Supersymmetries in Schrödinger–Pauli Equations and in Schrödinger Equations with Position Dependent Mass

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Dedicated to Véronique Hussin.

Abstract The contemporary results concerning supersymmetries in generalized Schrödinger equations are presented. Namely, position dependent mass Schrödinger equations are discussed as well as the equations with matrix potentials. An extended number of realistic quantum mechanical problems admitting extended supersymmetries are described.

Keywords Position dependent mass · Schrödinger–Pauli equations · Extended supersymmetries · Matrix potentials · Integrable systems

1 Introduction

In seventieth of the previous century a qualitatively new symmetry in physics had been discovered and called *supersymmetry* (SUSY) see, e.g. [1] but also [2] where the idea of SUSY was formulated in somewhat rudimentary form. Its rather specific property is the existence of symmetry transformations mixing bosonic and fermionic states. In other words transformations which connect fields with different statistics have been introduced.

Among the many attractive features of SUSY is that it provides an effective mechanism for the cancelation of the ultraviolet divergences in quantum field theory. In addition, it opens new ways to unify space-time symmetries (i.e., relativistic invariance) with internal symmetries and to construct unified field theories, including all types of interactions, refer, e.g. [3, 4] and [5].

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Mathematically, SUSY requests using of the graded Lie algebras instead of the usual ones, and the corresponding group parameters are not numbers but Grassmann variables. The essential progress in the related fields of mathematics was induced exactly by the needs of SUSY.

Unfortunately, till now we do not have convincing experimental arguments for introducing SUSY as a universal symmetry principle realized in Nature. Nevertheless, it is possible to find a number of realistic physical systems which admit this nice symmetry. Moreover, SUSY presents effective tools for understanding the relations between spectra of different Hamiltonians as well as for explaining degeneracy of their spectra, for constructing exactly or quasi-exactly solvable systems, for justifying formulations of initial and boundary problems, etc.; see, e.g., surveys [3, 6] and [7]. In other words, SUSY is realized in Nature at least in a rather extended number of particular physical systems.

The present work is concentrated on quantum mechanical systems since they provide a ground for testing the principal question: whether SUSY is realized in Nature or not, free of the complexities of field theories. Examples of such systems (like interaction of spin 1/2 particle with the Coulomb or constant and homogeneous magnetic field) which admit exact $N = 2$ SUSY are well known [8, 9] (see also Refs. [6, 7] and the references therein). However, we will concentrate on systems admitting more extended SUSY.

Let us remain that the supersymmetric quantum mechanics was created by Witten [10] as a toy model for illustration of global properties of the quantum field theory. But rather quickly it becomes a fundamental field attracting the interest of numerous physicists and mathematicians. In particular the SSQM presents powerful tools for explicit solution of quantum mechanical problems using the shape invariance approach [11]. The number of problems satisfying the shape invariance condition is rather restricted but includes the majority of exactly solvable Schrödinger equations. The well-known exceptions are exactly solvable Schrödinger equations with Natanzon potentials [12] which are formulated in terms of implicit functions.

A very important application of SUSY in quantum mechanics is classification of families of isospectral Hamiltonians. And there is a number of systems isospectral with the basic exactly solvable SEs. In the standard SUSY approach with the first order intertwining operators the problem of description of such families is reduced to constructing general solutions of the Riccati equations. More refined approaches can include intertwining operators of higher order [13], the N-fold supersymmetry [14], and the hidden nonlinear supersymmetry [15]. One more relevant subject of contemporary SUSY are the so-called exceptional orthogonal polynomials [16, 17].

Let us mention that other generalized supersymmetries which include the usual SUSY have been discussed also, among them the so-called parasupersymmetry [18–20], which also has good ruts in real physical problems. However, the standard SUSY is seemed to be more fundamental.

Just in quantum mechanics SUSY presents powerful tools for constructing exact solutions of Schrödinger equation (SE). And we will present a survey of contemporary results belonging to this field. We will not discuss generalizations of the standard SUSY in quantum mechanics like the ones mentioned above, but restrict ourselves to the standard SUSY quantum mechanics with the first order intertwining operators [21]. However, the systems with extended SUSY as well as systems including SEs with Pauli and spin-orbit couplings, with position dependent mass and with abstract matrix potentials will be considered. Notice that just these fields are the subjects of current interest of numerous investigators.

Let us stress that there are two faces of SUSY in quantum mechanics. First, there exist QM systems like the charged particle with spin 1/2 in the constant and homogeneous magnetic field which admit exact SUSY. Such systems admit constants of motion forming superalgebras. Second, it is possible to indicate the QM systems with "hidden" SUSY like the hydrogen atom, and just these systems can be solved exactly using the shape invariance of the related Schrödinger equations. We will discuss both types of SUSY. The realistic physical systems which admit exact SUSY will be considered in the next section, while the shape invariant systems are discussed in Sects. 3–6.

An inspiring example of QM problem with a shape invariant potential was discovered by Pron'ko and Stroganov [22] who studied a motion of a neutral non-relativistic fermion, e.g., neutron, interacting with the magnetic field generated by a current carrying wire. A relativistic version of such problem was discussed in [23].

The specificity of the PS problem is that it includes a *matrix superpotential*, while in the standard SUSY in quantum mechanics the superpotential is a scalar function. Matrix potentials and superpotentials naturally appear in quantum mechanical models including particles with spin (see, e.g. [24], Sections 10 and 11) and in multidimensional models of SSQM [25, 26]. Particular examples of such superpotentials were discussed in [27–31]. In papers [32] such superpotentials were used for modeling the motion of a spin $\frac{1}{2}$ particle in superposed magnetic and scalar fields. In paper [29] a certain class of such superpotentials was described, while more extended classes of them were classified in [33, 34]. In any case just systems matrix superpotentials belong to an interesting research field which makes it possible to find new coupled systems of exactly solvable Schrödinger equations. The contemporary results in this field will be discussed in the following.

In addition to SUSY, some SEs can possess one more nice property called superintegrability (SI). By definition, the quantum system is called superintegrable if it admits more integrals of motion than the degrees of freedom. Like SUSY, the SI can cause the exact solvability of the related SE, especially in the case when it is the maximal SI when the number of integrals of motion is equal to $2n + 1$ where n is the number of degrees of freedom.

There exists a tight connection between the SI and SUSY, and many QM systems are both supersymmetric and superintegrable. In fact the maximal SI induces SUSY and vice versa, in spite of that this fact was never proven for generic QM systems.

The superintegrable systems which are also supersymmetric will be a special subject of our discussion. Moreover, there will be systems with position dependent masses which are discussed in Sect. 6.

2 QM Systems with Exact SUSY

2.1 System with $N = 2$ SUSY

Let us start with the well-known and important physical system, i.e., the spinning and charged particle interacting with an external magnetic field. The corresponding QM Hamiltonian can be written in the following form:

$$H = \frac{\pi^2}{2m} + \frac{e}{2m}\sigma_i B_i, \tag{2.1}$$

where $\pi^2 = \pi_1^2 + \pi_2^2 + \pi_3^2$, $\pi_i = -\mathrm{i}\frac{\partial}{\partial x_i} - eA_i$, $i = 1, 2, 3$, $B_i = \varepsilon_{ijk}\frac{\partial A_j}{\partial k}$, σ_i are Pauli matrices, B_i and A_i are components of the external magnetic field and the corresponding vector-potential, and summation is imposed over the repeating index i.

In contrast with the standard Schrödinger Hamiltonian, operator (2.1) includes the Pauli term $\frac{e}{2m}\sigma_i B_i$ describing the interaction of the particle spin with the external magnetic field. The related stationary Schrödinger equation has the standard form:

$$H\psi = E\psi \tag{2.2}$$

with E being the Hamiltonian eigenvalues.

In the case of the constant and homogeneous magnetic field directed along the third coordinate axis the vector-potential can be reduced to the form:

$$A_1 = -\frac{1}{2}x_2 B_3, \quad A_2 = \frac{1}{2}x_1 B_3 \quad A_3 = 0, \tag{2.3}$$

and by definition $B_1 = B_2 = 0$, $B_3 = B = const$. Thus Hamiltonian (2.1) can be rewritten in the following form:

$$H = H_1 + H_2, \quad H_1 = \frac{p_3^2}{2m}, \quad H_2 = \frac{(\sigma_1\pi_1 + \sigma_2\pi_2)^2}{2m} \tag{2.4}$$

with $p_3 = -\mathrm{i}\frac{\partial}{\partial x_3}$.

The immediate consequence of representation (2.4) is that our Hamiltonian commutes with the three operators:

$$Q_1 = \sigma_1\pi_1 + \sigma_2\pi_2, \ Q_2 = \mathrm{i}\sigma_3 Q_1, \ Q_3 = p_3, \tag{2.5}$$

which satisfy the following algebraic relations:

$$[Q_3, Q_1] = [Q_3, Q_2] = [Q_3, H] = 0, \tag{2.6}$$

$$\{Q_\mu, Q_\nu\} = 2\delta_{\mu\nu}H_2, \ [Q_\mu, H_2] = 0, \tag{2.7}$$

where μ, ν independently takes the values 1, 2, $\delta_{\mu\nu}$ is the Kronecker delta, and the symbols [.,.] and {.,.} denote the commutator and anticommutator correspondingly.

