Coherent States in Quantum Optics: An Oriented Overview

Jean-Pierre Gazeau

Abstract In this survey, various generalizations of Glauber–Sudarshan coherent states are described in a unified way, with their statistical properties and their possible role in non-standard quantizations of the classical electromagnetic field. Some statistical photon-counting aspects of Perelomov SU(2) and SU(1, 1) coherent states are emphasized.

Keywords Coherent states · Quantum optics · Quantization · Photon-counting statistics · Group theoretical approaches

1 Introduction

The aim of this contribution is to give a restricted review on coherent states in a wide sense (linear, non-linear, and various other types), and on their possible relevance to quantum optics, where they are generically denoted by $|\alpha\rangle$, for a complex parameter α, with $|\alpha| < R$, $R \in (0, \infty)$. Many important aspects of these states, understood here in a wide sense, will not be considered, like photon-added, intelligent, squeezed, dressed, "non-classical," all those cat superpositions of any type, involved into quantum entanglement and information, ... Of course, such a variety of features can be found in existing articles or reviews. A few of them [1–6] are included in the list of references in order to provide the reader with an extended palette of various other references.

We have attempted to give a minimal framework for all various families of $|\alpha\rangle$'s which are described in the present review. Throughout the paper we put

J.-P. Gazeau (✉)
APC, UMR 7164, Univ Paris Diderot, Sorbonne Paris Cité, Paris, France

Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, RJ, Brazil
e-mail: gazeau@apc.in2p3.fr

Ş. Kuru et al. (eds.), *Integrability, Supersymmetry and Coherent States*, CRM Series in Mathematical Physics, https://doi.org/10.1007/978-3-030-20087-9_3

$\hbar = 1 = c$, except if we need to make precise physical units. In Sect. 2 we recall the main characteristics of the Hilbertian framework (one-mode) Fock space with the underlying Weyl–Heisenberg algebra of its lowering and raising operators, and the basic statistical interpretation in terms of detection probability. In Sect. 3 we introduce coherent states in Fock space as superpositions of number states with coefficients depending on a complex number α. These "PHIN" states are requested to obey two fundamental properties, normalization and resolution of the identity in Fock space. The physical meaning of the parameter α is explained in terms of the number of photons, and may or not be interpreted in terms of classical optics quadratures. A first example is given in terms of holomorphic Hermite polynomials. We then define an important subclass AN in PHIN. Section 4 is devoted to the celebrated prototype of all CS in class AN, namely the Glauber–Sudarshan states. Their multiple properties are recalled, and their fundamental role in quantum optics is briefly described by following the seminal 1963 Glauber paper. We end the section with a description of the CS issued from unitary displacement of an arbitrary number eigenstate in place of the vacuum. The latter belong to the PHIN class, but not in the AN class. The so-called non-linear CS in the AN class are presented in Sect. 5, and an example of q-deformed CS illustrates this important extension of standard CS. In Sect. 6 we adapt the Gilmore–Perelomov spin or SU(2) CS to the quantum optics framework and we emphasize their statistical meaning in terms of photon counting. We extend them also these CS to those issued from an arbitrary number state. We follow a similar approach in Sect. 7 with Perelomov and Barut–Girardello SU(1, 1) CS. Section 8 is devoted to another type of AN CS, named Susskind–Glogower, which reveal to be quite attractive in the context of quantum optics. We end in Sect. 9 this list of various CS with a new type of non-linear CS based on deformed binomial distribution. In Sect. 10 we briefly review the statistical aspects of CS in quantum optics by focusing on their potential statistical properties, like sub- or super-Poissonian or just Poissonian. The content of Sect. 11 concerns the role of all these generalizations of CS belonging to the AN class in the quantization of classical solutions of the Maxwell equations and the corresponding quadrature portraits. Some promising features of this CS quantization are discussed in Sect. 12.

2 Fock Space

In their number or Fock representation, the eigenstates of the harmonic oscillator are simply denoted by kets $|n\rangle$, where $n = 0, 1, \dots,$ stands for the number of elementary quanta of energy, named photons when the model is applied to a quantized monochromatic electromagnetic wave. These kets form an orthonormal basis of the Fock Hilbert space $\mathcal{H}$. The latter is actually a physical model for all separable Hilbert spaces, namely the space $\ell^2(\mathbb{N})$ of square summable sequences. For such a basis (actually for any Hilbertian basis $\{e_n\ , \ n = 0, 1, \dots\}$), the *lowering* or *annihilation* operator a, and its adjoint $a^\dagger$, the *raising* or *creation* operator, are defined by

$$a|n\rangle = \sqrt{n}|n-1\rangle\,, \quad a^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle\,, \tag{2.1}$$

together with the action of a on the ground or "vacuum" state $a|0\rangle = 0$. They obey the so-called canonical commutation rule (ccr) $[a, a^{\dagger}] = I$. In this context, the *number* operator $\hat{N} = a^{\dagger}a$ is diagonal in the basis $\{|n\rangle,\ n \in \mathbb{N}\}$, with spectrum $\mathbb{N}$: $\hat{N}|n\rangle = n|n\rangle$.

3 General Setting for Coherent States in a Wide Sense

3.1 The PHIN Class

A large class of one-mode optical coherent states can be written as the following normalized superposition of photon number states:

$$|\alpha\rangle = \sum_{n=0}^{\infty} \phi_n(\alpha)|n\rangle\,, \tag{3.1}$$

where the complex parameter α lies in some bounded or unbounded subset $\mathfrak{S}$ of $\mathbb{C}$. Its physical meaning will be discussed below in terms of detection probability. Note that the adjective "coherent" is used in a generic sense and should not be understood in the restrictive sense it was given originally by Glauber [7]. The complex-valued functions $\alpha \mapsto \phi_n(\alpha)$, from which the name "PHIN class," obey the two conditions

$$1 = \sum_{n=0}^{\infty} |\phi_n(\alpha)|^2\,, \quad \alpha \in \mathfrak{S}\,, \quad \text{(normalisation)} \tag{3.2}$$

$$\delta_{nn'} = \int_{\mathfrak{S}} \mathrm{d}^2\alpha\, \mathfrak{w}(\alpha)\, \overline{\phi_n(\alpha)}\, \phi_{n'}(\alpha)\,, \quad \text{(orthonormality)}\,, \tag{3.3}$$

where $\mathfrak{w}(\alpha)$ is a weight function, with support $\mathfrak{S}$ in $\mathbb{C}$. While Eq. (3.2) is necessary, Eq. (3.3) might be optional, except if we request resolution of the identity in the Fock Hilbert space spanned by the number states:

$$\int_{\mathfrak{S}} \mathrm{d}^2\alpha\, \mathfrak{w}(\alpha)\; |\alpha\rangle\langle\alpha| = I\,. \tag{3.4}$$

A finite sum in (3.1) due to $\phi_n = 0$ for all n larger than a certain n_{max} may be considered in this study.

If the orthonormality condition (3.3) is satisfied with a positive weight function, it allows us to interpret the map

$$\alpha \mapsto |\phi_n(\alpha)|^2 \equiv \varpi_n(\alpha) \tag{3.5}$$

as a probability distribution, with parameter n, on the support $\mathfrak{S}$ of $\mathfrak{w}$ in $\mathbb{C}$, equipped with the measure $\mathfrak{w}(\alpha)\ \mathrm{d}^2\alpha$.

On the other hand, the normalization condition (3.2) allows to interpret the discrete map

$$n \mapsto \varpi_n(\alpha) \tag{3.6}$$

as a probability distribution on $\mathbb{N}$, with parameter α, precisely the probability to detect n photons when the quantum light is in the coherent state $|\alpha\rangle$. The average value of the number operator

$$\bar{n} = \bar{n}(\alpha) := \langle\alpha|\hat{N}|\alpha\rangle = \sum_{n=0}^{\infty} n\, \varpi_n(\alpha) \tag{3.7}$$

can be viewed as the intensity (or energy up to a physical factor like $\hbar\omega$) of the state $|\alpha\rangle$ of the quantum monochromatic radiation under consideration. An optical phase space associated with this radiation may be defined as the image of the map

$$\mathfrak{S} \ni \alpha \mapsto \xi_\alpha = \sqrt{\bar{n}(\alpha)}\, e^{\mathrm{i}\arg\alpha} \in \mathbb{C}\,. \tag{3.8}$$

A statistical interpretation of the original set $\mathfrak{S}$ is made possible if one can invert the map (3.8). Two examples of such an inverse map will be given in Sects. 6 and 7.1, respectively, with interesting statistical interpretations.

3.2 A First Example of PHIN CS with Holomorphic Hermite Polynomials

These coherent states were introduced in [8]. Given a real number $0 < s < 1$, the functions $\phi_{n;s}$ are defined as

$$\phi_{n;s}(\alpha) := \frac{1}{\sqrt{b_n(s)\mathcal{N}_s(\alpha)}}\, e^{-\alpha^2/2}\, H_n(\alpha)\,, \quad \alpha \in \mathbb{C}\,. \tag{3.9}$$

The non-holomorphic part lies in the expression of $\mathcal{N}_s$

$$\mathcal{N}_s(\alpha) = \frac{s^{-1} - s}{2\pi}\, e^{-s\,X^2 + s^{-1}\,Y^2}\,,\ \alpha = X + \mathrm{i}Y\,.$$

The constant $b_n(s)$ is given by

$$b_n(s) = \frac{\pi\sqrt{s}}{1-s}\left(2\frac{1+s}{1-s}\right)^n n!\,.$$

The function $H_n(\alpha)$ is the usual Hermite polynomial of degree n [9], considered here as a holomorphic polynomial in the complex variable α. The corresponding normalized coherent states

$$|\alpha; s\rangle = \sum_{n=0}^{\infty} \phi_{n;s}(\alpha)|n\rangle \tag{3.10}$$

solve the identity in $\mathcal{H}$,

$$\frac{s^{-1}-s}{2\pi}\int_{\mathbb{C}} \mathrm{d}^2\alpha\, |\alpha; s\rangle\langle\alpha; s| = I\,. \tag{3.11}$$

Thus, in the present case we have the constant weight $\varpi(\alpha) = \frac{s^{-1}-s}{2\pi}$. This resolution of the identity results from the orthogonality relations verified by the holomorphic Hermite polynomials in the complex plane:

$$\int_{\mathbb{C}} \mathrm{d}X\,\mathrm{d}Y\,\overline{H_n(X+\mathrm{i}Y)}\,H_{n'}(X+\mathrm{i}Y)\exp\left[-(1-s)X^2-\left(\frac{1}{s}-1\right)Y^2\right] = b_n(s)\delta_{nn'}\,. \tag{3.12}$$

Note that the map $\alpha \mapsto \bar{n}(\alpha) = \sum_n n\left|e^{-\alpha^2/2}H_n(\alpha)\right|^2$ is not rotationally invariant.

