as a context/model for a variety of activities whereby students could attend to, both, the cardinal and ordinal aspect of numbers (37 beads, the 37th bead) as they worked on locating, comparing, and decomposing numbers; counting and calculating; grouping, and so on.

Figure 9.12 shows diagrammatically the way a student located the number 85 on the 100-bead necklace.

Figure 9.13 illustrates the use of the necklace as a tool for solving the following subtraction (difference as distance on the line) problem: “Today is the 5th of March. School begins on the 29th. How many vacation days do you still have?”

Figure 9.14 shows how two 2nd grade students use the open number line to solve $47 + 12 + 21$.

In 4th, 5th, and 6th grade GPDM classrooms necklaces and number lines are routinely used for representing, working on, and thinking about problematic situations involving ratio and proportion (Freudenthal, 1983), alongside other tools such as the ratio table, the double number line, and the bar model. On their part, when teaching functions, sequences, and series, upper elementary and secondary grade GPDM teachers take advantage of necklaces as artefacts to support algebraising, for example, describing, symbolising, and generalising repeating as well as recursive patterns.

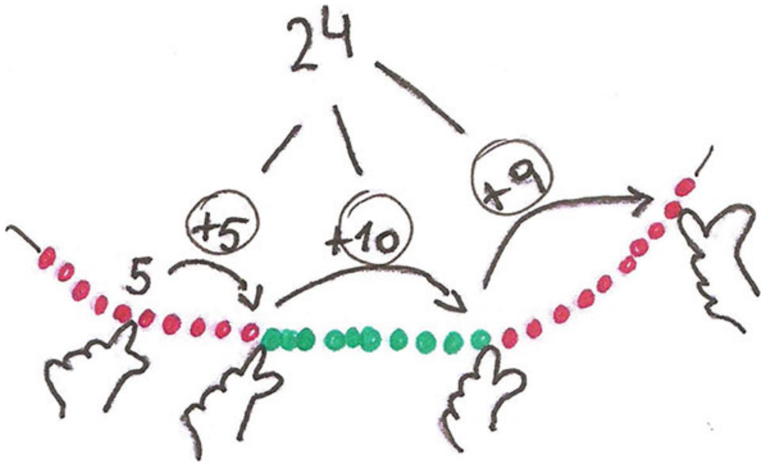


Fig. 9.13 Bead necklace as a tool for subtraction

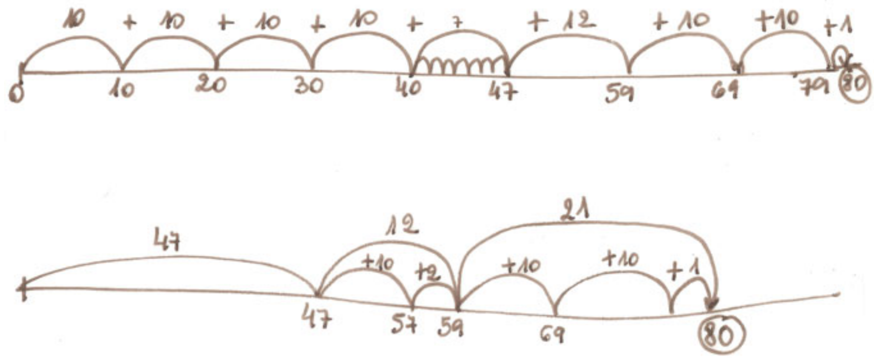


Fig. 9.14 Adding numbers on the open number line

9.2.4 The Function of Contexts in RME

The meaning of 'realistic' and the mathematical-didactical value of realistic contexts and situations intrigued most GPDM teachers. In particular, they debated the issue of how to find 'good' contexts and verify that those are actually fruitful mathematising. These concerns motivated several classroom experiments. One of these (Rabino, Bressan, & Zolkower, 2001), in 8th grade, involved comparing how students solved bare number calculations (multiplying and dividing rational numbers) with how they solved, a week later, context problems that involved those same calculations. The results confirmed the hypothesis that context problems were easier for the students to solve than bare number ones (Van den Heuvel-Panhuizen, 2005b), except when the situations depicted were unfamiliar to them. As shown in Fig. 9.15, the percentage

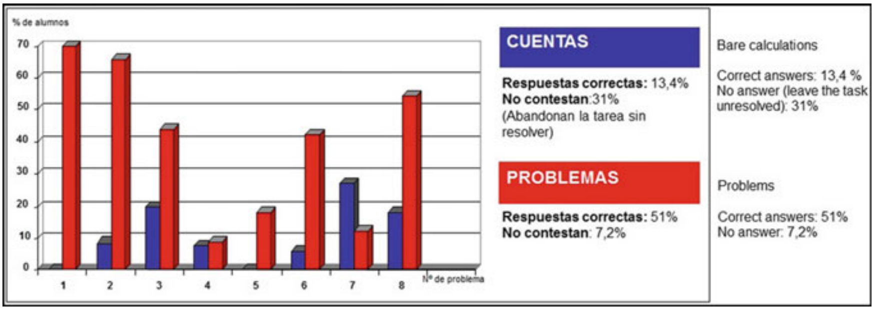


Fig. 9.15 Percentage of correct answers for bare number problems and for the context problems

of correct answers increased greatly for problems involving meaningful situations in familiar contexts. For example, for 60 ÷ 1/2, 83% of the students ($n = 35$) gave a wrong answer whereas for the corresponding context problem (60 L of beer packed in 1/2-L bottles; how many bottles?) almost 70% responded correctly. This experiment was replicated in other classrooms with similar results further confirming the hypothesis that realistic contextualising supports the meaningful construal of mathematical meanings.

Another experiment, in two 5th grade classrooms (Martínez Pérez, da Valle, Zolkower, & Bressan, 2002), aimed at developing in students a disposition to attend to the specifics of the problematic situations and activate their common sense and recall prior experiences about those situations. A third example is "¿Seño, es cierto eso?" (Is that true, miss?) (Pérez, Bressan, & Zolkower, 2001), an essay describing changes in two 6th grade classrooms which resulted from the move away from stereotypical and contrived word problems towards open-ended problematic situations embedded in realistic contexts.