Thus the considered Hamiltonian admits three constants of motion, one of which, i.e., Q_3, commutes with the two others. On the other hand, Q_1 and Q_2 are not in involution, but satisfy more complicated relations (2.7), which characterize a *Lie superalgebra*.

Just this specific supersymmetry can be treated as the reason of the twofold degeneration of the Landau levels, i.e., the non-ground energy levels of a spin 1/2 particle interacting with the constant and homogeneous magnetic field.

Generally speaking, superalgebra is a graded algebra. In the simplest case of the Z_2 grading the elements of the superalgebra belong to two different classes, say, odd or even. The multiplication laws for even and odd elements are different. In our case Q_1 and Q_2 are odd, while Q_3, H_1, and H_2 are even. The product of two algebra elements is defined as the commutator if at least one of them is even and as the anticommutator if both the elements are odd. In SUSY quantum mechanics the odd elements are called supercharges. Since we have indicated two supercharges then it is possible to say about $N = 2$ SUSY.

2.2 Extended SUSY

The considered system is only a particular (albeit very important) example of realistic physical problem admitting exact supersymmetry. In particular, it is obvious that the presented SUSY is valid for arbitrary Hamiltonian admitting representation (2.1) provided one component of the vector-potential of the external field is identically zero.

We will discuss also another examples, but first let us note that in fact Eq. (2.2) with Hamiltonian (2.4) admits a more extended SUSY.

In analogy with the above we can construct a supercharge valid for Eq. (2.1) in the case of arbitrary external magnetic field:

$$\tilde{Q}_1 = \sigma_i \pi_i \tag{2.8}$$

since $\tilde{Q}_1^2 = H$.

Let us show that it is possible to find three more supercharges provided the external field is given by relations (2.3). To do it we exploit the fact that Eq. (2.4) is invariant w.r.t. the following three discrete transformations:

$$\psi \to R_3\psi, \qquad \psi \to CR_1\psi, \qquad \psi \to CR_2\psi, \tag{2.9}$$

where R_a $(a = 1, 2, 3)$ are the space reflection transformations

$$R_a\psi(\mathbf{x}) = \sigma_a\theta_a\psi(\mathbf{x}), \qquad \theta_a\psi(\mathbf{x}) = \psi(r_a\mathbf{x}). \tag{2.10}$$

Here

$$r_1\mathbf{x} = (-x_1, x_2, x_3), \quad r_2\mathbf{x} = (x_1, -x_2, x_3), \quad r_3\mathbf{x} = (x_1, x_2, -x_3), \tag{2.11}$$

and $C = i\sigma_2 c$, where c is the operator of complex conjugation

$$c\psi(\mathbf{x}) = \psi^*(\mathbf{x}). \tag{2.12}$$

Note that operators (2.9) satisfy the following relations:

$$\begin{aligned} &\{R_a, \sigma_i\pi_i\} = \{R_a, C\} = \{CR_1, \sigma_i\pi_i\} = \{CR_2, \sigma_i\pi_i\} = 0, \\ &R_a^2 = -C^2 = 1, \qquad a = 1, 2, 3. \end{aligned} \tag{2.13}$$

Using (2.8), (2.13) we can see that the operators

$$\tilde{Q}_1 = \sigma_i\pi_i \quad Q_2 = iR_3\tilde{Q}_1, \quad Q_3 = CR_1\tilde{Q}_1, \quad Q_4 = CR_2\tilde{Q}_1 \tag{2.14}$$

fulfill the following relations:

$$\{Q_k, Q_l\} = 2g_{kl}\hat{H}, \qquad \left[Q_k, \hat{H}\right] = 0, \tag{2.15}$$

where $k, l = 1, 2, 3, 4$, $g_{11} = g_{22} = -g_{33} = -g_{44} = 1$; $g_{kl} = 0, k \neq l$. In other words, operators (2.14) are supercharges generating the $N = 4$ extended SUSY.

Let us note that the main trick for constructing the extended SUSY was using the discrete involutive symmetries, i.e., reflections (2.10), (2.11). We will see that in analogous way it is possible to find extended SUSY for rather generic Eqs. (2.2).

2.3 *Extended SUSY with Arbitrary Vector-Potentials*

The results of the previous section can be generalized to extended class of arbitrary potentials with well-defined parities. Starting with reflections (2.10) we find that the corresponding parity properties of vector-function $\mathbf{A}(\mathbf{x})$ (2.3) are of the form:

$$\mathbf{A}(r_1\mathbf{x}) = -r_1\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_2\mathbf{x}) = -r_2\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_3\mathbf{x}) = r_3\mathbf{A}(\mathbf{x}). \tag{2.16}$$

Relations (2.16) are satisfied by a large class of potentials which includes (2.3) as a particular case. For all such potentials the corresponding Eq. (2.2) is invariant w.r.t. involutions (2.9) and so admits the extended SUSY generated by supercharges (2.14). Moreover, Eq. (2.1) for $g = 2$ and an arbitrary uniform magnetic field, i.e., the field

$$A_1 = A_1(x_1, x_2), \quad A_2 = A_2(x_1, x_2), \quad A_3 = 0, \tag{2.17}$$

admits all internal symmetries described in the previous section provided $\mathbf{A}(\mathbf{x})$ satisfies relations (2.16).

Other systems with extended SUSY can be found by extending reflections (2.11) to the eight-dimensional group of involutions, i.e., by adding the fixed *rotation* transformations

$$\begin{aligned} &r_{12}\mathbf{x} = (-x_1, -x_2, x_3), \quad r_{31}\mathbf{x} = (-x_1, x_2, -x_3), \\ &r_{23}\mathbf{x} = (x_1, -x_2, -x_3), \quad r_{123}\mathbf{x} = (-x_1, -x_2, -x_3), \quad I\mathbf{x} = \mathbf{x}. \end{aligned} \tag{2.18}$$

Let the vector-potential $\mathbf{A}(\mathbf{x})$ has definite parities w.r.t. a subset of transformations (2.11) and (2.18). Then it is possible to construct supercharges which generate extended $N = 4$ and even $N = 5$ SUSY [35].

Thus we present a number of SE admitting extended SUSY. Let us stress then among them there is a lot of systems with a clear exact physical meaning, see [36] for discussion of this aspect.

3 SUSY in One Dimension and Shape Invariance

The models considered in the above were two or three dimensional in spatial variables and include systems of coupled Shcrödinger equations. However, many of them can be reduced to one dimensional systems using the separation of variables. Moreover, these systems can be decoupled.

Returning to Eq. (2.2) for a charged particle interacting with the constant and homogeneous magnetic field we can exploit its rotational invariance and search for solutions in separated radial and angular variables, i.e., to represent the wave function ψ as

$$\psi = \frac{1}{\tilde{r}} R(\tilde{r}) \mathrm{e}^{n\varphi}, \tag{3.1}$$

where $\tilde{r} = \sqrt{x_1^2 + x_2^2}$, $\varphi = \arctan \frac{x_2}{x_1}$. As a result we come to the following equation for the radial functions:

$$\tilde{H} R \equiv \left(-\frac{\partial^2}{\partial \tilde{r}^2} - \frac{m(m+1)}{\tilde{r}^2} + \omega\sigma_3 + \omega^2 \tilde{r}^2 \right) R = \tilde{E} R, \tag{3.2}$$

where $m = n - \frac{1}{2}$, $\omega = 2m\alpha$, and $\tilde{E} = 2mE + q_3^2 + \omega n$.

Alternatively, using the gauge transformation it is possible to pass from vector-potential (2.3) to the following ones: $A_1 = eHx_2$, $A_2 = A_3 = 0$. Then, representing the wave function in the form $\psi = \exp[\mathrm{i}(p_1 x_1 + p_3 x_3)]\phi(x_2)$ and setting $x_2 = \frac{1}{\alpha B}(p_1 + \sqrt{\alpha B} y)$ we obtain the following equation for ϕ:

$$\hat{H}\phi = \hat{E}\phi, \tag{3.3}$$

where

$$\hat{H} = -\frac{\partial^2}{\partial y^2} + \sigma_3\omega + \omega^2 x^2, \quad \hat{E} = 2mE - p_3^2. \tag{3.4}$$

Equation (3.4) defines the supersymmetric oscillator, while (3.2) is rather similar to the "3d supersymmetric oscillator" but includes half-integer parameter m while in the $3d$ oscillator this parameter is integer. Both the mentioned equations are decoupled to direct sums of equations since the related Hamiltonians $\tilde{H}$ and $\hat{H}$ have the following form:

$$\tilde{H} = \begin{pmatrix} \tilde{H}_+ & 0 \\ 0 & \tilde{H}_- \end{pmatrix}, \qquad \hat{H} = \begin{pmatrix} \hat{H}_+ & 0 \\ 0 & \hat{H}_- \end{pmatrix}, \tag{3.5}$$

where

$$\tilde{H}_\pm = -\frac{\partial^2}{\partial \tilde{r}^2} + \frac{n(n \mp 1)}{\tilde{r}^2} + \omega^2\tilde{r}^2 \pm \omega, \quad \hat{H}_\pm = -\frac{\partial^2}{\partial \tilde{y}^2} + \omega^2\tilde{y}^2 \pm \omega. \tag{3.6}$$

Hamiltonians $\hat{H}_\pm$ have two nice properties. First, they can be factorized:

$$\hat{H}_+ = a^+ a, \quad \hat{H}_- = a a^+, \tag{3.7}$$

where a^+ and a^- are the first order differential operators:

$$a^+ = -\frac{\partial}{\partial y} + W, \quad a^- = \frac{\partial}{\partial y} + W$$

with $W = \omega y$. Second, these Hamiltonians coincide up to a constant term: $\hat{H}_+ = \hat{H}_- + 2\omega$.