3.3 *The AN Class*

Particularly convenient to manage and mostly encountered are coherent states $|\alpha\rangle$ for which the functions ϕ_n factorize as

$$\phi_n(\alpha) = \alpha^n\, h_n(|\alpha|^2)\,, \quad \sum_{n=0}^{\infty}|\alpha|^{2n}|h_n(\alpha)|^2 = 1\,, \quad |\alpha| < R\,, \tag{3.13}$$

where R can be finite or infinite. All coherent states of the above type lie in the so-called AN class (AN for "αn"). Then, due to Fourier angular integration in (3.3), the orthonormality condition holds if there exists an isotropic weight function w such that the h_n's solve the following kind of moment problem on the interval $[0, R^2]$:

$$\int_0^{R^2} \mathrm{d}u\, w(u)\, u^n|h_n(u)|^2 = 1\,, \quad n \in \mathbb{N}\,. \tag{3.14}$$

This w is related to the above $\mathfrak{w}$ through

$$\mathfrak{w}\,(\alpha) = \frac{w(|\alpha|^2)}{\pi}\,. \tag{3.15}$$

Note that the probability (3.6) to detect n photons when the quantum light is in such a AN coherent state $|\alpha\rangle$ is expressed as a function of $u = |\alpha|^2$ only

$$n \mapsto \varpi_n(\alpha) \equiv \mathsf{P}_n\,(u) = u^n\ \left(h_n(u)\right)^2\,. \tag{3.16}$$

Hence, the map $\alpha \mapsto \bar{n}$ is here rotationally invariant: $\bar{n} = \bar{n}(u)$. On the other hand, the probability distribution on the interval $[0, R^2]$, for a detected n, that CS $|\alpha\rangle$ have classical intensity u is given by

$$u \mapsto \varpi_n(\alpha) \equiv \mathsf{P}_n\,(u)\,. \tag{3.17}$$

4 Glauber–Sudarshan CS

4.1 Definition and Properties

They are the most popular, of course, among the AN families, and historically the first ones to appear in QED with Schwinger [10], and in quantum optics with the 1963 seminal papers by Glauber [7, 11, 12] and Sudarshan [13]. See also some key papers like [14–16] for further developments in quantum optics and quantum field theory. They were introduced in quantum mechanics by Schrödinger [17] and later by Klauder [18–20]. They correspond to the Gaussian

$$h_n(u) = \frac{e^{-u/2}}{\sqrt{n!}}\,, \tag{4.1}$$

and read

$$|\alpha\,\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle. \tag{4.2}$$

Here, the parameter, i.e., the *amplitude*, $\alpha = X + \mathsf{i}Y$ represents an element of the optical phase space. Its Cartesian components X and Y in the Euclidean plane are called quadratures. In complete analogy with the harmonic oscillator model, the quantity $u = |\alpha|^2$ is considered as the classical *intensity* or *energy* of the coherent state $|\alpha\rangle$. The corresponding detection distribution is the familiar Poisson distribution

$$n \mapsto \mathsf{P}_n(u) = e^{-u}\,\frac{u^n}{n!}\,, \tag{4.3}$$

and the average value of the number operator is just the intensity.

$$\bar{n}(\alpha) = |\alpha|^2 = u\,. \tag{4.4}$$

Hence, the detection distribution is written in terms of this average value as

$$\mathsf{P}_n(u) = e^{-\bar{n}}\,\frac{\bar{n}^n}{n!}\,. \tag{4.5}$$

From now on the states (4.2) will be called *standard coherent states*. They are called harmonic oscillator CS when we consider the $|n\rangle$'s as eigenstates of the corresponding quantum Hamiltonian $H_{\rm osc} = \left(P^2 + Q^2\right)/2 = \hat{N} + 1/2$ with $Q = \dfrac{a + a^\dagger}{\sqrt{2}}$ and $P = \dfrac{a - a^\dagger}{\mathrm{i}\sqrt{2}}$. They are exceptional in the sense that they obey the following long list of properties that give them, on their whole own, a strong status of uniqueness.

P$_0$ *The map* $\mathbb{C} \ni \alpha \to |\alpha\rangle \in \mathcal{H}$ *is continuous.*

P$_1$ $|\alpha\rangle$ *is eigenvector of annihilation operator:* $a|\alpha\rangle = \alpha|\alpha\rangle$.

P$_2$ *The CS family resolves the unity:* $\int_{\mathbb{C}} \frac{\mathrm{d}^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha| = I$.

P$_3$ *The CS saturate the Heisenberg inequality :* $\Delta X\,\Delta Y = \Delta Q\,\Delta P = 1/2$.

P$_4$ *The CS family is temporally stable :* $e^{-\mathrm{i}H_{\rm osc}t}|\alpha\rangle = e^{-\mathrm{i}t/2}|e^{-\mathrm{i}t}\alpha\rangle$.

P$_5$ *The mean value (or "lower symbol") of the Hamiltonian* $H_{\rm osc}$ *mimics the classical relation energy-action:* $\check{H}_{\rm osc}(\alpha) := \langle\alpha|H_{\rm osc}|\alpha\rangle = |\alpha|^2 + \frac{1}{2}$.

P$_6$ *The CS family is the orbit of the ground state under the action of the Weyl displacement operator:* $|\alpha\rangle = e^{(\alpha a^\dagger - \bar{\alpha} a)}|0\rangle \equiv D(\alpha)|0\rangle$.

P$_7$ *The unitary Weyl–Heisenberg covariance follows from the above:*

$$\mathcal{U}(s,\zeta)|\alpha\rangle = e^{\mathrm{i}(s + \mathrm{Im}(\zeta\bar{\alpha}))}|\alpha + \zeta\rangle,\ \textit{where}\ \mathcal{U}(s,\zeta) := e^{\mathrm{i}s}\,D(\zeta).$$

P$_8$ *From* P_2 *the coherent states provide a straightforward quantization scheme:*

$$\textit{Function}\ f(\alpha) \to \textit{Operator}\ A_f = \int_{\mathbb{C}} \frac{\mathrm{d}^2\alpha}{\pi}\, f(\alpha)\,|\alpha\rangle\langle\alpha|\,.$$

These properties cover a wide spectrum, starting from the "wave-packet" expression (4.2) together with Properties P_3 and P_4, through an algebraic side (P_1), a group representation side (P_6 and P_7), a functional analysis side (P_2) to end with the ubiquitous problematic of the relationship between classical and quantum models (P_5 and P_8). Starting from this exceptional palette of properties, the game over the past almost seven decades has been to build families of CS having some of these properties, if not all of them, as it can be attested by the huge literature, articles, proceedings, special issues, and author(s) or collective books, a few of them being [21–32].

4.2 Why the Adjective **Coherent***? (Partially Extracted from [30])*

Let us compare the two equations :

$$a|\alpha\rangle = \alpha|\alpha\rangle\,, \qquad a|n\rangle = \sqrt{n}|n-1\rangle\,. \tag{4.6}$$

Hence, *an infinite superposition of number states* $|n\rangle$*, each of the latter describing a determinate number of elementary quanta, describes a state which is left unmodified (up to a factor) under the action of the operator annihilating an elementary quantum. The factor is equal to the parameter* α *labeling the considered coherent state.*

More generally, we have $f(a)|\alpha\rangle = f(\alpha)|\alpha\rangle$ for an analytic function f. This is precisely the idea developed by Glauber [7, 11, 12]. Indeed, an electromagnetic field in a box can be assimilated to a countably infinite assembly of harmonic oscillators. This results from a simple Fourier analysis of Maxwell equations. The (canonical) quantization of these classical harmonic oscillators yields the Fock space $\mathcal{F}$ spanned by all possible tensor products of number eigenstates $\bigotimes_k |n_k\rangle \equiv |n_1, n_2, \dots, n_k, \dots\rangle$, where "$k$" is a shortening for labeling the mode (including the photon polarization)

$$k \equiv \begin{cases} \mathbf{k} & \text{wave vector,} \\ \omega_k = \|\mathbf{k}\| c & \text{frequency,} \\ \lambda = 1, 2 & \text{helicity,} \end{cases} \tag{4.7}$$

and n_k is the number of photons in the mode "k." The Fourier expansion of the quantum vector potential reads as

$$\overrightarrow{A}(\mathbf{r}, t) = c \sum_k \sqrt{\frac{\hbar}{2\omega_k}} \left(a_k \mathbf{u}_k(\mathbf{r}) e^{-\mathrm{i}\omega_k t} + a_k^\dagger \overline{\mathbf{u}_k(\mathbf{r})} e^{\mathrm{i}\omega_k t} \right)\,. \tag{4.8}$$

As an operator, it acts (up to a gauge) on the Fock space $\mathcal{F}$ via a_k and $a_k^\dagger$ defined by

$$a_{k_0} \prod_k |n_k\rangle = \sqrt{n_{k_0}} |n_{k_0} - 1\rangle \prod_{k \neq k_0} |n_k\rangle\,, \tag{4.9}$$

and obeying the canonical commutation rules

$$[a_k, a_{k'}] = 0 = [a_k^\dagger, a_{k'}^\dagger]\,, \qquad [a_k, a_{k'}^\dagger] = \delta_{kk'}\, I\,. \tag{4.10}$$

Let us now give more insights on the modes, observables, and Hamiltonian. On the level of the mode functions $\mathbf{u}_k$ the Maxwell equations read as

$$\Delta \mathbf{u}_k(\mathbf{r}) + \frac{\omega_k^2}{c^2}\mathbf{u}_k(\mathbf{r}) = \mathbf{0}\,. \tag{4.11}$$

When confined to a cubic box C_L with size L, these functions form an orthonormal basis

$$\int_{C_L} \overline{\mathbf{u}_k(\mathbf{r})} \cdot \mathbf{u}_l(\mathbf{r})\, \mathrm{d}^3\mathbf{r} = \delta_{kl}\,,$$

with obvious discretization constraints on "k." By choosing the gauge $\nabla \cdot \mathbf{u}_k(\mathbf{r}) = 0$, their expression is

$$\mathbf{u}_k(\mathbf{r}) = L^{-3/2}\widehat{e}^{(\lambda)} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\,, \quad \lambda = 1 \text{ or } 2\,, \quad \mathbf{k}\cdot\widehat{e}^{(\lambda)} = 0\,, \tag{4.12}$$

where the $\widehat{e}^{(\lambda)}$'s stand for polarization vectors. The respective expressions of the electric and magnetic field operators are derived from the vector potential:

$$\overrightarrow{E} = -\frac{1}{c}\frac{\partial \overrightarrow{A}}{\partial t}\,, \quad \overrightarrow{B} = \overrightarrow{\nabla} \times \overrightarrow{A}\,.$$

Finally, the electromagnetic field Hamiltonian is given by

$$H_{\text{e.m.}} = \frac{1}{2}\int \left(\|\overrightarrow{E}\|^2 + \|\overrightarrow{B}\|^2\right) \mathrm{d}^3\mathbf{r} = \frac{1}{2}\sum_k \hbar\omega_k \left(a_k^\dagger a_k + a_k a_k^\dagger\right)\,.$$

Let us now decompose the electric field operator into positive and negative frequencies

$$\overrightarrow{E} = \overrightarrow{E}^{(+)} + \overrightarrow{E}^{(-)},\ \overrightarrow{E}^{(-)} = \overrightarrow{E}^{(+)\dagger}\,,$$

$$\overrightarrow{E}^{(+)}(\mathbf{r},t) = \mathrm{i}\sum_k \sqrt{\frac{\hbar\omega_k}{2}}\, a_k \mathbf{u}_k(\mathbf{r}) e^{-\mathrm{i}\omega_k t}\,. \tag{4.13}$$

We then consider the field described by the density (matrix) operator :

$$\rho = \sum_{(n_k)} c_{(n_k)} \prod_k |n_k\rangle\langle n_k|\,, \quad c_{(n_k)} \geq 0\,, \quad \mathrm{tr}\,\rho = 1\,, \tag{4.14}$$

and the derived sequence of correlation functions $G^{(n)}$. The Euclidean tensor components for the simplest one read as

$$G_{ij}^{(1)}(\mathbf{r},t;\mathbf{r}',t') = \mathrm{tr}\left\{\rho E_i^{(-)}(\mathbf{r},t) E_j^{(+)}(\mathbf{r}',t')\right\}\,, \quad i,j = 1,2,3\,. \tag{4.15}$$