9.2.5 Mental Arithmetic: Models and Strategies

Alongside a continued focus on contextualising and recontextualising, it became a central for GPDM teachers to deepen their own understanding of models in RME (in particular, the function of models in facilitating the transition from situational, to referential, general, and formal level) as well as develop their ability to use those models spontaneously and flexibly. This interest emerged organically, as teachers reflected on their students' use of models emerging in their students' activities around the MiC units they were studying.

In parallel with the work on models, the group focused on developing and strengthening their own as well as their students' mental arithmetic skills. The latter included cycles of design, implementation, documentation (video-recording and transcription), interpretative analysis, and reflection on mental math activities. This effort is in line with a central tenet of RME, namely, that the ability to use a variety of strategies, properties, and tools when solving arithmetic calculations is a central component of number sense. Strings (i.e., sets of interrelated bare number calculation presented horizontally to discourage the mechanical use of standard algorithms and, instead, promote the noticing and taking advantage of those relationships as well as using strategies that are suited to the numbers at hand) and other mental math activities became ubiquitous in GPDM classrooms at all grade levels. The above resulted in the publication of three booklets (for Grades 1–2, 3–4, and 5 through 7) with a wide range of activities for strengthening students' number sense that are currently in use in many classrooms within and beyond the GPDM.

Performing mental computations and comparing and contrasting alternative strategies solidified students' understanding of number and operations serving as a foundation for appropriating standard algorithms via progressive schematising and formalising. As an illustration, Fig. 9.16 shows multiplications of fraction strings done in a 6th grade classroom, with annotations of student strategies (on the left column: arrows linking the various problems; on the right column: description of strategies and supporting calculations) by the teacher, María de los Angeles Biedma.

Another example, shown in Fig. 9.17, presents a sequence of interrelated percentages of 360 supported by the bar model. This is a hybrid tool (double number line plus area model) that allows for finding equivalent ratios while keeping the part-whole relationship in view.

Figure 9.18 shows how a student calculated a series of percentages of 350 by using the double number line.

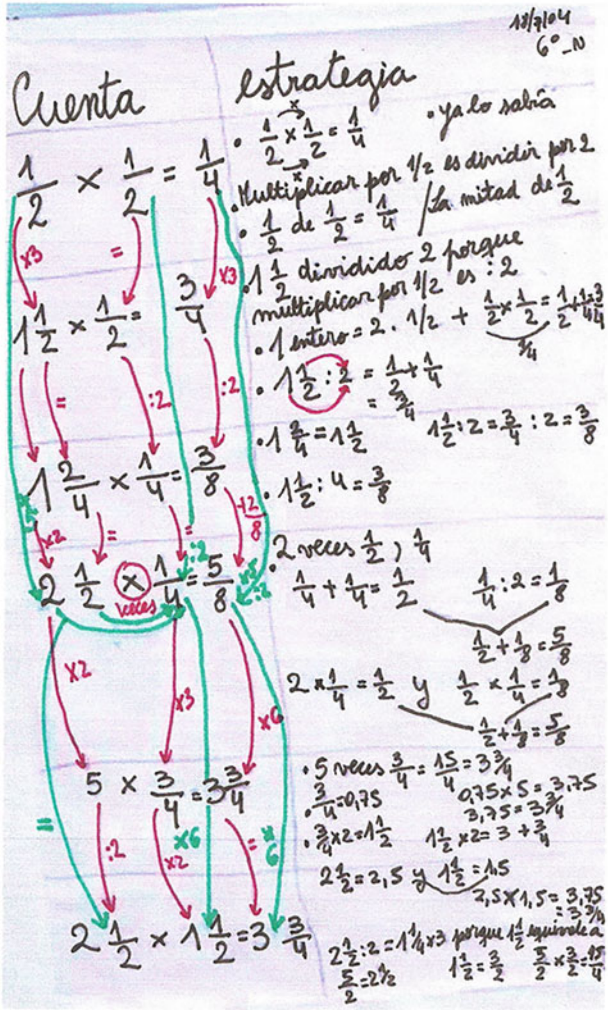
9.3 Second Phase (2005–2009): Deepening and Solidifying

9.3.1 Mathematising Within the GPDM

In 2005, Oscar Bressan, an Atomic Centre physicist and professor at the Balseiro Institute joined the GPDM. His involvement in the group contributed greatly to strengthen the mathematising abilities of participants with regard to selected topics in number theory, geometry and measurement, and probability and statistics (Bressan & Bressan, 2008). For example, in one of the sessions facilitated by Oscar, the focus was the following question: "Approximately how many digits are there in the product $20 \times 21 \times 22 \times 23 \times 24 \times 25 \times 26 \times 27 \times 28 \times 29 \times 30$?" Figure 9.19 shows the variety of strategies used by teachers to solve the problem.

Another problem tackled by the group session was 'Flowers and grass' (Fried & Amit, 2005):

We want to plant flowers and grass in a 6 m by 10 m rectangular garden. The grass will be planted in the four corners, in the shape of four isosceles right triangles with the right-angle vertex of each coinciding with the angles of the rectangle. The condition is that the two triangles, the one with vertex in *D* and the one with vertex in



Cuenta:
bare number problem
estrategia:
strategy

I knew it
Multiplying by ½ is (the same as) dividing by 2
Half of ½

(This is like) 1½ divided by 2 because multiplying by ½ is like dividing by 2
1 whole divided by 2 is ½+½×½=¼
1½÷2=½+¼

veces: times

Fig. 9.16 Multiplications of fraction strings

B need to be congruent. The flowers will be planted in the remaining parallelogram-shaped area.