Hamiltonians $\tilde{H}_\pm$ are factorizable too:

$$\tilde{H}_- = a_\kappa^+ a_\kappa^- + c_\kappa, \quad \tilde{H}_+ = a_\kappa^- a_\kappa^+ + c_{\kappa+1}, \tag{3.8}$$

where

$$a_\kappa^- = \frac{\partial}{\partial x} + W_\kappa, \quad a_\kappa^+ = -\frac{\partial}{\partial x} + W_\kappa, \tag{3.9}$$

and $c_\kappa = (2\kappa - 1)\omega$. Moreover these Hamiltonians satisfy the following relation:

$$\tilde{H}_+(\kappa) = \tilde{H}_-(\kappa + 1) + C_\kappa \tag{3.10}$$

with $C_\kappa = 2\omega$. In other words, Hamiltonians $\tilde{H}_\pm(\kappa)$ are *shape invariant* [11]. The same is true for Hamiltonians $\hat{H}_\pm(\kappa)$, which, however, do not include variable parameter κ.

Thus our analysis of the realistic quantum mechanical system having a clear physical meaning (charged particle with spin 1/2 interacting with the constant and homogeneous magnetic field) makes it possible to discover its nice hidden symmetry, i.e., the shape invariance. It happens that this symmetry is valid for many other important QM systems like the hydrogen atom, and causes their exact solvability [11].

To be shape invariant, Hamiltonian should be factorizable, i.e., to admit representation (3.8), (3.9) for $\tilde{H}_-(\kappa)$ with some function W called *superpotential*. In addition, it should satisfy condition (3.10) together with the corresponding Hamiltonian $\tilde{H}_+(\kappa)$ which is called *superpartner*. If so, the related eigenvalue problem (2.2) is exactly solvable, and its solutions can be found algorithmically.

The shape invariance condition can be formulated as a condition for the potential. Considering the 1d Hamiltonian $H = -\frac{\partial 2}{\partial x^2} + V(\kappa, x)$ with a given potential V dependent on x and parameter κ and representing $V(\kappa, x)$ as

$$V = W_\kappa^2 + W_\kappa', \tag{3.11}$$

where $W_\kappa' = \frac{\partial W_\kappa}{\partial x}$, and superpotential is a solution of the Riccati equation (3.11). Then we construct a superpartner potential

$$\tilde{V} = W_\kappa^2 - W_\kappa'. \tag{3.12}$$

The corresponding stationary Schrödinger equation is shape invariant provided $\tilde{V}(\kappa, x) = V(\kappa + 1) + C_\kappa$, where C_κ is a constant. In terms of the superpotential this condition looks as follows:

$$W_\kappa^2 - W_\kappa' = W_\kappa^2 + W_\kappa' + C_\kappa. \tag{3.13}$$

A natural question arises whether it is possible to formulate the shape invariance condition with another transformation law for potential parameters. The answer is yes, but the rule $\kappa \to \kappa + 1$ can be treated as general up to redefinition of these parameters. In other words, we always can change these parameters by some functions of them in such a way that their transformations will be reduced to shifts [37].

4 Matrix Superpotentials

4.1 Pron'ko–Stroganov Problem

The supersymmetric systems considered in the above include matrix potentials. However, when speaking about shape invariance, we deal with scalar potentials and superpotentials, refer to Eqs. (3.6). Let us show that the concept of shape invariance can be extended to the case of matrix superpotentials.

Like in Sect. 2 we will start with a well-defined QM system which includes a matrix potential and appears to be shape invariant. Namely, let us consider a neutral QM particle with non-trivial dipole momentum (e.g., neutron), interacting with the magnetic field generated by a straight line current directed along the third coordinate axis (Pron'ko–Stroganov problem [22]) The corresponding Schrodinger–Pauli Hamiltonian looks as follows:

$$\mathcal{H} = \frac{p_1^2 + p_2^2}{2m} + \lambda \frac{\sigma_1 x_2 - \sigma_2 x_1}{\tilde{r}^2}, \tag{4.1}$$

where λ is the integrated coupling constant, and σ_1 and σ_2 are Pauli matrices.

The last term in (4.1) is the Pauli interaction term $\lambda\sigma_i H_i$ where the magnetic field $\mathbf{H}$ has the following components which we write ignoring the constant multiplier included into the parameter λ:

$$H_1 \sim \frac{y}{r^2}, \quad H_2 \sim -\frac{x}{r^2}, \quad H_3 = 0. \tag{4.2}$$

Hamiltonian (4.1) commutes with the third component of the total orbital momentum $J_3 = x_1 p_2 - x_2 p_1 + 1/2\sigma_3$; thus, the corresponding stationary Schrödinger equation (2.2) admits solutions in separated variables. Moreover, the equation for radial functions takes the following form:

$$\hat{H}_\kappa \psi = E_\kappa \psi, \tag{4.3}$$

where $\hat{H}_\kappa$ is a Hamiltonian with a matrix potential, E_κ and ψ are its eigenvalue and eigenfunction correspondingly, moreover, ψ is a two-component spinor. Up to normalization of the radial variable $\tilde{r}$ the Hamiltonian $\hat{H}_\kappa$ can be represented as

$$\hat{H}_\kappa = -\frac{\partial^2}{\partial \tilde{r}^2} + \kappa(\kappa - \sigma_3)\frac{1}{\tilde{r}^2} + \sigma_1 \frac{1}{\tilde{r}}, \tag{4.4}$$

where σ_1 and σ_3 are Pauli matrices and κ is a natural number. In addition, solutions of Eq. (4.3) must be normalizable and vanish at $x = 0$.

Hamiltonian $\hat{H}_\kappa$ can be factorized as in (3.8) where

$$a_\kappa^- = \frac{\partial}{\partial x} + W_\kappa, \quad a_\kappa^+ = -\frac{\partial}{\partial x} + W_\kappa, \quad c_\kappa = -\frac{1}{(2\kappa + 1)^2}$$

and W is a *matrix superpotential*

$$W_\kappa = \frac{1}{2x}\sigma_3 - \frac{1}{2\kappa+1}\sigma_1 - \frac{2\kappa+1}{2x}. \tag{4.5}$$

It is easily verified that the superpartner of Hamiltonian $\hat{H}_\kappa$ satisfies relation (3.10). In other words, Eq. (4.3) admits supersymmetry with shape invariance and can be solved using the standard technique of SSQM exposed, e.g., in survey [24].

4.2 Generic Matrix Superpotentials

Following a natural desire to find other shape invariant matrix potentials we return to conditions (3.13) which should be satisfied by the corresponding matrix superpotentials.

Assume $W_k(x)$ is Hermitian. Then the corresponding potential $V_k(x)$ and its superpartner $V_k^+(x)$ are Hermitian too.

The problem of classification of shape invariant superpotentials, i.e., $n \times n$ matrices whose elements are functions of x, k satisfying conditions (3.13), was formulated and partially solved in papers [33] and [34]. Here we present the completed classification results for a special class of superpotentials being 2×2 matrices.

Consider superpotentials of the following special form:

$$W_k = kQ + \frac{1}{k}R + P, \tag{4.6}$$

where P, R, and Q are Hermitian matrices depending on x.

Substituting (4.6) into (3.13) we obtain the following equations for P, R, and Q:

$$Q' = \alpha(Q^2 + \nu I), \tag{4.7}$$

$$P' - \frac{\alpha}{2}\{Q, P\} + \varkappa I = 0, \tag{4.8}$$

$$\{R, P\} + \lambda I = 0, \tag{4.9}$$

$$R^2 = \omega^2 I, \tag{4.10}$$

where $Q' = \frac{dQ}{dx}$, $\{Q, P\} = QP + PQ$ is an anticommutator of matrices Q and P, I is the unit matrix, and $\varkappa,\ \lambda,\ \omega$ are constants. Thus the problem of classification of matrix superpotentials is reduced to solution of Eqs. (4.8)–(4.10) for unknown matrices Q and P, R.