They measure the correlation of the field state at different space-time points. A *coherent state* or *coherent radiation* $|\text{c.r.}\rangle$ for the electromagnetic field is then defined by

$$|\text{c.r.}\rangle = \prod_k |\alpha_k\rangle\,, \tag{4.16}$$

where $|\alpha_k\rangle$ is precisely the standard coherent state for the "k" mode :

$$|\alpha_k\rangle = e^{-\frac{|\alpha_k|^2}{2}} \sum_{n_k} \frac{(\alpha_k)^{n_k}}{\sqrt{n_k!}} |n_k\rangle\,, \quad a_k|\alpha_k\rangle = \alpha_k|\alpha_k\rangle\,, \tag{4.17}$$

with $\alpha_k \in \mathbb{C}$. The particular status of the state $|\text{c.r.}\rangle$ is well understood through the action of the positive frequency electric field operator

$$\overrightarrow{E}^{(+)}(\mathbf{r},t)|\text{c.r.}\rangle = \overrightarrow{\mathcal{E}}^{(+)}(\mathbf{r},t)|\text{c.r.}\rangle\,. \tag{4.18}$$

The expression $\overrightarrow{\mathcal{E}}^{(+)}(\mathbf{r},t)$ which shows up is precisely the classical field expression, solution to the Maxwell equations

$$\overrightarrow{\mathcal{E}}^{(+)}(\mathbf{r},t) = \mathsf{i}\sum_k \sqrt{\frac{\hbar\omega_k}{2}}\alpha_k \mathbf{u}_k(\mathbf{r})e^{-\mathsf{i}\omega_k t}\,. \tag{4.19}$$

Now, if the density operator is chosen as a pure coherent state, i.e.,

$$\rho = |\text{c.r.}\rangle\langle\text{c.r.}|\,, \tag{4.20}$$

then the components (4.15) of the first order correlation function factorize into independent terms :

$$G^{(1)}_{ij}(\mathbf{r},t;\mathbf{r}',t') = \overline{\mathcal{E}^{(-)}_i(\mathbf{r},t)}\mathcal{E}^{(+)}_j(\mathbf{r}',t')\,. \tag{4.21}$$

An electromagnetic field operator is said "fully coherent" in the Glauber sense if all of its correlation functions factorize like in (4.21). Nevertheless, one should notice that such a definition does not imply monochromaticity.

A last important point concerns the production of such states in quantum optics. They can be manufactured by adiabatically coupling the e.m. field to a classical source, for instance, a radiating current $\mathbf{j}(\mathbf{r},t)$. The coupling is described by the Hamiltonian

$$H_{\text{coupling}} = -\frac{1}{c}\int \mathrm{d}\mathbf{r}\, \overrightarrow{j}(\mathbf{r},t)\cdot\overrightarrow{A}(\mathbf{r},t)\,. \tag{4.22}$$

From the Schrödinger equation, the time evolution of a field state supposed to be originally, say at t_0, the state $|\text{vacuum}\rangle$(no photons) is given by

$$|t\rangle = \exp\left[\frac{\mathsf{i}}{\hbar c}\int_{t_0}^{t} \mathrm{d}t' \int \mathrm{d}\mathbf{r}\, \overrightarrow{j}(\mathbf{r}, t') \cdot \overrightarrow{A}(\mathbf{r}, t') + \mathsf{i}\varphi(t)\right] |\text{vacuum}\rangle\,, \tag{4.23}$$

where $\varphi(t)$ is some phase factor, which cancels if one deals with the density operator $|t\rangle\langle t|$ and can be dropped. From the Fourier expansion (4.8) we easily express the above evolution operator in terms of the Weyl displacement operators corresponding to each mode

$$\exp\left[\frac{\mathsf{i}}{\hbar c}\int_{t_0}^{t} \mathrm{d}t' \int \mathrm{d}\mathbf{r}\, \overrightarrow{j}(\mathbf{r}, t') \cdot \overrightarrow{A}(\mathbf{r}, t')\right] = \prod_k D(\alpha_k(t))\,, \tag{4.24}$$

where the complex amplitudes are given by

$$\alpha_k(t) = \frac{\mathsf{i}}{\hbar c}\int_{t_0}^{t} \mathrm{d}t' \int \mathrm{d}\mathbf{r}\, \overrightarrow{j}(\mathbf{r}, t') \cdot \overline{\mathbf{u}_k(\mathbf{r})} e^{\mathsf{i}\omega_k t'}\,. \tag{4.25}$$

Hence, we obtain the time-dependent e.m. CS

$$|t\rangle = \otimes_k |\alpha_k(t)\rangle\,. \tag{4.26}$$

4.3 Weyl–Heisenberg CS with Laguerre Polynomials

The construction of the standard CS is minimal from the point of view of the action of the Weyl unitary operator $D(\alpha)$ on the vacuum $|0\rangle$ (Property $\mathbf{P}_6$). More elaborate states are issued from the action of $D(\alpha)$ on other states $|s\rangle$, $s = 1, 2, \ldots,$ of the Fock basis, which might be considered as initial states in the evolution described by (4.23). Hence, let us define the family of CS

$$|\alpha; s\rangle = D(\alpha)|s\rangle = \sum_{n=0}^{\infty} D_{ns}(\alpha)|n\rangle\,. \tag{4.27}$$

The coefficients in this Fock expansion are the matrix elements $D_{ns} = \langle n|D(\alpha)|s\rangle$ of the displacement operator. They are given in terms of the generalized Laguerre polynomials [9] as

$$\begin{aligned} D_{ns}(\alpha) &:= \sqrt{\frac{s!}{n!}}\, e^{-\frac{|\alpha|^2}{2}}\, \alpha^{n-s}\, L_s^{(n-s)}\left(|\alpha|^2\right) \quad \text{for} \quad s \leq n\,, \\ &= \sqrt{\frac{n!}{s!}}\, e^{-\frac{|\alpha|^2}{2}}\, (-\bar{\alpha})^{s-n}\, L_n^{(s-n)}\left(|\alpha|^2\right) \quad \text{for} \quad s > n\,. \end{aligned} \tag{4.28}$$

As matrix elements of a projective square-integrable UIR of the Weyl–Heisenberg group they obey the orthogonality relations

$$\int_{\mathbb{C}} \frac{d^2\alpha}{\pi}\, \overline{D_{ns}(\alpha)}\, D_{n's'}(\alpha) = \delta_{nn'}\, \delta_{ss'}\,. \tag{4.29}$$

Like for the general case presented in (3.3)–(3.4) this property validates the resolution of the identity

$$\int_{\mathbb{C}^2} \frac{d^2\alpha}{\pi}\, |\alpha; s\rangle\langle\alpha; s| = I\,. \tag{4.30}$$

The corresponding detection distribution is the "Laguerre weighted" Poisson distribution

$$n \mapsto \mathsf{P}_n(u) = \begin{cases} e^{-u}\, \dfrac{u^{s-n}}{(s-n)!}\, \dfrac{\left(L_n^{(s-n)}(u)\right)^2}{\binom{s}{n}} & n \leq s \\ e^{-u}\, \dfrac{u^{n-s}}{(n-s)!}\, \dfrac{\left(L_s^{(n-s)}(u)\right)^2}{\binom{n}{s}} & n \geq s \end{cases}\,. \tag{4.31}$$

Of course, the optical phase space made of the complex $\sqrt{\bar{n}(\alpha)}e^{\mathrm{i}\arg\alpha}$ is here less immediate.

We notice that for $s > 0$, these CS $|\alpha; s\rangle$ do not pertain to the AN class, since we find in the expansion a finite number of terms in $\bar{\alpha}^n$ besides an infinite number of terms in α^n. On the other hand, there exist families of coherent states in the AN class (or their complex conjugate) which are related to the generalized Laguerre polynomials in a quasi-identical way [33, 34].

5 Non-linear CS

5.1 General

We define as non-linear CS those AN CS for which the functions $h_n(u)$ assume the simple form

$$h_n(u) = \frac{\lambda_n}{\sqrt{\mathcal{N}(u)}}\,, \quad \mathcal{N}(u) = \sum_{n=0}^{\infty} |\lambda_n|^2 u^n\,. \tag{5.1}$$

5.2 Deformed Poissonian CS

They are particular cases of the above. All λ_n form a strictly decreasing sequence of positive numbers tending to 0:

$$\lambda_0 = 1 > \lambda_1 > \cdots \lambda_n > \lambda_{n+1} > \cdots , \quad \lambda_n \to 0 . \tag{5.2}$$

We now introduce the strictly increasing sequence

$$x_n = \left(\frac{\lambda_{n-1}}{\lambda_n}\right)^2 , \quad x_0 = 0 . \tag{5.3}$$

It is straightforward to check that

$$\lambda_n = \frac{1}{\sqrt{x_n!}} , \quad \text{with} \quad x_n! := x_1 x_2 \cdots x_n . \tag{5.4}$$

Then $\mathcal{N}(u)$ is the generalized exponential with convergence radius R^2

$$\mathcal{N}(u) = \sum_{n=0}^{\infty} \frac{u^n}{x_n!} , \tag{5.5}$$

and the corresponding CS take the form extending to the non-linear case the familiar Glauber–Sudarshan one

$$|\alpha\rangle = \frac{1}{\sqrt{\mathcal{N}(|\alpha)|^2)}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{x_n!}} |n\rangle . \tag{5.6}$$

The orthonormality condition (3.3) is completely fulfilled if there exists a weight $w(u)$ solving the moment problem for the sequence $(x_n!)_{n\in\mathbb{N}}$

$$x_n! = \int_0^{R^2} \mathrm{d}u \, \frac{w(u)}{\mathcal{N}(u)} \, u^n . \tag{5.7}$$

The detection probability distribution is the deformed Poisson distribution:

$$n \mapsto \mathsf{P}_n(u) = \frac{1}{\mathcal{N}(u)} \frac{u^n}{x_n!} . \tag{5.8}$$

The average value of the number operator $\bar{n}$ is given by

$$\bar{n}\left(|\alpha|^2\right)) = \langle\alpha|\hat{N}|\alpha\rangle = u \left. \frac{\mathrm{d}\log\mathcal{N}(u)}{\mathrm{d}u} \right|_{u=|\alpha|^2} . \tag{5.9}$$

5.3 Example with q Deformations of Integers

These coherent states have been studied by many authors, see [35], that we follow here, and the references therein. They are built from the symmetric or bosonic q-deformation of natural numbers:

$$x_n = {}^{[s]}[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = {}^{[s]}[n]_{q^{-1}}\,, \quad q > 0\,. \tag{5.10}$$

$$|\alpha\rangle_q = \frac{1}{\sqrt{\mathcal{N}_q(|\alpha|^2)}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{{}^{[s]}[n]_q!}}\,|n\rangle\,, \tag{5.11}$$

where its associated exponential is one of the so-called q exponentials [36]

$$\mathcal{N}_q(u) = \mathfrak{e}_q(u) \equiv= \sum_{n=0}^{+\infty} \frac{u^n}{{}^{[s]}[n]_q!}\,. \tag{5.12}$$

This series defines the analytic entire function $\mathfrak{e}_q(z)$ in the complex plane for any positive q. The CS $|\alpha\rangle_q$ in the limit $q \to 1$ goes to the standard CS $|\alpha\rangle$. The solution to the moment problem (3.14) for $0 < q < 1$ is given by

$$\int_0^{\infty} \mathrm{d}u\, w_q(u)\, \frac{u^n}{\mathfrak{e}_q(u)\, {}^{[s]}[n]_q!} = 1$$

with positive density

$$w_q(t) = (q^{-1} - q) \sum_{j=0}^{\infty} g_q\left(t\,\frac{q^{-1} - q}{q^{2j}}\right) \mathfrak{E}_q\left(-\frac{q^{2j}}{q^{-1} - q}\right)\,.$$

The function g_q is given by

$$g_q(u) = \frac{1}{\sqrt{2\pi\,|\ln q|}}\, \exp\left[-\frac{\left[\ln\left(\frac{u}{\sqrt{q}}\right)\right]^2}{2|\ln q|}\right]\,,$$

and a second q-exponential [36] appears here

$$\mathfrak{E}_q(u) := \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}\, \frac{u^n}{{}^{[s]}[n]_q!}\,.$$

Its radius of convergence is ∞ for $0 < q \leq 1$ (it is equal to $1/(q - q^{-1})$ for $q > 1$). There results the resolution of the identity

$$\int_{\mathbb{C}} \mathrm{d}^2\alpha \, \mathfrak{w}_q\,(\alpha)\; |\alpha\rangle_{q\,q}\langle\alpha| = I\,, \quad \mathfrak{w}_q\,(\alpha) = \frac{w_q(|\alpha|^2)}{\pi}\,. \tag{5.13}$$

More exotic families of non-linear CS are, for instance, presented in [37].