Group participants were presented with, both, the verbal description of the problematic situation and the accompanying diagram (Fig. 9.20) and the request that they write down observations and formulate as many meaningful questions about it as they could. Next the group selected the following three questions to focus on: Assuming that there are different possibilities for the parallelogram to plant flowers in, depending on the length of segment DE, would all of those yield the same area? If not, which

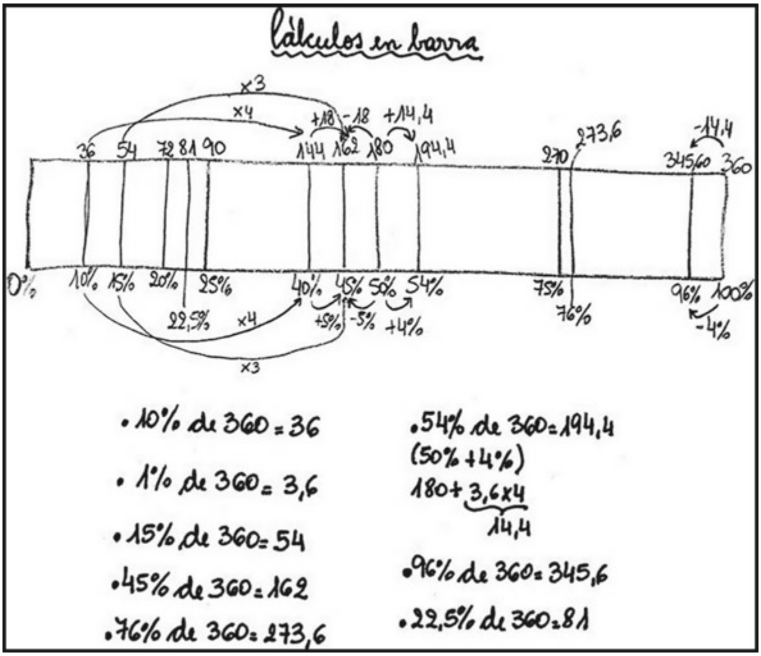


Fig. 9.17 Calculating percentages of 360 with the bar model (Van den Heuvel-Panhuizen, 2003)

parallelogram would maximise the area for planting the flowers? What happens to the area of the triangles in the corners as the area of the parallelogram changes? Figure 9.21 shows the different approaches followed by participants (Zolkower & Bressan, 2012).

Synergically, the heterogeneity of the group allowed participants to appreciate different levels and manners of mathematising including how concrete material (Fig. 9.21a), graph paper diagrams (Fig. 9.21b), function tables (Fig. 9.21c), and calculus tools (function, derivative, graphing the inverted parabola, Fig. 9.21d) all served as tools for representing and/or solving the problem. This modality of working with “spiral tasks” (Fried & Amit, 2005, p. 432) was extended, with great success, to workshops attended by other teachers and teachers-in-training in Bariloche and in surrounding towns and cities.

The pressing need to deepen participants’ mathematical-didactical abilities and, at the same time, generate material for the increasing number of courses offered by the GPDM led to translating papers by RME specialists, adapt (recontextualised) MiC units, and design activities and instructional sequences. Among the latter, worth mentioning is the design of materials for: linear functions; ratio and proportion, fractions, decimals, and percentages; congruency and similarity; patterns, symbols, and rules; mental arithmetic; side and top views; and polygons. In designing the above, emphasis was placed on using suitable, familiar, and meaningful contexts,

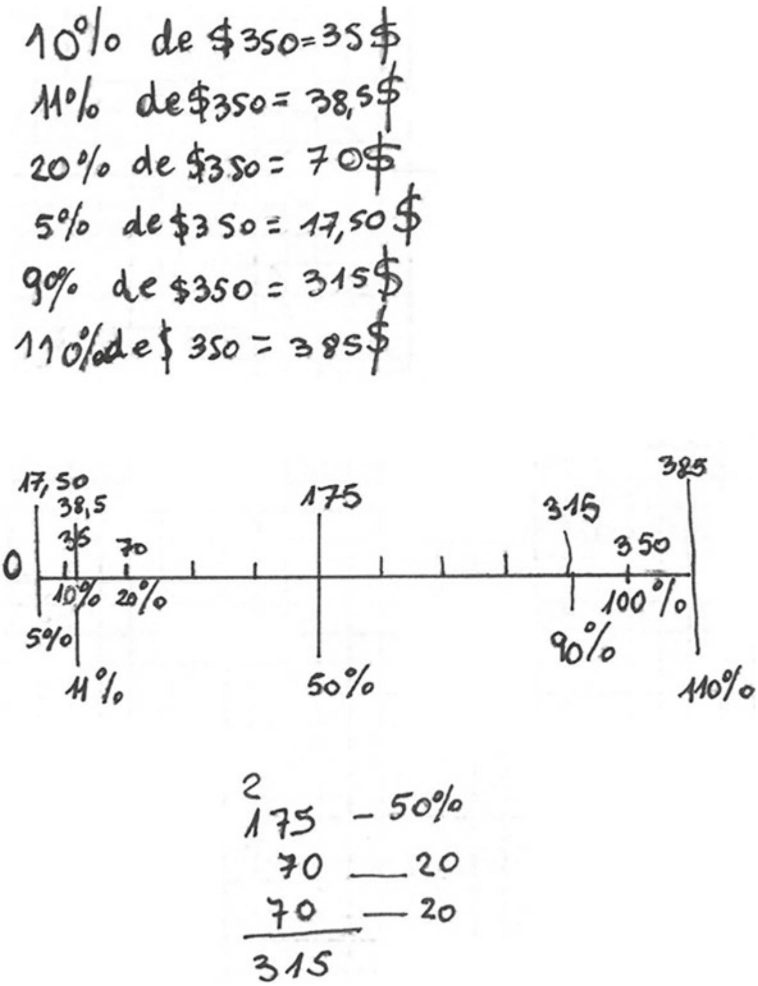


Fig. 9.18 Double number line as a tool for calculating percentages of 350

situations, and artefacts to support mathematising, for example, advertisements, price lists, mandalas, clippings from newspapers and other printed media, photographs, hiking and elevation maps, bus schedules, restaurant menus, dissection and edge-matching puzzles.

The second phase saw an increase in the number of courses offered by the GPDM in Bariloche, rural areas of the provinces of Río Negro, and cities in the provinces of Neuquén, Mendoza, and Buenos Aires. In Neuquén, several GPDM participants collaborated in designing and teaching the post-degree mathematics unit Teaching in Schools Located in Diverse Urban Contexts. This required adapting RME-inspired

Rocío (elementary school teacher) paired up the numbers whose one-digits add to 10 (i.e, 23 x 27, 24 x 26, 22 x 28, and 21 x 29). She noted that those four products, which are a bit more than 600 each, add 3 digits to the overall product but got stuck there.