4.3 Scalar Superpotentials

First we consider the scalar case when Q, P, and R in (4.6) are 1×1 "matrices." The corresponding Eqs. (4.7)–(4.10) can be integrated rather easily, refer to [34] for detailed calculations. As a result we obtain the well-known list of scalar superpotentials:

$$W = -\frac{\kappa}{x} + \frac{\omega}{\kappa} \;\texttt{(Coulomb)}, \tag{4.11}$$

$$W = \lambda\kappa \tan\lambda x + \frac{\omega}{\kappa} \;\texttt{(Rosen 1)}, \tag{4.12}$$

$$W = \lambda\kappa \tanh\lambda x + \frac{\omega}{\kappa} \;\texttt{(Rosen 2)}, \tag{4.13}$$

$$W = -\lambda\kappa \coth\lambda x + \frac{\omega}{\kappa} \;\texttt{(Eckart)}, \tag{4.14}$$

$$W = \mu x \;\texttt{(Harmonic Oscillator)}, \tag{4.15}$$

$$W = \mu x - \frac{\kappa}{x} \;\texttt{(3D Oscillator)}, \tag{4.16}$$

$$W = \lambda\kappa \tan\lambda x + \mu \sec\lambda x \;\texttt{(Scarf I)}, \tag{4.17}$$

$$W = \lambda\kappa \tanh\lambda x + \mu\texttt{sech}\lambda x \;\texttt{(Scarf 2)}, \tag{4.18}$$

$$W = \lambda\kappa \coth\lambda x + \mu\texttt{cosech}\lambda x \;\texttt{(Pöschl-Teller)}, \tag{4.19}$$

$$W = \kappa - \mu \exp(-x) \;\texttt{(Morse)}. \tag{4.20}$$

Thus we recover the known list of superpotentials (4.11)–(4.20) which generate classical additive shape invariant potentials, in a straightforward and very simple way. The corresponding potentials V_κ can be found using definition (3.11).

4.4 Matrix Superpotentials of Dimension 2×2

Here we consider the case when superpotentials are x-dependent 2×2 matrices of form (4.6).

Supposing that $Q(x)$ is diagonal (like in (4.5)), it is possible to specify five inequivalent solutions of Eqs. (3.13):

$$W_{\kappa,\mu} = ((2\mu + 1)\,\sigma_3 - 2\kappa - 1)\,\frac{1}{2x} + \frac{\omega}{2\kappa + 1}\sigma_1, \quad \mu > -\frac{1}{2}, \tag{4.21}$$

$$W_{\kappa,\mu} = \lambda\left(-\kappa + \mu\exp(-\lambda x)\sigma_1 - \frac{\omega}{\kappa}\sigma_3\right), \tag{4.22}$$

$$W_{\kappa,\mu} = \lambda\left(\kappa\tan\lambda x + \mu\sec\lambda x\sigma_3 + \frac{\omega}{\kappa}\sigma_1\right), \tag{4.23}$$

$$W_{\kappa,\mu} = \lambda \left(-\kappa \coth \lambda x + \mu \operatorname{csch} \lambda x \sigma_3 - \frac{\omega}{\kappa}\sigma_1 \right), \quad \mu < 0, \ \omega > 0, \tag{4.24}$$

$$W_{\kappa,\mu} = \lambda \left(-\kappa \tanh \lambda x + \mu \operatorname{sech} \lambda x \sigma_1 - \frac{\omega}{\kappa}\sigma_3 \right), \tag{4.25}$$

where we introduce the rescaled parameter $\kappa = \frac{k}{\alpha}$. These superpotentials are defined up to translations $x \to x + c$, $\kappa \to \kappa + \gamma$, and up to unitary transformations $W_{\kappa,\mu} \to U_a W_{\kappa,\mu} U_a^\dagger$, where $U_1 = \sigma_1$, $U_2 = \frac{1}{\sqrt{2}}(1 \pm \mathrm{i}\sigma_2)$, and $U_3 = \sigma_3$. In particular these transformations change signs of parameters μ and ω in (4.22)–(4.25) and of $\mu + \frac{1}{2}$ in (4.21), thus without loss of generality we can set

$$\omega > 0, \quad \mu > 0 \tag{4.26}$$

in superpotentials (4.22)–(4.25).

Notice that the transformations $k \to k' = k + \alpha$ correspond to the following transformations for κ:

$$\kappa \to \kappa' = \kappa + 1. \tag{4.27}$$

If $\mu = 0$ and $\omega = 1$, then operator (4.21) coincides with the superpotential for PS problem given by Eq. (4.5). For $\mu \neq 0$ superpotential (4.21) is not equivalent to (4.5). The other presented matrix superpotentials were found in [33] for the first time.

The corresponding potentials V_κ can be found starting with (4.21)–(4.24) and using definition (3.11):

$$\hat{V}_\kappa = \left(\mu(\mu+1) + \kappa^2 - \kappa(2\mu+1)\sigma_3 \right) \frac{1}{x^2} - \frac{\omega}{x}\sigma_1, \tag{4.28}$$

$$\hat{V}_\kappa = \lambda^2 \left(\mu^2 \exp(-2\lambda x) - (2\kappa - 1)\mu \exp(-\lambda x)\sigma_1 + 2\omega\sigma_3 \right), \tag{4.29}$$

$$\begin{aligned}\hat{V}_\kappa = \lambda^2 \Big(&(\kappa(\kappa-1) + \mu^2) \sec^2 \lambda x + 2\omega \tan \lambda x \sigma_1 \\ &+ \mu(2\kappa - 1) \sec \lambda x \tan \lambda x \sigma_3 \Big),\end{aligned} \tag{4.30}$$

$$\begin{aligned}\hat{V}_\kappa = \lambda^2 \Big(&(\kappa(\kappa-1) + \mu^2) \operatorname{csch}^2(\lambda x) + 2\omega \coth \lambda x \sigma_1 \\ &+ \mu(1 - 2\kappa) \coth \lambda x \operatorname{csch} \lambda x \sigma_3 \Big),\end{aligned} \tag{4.31}$$

$$\begin{aligned}\hat{V}_\kappa = \lambda^2 \Big(&(\mu^2 - \kappa(\kappa-1)) \operatorname{sech}^2 \lambda x + 2\omega \tanh \lambda x \sigma_3 \\ &- \mu(2\kappa - 1) \operatorname{sech} \lambda x \tanh \lambda x \sigma_1 \Big).\end{aligned} \tag{4.32}$$

Potentials (4.28), (4.29), (4.30), (4.31), and (4.32) are generated by superpotentials (4.21), (4.22), (4.23), (4.24), and (4.25), respectively. All the above potentials are shape invariant and give rise to exactly solvable problems for systems of Schrödinger–Pauli type.

It was proven in [33] that $n \times n$ matrix superpotentials of the form (4.6) with a diagonal matrix Q and $n > 2$ can be reduced to direct sums of operators fixed in (4.21) and scalar superpotentials specified in Eqs. (4.11)–(4.20). Thus in fact we present a complete description of superpotentials (4.6) being matrices of arbitrary dimension, provided matrix Q is diagonal.

The case of non-diagonal matrices Q has been examined in paper [34]. The classifying Eqs. (4.7)–(4.10) have been solved for the cases of superpotentials being 2×2 or 3×3 matrices. In the first case the following list of superpotentials was obtained:

$$W_\kappa^{(1)} = \lambda\Big(\kappa\,(\sigma_+ \tan(\lambda x + c) + \sigma_- \tan(\lambda x - c)) \tag{4.33}$$

$$+ \mu\sigma_1\sqrt{\sec(\lambda x - c)\sec(\lambda x + c)} + \frac{1}{\kappa}R\Big), \tag{4.34}$$

$$W_\kappa^{(2)} = \lambda\Big(-\kappa(\sigma_+ \coth(\lambda x + c) + \sigma_- \coth(\lambda x - c)) \tag{4.35}$$

$$+ \mu\sigma_1\sqrt{\operatorname{csch}(\lambda x - c)\operatorname{csch}(\lambda x + c)} + \frac{1}{\kappa}R\Big), \tag{4.36}$$

$$W_\kappa^{(3)} = \lambda\Big(-\kappa(\sigma_+ \tanh(\lambda x + c) + \sigma_- \tanh(\lambda x - c)) \tag{4.37}$$

$$+ \mu\sigma_1\sqrt{\operatorname{sech}(\lambda x - c)\operatorname{sech}(\lambda x + c)} + \frac{1}{\kappa}R\Big), \tag{4.38}$$

$$W_\kappa^{(4)} = \lambda\Big(-\kappa(\sigma_+ \tanh(\lambda x + c) + \sigma_+ \coth(\lambda x - c)) \tag{4.39}$$

$$+ \mu\sigma_1\sqrt{\operatorname{sech}(\lambda x + c)\operatorname{csch}(\lambda x - c)} + \frac{1}{\kappa}R\Big), \tag{4.40}$$

$$W_\kappa^{(5)} = \lambda\left(-\kappa(\sigma_+ \tanh(\lambda x) + \sigma_-) + \mu\sigma_1\sqrt{\operatorname{sech}(\lambda x)\exp(-\lambda x)} + \frac{1}{\kappa}R\right),$$

$$W_\kappa^{(6)} = \lambda\left(-\kappa(\sigma_+ \coth(\lambda x) + \sigma_-) + \mu\sigma_1\sqrt{\operatorname{csch}(\lambda x)\exp(-\lambda x)} + \frac{1}{\kappa}R\right),$$

$$W_\kappa^{(7)} = -\kappa\left(\frac{\sigma_+}{x + c} + \frac{\sigma_-}{x - c}\right) + \frac{\mu\sigma_1}{\sqrt{x^2 - c^2}} + \frac{1}{\kappa}R, \tag{4.41}$$

$$W_\kappa^{(8)} = -\kappa\frac{\sigma_+}{x} + \mu\sigma_1\frac{1}{\sqrt{x}} + \frac{1}{\kappa}R, \tag{4.42}$$

$$W_\kappa^{(9)} = \lambda\left(-\kappa I + \mu\exp(-\lambda x)\sigma_1 - \frac{\omega}{\kappa}\sigma_3\right). \tag{4.43}$$

Here

$$\sigma_\pm = \frac{1}{2}(\sigma_0 \pm \sigma_3), \quad R = r_3\sigma_3 + r_2\sigma_2, \tag{4.44}$$

r_a are constants satisfying $r_2^2 + r_3^2 = \omega^2$, and κ, μ, λ, and $c \neq 0$ are arbitrary parameters.