6 Spin CS as Optical CS

These states are an adaptation to the quantum optical context of the well-known Gilmore or Perelomov SU(2)-CS, also called spin CS [22, 23]. The Fock space reduces to the finite-dimensional subspace $\mathcal{H}_j$, with dimension $n_j + 1 := 2j + 1$, for j positive integer or half-integer, consistently with the fact that the functions h_n, given here by

$$h_n(u) = \sqrt{\binom{n_j}{n}}\,(1+u)^{-\frac{n_j}{2}}\,, \quad \binom{n_j}{n} = \frac{n_j!}{n!(n_j - n)!}\,, \tag{6.1}$$

cancel for $n > n_j$. The corresponding spin CS read

$$|\alpha; n_j\rangle = \left(1 + |\alpha|^2\right)^{-\frac{n_j}{2}} \sum_{n=0}^{n_j} \sqrt{\binom{n_j}{n}}\,\alpha^n\,|n\rangle\,. \tag{6.2}$$

They resolve the unity in $\mathcal{H}_{n_j}$ in the following way:

$$\frac{n_j + 1}{\pi} \int_{\mathbb{C}} \frac{\mathrm{d}^2\alpha}{(1 + |\alpha|^2)^2}\,|\alpha; n_j\rangle\langle\alpha; n_j| = I\,. \tag{6.3}$$

The detection probability distribution is binomial:

$$n \mapsto \mathsf{P}_n(u) = (1+u)^{-n_j}\binom{n_j}{n}u^n\,. \tag{6.4}$$

There results the average value of the number operator

$$\bar{n}(u) = n_j\,\frac{u}{1+u} \;\Leftrightarrow\; u = \frac{\bar{n}/n_j}{1 - \bar{n}/n_j}\,. \tag{6.5}$$

Thus the probability (6.4) is expressed in terms of the ratio $p := \bar{n}/n_j$ as

$$\mathsf{P}_n(u) \equiv \widetilde{\mathsf{P}}_n(p) = \binom{n_j}{n} (1-p)^{n_j - n}\, p^n \,, \tag{6.6}$$

which allows to define the optical phase space as the open disk of radius $\sqrt{n_j}$, $\mathcal{D}_{\sqrt{n_j}} = \left\{ \xi_\alpha = \sqrt{\bar{n}\left(|\alpha|^2\right)} e^{\mathrm{i}\arg\alpha} \,,\ |\xi_\alpha| < \sqrt{n_j} \right\}$.

The interpretation of $\mathsf{P}_n(u)$ together with the number n_j in terms of photon statistics (see Sect. 10 for more details) is luminous if we consider a beam of perfectly coherent light with a constant intensity. If the beam is of finite length L and is subdivided into n_j segments of length L/n_j, then $\widetilde{\mathsf{P}}_n(p)$ is the probability of finding n subsegments containing one photon and $(n_j - n)$ containing no photons, in any possible order [38]. A more general statistical interpretation of (6.4) or (6.6) is discussed in [39].

Note that the standard coherent states are obtained from the above CS at the limit $n_j \to \infty$ through a contraction process. The latter is carried out through a scaling of the complex variable α, namely $\alpha \mapsto \sqrt{n_j}\,\alpha$. Then the binomial distribution $\widetilde{\mathsf{P}}_n(p)$ becomes the Poissonian (4.5), as expected.

Actually, these states are the simplest ones among a whole family issued from the Perelomov construction [22, 30, 40], and based on spin spherical harmonics. For our present purpose we modify their definition by including an extra phase factor and delete the factor $\sqrt{\frac{2j+1}{4\pi}}$. For $j \in \mathbb{N}/2$ and a given $-j \leq \sigma \leq j$, the spin spherical harmonics are the following functions on the unit sphere $\mathbb{S}^2$:

$$\begin{aligned} {}_\sigma\mathfrak{Y}_{j\mu}(\Omega) := (-1)^{(j-\mu)} \sqrt{\frac{(j-\mu)!(j+\mu)!}{(j-\sigma)!(j+\sigma)!}} \times \\ \times \frac{1}{2^\mu} (1+\cos\theta)^{\frac{\mu+\sigma}{2}} (1-\cos\theta)^{\frac{\mu-\sigma}{2}} P_{j-\mu}^{(\mu-\sigma,\mu+\sigma)}(\cos\theta)\, e^{-\mathrm{i}(j-\mu)\varphi} \,, \end{aligned} \tag{6.7}$$

where $\Omega = (\theta, \varphi)$ (polar coordinates), $-j \leq \mu \leq j$, and the $P_n^{(a,b)}(x)$ are Jacobi polynomials [9] with $P_0^{(a,b)}(x) = 1$. Singularities of the factors at $\theta = 0$ (resp. $\theta = \pi$) for the power $\mu - \sigma < 0$ (resp. $\mu + \sigma < 0$) are just apparent. To remove them it is necessary to use alternate expressions of the Jacobi polynomials based on the relations:

$$P_n^{(-a,b)}(x) = \frac{\binom{n+b}{a}}{\binom{n}{a}} \left(\frac{x-1}{2}\right)^a P_{n-a}^{(a,b)}(x) \,. \tag{6.8}$$

The functions (6.7) obey the two conditions required in the construction of coherent states

$$\frac{2j+1}{4\pi}\int_{\mathbb{S}^2} \mathrm{d}\Omega \, \overline{{}_{\sigma}\mathfrak{Y}_{j\mu}(\Omega)}\, {}_{\sigma}\mathfrak{Y}_{j\mu'}(\Omega) = \delta_{\mu\mu'} \quad \text{(orthogonality)} \tag{6.9}$$

$$\sum_{\mu=-j}^{j} |{}_{\sigma}\mathfrak{Y}_{j\mu}(\Omega)|^2 = 1 \quad \text{(normalisation)}\,. \tag{6.10}$$

At $j = l$ integer and $\sigma = 0$, $\mu = m$ we recover the spherical harmonics $Y_{lm}(\Omega)$ (up to the factor $(-1)^l e^{-\mathrm{i}j\varphi}\sqrt{\frac{2l+1}{4\pi}}$). We now consider the parameter α in (6.2) as issued from the stereographic projection $\mathbb{S}^2 \ni \Omega \mapsto \alpha \in \mathbb{C}$:

$$\alpha = \tan\frac{\theta}{2}\, e^{-\mathrm{i}\varphi}\,, \quad \text{with} \quad \mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\varphi = \frac{4\mathrm{d}^2\alpha}{(1+|\alpha|^2)^2}\,. \tag{6.11}$$

In this regard, the probability $p = \bar{n}/n_j$ is equal to $\sin\theta/2$, while $\varphi = \arg\alpha$. With the notations $n_j = 2j \in \mathbb{N}$, $n = j - \mu = 0, 1, 2, \dots, n_j$, $0 \le s = j - \sigma \le n_j$, adapted to the content of the present paper, and from the expression of the Jacobi polynomials, we get the functions (6.7) in terms of $\alpha \in \mathbb{C}$:

$${}_{\sigma}\mathfrak{Y}_{j\mu}(\Omega) = \alpha^n \, h_{n;s}\left(|\alpha|^2\right)\,, \tag{6.12}$$

where

$$h_{n;s}(u) = \sqrt{\frac{n!(n_j-n)!}{s!(n_j-s)!}}\,(1+u)^{-\frac{n_j}{2}} \sum_{r=\max(0,n+s-n_j)}^{\min(n,s)} \binom{s}{r}\binom{n_j-s}{n-r}(-1)^r\, u^{s/2-r}\,. \tag{6.13}$$

The corresponding "Jacobi" CS are in the AN class and read

$$|\alpha; n_j; s\rangle = \sum_{n=0}^{n_j} \alpha^n \, h_{n;s}\left(|\alpha|^2\right)\, |n\rangle\,. \tag{6.14}$$

They solve the identity as

$$\frac{n_j+1}{\pi}\int_{\mathbb{C}} \frac{\mathrm{d}^2\alpha}{(1+|\alpha|^2)^2}\, |\alpha; n_j; s\rangle\langle\alpha; n_j; s| = I. \tag{6.15}$$

The states (6.2) are recovered for $s = 0$. Similarly to CS (4.27) states (6.14) can be also viewed as displaced occupied states. Indeed, they can be written in the

Perelomov way as

$$|\alpha; n_j; s\rangle = \mathcal{D}^{n_j/2}(\zeta_\alpha)\,|s\rangle\,, \tag{6.16}$$

where $\zeta_\alpha = \begin{pmatrix} \left(1+|\alpha|^2\right)^{-1/2} & \left(1+|\alpha|^2\right)^{-1/2}\alpha \\ -\left(1+|\alpha|^2\right)^{-1/2}\bar{\alpha} & \left(1+|\alpha|^2\right)^{-1/2} \end{pmatrix}$ is the element of SU(2) which brings 0 to α under the homographic action

$$\alpha \mapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \cdot \alpha := \frac{a\alpha + b}{-\bar{b}\alpha + \bar{a}}$$

of this group on the complex plane, and $\mathcal{D}^{n_j/2}$ is the corresponding $n_j + 1$-dimensional UIR of SU(2). Let us write $\mathcal{D}^{n_j/2}(\zeta_\alpha)$ as a displacement operator similar to the Weyl–Heisenberg one (propriety $\mathbf{P}_6$) and involving the usual angular momentum generators $J_\pm$ for the representation $\mathcal{D}^{n_j/2}$

$$\mathcal{D}^{n_j/2}(\zeta_\alpha) = e^{\varsigma_\alpha J_+ - \bar{\varsigma}_\alpha J_-} \equiv D_{n_j}(\varsigma_\alpha)\,, \quad \varsigma_\alpha = -\tan^{-1}|\alpha|\,e^{-\mathrm{i}\arg\alpha}\,. \tag{6.17}$$

Note that we could have adopted here the historical approaches by Jordan, Holstein, Primakoff, Schwinger [41–43] in transforming these angular momentum operators in terms of "bosonic" a and $a^\dagger$. Nevertheless this QFT artificial flavor is not really useful in the present context.