Oscar continued Rocio's idea:

24x26= (30-6) (30-4) = 900 – (6+4) x 30 + 6x4= 600 +24

23x27= (30-7) (30-3) = 900 – (7+3) x 30 +7x3 = 600 + 21

… and so on with 22 x 28, 21 x 29 and 20 x 30, which gives about 600^5.

600^5 can be expressed as 6^5x10^{10} that is close to 10^4x10^{10}. Therefore, the product has 14 digits.

Adriana (secondary school teacher) solved the problem by approximation. Using the calculator, she did 20^{10} and got 14 digits; then she did 30^{10} and got 15 digits; then 25^{10} would give between 14 and 15 digits.

Patricia (secondary) also used the calculator but she transformed the expression using factorials. She divided 30! by 19!

(Thus: 30x29x28x27x….x1) ÷ (19x18x17x…x1) and obtained 14 digits.

Oscar worked with logs. He knows that the log of 2 is 0.30, thus the log of 20 is 1.30. This allowed him to estimate 1.30 x 10 (20 appears 10 times in the product) and get 13. He adds 1 to that and concludes that the number of digits is approximately 14.

Ana María made groups of 20 x 20 ignoring the one's place and got 400^5x30. That can be expressed as 4^5x100^5x30 = 4^5x10^{10}x30, which gives a number with approximately 13 digits. If we add back to that 20 times the sum of the ones digits, from 1 to 9, which we ignored initially, we get 900. So, the product in question has about 14 digits.

Fig. 9.19 Solving the 'Large product digits' problem

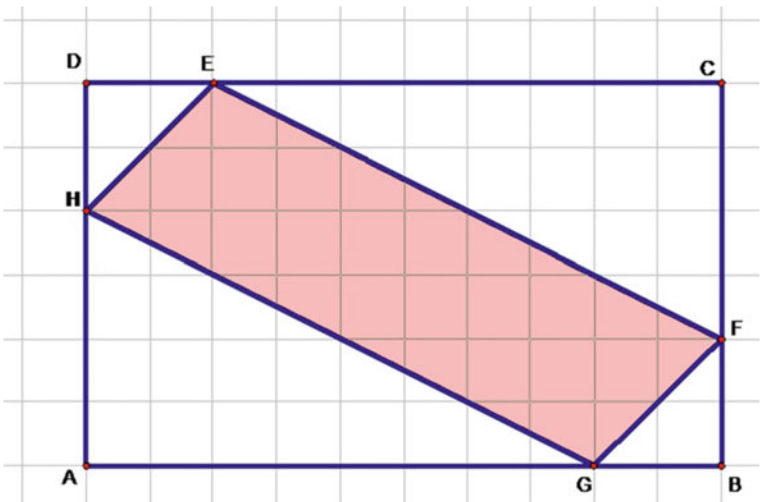


Fig. 9.20 Diagram accompanying the 'Flowers and grass' problem

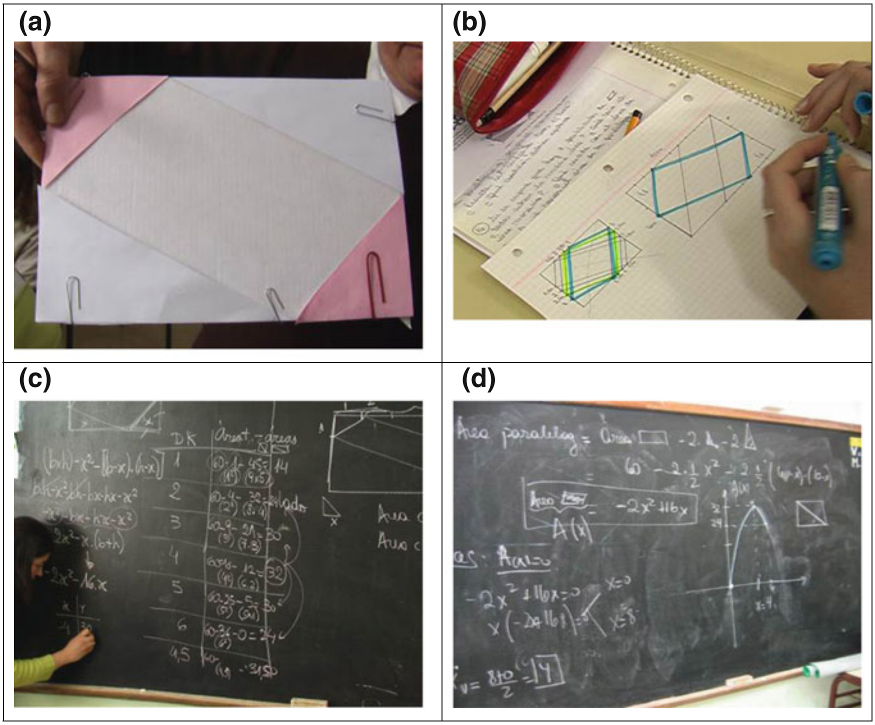


Fig. 9.21 Solving the 'Flowers and grass' problem

materials for working in those schools, trying those out, and gathering classroom-generated artefacts.

9.3.2 Making Connections

In line with the RME principle that intertwining the four curriculum strands and making connections between mathematics and other school subjects gives coherence to instruction and promotes different ways of and tools for mathematising, GPDM participants designed, tried out, and documented many interdisciplinary instructional sequences (see gpdmatematica.org.ar).

For example, a lesson that gave students the opportunity to link mathematics with natural sciences centred on a postcard image of a tiny hummingbird. The experience was framed within the broader theme of using visual images, a sub-type of 'rich contexts' (Freudenthal, 1991), as springboard for students to formulate and tackle meaningful questions in contrast with the ubiquitous reliance on ready-made, stereotypical

Fig. 9.22 Bird on a # 2 pencil

pseudo-narratives known as word or story problems. Figure 9.22 shows the postcard Silvia Pérez presented to her 5th grade students (Pérez, Bressan, & Zolkower, 2006).