4.5 Matrix Superpotentials of Dimension 3 × 3

In analogy with the above we can find superpotentials realized by irreducible 3×3 matrices which are presented in the following formulae:

$$\begin{aligned}
W &= (S_1^2 - 1)\frac{\kappa}{x + c_1} + (S_2^2 - 1)\frac{\kappa}{x + c_2} + (S_3^2 - 1)\frac{\kappa}{x} \\
&\quad + S_1\frac{\mu_1}{\sqrt{x(x + c_1)}} + S_2\frac{\mu_2}{\sqrt{x(x + c_2)}} + \frac{\omega}{\kappa}(2S_3^2 - 1), \\
W &= (S_1^2 - 1)\frac{\kappa}{x} + (S_2^2 - 1)\frac{\kappa}{x + c_1} + S_1\frac{\mu_2}{\sqrt{x}} + S_2\frac{\mu_1}{\sqrt{x + c_1}} + \frac{\omega}{\kappa}(2S_3^2 - 1), \\
W &= (S_1^2 - 1)\frac{\kappa}{x + c_1} + (S_3^2 - 1)\frac{\kappa}{x} + S_1\frac{\mu_2}{\sqrt{x}} + S_3\frac{\mu_1}{\sqrt{x(x + c_1)}} + \frac{\omega}{\kappa}(2S_3^2 - 1), \\
W &= (S_1^2 - 1)\frac{\kappa}{x} + S_1 c + S_2\frac{\mu_1}{\sqrt{x}} + \frac{\omega}{\kappa}(2S_3^2 - 1), \\
W &= (S_1^2 - 1)\frac{\kappa}{x + c_1} + (S_2^2 - 1)\frac{\kappa}{x + c_2} + (S_3^2 - 1)\frac{\kappa}{x} \\
&\quad + S_1\frac{\mu_1}{\sqrt{x(x + c_1)}} + S_2\frac{\mu_2}{\sqrt{x(x + c_2)}} + S_3\frac{\mu_3}{\sqrt{(x + c_1)(x + c_2)}}, \\
W &= (S_1^2 - 1)\frac{\kappa}{x} + (S_2^2 - 1)\frac{\kappa}{x + c_2} + S_1\frac{\mu_1}{\sqrt{x}} + S_2\frac{\mu_2}{\sqrt{x + c_2}} + S_3\frac{\mu_3}{\sqrt{x(x + c_2)}}, \\
W &= (S_1^2 - 1)\frac{\kappa}{x} + S_1 c + S_3\frac{\mu_1}{\sqrt{x}} + S_2\frac{\mu_2}{\sqrt{x}},
\end{aligned}$$

where c, c_1, c_2, μ_1, and μ_2 are integration constants, and

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.45}$$

are matrices of spin $s = 1$.

The hermiticity condition generates the following restrictions:

$$x > 0, \quad \texttt{if} \quad \mu_1^2 + \mu_2^2 > 0; \qquad c_i < 0 \quad \texttt{if} \quad \mu_i \neq 0. \tag{4.46}$$

Formulae (4.34)–(4.43) give the completed list of the certain class of matrix potentials. Note that they give rise to many realistic QM models described by coupled systems of Schrödinger equations, see the following section.

4.6 Shape Invariant QM Systems with Matrix Potentials

The discussed matrix superpotentials naturally appear in realistic QM systems. The entire collection of such system can be found in [38, 39] and [40]. Here we present two examples only.

Consider the following Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{\lambda}{2m}\sigma_i B_i + V, \tag{4.47}$$

where σ_i are Pauli matrices, $B_i = B_i(\mathbf{x})$ are vector components of magnetic field strength, $V = V(\mathbf{x})$ is a potential, and vector $\mathbf{x}$ represents independent variables. In addition, λ denotes the constant of anomalous coupling which is usually represented as $\lambda = g\mu_0$, where μ_0 is the Bohr magneton and g is the Landé factor.

Formula (4.47) presents a generalization of the Pron'ko–Stroganov Hamiltonian for the case of arbitrary external field. And some Schrödinger equations with Hamiltonians (4.47) appear to be shape invariant. The example is given by the following equation:

$$\begin{aligned} H\psi \equiv (-\nabla^2 + \lambda(1-2\kappa)\exp(-x_2)(\sigma_1\cos x_1 - \sigma_2 \sin x_1) \\ + \lambda^2 \exp(-2x_2))\psi = \hat{E}\psi. \end{aligned} \tag{4.48}$$

Here λ is the integrated coupling constant, and independent variables are rescaled to obtain more compact formulae.

Hamiltonian H in (4.48) admits integral of motion $Q = p_1 - \frac{\sigma_3}{2}$. Thus it is possible to expand solutions of (4.48) via eigenvectors of Q which look as follows:

$$\psi_p = \begin{pmatrix} \exp(i(p+\frac{1}{2})x_1)\varphi(x_2) \\ \exp(i(p-\frac{1}{2})x_1)\xi(x_2) \end{pmatrix} \tag{4.49}$$

and satisfy the condition $Q\psi_p = p\psi_p$.

Substituting (4.49) into (4.48) we come to Eq. (5.1) where

$$\hat{V}_\kappa = \lambda^2 \exp(-2y) - \lambda(2\kappa - 1)\exp(-y)\sigma_1 - p\sigma_3,$$
$$y = x_2, \quad E = \tilde{E} - p^2 - \frac{1}{4}, \quad \psi = \begin{pmatrix} \varphi \\ \xi \end{pmatrix}. \tag{4.50}$$

Potential $\hat{V}_\kappa$ (4.50) belongs to the list of shape invariant matrix potentials presented in the above, see Eq. (4.29). Thus Eq. (4.48) can be solved exactly using tools of SUSY quantum mechanics [39]. Notice that this equation is also superintegrable [38].

Let us present an analog of the PS model for particle of spin 1. This model is both superintegrable and shape invariant. It is based on the following Hamiltonian:

$$\mathcal{H}_s = \frac{p_1^2 + p_2^2}{2m} + \frac{1}{r}\mu_s(\mathbf{n}), \tag{4.51}$$

where

$$\mu_s(\mathbf{n}) = \mu_1(\mathbf{n}) = \mu(2(\mathbf{S} \times \mathbf{n})^2 - 1) + \lambda(2(\mathbf{S} \cdot \mathbf{n})^2 - 1). \tag{4.52}$$

Here μ and λ are arbitrary real parameters, $\mathbf{S} \cdot \mathbf{n} = S_1 n_2 + S_2 n_1$, and $\mathbf{S} \times \mathbf{n} = S_1 n_2 - S_2 n_1$, $n_1 = \frac{x_1}{\sqrt{x_1^2+x_2^2}}$, $n_2 = \frac{x_2}{\sqrt{x_1^2+x_2^2}}$, S_1 and S_2 are matrices of spin 1 given by formula (4.45).

It is the Hamiltonian defined by Eqs. (4.1) and (4.2) that generalized the Pron'ko–Stroganov model for the case of spin one. This Hamiltonian leads to shape invariant radial equations with matrix potential being the direct sum of a modified Coulomb potential and potential (4.28).

4.7 *Dual Shape Invariance*

Starting with (4.21)–(4.24) we found the related potentials (4.28)–(4.31) in a unique fashion. But there is an interesting inverse problem: to find possible superpotentials corresponding to given potentials. Formally speaking, this means to find all solutions of the Riccati equation (3.11) for W. However, such solutions depend on two arbitrary parameters (κ and the integration constant), and there is some ambiguity in choosing such of them which should be changing to generate the superpartner potential. Notice that the mentioned inverse problem is very interesting since it opens a way to generate families of isospectral Hamiltonians [24].

In the case of matrix superpotentials this business is even more important since in some cases there exist two superpotentials compatible with the shape invariance condition. And both these superpotential can be requested to generate solutions of the related eigenvalue problem.