7 SU(1, 1)-CS as Optical CS

7.1 Perelomov CS

These states are also an adaptation to the quantum optical context of the Perelomov SU(1, 1)-CS [22, 23, 30, 44]. They are yielded through a SU(1, 1) unitary action on a number state. The Fock Hilbert space $\mathcal{H}$ is infinite-dimensional, while the complex number α is restricted to the open unit disk $\mathcal{D} := \{\alpha \in \mathbb{C}\,,\ |\alpha| < 1\}$. Let $\varkappa > 1/2$ and $s \in \mathbb{N}$. We then define the $(\varkappa; s)$-dependent CS family as the "SU(1, 1)-displaced s-th state"

$$|\alpha; \varkappa; s\rangle = U^{\varkappa}(p(\bar{\alpha}))|s\rangle = \sum_{n=0}^{\infty} U^{\varkappa}_{ns}(p(\bar{\alpha}))|n\rangle \equiv \sum_{n=0}^{\infty} \phi_{n;\varkappa;s}(\alpha)\,|n\rangle\,, \tag{7.1}$$

where the $U^{\varkappa}_{ns}(p(\bar{\alpha}))$'s are matrix elements of the UIR $U^{\varkappa}$ of SU(1, 1) in its discrete series and $p(\bar{\alpha})$ is the particular matrix

$$\begin{pmatrix} \left(1-|\alpha|^2\right)^{-1/2} & \left(1-|\alpha|^2\right)^{-1/2}\bar{\alpha} \\ \left(1-|\alpha|^2\right)^{-1/2}\alpha & \left(1-|\alpha|^2\right)^{-1/2} \end{pmatrix} \in \mathrm{SU}(1,1)\,. \tag{7.2}$$

They are given in terms of Jacobi polynomials as

$$U^{\varkappa}_{ns}(p(\bar{\alpha})) = \left(\frac{n_<!\,\Gamma(2\varkappa + n_>)}{n_>!\,\Gamma(2\varkappa + n_<)}\right)^{1/2} \left(1 - |\alpha|^2\right)^{\varkappa} (\mathrm{sgn}(n-s))^{n-s} \times$$
$$\times\, P^{(n_> - n_<\,,\,2\varkappa-1)}_{n_<}\left(1 - 2|\alpha|^2\right) \times \begin{cases} \alpha^{n-s} & \text{if } n_> = n \\ \bar{\alpha}^{s-n} & \text{if } n_> = s \end{cases} \tag{7.3}$$

with $n_{\gtrless} = \begin{cases} \max \\ \min \end{cases} (n, s) \geq 0$. The states (7.1) solve the identity:

$$\frac{2\varkappa - 1}{\pi} \int_{\mathcal{D}} \frac{\mathrm{d}^2\alpha}{\left(1 - |\alpha|^2\right)^2}\, |\alpha; \varkappa; s\rangle\langle\alpha; \varkappa; s| = I\,. \tag{7.4}$$

The simplest case $s = 0$ pertains to the AN class

$$|\alpha; \varkappa; 0\rangle \equiv |\alpha; \varkappa\rangle = \sum_{n=0}^{\infty} \alpha^n\, h_{n;\varkappa}\left(|\alpha|^2\right) |n\rangle\,,\ \ h_{n;\varkappa}(u) := \sqrt{\binom{2\varkappa - 1 + n}{n}}\,(1-u)^{\varkappa}\,. \tag{7.5}$$

The corresponding detection probability distribution is negative binomial

$$n \mapsto \mathsf{P}_n(u) = (1-u)^{2\varkappa} \binom{2\varkappa - 1 + n}{n} u^n\,. \tag{7.6}$$

The average value of the number operator reads as

$$\bar{n}(u) = 2\varkappa\, \frac{u}{1-u} \;\Leftrightarrow\; u = \frac{\bar{n}/2\varkappa}{1 + \bar{n}/2\varkappa}\,. \tag{7.7}$$

By introducing the "efficiency" $\eta := 1/2\varkappa \in (0, 1)$ the probability (7.6) is expressed in terms of the corrected average value $\bar{N} := \eta\bar{n}$ as

$$\mathsf{P}_n(u) \equiv \widetilde{\mathsf{P}}_n(\bar{N}) = (1 + \bar{N})^{-1/\eta} \binom{1/\eta - 1 + n}{n} \left(\frac{\bar{N}}{1 + \bar{N}}\right)^n\,. \tag{7.8}$$

It is remarkable that such a distribution reduces to the celebrated Bose–Einstein one for the thermal light at the limit $\eta = 1$, i.e., at the lowest bound $\varkappa = 1/2$ of the discrete series of $SU(1,1)$. For $\eta < 1$, the difference might be understood from the fact that we consider the average photocount number $\bar{N}$ instead of the mean photon number $\bar{n}$ impinging on the detector in the same interval [38]. For a related interpretation within the framework of thermal equilibrium states of the oscillator see [45].

Note that the above CS, built from the negative binomial distribution, were also discussed in [39].

Like for CS (4.27), the CS $|\alpha;\varkappa;s\rangle$ in (7.1) do not pertain to the AN class for $s>0$. In their expansion there are s terms in $\bar{\alpha}^{s-n}$, $s>n$, besides an infinite number of terms in α^{n-s}, $s\leq n$. Finally, like for the Weyl–Heisenberg and SU(2) cases, the representation operator $U^{\varkappa}(p(\bar{\alpha}))$ used in (7.1) to build the SU(1, 1) CS can be given the following form of a displacement operator involving the generators $K_{\pm}$ for the representation U^{κ} [23]:

$$U^{\kappa}(p(\bar{\alpha}))=e^{\varrho_\alpha K_+-\bar{\varrho}_\alpha K_-}\equiv D_\kappa(\varrho_\alpha)\,,\quad \varrho_\alpha=\tanh^{-1}|\alpha|\,e^{\mathrm{i}\arg\alpha}\,. \tag{7.9}$$

7.2 Barut–Girardello CS

These non-linear CS states [46, 47] pertain to the AN class. They are requested to be eigenstates of the SU(1, 1) lowering operator in its discrete series representation $U^{\varkappa}$, $\varkappa>1/2$. The Fock Hilbert space $\mathcal{H}$ is infinite-dimensional, while the complex number α has no domain restriction in $\mathbb{C}$. With the notations of (5.6) they read

$$|\alpha;\varkappa\rangle_{\mathrm{BG}}=\frac{1}{\sqrt{\mathcal{N}_{\mathrm{BG}}(|\alpha|^2)}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{x_n!}}|n\rangle\,,\ x_n=n(2\varkappa+n-1)\,,\ x_n!=n!\frac{\Gamma(2\varkappa+n)}{\Gamma(2\varkappa)}\,, \tag{7.10}$$

with

$$\mathcal{N}_{\mathrm{BG}}(u)=\Gamma(2\varkappa)\sum_{n=0}^{\infty}\frac{u^n}{n!\Gamma(2\varkappa+n)}=\Gamma(2\varkappa)\,u^{-\varkappa}\,I_{2\varkappa-1}(2\sqrt{u}), \tag{7.11}$$

where I_ν is a modified Bessel function [9]. In the present case the moment problem (3.14) is solved as

$$\int_0^{\infty}\mathrm{d}u\,w_{\mathrm{BG}}(u)\,\frac{u^n}{\mathcal{N}_{\mathrm{BG}}(u)\,x_n!}=1\,,\ w_{\mathrm{BG}}(u)=\mathcal{N}_{\mathrm{BG}}(u)\,\frac{2}{\Gamma(2\varkappa)}\,u^{\varkappa-1/2}\,K_{2\varkappa-1}(2\sqrt{u})\,, \tag{7.12}$$

where K_ν is the second modified Bessel function. The resolution of the identity follows:

$$\int_{\mathbb{C}}\mathrm{d}^2\alpha\,\mathfrak{w}_{\mathrm{BG}}(\alpha)\ |\alpha;\varkappa\rangle_{\mathrm{BG}\,\mathrm{BG}}\langle\alpha;\varkappa|=I\,,\quad \mathfrak{w}_{\mathrm{BG}}(u)=\frac{w_{\mathrm{BG}}(u)}{\pi}\,. \tag{7.13}$$

8 Adapted Susskind–Glogower CS

Let us examine the Susskind–Glogower CS [48] presented in [49]. These normalized states read for real $\alpha \equiv x \in \mathbb{R}$

$$|x\rangle_{\mathrm{SG}} = \sum_{n=0}^{\infty}(n+1)\,\frac{J_{n+1}(2x)}{x}\,|n\rangle\,, \tag{8.1}$$

where the Bessel function J_ν is given by

$$J_\nu(z) = \left(\frac{z}{2}\right)^{\nu}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m}}{m!\,\Gamma(\nu+m+1)}\,. \tag{8.2}$$

The normalization implies the interesting identity (E. Curado, private communication)

$$\sum_{n=1}^{\infty} n^2\,\left(J_n(2x)\right)^2 = x^2\,. \tag{8.3}$$

The above expression allows us to extend the formula (8.1) in a non-analytic way to complex α as

$$(n+1)\frac{J_{n+1}(2x)}{x} \mapsto \alpha^n\,(n+1)\sum_{m=0}^{\infty}\frac{(-1)^m|\alpha|^{2m}}{m!\,\Gamma(n+m+2)} \equiv \alpha^n\,h_n^{\mathrm{SG}}(|\alpha|^2)\,, \tag{8.4}$$

i.e.,

$$h_n^{\mathrm{SG}}(u) = (n+1)\,\frac{1}{u^{\frac{n+1}{2}}}\,J_{n+1}(2\sqrt{u})\,, \tag{8.5}$$

and thus

$$|\alpha\rangle_{\mathrm{SG}} = \sum_{n=0}^{\infty}\alpha^n\,h_n^{\mathrm{SG}}(|\alpha|^2)\,|n\rangle\,. \tag{8.6}$$

The moment Eq. (3.14) reads here

$$\int_0^{\infty}\mathrm{d}u\,\frac{w(u)}{u}\,\left(J_n(2\sqrt{u})\right)^2 = 2\int_0^{\infty}\mathrm{d}t\,\frac{w(t^2)}{t}\,\left(J_n(2t)\right)^2 = \frac{1}{n^2}\,. \tag{8.7}$$

Let us examine the following integral formula for Bessel functions [9]:

$$\int_0^\infty \frac{\mathrm{d}t}{t}\, (J_n(2t))^2 = \frac{1}{2n}\,. \tag{8.8}$$

This leads us to replace the SG-CS of (8.1) by the modified

$$|\alpha\rangle_{\mathrm{SGm}} = \sum_{n=0}^{\infty} \alpha^n\, h_n^{\mathrm{SGm}}(|\alpha|^2)\, |n\rangle\,, \quad h_n^{\mathrm{SGm}}(u) = \sqrt{\frac{n+1}{\mathcal{N}(u)}}\, \frac{1}{u^{\frac{n+1}{2}}}\, J_{n+1}(2\sqrt{u})\,, \tag{8.9}$$

with

$$\mathcal{N}(u) = \frac{1}{u}\sum_{n=1}^{\infty} n\, \left(J_n(2\sqrt{u})\right)^2\,. \tag{8.10}$$

Then the formula (8.8) allows us to prove that the resolution of the identity is fulfilled by these $|\alpha\rangle_{\mathrm{SGm}}$ with $w(u) = \mathcal{N}(u)$. More details, particularly those concerning statistical aspects, are given in [50].