This image, a grandmother's gift to the class on the occasion of the ongoing natural science project, generated plenty of comments and questions, for example: "It can't be so small!", "Is that a trick?", "How did they take that picture?" As Silvia asked her students to organise those questions in some manner, they did so according to three categories they themselves formulated: questions that could be answered using information retrievable from the postcard image itself; questions that called for information not included in the postcard yet available if searched elsewhere; and questions that could not be answered at all due to the impossibility of accessing the needed information.

Next, the class agreed to address the following questions: "About how much does the hummingbird measure?", "Exactly how much does it measure?", "What is the relationship between the size of the bird and the size of the (#2) pencil?" As they addressed these questions, students explored the relationship between lengths in reality and in the picture and, via measuring, estimating, and using the ratio table, arrived at 4.9 cm. The bird is the *Mellisuga helenae*, the smallest type of hummingbird in the world, and measures less than 5 cm. As a spin-off of this project, the class visited a local radio station to talk to listeners about these birds. Along these same lines, namely the intertwining of mathematics and natural sciences, there was a project, done in 2005 in two 4th grade mathematics/science classes, around the measuring of moss (Pérez, 2007).

Similarly, in the winter of 2008 Rocío Alvarez engaged her 7th graders (school attended by low SES students) in a mathematics/science inquiry around the theme of snow. The inquiry emerged spontaneously as several students expressed concern about the shortage of snow expected for that winter, a central preoccupation given that the seasonal work and income of many Bariloche inhabitants depends heavily on snow attracting skiers to the city and its surroundings. While many questions posed by the students concerned snow as a physical-chemical phenomenon, a few expressed curiosity of a geometric kind, as in: "What does a snowflake look like when you look at it close up?" Rocío asked them to draw snowflakes, next those diagrams were checked experimentally (using magnifying glasses) and, honing into the geometric structure of snowflakes, including its multiple symmetries, the work centred the properties of regular hexagons, and how those properties can be used to construct them with ruler and compass (Fig. 9.23).[9] This experience was replicated years later with other groups of 7th grade students (Álvarez, 2015).

Invited to apply what they had learned about how to construct regular hexagons with geometric tools, students generated designs such as the following (Fig. 9.24).

The following year, when her 7th graders expressed interest in ergonomics and its application to the design of furniture, Rocío seised the opportunity to use that as a context for doing some geometry. Motivated by the question of what would be the ideal couch/chair for watching TV, the students tried different seating positions, searched online for types of chairs, and compared and contrasted various models measuring different heights and angles. Finally, they concluded that the best angle formed by the back and the seat is one ranging between 100° and 120°, and that the angle between the top of the back and the edge of the seat should be between 110° and 130°. After working on "Part D: Angles" of the MiC unit *Made to Measure* (De Lange & Wijers, 1997–1998), the students were able to confirm their predictions. Finally, they designed their own couch/chairs, taking into account angle measurement constraints, as shown below (Fig. 9.25).

9.3.3 Fall Seminar: Teachers Teaching Teacher Educators

The increasing visibility and impact of the GPDM soon caught the attention of the education authorities at the national level. In May of 2009, with funding and logistical support from the National Ministry of Education, the group offered in Bariloche a five-day Fall Seminar on RME for teacher educators representing of all of the 23 Argentinean provinces. The seminar was attended by 50 teachers, 38 of them with scholarships given by the INFD,[10] as well as curriculum specialists from the latter. In addition, the seminar benefited from the participation of Willem Uittenbogaard (and his wife Sylvia Eerhart) and Diana Rosenberg, the latter playing a crucial role as guide and simultaneous translator.

[9] An additional resource for this inquiry was the collection of snowflake images by Wilson Bentley (1865–1931), http://snowflakebentley.com.

[10] The INFD is part of the National Ministry of Culture and Education and has the function of directing and coordinating teacher education policies and programmes.