To find the mentioned additional superpotentials we use the invariance of potentials (4.28), (4.30), and (4.31) with respect to the simultaneous change of arbitrary parameters:

$$\mu \to \kappa - \frac{1}{2}, \quad \kappa \to \mu + \frac{1}{2}. \tag{4.53}$$

This means that in addition to the shape invariance w.r.t. shifts of κ potentials (4.28), (4.30), and (4.31) should be shape invariant w.r.t. shifts of parameter μ too.

Thus, it is possible to represent potentials (4.21), (4.23), and (4.24) in the following alternative form:

$$\widetilde{W}^2_{\mu,\kappa} - \widetilde{W}'_{\mu,\kappa} = \hat{V}_\mu + c_\mu, \tag{4.54}$$

where $\hat{V}_\mu = \hat{V}_\kappa$, and

$$\widetilde{W}_{\mu,\kappa} = \frac{\kappa\sigma_3 - \mu - 1}{x} + \frac{\omega}{2(\mu+1)}\sigma_1, \quad c_\mu = \frac{\omega^2}{4(\mu+1)^2} \tag{4.55}$$

for $\hat{V}_k$ given by Eq. (4.28)

$$\widetilde{W}_{\mu,\kappa} = \frac{\lambda}{2}\left((2\mu+1)\tan\lambda x + (2\kappa-1)\sec\lambda x\sigma_3 + \frac{4\omega}{2\mu+1}\sigma_1\right) \tag{4.56}$$

for potential (4.30), and

$$\widetilde{W}_{\mu,\kappa} = \frac{\lambda}{2}\left(-(2\mu+1)\coth\lambda x + (2\kappa-1)\,\mathrm{csch}\,\lambda x\sigma_3 - \frac{4\omega}{2\mu+1}\sigma_1\right) \tag{4.57}$$

for potential (4.31). The related constant c_μ is

$$c_\mu = \lambda^2\left(\pm\frac{1}{4}(2\mu+1)^2 + \frac{4\omega^2}{(2\mu+1)^2}\right), \tag{4.58}$$

where the sign "+" and "−" correspond to the cases (4.56) and (4.57), respectively.

We stress that superpartners of potentials (4.54) constructed using superpotentials $\widetilde{W}_{\mu,\kappa}$, i.e.,

$$\hat{V}^+_\mu = \widetilde{W}^2_{\mu,\kappa} + \widetilde{W}'_{\mu,\kappa}, \tag{4.59}$$

satisfy the shape invariance condition since

$$\hat{V}^+_\mu = \hat{V}_{\mu+1} + C_\mu$$

with $C_\mu = c_{\mu+1} - c_\mu$.

Thus potentials are shape invariant w.r.t. shifts of two parameters, namely, κ and μ. More exactly, superpartners for potentials (4.28), (4.30), and (4.31) can be obtained either by shifts of κ or by shifts of μ, while simultaneous shifts are forbidden. We call this phenomena *dual shape invariance*.

5 Exact Solutions of Shape Invariant Schrödinger Equations

5.1 Generic Approach and Energy Values

An important consequence of the shape invariance is the nice possibility to construct exact solutions of the related stationary Schrödinger equation. The procedure of construction of exact solutions for the case of scalar shape invariant potentials is described in various surveys, see, e.g. [24]. Here we present this procedure for the more general case of matrix potentials.

Consider the stationary Schrödinger equation

$$\hat{H}_\kappa \psi \equiv \left(-\frac{\partial^2}{\partial x^2} + \hat{V}_\kappa\right)\psi = E_\kappa \psi, \tag{5.1}$$

where $\hat{H}_\kappa = a^+_{\kappa,\mu} a^-_{\kappa,\mu} + c_\kappa$ and $\hat{V}_\kappa$ is a shape invariant potential. An algorithm for construction of exact solutions of supersymmetric and shape invariant Schrödinger equations includes the following steps (see, e.g. [24]):

– To find the ground state solutions $\psi_0(\kappa, \mu, x)$ which are proportional to square integrable solutions of the first order equation

$$a^-_{\kappa,\mu}\psi_0(\kappa, \mu, x) \equiv \left(\frac{\partial}{\partial x} + W_{\kappa,\mu}\right)\psi_0(\kappa, \mu, x) = 0. \tag{5.2}$$

Function $\psi_0(\kappa, \mu, x)$ solves Eq. (5.1) with

$$E_\kappa = E_{\kappa,0} = -c_\kappa. \tag{5.3}$$

– To find a solution $\psi_1(\kappa, \mu, x)$ for the first excited state which is defined by the following relation:

$$\psi_1(\kappa, \mu, x) = a^+_{\kappa,\mu}\psi_0(\kappa+1, \mu, x) \equiv \left(-\frac{\partial}{\partial x} + W_{\kappa,\mu}\right)\psi_0(\kappa+1, \mu, x). \tag{5.4}$$

Since $a^\pm_\kappa$ and $\hat{H}_\kappa$ satisfy the intertwining relations

$$\hat{H}_\kappa a^+_{\kappa,\mu} = a^+_{\kappa,\mu}\hat{H}_{\kappa+1} \tag{5.5}$$

function (5.4) solves Eq. (5.1) with $E_\kappa = E_{\kappa,1} = -c_{\kappa+1}$.

– Solutions for the second excited state can be found as $\psi_2(\kappa, \mu, x) = a^+_{\kappa,\mu}\psi_1(\kappa + 1, \mu, x)$, etc. Finally, solutions which correspond to nth exited state for any admissible natural number $n > 0$ can be represented as

$$\psi_n(\kappa, \mu, x) = a^+_{\kappa,\mu}a^+_{\kappa+1,\mu}\cdots a^+_{\kappa+n-1,\mu}\psi_0(\kappa + n, \mu, x). \tag{5.6}$$

The corresponding eigenvalue $E_{\kappa,n}$ is equal to $-c_{\kappa+n}$.

– For systems admitting the dual shape invariance it is necessary to repeat the steps enumerated above using alternative (or additional) superpotentials.

All matrix potentials presented in the above generate integrable models with Hamiltonian (5.1). However, it is necessary to examine their consistency, in particular, to verify that there exist square integrable solutions of Eq. (5.2) for the ground states.

In the following sections we find such solutions for all superpotentials given by Eqs. (4.21)–(4.24) and (4.55)–(4.57). However, to obtain normalizable ground state solutions it is necessary to impose certain conditions on parameters of these superpotentials.

Let us present the energy spectra for models (5.1) with potentials (4.28)–(4.31) which can be found by applying the presented algorithm:

$$E = -\frac{\omega^2}{(2N + 1)^2} \tag{5.7}$$

for potential (4.28),

$$E = -\lambda^2\left(N^2 + \frac{\omega^2}{N^2}\right) \tag{5.8}$$

for potentials (4.29), (4.31), (4.32), and

$$E = \lambda^2\left(N^2 - \frac{\omega^2}{N^2}\right) \tag{5.9}$$

for potentials (4.30).

Here N is the spectral parameter which can take the following values:

$$N = n + \kappa, \tag{5.10}$$

and (or)

$$N = n + \mu + \frac{1}{2}, \tag{5.11}$$

where $n = 0, 1, 2, \ldots$ are natural numbers which can take any values for potentials (4.28)–(4.30). For potentials (4.29), (4.32), and (4.31) *with a fixed* $k < 0$ the admissible values of n are bound by the condition $(k+n)^2 > |\omega|$.

5.2 Ground State Solutions

To find the ground state solutions for Eqs. (5.1) with potentials (4.28)–(4.31) it is sufficient to solve Eqs. (5.2), where $W_{\kappa,\mu}$ are superpotentials (4.21)–(4.24), and analogous equation with superpotentials (4.55)–(4.57). This can be done for all the mentioned cases, but we present here only two of them.

The corresponding solutions should be square integrable two-component functions which we denote as

$$\psi_0(\kappa, \mu, x) = \begin{pmatrix} \varphi \\ \xi \end{pmatrix}. \tag{5.12}$$

Consider the superpotential defined by Eq. (4.21). Substituting (4.21) and (5.12) into (5.2) we obtain

$$\frac{\partial \varphi}{\partial x} + (\mu - \kappa) \frac{\varphi}{x} + \frac{\omega}{2\kappa + 1} \xi = 0, \tag{5.13}$$

$$\frac{\partial \xi}{\partial x} - (\mu + \kappa + 1) \frac{\xi}{x} + \frac{\omega}{2\kappa + 1} \varphi = 0. \tag{5.14}$$

Solving (5.14) for φ, substituting the solution into (5.13) and making the change

$$\xi = y^{\kappa+1} \hat{\xi}(y), \quad y = \frac{\omega x}{2k + 1}, \tag{5.15}$$

we obtain the equation

$$y^2 \frac{\partial^2 \hat{\xi}}{\partial y^2} + y \frac{\partial \hat{\xi}}{\partial y} - \left(y^2 + \mu^2 \right) \hat{\xi} = 0, \tag{5.16}$$

whose square integrable solution is proportional to the modified Bessel function:

$$\hat{\xi} = c K_\mu(y). \tag{5.17}$$

Substituting (5.17) into (5.15) and using (5.14) we obtain

$$\varphi = y^{\kappa+1} K_{\mu+1}(y), \quad \xi = y^{\kappa+1} K_{|\mu|}(y), \tag{5.18}$$

where y is the variable defined in (5.15), $\omega x/(2\kappa + 1) \geq 0$.