9 CS from Symmetric Deformed Binomial Distributions (DFB)

In [51] (see also the related works [52–54]) was presented the following generalization of the binomial distribution:

$$\mathfrak{p}_k^{(n)}(\xi) = \frac{x_n!}{x_{n-k}!x_k!} q_k(\xi) q_{n-k}(1-\xi)\,, \tag{9.1}$$

where the $\{x_n\}$'s form a non-negative sequence and the $q_k(\xi)$ are polynomials of degree k, while ξ is a running parameter on the interval [0, 1]. The $\mathfrak{p}_k^{(n)}(\xi)$ are constrained by

(a) the normalization

$$\forall n \in \mathbb{N}, \quad \forall \xi \in [0,1], \quad \sum_{k=0}^{n} \mathfrak{p}_k^{(n)}(\xi) = 1, \tag{9.2}$$

(b) the non-negativeness condition (requested by statistical interpretation)

$$\forall n, k \in \mathbb{N}, \quad \forall \xi \in [0,1], \quad \mathfrak{p}_k^{(n)}(\xi) \geq 0. \tag{9.3}$$

These conditions imply that $q_0(\xi) = \pm 1$. With the choice $q_0(\xi) = 1$ one easily proves that the non-negativeness condition (9.3) is equivalent to the non-negativeness of the polynomials q_n on the interval [0, 1]. Hence the quantity $\mathfrak{p}_k^{(n)}(\xi)$ can be interpreted as the probability of having k wins and $n-k$ losses in a sequence of *correlated* n trials. Besides, as we recover the invariance under $k \to n-k$ and $\xi \to 1-\xi$ of the binomial distribution, no bias (in the case $\xi = 1/2$) can exist favoring either win or loss. The polynomials $q_n(\xi)$ are viewed here as *deformations* of ξ^n. We now suppose that the generating function for the polynomials q_n, defined as

$$F(\xi; t) := \sum_{n=0}^{\infty} \frac{q_n(\xi)}{x_n!} t^n \,, \tag{9.4}$$

can be expressed as

$$F(\xi; t) = e^{\sum_{n=1}^{\infty} a_n t^n} \quad \text{with} \quad a_1 = 1\,,\ a_n = a_n(\xi) \geq 0\,,\ \sum_{n=1}^{\infty} a_n < \infty\,. \tag{9.5}$$

It is proved in [51] that conditions of normalization (a) and non-negativeness (b) on $\mathfrak{p}_k^{(n)}(\xi)$ are satisfied. We now define

$$f_n = \int_0^{\infty} q_n(\xi)\, e^{-\xi}\, d\xi \quad \text{and} \quad b_{m,n} = \int_0^1 q_m(\xi)\, q_n(1-\xi)\, d\xi\,. \tag{9.6}$$

The f_n and $b_{m,n}$ are deformations of the usual factorial and beta function, respectively, deduced from their usual integral definitions through the substitution $\xi^n \mapsto q_n(\xi)$. The following properties are proven in [51]:

$$\begin{aligned} & q_n(\xi) \geq 0\ \forall \xi \in \mathbb{R}^+\,, \quad x_n! \leq f_n\,, \\ & \sum_{n=0}^{\infty} \frac{q_n(\xi)}{f_n} < \infty\ \forall \xi \in \mathbb{R}^+\,, \quad \text{and} \quad b_{m,n} \geq \frac{x_m! x_n!}{(m+n+1)!}\,. \end{aligned} \tag{9.7}$$

Then let us introduce the function $\mathcal{N}(z)$ defined on $\mathbb{C}$ as

$$\forall z \in \mathbb{C} \quad \mathcal{N}(z) = \sum_{n=0}^{\infty} \frac{q_n(z)}{f_n}\,. \tag{9.8}$$

This definition makes sense since from Eq. (9.7)

$$\sum_{n=0}^{\infty} \left| \frac{q_n(z)}{f_n} \right| \leq \sum_{n=0}^{\infty} \frac{q_n(|z|)}{f_n} < \infty. \tag{9.9}$$

The above material allows us to present below two new generalizations of standard and spin coherent states.

9.1 DFB Coherent States on the Complex Plane

They are defined in the Fock space as

$$|\alpha\rangle_{\mathrm{dfb}} = \frac{1}{\sqrt{\mathcal{N}(|\alpha|^2)}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{f_n}} \sqrt{q_n(|\alpha|^2)}\, e^{\mathrm{i}\, n \arg(\alpha)} |n\rangle \,. \tag{9.10}$$

These states verify the following resolution of the unity:

$$\int_{\mathbb{C}} \frac{\mathrm{d}^2\alpha}{\pi} e^{-|\alpha|^2} \mathcal{N}(|\alpha|^2)\, |\alpha\rangle_{\mathrm{dfb\,dfb}}\langle\alpha| = I \,. \tag{9.11}$$

They are a natural generalization of the standard coherent states that correspond to the special polynomials $q_n(\xi) = \xi^n$. The latter are associated to the generating function $F(t) = e^t$ that gives the usual binomial distribution.

9.2 DFB Spin Coherent States

These states can be considered as generalizing the spin coherent states (6.2)

$$|\alpha; n_j\rangle_{\mathrm{dfb}} = \frac{1}{\sqrt{\mathcal{N}(|\alpha|^2)}} \sum_{n=0}^{n_j} \sqrt{\frac{q_n\left(\frac{1}{1+|\alpha|^2}\right) q_{n_j-n}\left(\frac{|\alpha|^2}{1+|\alpha|^2}\right)}{b_{n,n_j-n}}} e^{\mathrm{i}\arg(\alpha)} |n\rangle \,, \tag{9.12}$$

where the $b_{m,n}$ are defined in Eq. (9.6) and $\mathcal{N}(u)$ is given by

$$\mathcal{N}(u) = \sum_{n=0}^{n_j} \frac{q_n\left(\frac{1}{1+u}\right) q_{n_j-n}\left(\frac{u}{1+u}\right)}{b_{n,n_j-n}} \,. \tag{9.13}$$

The family of states (9.12) resolves the unity:

$$\int_{\mathbb{C}} \mathrm{d}^2\alpha\, \varpi(\alpha)\; |\alpha; n_j\rangle_{\mathrm{dfb\,dfb}}\langle\alpha; n_j| = I \,, \quad \varpi(\alpha) = \frac{\mathcal{N}\left(|\alpha|^2\right)}{\pi\left(1+|\alpha|^2\right)^2} \,. \tag{9.14}$$

10 Photon Counting: Basic Statistical Aspects

In this section, we mainly follow the inspiring chapter 5 of Ref. [38] (see also the seminal papers [55–57] on the topic, the renowned [58], the pedagogical [59], and the more recent [60–62]). In quantum optics one views a beam of light as a stream of discrete energy packets named “photons” rather than a classical wave. With a photon counter the average count rate is determined by the intensity of the light beam, but the actual count rate fluctuates from measurement to measurement. Whence, one easily understands that two statistics are in competition here, on one hand the statistical nature of the photodetection process, and on the other hand, the intrinsic photon statistics of the light beam, e.g., the average $\bar{n}(\alpha)$ for a CS $|\alpha\rangle$. Photon-counting detectors are specified by their quantum efficiency η, which is defined as the ratio of the number of photocounts to the number of incident photons. For a perfectly coherent monochromatic beam of angular frequency ω, constant intensity I, and area A, and for a counting time T

$$\eta = \frac{N(T)}{\Phi T}\,, \tag{10.1}$$

where the photon flux is $\Phi = \dfrac{IA}{\hbar\omega} \equiv \dfrac{P}{\hbar\omega}$, P being the power. Thus the corresponding count rate is $\mathcal{R} = \dfrac{\eta P}{\hbar\omega}$ counts s^{-1}. Due to a "dead time" of $\sim 1\,\mu s$ for the detector reaction, the count rate cannot be larger than $\sim 10^6$ counts s^{-1}, and due to weak values $\eta \sim 10\%$ for standard detectors, photon counters are only useful for analyzing properties of very faint beams with optical powers of $\sim 10^{-12}$W or less. The detection of light beams with higher powers requires other methods.

Although the average photon flux can have a well-defined value, the photon number on short time-scales fluctuates due to the discrete nature of the photons. These fluctuations are described by the photon statistics of the light.

One proves that the photon statistics for a coherent light wave with constant intensity (e.g., a light beam described by the electric field $\mathcal{E}(x,t) = \mathcal{E}_0 \sin(kx - \omega t + \phi)$ with constant angular frequency ω, phase ϕ, and intensity $\mathcal{E}_0$) is encoded by the Poisson distribution

$$n \mapsto \mathsf{P}_n(\bar{n}) = e^{-\bar{n}}\,\frac{(\bar{n})^n}{n!}\,, \tag{10.2}$$

This randomness of the count rate of a photon-counting system detecting individual photons from a light beam with constant intensity originates from chopping the continuous beam into discrete energy packets with an equal probability of finding the energy packet within any given time subinterval.

Let us introduce the variance as the quantity

$$\mathrm{Var}_n(\bar{n}) \equiv (\Delta n)^2 = \sum_{n=0}^{\infty} (n-\bar{n})^2 \mathsf{P}_n(\bar{n})\,.$$

Thus, for a Poissonian coherent beam, $\Delta n = \sqrt{\bar{n}}$. There results that three different types of photon statistics can occur: Poissonian, super-Poissonian, and sub-Poissonian. The two first ones are consistent as well with the classical theory of light, whereas sub-Poissonian statistics is not and constitutes direct confirmation of the photon nature of light. More precisely

(i) if the Poissonian statistics holds, e.g., for a perfectly coherent light beam with constant optical power P, we have

$$\Delta n = \sqrt{\bar{n}}\,, \tag{10.3}$$

(ii) if the super-Poissonian statistics, e.g., classical light beams with time-varying light intensities, like thermal light from a black-body source, or like partially coherent light from a discharge lamp, we have

$$\Delta n > \sqrt{\bar{n}}\,, \tag{10.4}$$

(iii) finally, the sub-Poissonian statistics is featured by a narrower distribution than the Poissonian case

$$\Delta n < \sqrt{\bar{n}}\,. \tag{10.5}$$

This light is "quieter" than the perfectly coherent light. Since a perfectly coherent beam is the most stable form of light that can be envisaged in classical optics, sub-Poissonian light has no classical counterpart.

In this context popular useful parameters are introduced to account for CS statistical properties, e.g., the Mandel parameter $Q = (\Delta n)^2/\bar{n} - 1$, where $(\Delta n)^2 = \overline{n^2} - \bar{n}^2$, which is <0 (resp. >0, $=0$) for sub-Poissonian (resp. super-Poissonian, Poissonian), the parameter $Q/\bar{n} + 1$ which is >1 for "bunching" CS and <1 for "anti-bunching" CS, etc.

The aim of the quantum theory of photodetection is to relate the photocount statistics observed in a particular experiment to those of the incoming photons, more precisely the average photocount number $\bar{N}$ to the mean photon number $\bar{n}$ incident on the detector in a same time interval. The quantum efficiency η of the detector, defined as $\eta = \bar{N}/\bar{n}$ is the critical parameter that determines the relationship between the photoelectron and photon statistics. Indeed, consider the relation between variances $(\Delta N)^2 = \eta^2\,(\Delta n)^2 + \eta\,(1-\eta)\,\bar{n}$.

- If $\eta = 1$, we have $\Delta N = \Delta n$: the photocount fluctuations faithfully reproduce the fluctuations of the incident photon stream.
- If the incident light has Poissonian statistics $\Delta n = \sqrt{\bar{n}}$, then $(\Delta N)^2 = \eta\,\bar{n}$ for all values of η: photocount is Poisson.
- If $\eta \ll 1$, the photocount fluctuations tend to the Poissonian result with $(\Delta N)^2 = \eta\,\bar{n} = \bar{N}$ irrespective of the underlying photon statistics.