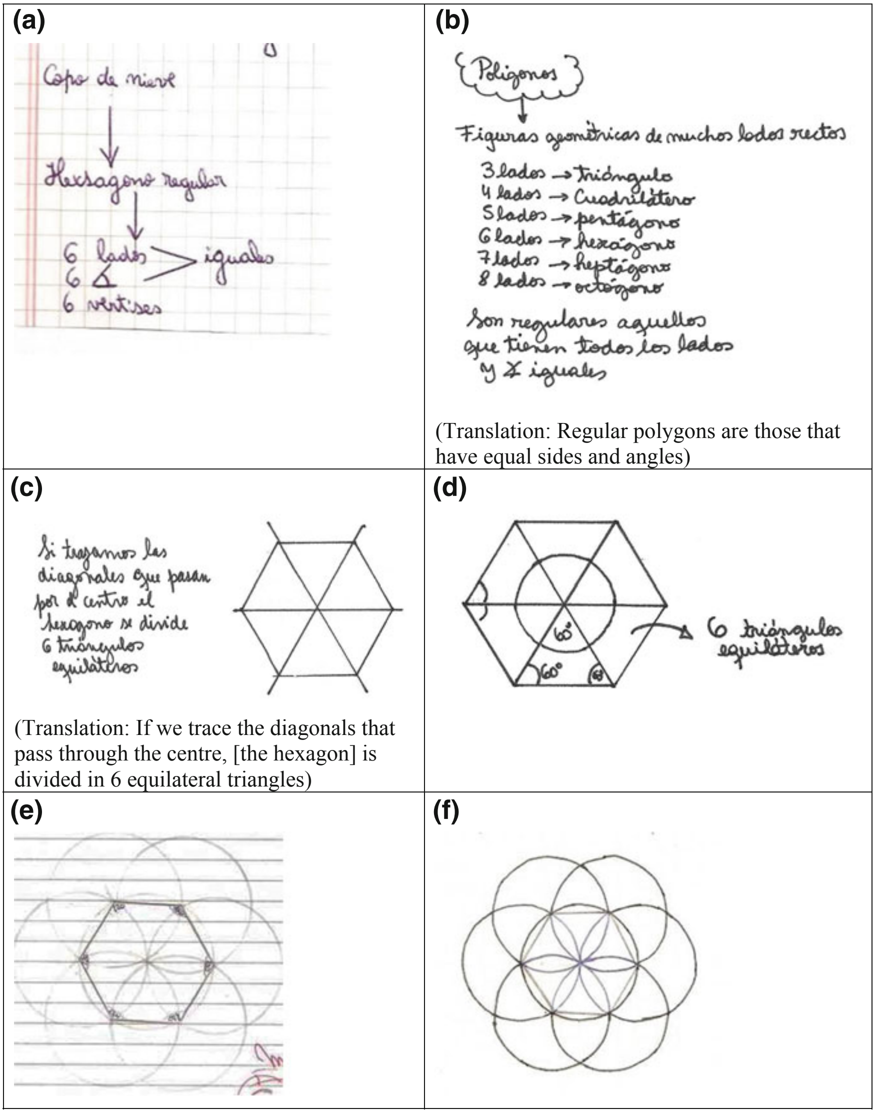


Fig. 9.23 From snowflakes to the properties and construction of regular hexagons

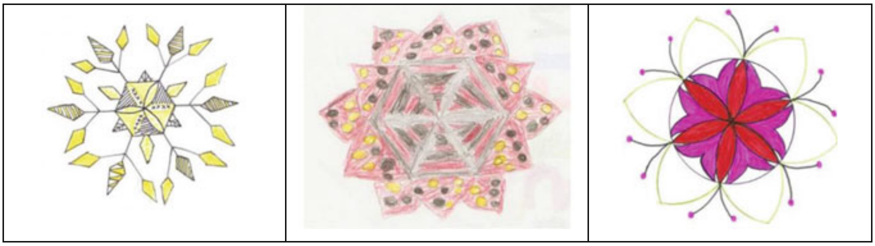


Fig. 9.24 Students' own hexagonal designs

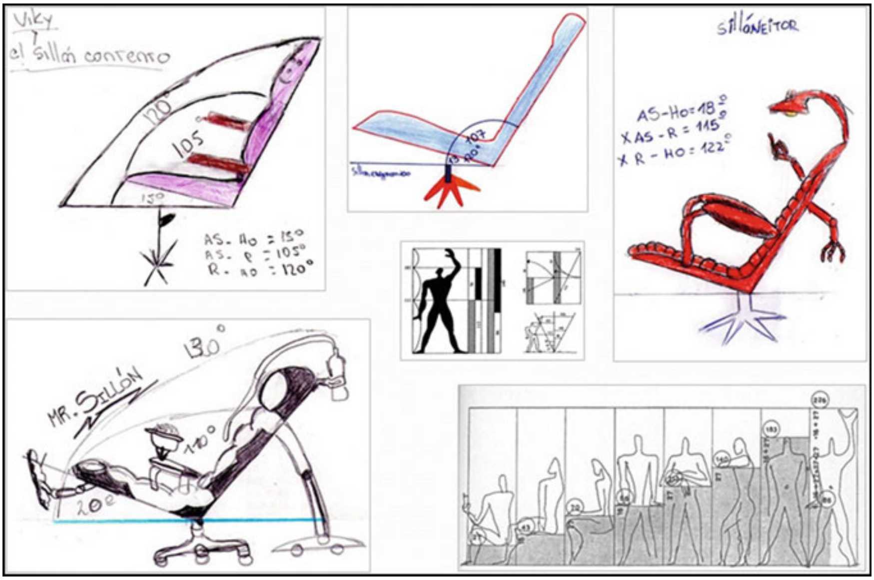


Fig. 9.25 Students' designs for an ergonomic couch/chair

The Fall Seminar was organised as follows: in the morning, teacher-led presentations and thematic workshops on the RME approach to whole number and rational number operations, geometry and measurement, and algebra (specifically, the patterns, symbols, and equations sub-strand); in the afternoon, guided visits to selected GPDM classrooms as well as to the teacher training institute, followed by post-visit debriefing sessions. In line with Freudenthal's (1991) emphasis on the parallel between mathematising and didactising, the seminar was organised around teaching experiments in GPDM classrooms (in Grades 1 through 10 as well as in teacher training courses), with guided opportunities for participants to observe learning and teaching processes, analyse student work, and reflect on all of the above.

Among the lessons planned for and tried out for the occasion were: an adaptation of the 'Barter' problem from *Comparing Quantities* (Kindt, Abels, Meyer, & Pligge, 1997–1998); brick pyramid problems (from the Dutch textbook series *Wis en Reken*;

see also, Zolkower & Rubel, 2015; Abrahamson, Zolkower, & Stone, this volume); an 18-piece regular hexagon dissection puzzle (Bressan, Rabino, & Zolkower, 2014); mental arithmetic mini-lessons (Pérez, Zolkower, & Bressan, 2014); side and top views (inspired by *Side Seeing*, Jonker, Querelle, Clarke, & Cole, 1997–1998); and exploring similarity within the context of making a triangular patchwork quilt for a sofa (adapted from *Triangles and Patchwork*, Roodhardt et al., 1997–1998). As an example, Fig. 9.26 shows the work of 7th grade students' working on the 'Barter' problem:

> Paulo goes to the market with 2 sheep and 1 goat which he wants to barter for corn to bring home for his family. They offer him: 1 bag of salt for 2 chickens, 2 bags of corn for 3 bags of salt, and 6 bags of salt for each sheep. What can Paulo do with his sheep and goat in order to come home with corn?

These and other student productions were posted simultaneously on the board and then compared and contrasted; also, errors were spotted and corrected (Fig. 9.26c). The teacher then engaged her students in a whole-class conversation focused on progressive symbolising which culminated in the conjoined writing, on a new poster (Fig. 9.27), of expressions that included letter symbols signifying the items bartered, arrows connecting fair share exchanges (exchange this for that), and the formal algebraic language of equations.

The Fall Seminar contributed greatly to further disseminating RME throughout Argentina, which can be inferred from the significant increase in the number of visitors to our webpage as well as the plethora of invitations for GPDM members to lead teacher-training seminars, offer thematic workshops, present at research conferences, and elaborate or evaluate curriculum documents and instructional materials.