Functions (5.18) are square integrable provided parameter κ is positive and satisfies the following relation:

$$\kappa - \mu > 0. \tag{5.19}$$

If this condition is violated, i.e., $\kappa - \mu \leq 0$ solutions (5.18) are not square integrable. But since potential (4.28) admits the dual shape invariance, it is possible to make an alternative factorization of Eq. (5.1) using superpotential (4.55) and search for normalizable solutions of the following equation:

$$\tilde{a}^-_{\mu,\kappa}\tilde{\psi}_0(\mu,\kappa,x)\tilde{\psi}_0(\mu,\kappa,x) = 0. \tag{5.20}$$

where $\tilde{a}^-_{\mu,\kappa} = \frac{\partial}{\partial x} + \widetilde{W}_{\mu,\kappa}$. Indeed, solving (5.20) we obtain a perfect ground state vector:

$$\tilde{\psi}_0(\mu,\kappa,x) = \begin{pmatrix}\tilde{\varphi}\\ \tilde{\xi}\end{pmatrix}, \quad \tilde{\varphi} = y^{\mu+\frac{3}{2}}K_{|\nu|}(y), \quad \tilde{\xi} = y^{\mu+\frac{3}{2}}K_{|\nu-1|}(y), \tag{5.21}$$

where $y = \frac{\omega x}{2(\mu+1)}$ and $\nu = \kappa + 1/2$. The normalizability conditions for solution (5.21) are

$$\kappa - \mu < 1, \quad \text{if} \quad \kappa \geq 0, \text{ and } \quad \kappa + \mu > 1, \quad \text{if} \quad \kappa < 0. \tag{5.22}$$

Analogously, considering Eq. (5.2) with superpotential (4.22) and representing its solution in the form (5.12) with

$$\xi = y^{\frac{1}{2}-\kappa}\hat{\xi}(y), \quad \varphi = y^{\frac{1}{2}-\kappa}\hat{\varphi}(y), \quad y = \mu\exp(-\lambda x),$$

we find the following solutions:

$$\varphi = y^{\frac{1}{2}-\kappa}K_{|\nu|}(y), \quad \xi = -y^{\frac{1}{2}-\kappa}K_{|\nu-1|}(y) \tag{5.23}$$

where $\nu = \omega/\kappa + 1/2$ and parameters ω and κ should satisfy the conditions

$$\kappa < 0, \quad \kappa^2 > \omega. \tag{5.24}$$

Since potential (4.29) does not admit the dual shape invariance, there are no other ground state solutions.

In analogous manner we find solutions of Eqs. (5.2) and (5.20) for the remaining superpotentials (4.22)–(4.24), refer to [33] for details. Solutions which correspond to nth energy level can be obtained by applying Eq. (5.6). Under certain conditions on spectral parameters all such solutions are square integrable and reduce to zero at $x = 0$ [33].

5.3 Isospectrality

Let us note that for some values of parameters μ and κ potentials (4.28)–(4.32) are isospectral with direct sums of known scalar potentials.

Considering potential (4.28) and using its dual shape invariance it is possible to show that for half-integer μ V_κ can be transformed to a direct sum of scalar Coulomb potentials. In analogous way we can show that potentials (4.30) with half-integer κ or integer μ is isospectral with the potential

$$\hat{V}_\kappa = \lambda^2 \left(r(r-1)\sec^2 \lambda x + 2\omega \tan \lambda x \sigma_1 \right), \quad r = \frac{1}{2} \pm \mu \quad \text{or} \quad r = \kappa, \qquad (5.25)$$

which is equivalent to the direct sum of two trigonometric Rosen–Morse potentials. Under the same conditions for parameters μ and κ potential (4.32) is isospectral with the direct sum of two Eckart potentials. Finally, potential (4.32) is isospectral with direct sum of two hyperbolic Rosen–Morse potentials.

In other words, for some special values of parameters μ and κ there exist the isospectrality relations of matrix potentials (4.28)–(4.32) with well-known scalar potentials. However, for another values of these parameters such relations do not exist.

6 Shape Invariant Systems with Position Dependent Mass

SE with position dependent mass are requested for description of various condensed-matter systems such as semiconductors, quantum liquids and metal clusters, quantum dots, etc. However, in contrast with standard QM systems, their symmetries, supersymmetries, and integrals of motion were never investigated systematically.

The systematic study of symmetries of the position dependent mass SEs was started recently. In particular, the completed group classification of such equations in two and three dimensions has been carried out in [41, 42] and [43]. Here we present the classification of all rotationally invariant systems admitting second order integrals of motion [44] which appear to be shape invariant and exactly solvable.

6.1 Rotationally Invariant Systems

We will study stationary Schrödinger equations with position dependent mass, which formally coincide with (4.3), but include Hamiltonians with variable mass parameters:

$$\hat{H} = p_a f(\mathbf{x}) p_a + \tilde{V}(\mathbf{x}). \qquad (6.1)$$

Here $V(\mathbf{x})$ and $f(\mathbf{x}) = \frac{1}{2m(\mathbf{x})}$ are arbitrary functions associated with the effective potential and inverse effective PDM, and summation from 1 to 3 is imposed over the repeating index a. In addition, $\mathbf{x} = (x^1, x^2, x^3)$ denotes a 3d space vector.

In paper [41] all Hamiltonians (6.1) admitting first order integrals of motion are classified. In particular, the rotationally invariant systems include the following functions f and V:

$$f = f(x), \quad \tilde{V} = \tilde{V}(x), \quad x = \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{6.2}$$

In accordance with [41] there are four Hamiltonians with a more extended symmetry. They are specified by the following inverse masses and potentials:

$$f = x^2, \quad \tilde{V} = 0, \tag{6.3}$$

$$f = (1 + x^2)^2, \quad \tilde{V} = -6x^2, \tag{6.4}$$

$$f = (1 - x^2)^2, \quad \tilde{V} = -6x^2, \tag{6.5}$$

$$f = x^4, \quad \tilde{V} = -6x^2. \tag{6.6}$$

PDM systems admitting second order integrals of motion are classified in [44]. There are two subclasses of such systems. One class includes the systems admitting vector integrals of motion while in the second one we have the tensor integrals. All these systems are shape invariant, and are presented in the classification Tables 1 and 2.

In the third columns of the tables the effective radial potentials are indicated which appear after the separation of variables. All radial potentials are scalar and shape invariant, i.e., can be expressed in the form (3.11) where the related superpotentials W_κ are enumerated in formulae (4.11)–(4.20). The kinds of the superpotentials is fixed in the fifth columns. The content of the terms presented in the fourth columns is explained in the next section.

We see that there exist exactly 20 superintegrable systems invariant with respect to 3d rotations. Moreover, the majority of them is defined up to one arbitrary parameter, while there exist four systems dependent on two parameters, see Items 9 and 10.

6.2 Two Strategies in Construction of Exact Solutions

Let us consider Eqs. (4.3) where H are Hamiltonians (6.1) whose mass and potential terms are specified in the presented tables. We will search for square integrable solutions of these systems vanishing at $x = 0$.

First let us transform (4.3) to the following equivalent form:

$$\tilde{H}\Psi = E\Psi, \tag{6.7}$$

Table 1 Functions f and V specifying non-equivalent Hamiltonians (6.1)

No	f	V	Solution approach	Effective potentials
1.	x	αx	Direct or two-step	3d oscillator or Coulomb
2.	x^4	αx	Direct or two-step	Coulomb or 3d oscillator
3.	$x(x-1)^2$	$\frac{\alpha x}{(x+1)^2}$	Direct or two-step	Eckart or hyperbolic Pöschl–Teller
4.	$x(x+1)^2$	$\frac{\alpha x}{(x-1)^2}$	Direct or two-step	Eckart or trigonometric Pöschl–Teller
5.	$(1+x^2)^2$	$\frac{\alpha(1-x^2)}{x}$	Direct	Trigonometric Rosen–Morse
6.	$(1-x^2)^2$	$\frac{\alpha(1+x^2)}{x}$	Direct	Eckart
7.	$\frac{x}{x+1}$	$\frac{\alpha x}{x+1}$	Two-step	Coulomb
8.	$\frac{x}{x-1}$	$\frac{\alpha x}{x-1}$	Two-step	Coulomb
9.	$\frac{(x^2-1)^2 x}{x^2-2\kappa x+1}$	$\frac{\alpha x}{x^2-2\kappa x+1}$	Two-step	Eckart
10.	$\frac{(x^2+1)^2 x}{x^2-2\kappa x-1}$	$\frac{\alpha x}{x^2-2\kappa x-1}$	Two-step	Trigonometric Rosen–Morse

where

$$\tilde{H} = \sqrt{f} H \frac{1}{\sqrt{f}} = f p^2 + V, \quad \Psi = \sqrt{f}\psi. \tag{6.8}$$