Observing sub-Poissonian statistics in the laboratory is a delicate matter since it depends on the availability of single-photon detectors with high quantum efficiencies.

11 AN CS Quantization

11.1 The Quantization Map and Its Complementary

If the resolution of the identity (3.4) is valid for a given family of AN CS determined by the sequence of functions $\mathsf{h} := (h_n(u))$, it makes the quantization of functions (or distributions) $f(\alpha)$ possible along the linear map

$$f(\alpha) \mapsto A^{\mathsf{h}}_f = \int_{|\alpha|<R} \frac{\mathrm{d}^2\alpha}{\pi}\, w(|\alpha|^2)\, f(\alpha)\, |\alpha\rangle\langle\alpha| \,, \tag{11.1}$$

together with its complementary map, likely to provide a "semi-classical" optical phase space portrait, or *lower symbol*, of A^{h}_f through the map (3.8)

$$\langle\alpha|A^{\mathsf{h}}_f|\alpha\rangle = \int_{|\beta|<R} \frac{\mathrm{d}^2\beta}{\pi}\, w(|\beta|^2)\, f(\beta)\, |\langle\alpha|\beta\rangle|^2 \equiv \check{f}^{\mathsf{h}}(\alpha)\,. \tag{11.2}$$

Since for fixed α the map $\beta \mapsto w(|\beta|^2)\,|\langle\alpha|\beta\rangle|^2$ is a probability distribution on the centered disk $\mathcal{D}_R$ of radius R, the map $f(\alpha) \mapsto \check{f}^{\mathsf{h}}(\alpha)$ is a local, generally regularizing, averaging, of the original f.

The quantization map (11.1) can be extended to cases comprising geometric constraints in the optical phase portrait through the map (3.8), and encoded by distributions like Dirac or Heaviside functions.

11.2 AN CS Quantization of Simple Functions

When applied to the simplest functions α and $\bar{\alpha}$ weighted by a positive $\mathfrak{n}\left(|\alpha|^2\right)$, the quantization map (11.1) yields lowering and raising operators

$$\alpha \mapsto a^{\mathsf{h}} = \int_{|\alpha|<R} \frac{\mathrm{d}^2\alpha}{\pi}\, \tilde{w}(|\alpha|^2)\, \alpha\, |\alpha\rangle\langle\alpha| = \sum_{n=1}^{\infty} a^{\mathsf{h}}_{n-1n} |n-1\rangle\langle n| \,, \tag{11.3}$$

$$\bar{\alpha} \mapsto \left(a^{\mathsf{h}}\right)^{\dagger} = \sum_{n=0}^{\infty} \overline{a^{\mathsf{h}}_{nn+1}} |n+1\rangle\langle n| \,, \tag{11.4}$$

where $\tilde{w}(u) := \mathfrak{n}(u)w(u)$. Their matrix elements are given by the integrals

$$a^{\mathsf{h}}_{n-1n} := \int_0^{R^2} \mathrm{d}u\, \tilde{w}(u)\, u^n\, h_{n-1}(u)\, \overline{h_n(u)} \,, \tag{11.5}$$

and $a^{\mathsf{h}}|0\rangle = 0$.

The lower symbol of a^{h} and its adjoint read, respectively:

$$\widecheck{a^{\mathsf{h}}}(\alpha) = \langle\alpha|a^{\mathsf{h}}|\alpha\rangle = \alpha\,\tau\left(|\alpha|^2\right)\,, \quad \widecheck{\left(a^{\mathsf{h}}\right)^{\dagger}}(\alpha) = \overline{\widecheck{a^{\mathsf{h}}}(\alpha)}\,, \tag{11.6}$$

in which the "weighting" factor is given by $\tau(u) = \sum_{n\geq 0} a^{\mathsf{h}}_{nn+1}\, u^n\, \overline{h_n(u)}\, h_{n+1}(u)$.

In the above, as it was mentioned in Sect. 3 and, as it occurred in the spin case, the involved sums can be finite, and a finite number of matrix elements (11.5) are not zero. As a generalization of the number operator we get in the present case

$$a^{\mathsf{h}}\left(a^{\mathsf{h}}\right)^{\dagger} = \mathsf{X}^{\mathsf{h}}_{\hat{N}+I}\,, \quad \left(a^{\mathsf{h}}\right)^{\dagger} a = \mathsf{X}^{\mathsf{h}}_{\hat{N}}\,, \quad \left[a^{\mathsf{h}}, \left(a^{\mathsf{h}}\right)^{\dagger}\right] = \mathsf{X}^{\mathsf{h}}_{\hat{N}+I} - \mathsf{X}^{\mathsf{h}}_{\hat{N}}\,, \tag{11.7}$$

with the notations

$$\mathsf{X}^{\mathsf{h}}_n = |a^{\mathsf{h}}_{n-1n}|^2\,, \quad \mathsf{X}^{\mathsf{h}}_0 = 0\,, \quad \mathsf{X}^{\mathsf{h}}_{\hat{N}}|n\rangle = \mathsf{X}^{\mathsf{h}}_n|n\rangle\,, \quad \mathsf{X}^{\mathsf{h}}_{\hat{N}+I}|n\rangle = \mathsf{X}^{\mathsf{h}}_{n+1}|n\rangle\,. \tag{11.8}$$

When all the h_n's are real, the diagonal elements in (11.7) are given by the product of integrals

$$\begin{aligned}\mathsf{X}^{\mathsf{h}}_{n+1} - \mathsf{X}^{\mathsf{h}}_n = &\left[\int_0^{R^2} \mathrm{d}u\, \tilde{w}(u)\, u^n\, h_n(u)\, \left(uh_{n+1}(u) - h_{n-1}(u)\right)\right] \\ &\times \left[\int_0^{R^2} \mathrm{d}u\, \tilde{w}(u)\, u^n\, h_n(u)\, \left(uh_{n+1}(u) + h_{n-1}(u)\right)\right].\end{aligned} \tag{11.9}$$

The quantum version of $u = |\alpha|^2$ and its lower symbol read as

$$\begin{aligned}A^{\mathsf{h}}_u &= \sum_n \langle u\rangle_n |n\rangle\langle n|\,, \quad \langle u\rangle_n := \int_0^{R^2} \mathrm{d}u\, \tilde{w}(u)\, u^{n+1}\, h_n(u) \\ \langle\alpha|A^{\mathsf{h}}_u|\alpha\rangle &= \langle\langle u\rangle_n\rangle_\alpha\,(u) := \sum_n \langle u\rangle_n\, u^n\, |h_n(u)|^2 = \sum_n \langle u\rangle_n\, \mathsf{P}^{\mathsf{h}}_n\,.\end{aligned} \tag{11.10}$$

We notice here an interesting duality between classical ($\langle\cdot\rangle_n$) and quantum ($\langle\cdot\rangle_\alpha$) statistical averages.

11.3 AN CS as a-Eigenstates

One crucial property of the Glauber–Sudarshan CS is that they are eigenstates of the lowering operator a. Imposing this property to AN CS leads to a supplementary condition on the functions h_n.

$$a^{\mathsf{h}}|\alpha\rangle = \alpha|\alpha\rangle \Rightarrow h_n(u) = h_{n+1}(u) \int_0^{R^2} \mathrm{d}t\, \tilde{w}(t)\, t^{n+1}\, h_n(t)\, \overline{h_{n+1}(t)}\,. \tag{11.11}$$

Let us examine the particular case of non-linear CS of the deformed Poissonian type (5.6). In this case, $\mathsf{X}_n = x_n$, and whence the construction formula

$$|\alpha\rangle = \frac{\mathcal{N}(\alpha {a^{\mathsf{h}}}^{\dagger})}{\sqrt{\mathcal{N}(|\alpha|^2)}}|0\rangle\,. \tag{11.12}$$

Moreover (11.11) imposes that the sequence $x_n!$ derives from the following moment problem:

$$x_n! = \int_0^{R^2} \mathrm{d}u\, \frac{w(u)}{\mathcal{N}(u)}\, u^n\,. \tag{11.13}$$

Now, instead of starting from a known sequence (x_n), one can reverse the game by choosing a suitable function $f(u) = \dfrac{w(u)}{\mathcal{N}(u)}$ to calculate the corresponding $x_n!$ (from which we deduce the x_n's), the resulting generalized exponential $\mathcal{N}(u)$ (and checking the finiteness of the convergence radius), and eventually the weight function $w(u) = f(u)\mathcal{N}(u)$. There are an infinity of "manufactured" products in this non-linear CS factory!

11.4 AN CS from Displacement Operator

One can attempt to build (other?) AN CS by following the standard procedure involving the unitary "displacement" operator built from a^{h} and ${a^{\mathsf{h}}}^{\dagger}$ and acting on the vacuum

$$|\breve{\alpha}\rangle_{\mathrm{disp}} := D_{\mathsf{h}}(\breve{\alpha})\,|0\rangle = \sum_{n=0}^{\infty} \breve{\alpha}^n\, h_n^{\mathrm{disp}}(|\breve{\alpha}|^2)\,|n\rangle\,, \quad D_{\mathsf{h}}(\breve{\alpha}) := e^{\breve{\alpha}{a^{\mathsf{h}}}^{\dagger} - \bar{\breve{\alpha}} a^{\mathsf{h}}}\,, \tag{11.14}$$

where the notation $\breve{\alpha}$ is used to make the distinction from the original α. Of course, ${D_{\mathsf{h}}}^{\dagger}(\breve{\alpha}) = {D_{\mathsf{h}}}^{-1}(\breve{\alpha})$ is not equal in general to $D_{\mathsf{h}}(-\breve{\alpha})$. Besides the two examples (6.17) and (7.9) encountered in the SU(2) and SU(1, 1) CS constructions, for which the respective weights $\mathfrak{n}(u)$ can be given explicitly, another recent interesting example is given in [63].

So an appealing program is to establish the relation between the original h_n's and these (new?) h_n^{disp}'s, through a suitable choice of the weight $\mathfrak{n}(u)$, actually a big challenge in the general case! More interesting yet is the fact that these new CS's might be experimentally produced in the Glauber's way (4.23), once we accept that the a^{h} and $a^{\mathsf{h}\dagger}$ appearing in the quantum version (4.8) of the classical e.m. field are yielded by a CS quantization different from the historical Dirac (canonical) one [64]. Hence one introduces a kind of duality between two families of coherent states, the first one used in the quantization procedure $f(\alpha) \mapsto A^{\mathsf{h}}_f$, producing the operators $\mathfrak{n}(u)\alpha \mapsto a^{\mathsf{h}}$ and $\mathfrak{n}(u)\bar{\alpha} \mapsto a^{\mathsf{h}\dagger}$, and so the unitary displacement $D^{\mathsf{h}}(\breve{\alpha}) := e^{\breve{\alpha}a^{\mathsf{h}\dagger} - \bar{\breve{\alpha}}a^{\mathsf{h}}}$, while the other one uses this $D_{\mathsf{h}}(\breve{\alpha})$ to build potentially experimental CS yielded in the Glauber's way.

12 Conclusion

We have presented in this paper a unifying approach to build coherent states in a wide sense that are potentially relevant to quantum optics. Of course, for most of them, their experimental observation or production comes close to being impossible with the current experimental physics. Nevertheless, when one considers the way quantum optics has emerged from the golden 1920s of quantum mechanics, nothing prevents us to enlarge the Dirac quantization of the classical e.m. field in order to include all these deformations (non-linear or others) by adopting the consistent method exposed in the previous section.