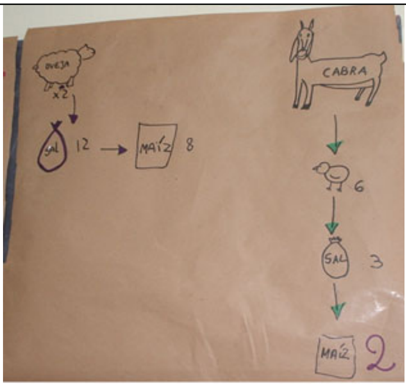
9.3.4 In the Meanwhile, in Pre-service Teacher Education

The Teacher Training Institute in Bariloche houses three programmes: elementary, early childhood, and special education. Since 2008, GPDM members teach many of the courses offered therein. Pre-service candidates arrive to those programmes with patchy and mostly procedural mathematical content knowledge and limited problem solving skills. Furthermore, many of them have a complex and often troubled relationship to mathematics, a "rigid and strict subject" that is "hard to understand," and which they have had "lots of difficulties" with.

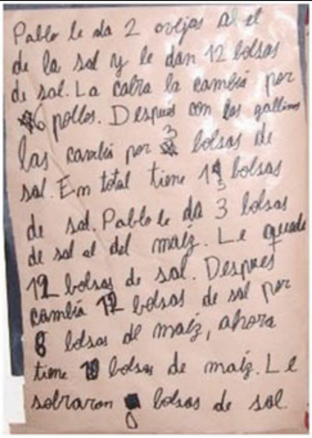
In view of the above, the initial preparation of mathematics teachers aims at helping candidates re-signify, expand, and deepen their mathematical knowledge as well as revisit and transform their beliefs about the subject, how students learn it, and how to teach it. In that respect, RME is a powerful tool for achieving such aims. As an illustration of the impact of RME in teacher education through the work of GPDM teacher educators, below is the testimony of a pre-service teacher candidate:

> My relationship to mathematics changed a lot. It used to be very hard for me. I would often get frustrated… I used to hate it. But this year, I think because of how we approached it in

(a)

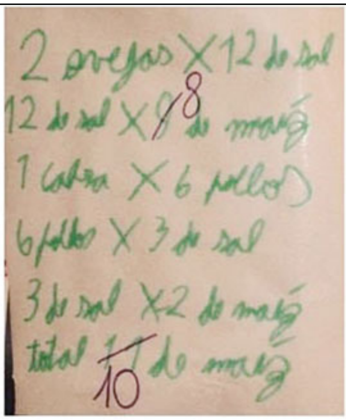


(b)



Pablo gives 2 sheep to the man from the salt and receives 12 bags of salt. He (ex)changes the chickens for 3 bags of salt. Altogether he has 15 bags of salt. Pablo gives 3 bags of salt to the man with the corn. He has 12 bags of salt left. Then he changes 12 salt bags for 8 corn bags; now he has 10 corn bags. He has 0 bags of salt left over.

(c)



2 sheep x 12 bags of salt
12 bags of salt x ~~9~~ 8 corn bags
1 goat x 6 chicken
6 chicken x 3 salt bags
3 salt bags x 2 corn bags
Total ~~11~~ 10 of corn

Fig. 9.26 Student work on the 'Barter' problem

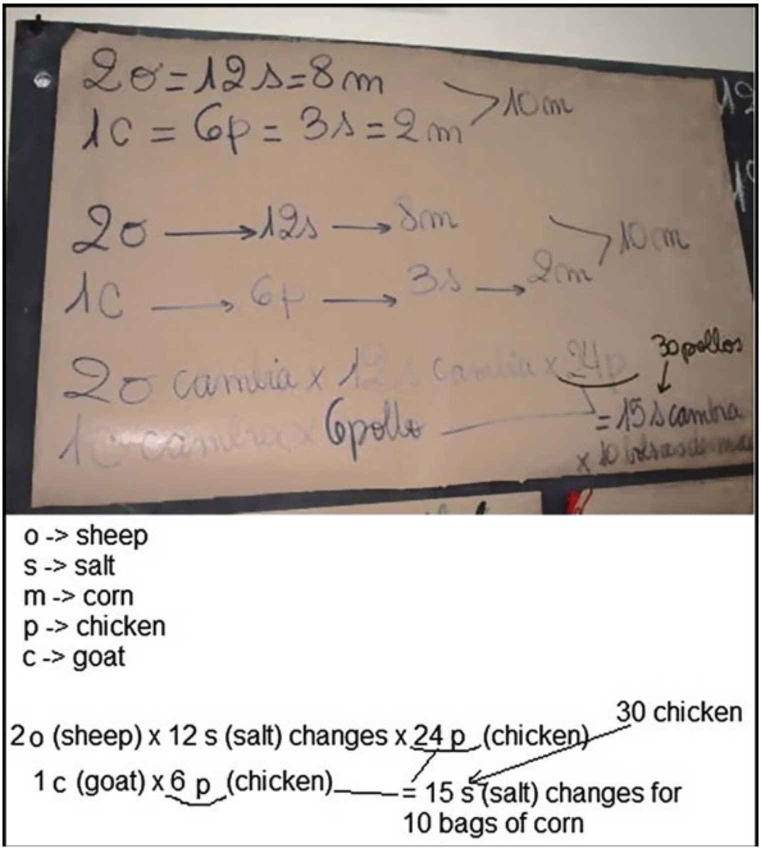


Fig. 9.27 Using formal algebraic language to solve the 'Barter' problem

> this class, focusing on learning and understanding, it changed completely my view of this subject. Now I put a lot of enthusiasm when doing math and when I make mistakes I do not feel frustrated because I know I can learn from those mistakes.

9.3.5 Thinking Aloud Together

Mathematising, interacting, and reflecting are tightly connected (Dekker & Elshout-Mohr, 2004; Elbers, 2003; Freudenthal, 1991; Goffree & Dolk, 1995). Yet for interaction and reflection to support progressive mathematising teachers ought to be skilful at guiding whole-class exchanges towards: posing questions about open-ended situations; sharing ideas, connections, productions, constructions, and further questions; practicing the use of models and strategies; and comparing, contrasting, and assessing alternative solution approaches or different ways of diagramming, symbolising, and generalising mathematical relationships.