Then, introducing spherical variables and expanding solutions via spherical functions Y_m^l

$$\Psi = \frac{1}{x} \sum_{l,m} \phi_{lm}(x) Y_m^l, \tag{6.9}$$

we obtain the following equation for radial functions:

$$-f \frac{\partial^2 \phi_{lm}}{\partial x^2} + \left(\frac{f l(l+1)}{x^2} + V \right) \phi_{lm} = E \phi_{lm}. \tag{6.10}$$

Let us present two possible ways to solve Eq. (6.10). They can be treated as particular cases of Liouville transformation (refer to [45] for definitions) and include

Table 2 Functions f and V specifying non-equivalent Hamiltonians (6.1) which admit tensor integrals of motion

No	f	V	Solution approach	Effective radial potential
1.	$\frac{1}{x^2}$	$\frac{\alpha}{x^2}$	Direct or two-step	Coulomb or 3d oscillator
2.	x^4	$-\frac{\alpha}{x^2}$	Direct or two-step	3d oscillator or Coulomb
3.	$(x^2-1)^2$	$\frac{\alpha x^2}{(x^2+1)^2}$	Direct or two-step	Eckart or hyperbolic Pöschl–Teller
4.	$(x^2+1)^2$	$\frac{\alpha x^2}{(x^2-1)^2}$	Direct or two-step	Eckart or trigonometric Pöschl–Teller
5.	$\frac{(x^4-1)^2}{x^2}$	$\frac{\alpha(x^4+1)}{x^2}$	Direct	Eckart
6.	$\frac{(x^4+1)^2}{x^2}$	$\frac{\alpha(x^4-1)}{x^2}$	Direct	Trigonometric Rosen–Morse
7.	$\frac{1}{x^2+1}$	$\frac{\alpha}{x^2+1}$	Two-step	3d oscillator
8.	$\frac{1}{x^2-1}$	$\frac{\alpha}{x^2-1}$	Two-step	3d oscillator
9.	$\frac{(x^4-1)^2}{x^4-2\kappa x^2+1}$	$\frac{\alpha x^2}{x^4-2\kappa x^2+1}$	Two-step	Eckart
10.	$\frac{(x^4+1)^2}{x^4-2\kappa x^2-1}$	$\frac{\alpha x^2}{x^4-2\kappa x^2-1}$	Two-step	Trigonometric Rosen–Morse

commonly known steps. But it is necessary to fix them as concrete algorithms to obtain shape invariant potentials presented in the tables.

The first way (which we call direct) includes consequent changes of independent and dependent variables:

$$\phi_{lm} \to \Phi_{lm} = f^{\frac{1}{4}}\phi_{lm}, \quad \frac{\partial}{\partial x} \to f^{\frac{1}{4}}\frac{\partial}{\partial x}f^{-\frac{1}{4}} = \frac{\partial}{\partial x} + \frac{f'}{4f} \tag{6.11}$$

and then

$$x \to y(x), \tag{6.12}$$

where y solves the equation $\frac{\partial y}{\partial x} = \frac{1}{\sqrt{f}}$. As a result Eq. (6.9) will be reduced to a more customary form:

$$-\frac{\partial^2 \Phi_{lm}}{\partial y^2} + \tilde{V}\Phi_{lm} = E\Phi_{lm}, \tag{6.13}$$

where $\tilde{V}$ is an effective potential

$$\tilde{V} = V + f\left(\frac{l(l+1)}{x^2} - \left(\frac{f'}{4f}\right)^2 - \left(\frac{f'}{4f}\right)'\right), \quad x = x(y). \tag{6.14}$$

Equations (6.7), (6.8) with functions f and V specified in Items 1–6 of both Tables 1 and 2 can be effectively solved using the presented reduction to radial Eq. (6.13). All the corresponding potentials (6.14) appear to be shape invariant, and just these potentials are indicated in the fifth columns of the tables. The related Eqs. (6.13) are shape invariant too and can be solved using the SUSY routine.

However, if we apply the direct approach to the remaining systems (indicated in Items 7–10 of both tables), we come to Eqs. (6.13) which are not shape invariant and are hardly solvable, if at all. To solve these systems we need a more sophisticated procedure which we call two-step approach. To apply it we multiply (6.10) by αV^{-1} and obtain the following equation:

$$-\tilde{f}\frac{\partial^2\phi_{lm}}{\partial x^2} + \left(\frac{\tilde{f}l(l+1)}{x^2} + \tilde{V}\right)\phi_{lm} = \mathcal{E}\phi_{lm}, \tag{6.15}$$

where $\tilde{f} = \frac{\alpha f}{V}$, $\tilde{V} = -\frac{\alpha E}{V}$, and $\mathcal{E} = -\alpha$. Then treating $\mathcal{E}$ as an eigenvalue and solving Eq. (6.15) we can find α as a function of E, which defines admissible energy values at least implicitly. To do it, it is convenient to make changes (6.11) and (6.12), where $f \to \tilde{f}$.

The presented trick with a formal changing the roles of constants α and E is well known. Our point is that *any of the presented superintegrable systems can be effectively solved using either the direct approach presented in Eqs.* (6.8)–(6.14) *or the two-step approach*. Moreover, some of the presented systems can be solved using both the direct and two-step approaches, as indicated in the fourth columns of Tables 1 and 2. In all cases we obtain shape invariant effective potentials and can use tools of SUSY quantum mechanics.

6.3 System Dependent on Two Parameters

Let us consider the systems specified in Item 10 of Table 2. The corresponding Hamiltonian (6.8) and radial Eq. (6.10) have the following form:

$$H = \frac{(x^4+1)^2}{x^4 - 2\kappa x^2 - 1}p^2 + \frac{\alpha x^2}{x^4 - 2\kappa x^2 - 1}$$

and

$$\left(-\frac{(x^4+1)^2}{x^4-2\kappa x^2-1}\left(\frac{\partial^2}{\partial x^2}-\frac{l(l+1)}{x^2}\right)+\frac{\alpha x^2}{x^4-2\kappa x^2-1}\right)\phi_{lm}=E\phi_{lm}. \tag{6.16}$$

Multiplying (6.16) from the left by $\frac{x^4-2\kappa x^2-1}{x^2}$ we come to the following equation:

$$\left(-\frac{(x^4+1)^2}{x^2}\left(\frac{\partial^2}{\partial x^2}-\frac{l(l+1)}{x^2}\right)+\frac{\tilde{\alpha}(x^4-1)}{x^2}\right)\phi_{lm}=\mathcal{E}\phi_{lm}, \tag{6.17}$$

where

$$\tilde{\alpha}=-E \quad \text{and} \quad \mathcal{E}=-\alpha-2\kappa E. \tag{6.18}$$

Notice that Eq. (6.17) with $\tilde{\alpha}\to\alpha$ and $\mathcal{E}\to E$ is needed also to find eigenvectors of the Hamiltonian whose mass and potential terms are specified in Item 6 of Table 2.

Making transformations (6.11) and (6.12) with $f=\frac{(x^4+1)^2}{x^2}$ and $y=\frac{1}{2}\arctan(x^2)$ we reduce Eq. (6.17) to the following form:

$$-\frac{\partial^2\Phi_{lm}}{\partial y^2}+\left(\mu(\mu-4)\csc^2(4y)+2\tilde{\alpha}\cot(4y)\right)\Phi_{lm}=\tilde{\mathcal{E}}\Phi_{lm}, \tag{6.19}$$

where

$$\tilde{\mathcal{E}}=\mathcal{E}+4, \quad \mu=2l+3. \tag{6.20}$$

Thus we come to equation with a shape invariant (Rosen–Morse I) potential. It is consistent provided parameters $\tilde{\alpha}$ and μ are positive. Solving this equation using the standard tools of SUSY QM we can easily find its eigenfunctions and eigenvalues; the corresponding eigenvalues for Eq. (6.16) are given by the following formula [44]:

$$E_n=(2l+3+4n)^2\left(\kappa-\sqrt{\kappa^2+1+\frac{\alpha-4}{(2l+3+4n)^2}}\right), \tag{6.21}$$

where both n and l are integers.

7 Discussion

To construct QM systems with extended SUSY we essentially use discrete symmetries, i.e., reflections and rotations to the fixed angles.

The idea itself to apply reflections to construct $N = 2$ SUSY was proposed in paper [46]. Then it was applied to generate extended supersymmetries [35, 36, 47, 48], moreover, in the latter paper the discrete rotations were applied also. In addition, using these discrete symmetries, it is possible to make a reduction of SUSY algebras as it was shown in paper [49] and some others.

We start our discussion with presenting of these old results in order to stress that SUSY has strong roots in quantum mechanics since a lot of important QM models do be supersymmetric. Moreover, even the simplest SUSY model, i.e., the charged particle interacting with the uniform magnetic field, in fact admits the extended supersymmetry with four supercharges [35].

But the main content of the present survey are some modern trends in SUSY quantum mechanics. They are the matrix formulation of the shape invariance which is requested for description of QM particles with spin interacting with external fields, and supersymmetries of Schrödinger equations with position dependent masses. And we believe that the presented results can be treated as a challenge to generalize various branches of SUSY to the case of matrix superpotentials. And it is nice that some elements of such generalizations can be already recognized in literature, see, e.g. [50–55].

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