Acknowledgements This research is supported in part by the Ministerio de Economía y Competitividad of Spain under grant MTM2014-57129-C2-1-P and the Junta de Castilla y León (grant VA137G18). The author is also indebted to the University of Valladolid. He thanks M. del Olmo (UVA) for helpful discussions about this review. He addresses special thanks to Y. Hassoumi (Rabat University) and to the Organizers of the Workshop QIQE'2018 in Al-Hoceima, Morocco, for valuable comments and questions which allowed to improve significantly the content of this review.

References

1. A.H. El Kinani, M. Daoud, Generalized intelligent states for an arbitrary quantum system. J. Phys. A Math. Gen. **34**, 5373–5387 (2001)
2. E.E. Hach III, P.M. Alsing, C.C. Gerry, Violations of a Bell inequality for entangled SU(1, 1) coherent states based on dichotomic observables. Phys. Rev. A **93**, 042104-1–042104-8 (2016)
3. S. Cruz y Cruz, Z. Gress, Group approach to the paraxial propagation of Hermite-Gaussian modes in a parabolic medium. Ann. Phys. **383**, 257–277 (2017)
4. S.E. Hoffmann, V. Hussin, I. Marquette, Y.-Z. Zhang, Non-classical behaviour of coherent states for systems constructed using exceptional orthogonal polynomials. J. Phys. A Math. Theor. **51**, 085202-1–085202-16 (2018)
5. K. Górska, A. Horzela, F.H. Szafraniec, Coherence, squeezing and entanglement: an example of peaceful coexistence, in J.-P. Antoine, F. Bagarello, J.P. Gazeau, eds. Coherent States and their applications: a contemporary panorama, in *Proceedings of the CIRM Workshop*, 13–18 Nov 2016. Springer Proceedings in Physics (SPPHY), vol. **205** (2018), pp. 89–117

6. E.E. Hach, R. Birrittella, P.M. Alsing, C.C. Gerry, SU(1, 1) parity and strong violations of a Bell inequality by entangled Barut-Girardello coherent states. J. Opt. Soc. Am. B **35**, 2433–2442 (2018)
7. R.J. Glauber, Photons correlations. Phys. Rev. Lett. **10**, 84–86 (1963)
8. J.-P. Gazeau, F.H. Szafraniec, Holomorphic Hermite polynomials and a non-commutative plane. J. Phys. A Math. Theor. **44**, 495201-1–495201-13 (2011)
9. W. Magnus, F. Oberhettinger, R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer, Berlin, 1966)
10. J. Schwinger, The theory of quantized fields. III. Phys. Rev. **91**, 728–740 (1953)
11. R.J. Glauber, The quantum theory of optical coherence. Phys. Rev. **130**, 2529–2539 (1963)
12. R.J. Glauber, Coherent and incoherent states of radiation field. Phys. Rev. **131**, 2766–2788 (1963)
13. E.C.G. Sudarshan, Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams. Phys. Rev. Lett. **10**, 277–279 (1963)
14. L. Mandel, E. Wolf, Coherence properties of optical fields. Rev. Mod. Phys. **37**, 231–287 (1965)
15. K.E. Cahill, R.J. Glauber, Ordered expansions in Boson amplitude operators. Phys. Rev. **177**, 1857–1881 (1969)
16. B.S. Agarwal, E. Wolf, Calculus for functions of noncommuting operators and general phase-space methods in quantum mechanics. Phys. Rev. D **2**, 2161–2186 (I), 2187–2205 (II), 2206–2225 (III) (1970)
17. E. Schrödinger, Der stetige Übergang von der Mikro- zur Makromechanik. Naturwiss **14**, 664 (1926)
18. J.R. Klauder, The action option and the Feynman quantization of spinor fields in terms of ordinary c-numbers. Ann. Phys. **11**, 123 (1960)
19. J.R. Klauder, Continuous-representation theory I. Postulates of continuous-representation theory. J. Math. Phys. **4**, 1055–1058 (1963)
20. J.R. Klauder, Continuous-representation theory II. Generalized relation between quantum and classical dynamics. J. Math. Phys. **4**, 1058–1073 (1963)
21. J.R. Klauder, B.S. Skagerstam (ed.), *Coherent States. Applications in Physics and Mathematical Physics* (World Scientific, Singapore, 1985)
22. A.M. Perelomov, Coherent states for arbitrary lie group. Commun. Math. Phys. **26**, 222–236 (1972)
23. A.M. Perelomov, *Generalized Coherent States and Their Applications* (Springer, Berlin, 1986)
24. W.-M. Zhang, D.H. Feng, R. Gilmore, Coherent states: theory and some applications. Rev. Mod. Phys. **26**, 867–927 (1990)
25. D.H. Feng, J.R. Klauder, M. Strayer (ed.) Coherent States: Past, Present and Future, in *Proceedings of the 1993 Oak Ridge Conference* (World Scientific, Singapore, 1994)
26. S.T. Ali, J.-P Antoine, J.-P. Gazeau, *Coherent States, Wavelets and their Generalizations* (2000), 2d edn., Theoretical and Mathematical Physics (Springer, New York, 2014)
27. V.V. Dodonov, 'Nonclassical' states in quantum optics: a 'squeezed' review of the first 75 years. J. Opt. B Quantum Semiclass. Opt. **4**, R1 (2002)
28. V.V. Dodonov, V.I. Man'ko (ed.), *Theory of Nonclassical States of Light* (Taylor & Francis, London, 2003)
29. A. Vourdas, Analytic representations in quantum mechanics. J. Phys. A **39**, R65 (2006)
30. J.-P. Gazeau, *Coherent States in Quantum Physics* (Wiley-VCH, Berlin, 2009)
31. S.T. Ali, J.P. Antoine, F. Bagarello, J.P. Gazeau, Special issue on coherent states: mathematical and physical aspects. J. Phys. A Math. Theor. **45** (2012)
32. J.-P. Antoine, F. Bagarello, J.P. Gazeau, Coherent States and their applications: a contemporary panorama, in *Proceedings of the CIRM Workshop*, 13–18 Nov 2016. Springer Proceedings in Physics (SPPHY), vol. **205** (2018)
33. N. Cotfas, J.-P. Gazeau, K. Górska, Complex and real Hermite polynomials and related quantizations. J. Phys. A Math. Theor. **43**, 305304-1–305304-14 (2010)

34. S.T. Ali, F. Bagarello, J.-P. Gazeau, Quantizations from reproducing kernel spaces. Ann. Phys. **332**, 127–142 (2012)
35. J.-P. Gazeau, M.A. del Olmo, Pisot q-coherent states quantization of the harmonic oscillator. Ann. Phys. **330**, 220–245 (2013)
36. A. De Sole, V. Kac, On integral representations of q-gamma and q-beta functions. Rend. Mat. Acc. Lincei **9**, 11–29 (2005). ArXiv: math.QA/0302032
37. M. El Baz, R. Fresneda, J.-P. Gazeau, Y. Hassouni, Coherent state quantization of paragrassmann algebras. J. Phys. A Math. Theor. **43**, 385202-1–385202-15 (2010); Corrigendum J. Phys. A Math. Theor. **45**, 079501-1–079501-2 (2012)
38. M. Fox, *Quantum Optics: An Introduction* (Oxford University, New York, 2006)
39. S.T. Ali, J.-P. Gazeau, B. Heller, Coherent states and Bayesian duality. J. Phys. A Math. Theor. **41**, 365302-1–365302-22 (2008)
40. J.-P. Gazeau, E. Huguet, M. Lachièze-Rey, J. Renaud, Fuzzy spheres from inequivalent coherent states quantizations. J. Phys. A Math. Theor. **40**, 10225–10249 (2007)
41. P. Jordan, Der Zusammenhang der symmetrischen und linearen Gruppen und das Mehrkörperproblem". Z. Phys. **94**, 531–535 (1935)
42. T. Holstein, H. Primakoff, Phys. Rev. **58**, 1098–1113 (1940)
43. J. Schwinger, On Angular Momentum, Unpublished Report, Harvard University, Nuclear Development Associates, Inc., United States Department of Energy (through predecessor agency the Atomic Energy Commission), Report Number NYO-3071 (1952).
44. J.-P. Gazeau, M. del Olmo, Covariant integral quantization of the unit disk, submitted (2018). ArXiv:1810.10399 [math-ph]
45. Y. Aharonov, E.C. Lerner, H.W. Huang, J.M. Knight, Oscillator phase states, thermal equilibrium and group representations. J. Math. Phys. **14**, 746–755 (2011)
46. A. O. Barut, L. Girardello, New "Coherent" states associated with non-compact groups. Commun. Math. Phys. **21**, 41–55 (1971)
47. J.-P. Antoine, J.-P. Gazeau, J.R. Klauder, P. Monceau, K.A. Penson, J. Math. Phys. **42**, 2349–2387 (2001)
48. L. Susskind, J. Glogower, Quantum mechanical phase and time operator. Phys. Phys. Fiz. 1 **1**, 49–61 (1964)
49. H.M. Moya-Cessa, F. Soto-Eguibar, *Introduction to Quantum Optics* (Rinton, Paramus, 2011)
50. E.M.F. Curado, S. Faci, J.-P. Gazeau, D. Noguera, *in progress*.
51. H. Bergeron, E.M.F. Curado, J.-P. Gazeau, Ligia M.C.S. Rodrigues, Symmetric generalized binomial distributions. J. Math. Phys. **54**, 123301-1–123301-22 (2013)
52. E.M.F. Curado, J.-P. Gazeau, Ligia M.C.S. Rodrigues, Nonlinear coherent states for optimizing quantum information. Phys. Scr. **82**, 038108-1–038108-9 (2010)
53. E.M.F. Curado, J.-P. Gazeau, Ligia M.C.S. Rodrigues, On a generalization of the binomial distribution and its Poisson-like limit. J. Stat. Phys. **146**, 264–280 (2012)
54. H. Bergeron, E.M.F. Curado, J.-P. Gazeau, Ligia M.C.S. Rodrigues, Generating functions for generalized binomial distributions. J. Math. Phys. **53**, 103304-1–103304-22 (2012)
55. L. Mandel, Fluctuations of photons beams and their correlations. Proc. Phys. Soc. (London) **72**, 1037–1048 (1958); Fluctuations of photon beams: the distribution of photoelectrons. Proc. Phys. Soc. **74**, 233–243 (1959)
56. L. Mandel, E. Wolf, *Selected Papers on Coherence and Fluctuations of Light*, vols. 1, 2 (Dover, New York, 1970)
57. D.N. Klyshko, Observable signs of nonclassical light. Phys. Lett. A **213**, 7–15 (1996)
58. R. Loudon, *The Quantum Theory of Light*, 3rd edn. (Oxford University, Oxford 2000)
59. P. Koczyk, P. Wiewior, C. Radzewicz, Photon counting statistics - undergraduate experiment. Am. J. Phys. **64**(1996), 240–245 (1996)
60. C. Gerry, P. Knight, *Introductory Quantum Optics* (Cambridge University, Cambridge, 2004)
61. H.A. Bachor, T.C. Ralph, *A Guide to Experiments in Quantum Optics* (Wiley-VCH, Weinheim, 2004)

62. M.D. Eisaman, J. Fan, A. Migdall, S.V. Polyakov, Single-photon sources and detectors (Invited Review Article). Rev. Sci. Instrum. **82**, 071101-25 (2011)
63. C. Huerta Alderete, Liliana Villanueva Vergara, B.M. Rodríguez-Lara, Nonclassical and semiclassical para-Bose states. Phys. Rev. A **95**, 043835-1–043835-7 (2017)
64. P.A.M. Dirac, The quantum theory of emission and absorption of radiation. Proc. R. Soc. Lond. A **114**, 243–265 (1927)