In most classrooms, the whole-class share and discussion portion of the lesson typically occurs after the students have finished solving the problem at hand, and the interaction focuses on individual students (or groups) presenting their work, with the teacher facilitating the exchanges via follow-up questions (e.g., revoicing, echoing) aimed at level raising, and then closing up with a more or less interactive (and more or less explicit) institutionalisation of the target mathematical idea. Interested in exploring a variation of this approach which we have witnessed in classrooms taught by experienced and highly effective teachers, we framed our research around the following question: "What is the effect on students learning when whole-class conversations are held not when students are done solving the problem but, instead, midway through that process?" The empirical evidence we found (Shreyar, Zolkower, & Pérez, 2010; Zolkower & Pérez, 2007, 2012; Zolkower & Shreyar, 2007; Zolkower, Shreyar, & Pérez, 2015) suggests that these thinking-aloud-together conversations, when skilfully conducted by the teacher, can serve as an interpersonal plane for sharing ideas in statu nascendi thus maximising opportunities for students to appropriate them.

9.4 Third Phase (2011–2015): The GPDM, an Ever-Expanding Endeavour

During this third phase, the GPDM established itself as an important referent for mathematics teacher education in Argentina. Its members are frequently invited to conduct thematic workshops and seminars and present at conferences in the provinces of Catamarca, Córdoba, Buenos Aires, Ciudad Autónoma de Buenos Aires, and Santa Fe as well as in Salto (Uruguay) and, most recently, in Lima (Perú). The GPDM is regularly called to participate in committees on teacher preparation, in-service training, and instruction and assessment design at the provincial and the national levels. Furthermore, input from the group and, through it, from RME ideas have found their way into the curriculum and standards documents for the provinces of Río Negro and Neuquén.

The group's webpage plays a paramount role in disseminating the realistic approach via papers and instructional materials, adaptations of the latter (recontextualised units and activities), newly designed material, and teacher narratives of classroom experiences (many of which are published in *Novedades Educativas*). Increasingly, the group's webpage serves as a venue for receiving requests to serve in thesis committees and evaluate programmes throughout Argentina as well as other countries in the region. All of the above evidenced the place of the GPDM as an important referent on RME within Spanish speaking South America.

9.4.1 More Publications and Translations

Since 2001, RME-inspired lessons and units designed and implemented by GPDM teachers are regularly published in magazines and journals of widespread distribution in Argentina and abroad (see webpage, publications list). Also, the number of translations into Spanish of RME-related material continues to increase. For instance, in 2006 Gallego translated the online publication *Great Assessment Problems* (Dekker & Querelle, 2002); in 2011, Gallego and Zolkower translated *Children Learn Mathematics* (Van den Heuvel-Panhuizen, 2008, 2010); and in 2012, Gallego and da Valle translated *Young Children Learn Measurement and Geometry* (Van den Heuvel-Panhuizen & Buys, 2008, 2012) for a Mexican publisher.

9.4.2 Research Projects

Worth mentioning here are two recent collaborative inquiry projects conducted by group members that bear a strongly RME imprint. The first project, framed within the larger initiative Mathematics For All, focused on using games (e.g., games for teaching divisibility and other number theory topics). The second project, which we have referred to in Sect. 3.5, concerns the study of a teacher's manner of conducting whole-class conversations.

The mathematics games project, financially supported by the INFD, involved an interdisciplinary team that included members of the GPDM who are also professors at the Instituto de Formación Docente Continua.[11] This work resulted in the publication *El Juego en la Enseñanza de la Matemática* (Brinnitzer et al., 2015), a volume consisting of 60 games that can be used to address a wide range of curriculum topics, with variations for different grade levels as well as for meeting the needs of students who perform at various levels in mathematics.

Regarding the second project, over the past eight years, Zolkower and Pérez, together with Sam Shreyar from Teachers College, Columbia University, have been studying whole-class interaction in mathematics classrooms within a theoretical-methodological framework that intertwines tools from Systemic Functional Linguistics (SFL) (Halliday & Matthiessen, 2004) with ideas from Vygotsky, Dewey, and Freudenthal. Treating whole-class conversations as multi-semiotic texts (O'Halloran, 2000; Zolkower & de Freitas, 2012), we use SFL tools to describe and explain the choices of grammar and vocabulary, in addition to gestures and diagrams, made by the teacher in conducting those exchanges (Shreyar et al., 2010; Zolkower & Pérez, 2012; Zolkower, Shreyar, & Pérez, 2015). The main goal of this research is to describe, at a fine-grained level of lexico-grammatical detail, the manner in which

[11] The team, called Ludomateca, includes professors who teach courses in different areas (mathematics, arts, student teaching practicum) and levels (early childhood and elementary) as well as students in the elementary education programme.

experienced teachers support progressive mathematising in thinking-aloud-together conversations that involve the class as a whole.

9.5 Closure

Over the past sixteen years, our study and implementation of RME has been an invaluable collective experience in that it allowed us to: (a) experience first-hand the meaning of mathematising; (b) improve our own number sense, symbol sense, and spatial sense; (c) appropriate a wide range of mathematising tools and ways of using those tools; (d) enhance our ability to make, modify, and use diagrams as mathematical thinking devices; (e) understand how to use realistic contexts and situations can be used to support the guided reinvention of mathematising in the different curriculum strands; (f) acquire practice in recognising, finding, inventing, and using realistic problematic situations, including those which involve inter-strand and inter-disciplinary connections; (g) make room for and use students' productions and constructions; (h) acquire practice in organising and guiding students' work attending to informal, semi-formal, and formal levels of mathematising; (i) appreciate and make good didactical use of heterogeneities (i.e., differences in social, cultural, linguistic background as well as academic performance); and (j) become aware of the pivotal role of reflection in mathematising and didactising.

Rather than applying the principles of RME top down as dogmas and using RME instructional materials as ready-made recipes, the GPDM engages in processes of design, try out, reflection, revision, new try outs, and so on, in spiral movements that inter-connected our own mathematising with that of students in Grades K–12 and in teacher preparation courses. Fuelled by the common goal of making mathematics accessible, meaningful, and relevant for all students, we are reconstructing realistic mathematics education from the bottom up and, in so doing, we are contributing, albeit locally, to breach the gap between teachers and mathematics education specialists and the dichotomy between theory and practice.

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