Chapter 32
Block Element with a Circular Boundary

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Abstract The block element method (Babeshko in Ecol Bull Res Centers Black Sea Econ Coop (BSEC) 2:13, 2014, [1]), which was elaborated by V. A. Babeshko and developed by his disciples, makes it possible to solve boundary-value problems stated for arbitrary convex domains and described in terms of arbitrary linear systems of differential equations in finite-order partial derivatives. The boundary-value problems are assumed to be properly stated. The domains can be bounded, semi-bounded and unbounded. Moreover, the method can be applied to the finite aggregate of domains, which interact over shared boundaries. According to the block element method, the boundary-value problem is set in the topological space and reduced to the system of functional equations using external analysis tools. From the pseudo-differential equations obtained due to differential factorization and automorphism, integral equations are extracted, which correspond to certain boundary conditions. A large number of variations of the block element method have been developed so far for the three-dimensional setting of boundary-value problems for various mathematical physics equations. The method is highly versatile, however, it requires the knowledge of topology, differential geometry, multidimensional complex analysis, external analysis, i.e. the disciplines, which are normally not included in the standard mathematics curricular at the university level. To increase the comprehension of the method, it is necessary for scientific and methodological purposes to study in detail how comparatively simple problems are solved by this method. The research provides a detailed algorithm of the block element for the model boundary-value problem, which is described by the Bessel equation for the circular domain.

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32.1 Statement and Analytical Solution of the Boundary-Value Problem

Let us consider for the circular domain Ω with the radius $r_0 > 0$, the equation:

$$\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} - k^2 \varphi = 0, \quad (x_1, x_2) \in \Omega. \tag{32.1}$$

Hereinafter, we will consider the boundary-value solutions:

$$\varphi|_{r=r_0} = f_1, \left.\frac{\partial \varphi}{\partial r}\right|_{r=r_0} = f_2, \tag{32.2}$$

which depend only on the radius $r = \sqrt{x_1^2 + x_2^2}$. Equation (32.1) for the function, which depends only on the radius r, can be written as follows:

$$\frac{\partial^2 \varphi}{\partial \mathrm{r}^2} + \frac{1}{r}\frac{\partial \varphi}{\partial \mathrm{r}} - k^2 \varphi = 0. \tag{32.3}$$

The general solution of the problem (32.3), (32.2) is written in terms of modified Bessel functions I_0, K_0 [2] as

$$\varphi(r) = C_1 I_0(kr) + C_2 K_0(kr). \tag{32.4}$$

Considering the known properties of functions I_0, K_0 [2]:

$$I_0'(z) = I_1(z), I_0(z) = J_0(iz), I_1(z) = -iJ_1(iz),$$
$$K_0'(z) = -K_1(z), K_0(z) = \frac{i\pi}{2} H_0^{(1)}(iz), K_1(z) = \frac{-\pi}{2} H_1^{(1)}(iz),$$

with respect to C_1, C_2, we obtain the equation system:

$$C_1 I_0(kr_0) + C_2 K_0(kr_0) = f_1, C_1 k I_1(kr_0) - C_2 k K_1(kr_0) = f_2,$$

whence it follows that

$$C_1 = \frac{-f_1 k K_1(kr_0) + f_2 K_0(kr_0)}{\Delta}, \quad C_2 = \frac{-f_1 k I_1(kr_0) + f_2 I_0(kr_0)}{\Delta},$$
$$\Delta = \Delta(k, r_0) = -k(I_0(kr_0) K_1(kr_0) + K_0(kr_0) I_1(kr_0)).$$

It is easily to show that

$$\Delta = k\left(\frac{\pi}{2}\right)\left(J_0(ikr_0)H_1^{(1)}(ikr_0) - J_1(ikr_0)H_1^{(1)}(ikr_0)\right) \equiv -\frac{1}{r_0}.$$

In this case, we obtain:

$$C_1 = r_0\left(\frac{i\pi}{2}\right)\left(f_1 ikH_1^{(1)}(ikr_0) + f_2H_0^{(1)}(ikr_0)\right), C_2 = r_0(-f_1 ikJ_1(ikr_0) - f_2J_0(ikr_0)).$$

The general representation of solving a boundary-value problem (32.3), (32.2) takes the form:

$$\begin{aligned}\varphi(r) = J_0(ikr)\left(\frac{i\pi r_0}{2}\right)\left(f_1 ikH_1^{(1)}(ikr_0) + f_2H_0^{(1)}(ikr_0)\right)\\ + H_0^{(1)}(ikr)\left(\frac{i\pi r_0}{2}\right)(-f_1 ikJ_1(ikr_0) - f_2J_0(ikr_0)).\end{aligned} \tag{32.5}$$

Solution (32.5) consists of two summands:

$$\varphi(r) = \varphi^{(1)}(r) + \varphi^{(2)}(r).$$

Solution $\varphi^{(1)}$ is bounded by

$$\varphi^{(1)}(r) = C_1J_0(ikr) = J_0(ikr)\left(\frac{i\pi r_0}{2}\right)\left(f_1 ikH_1^{(1)}(ikr_0) + f_2H_0^{(1)}(ikr_0)\right). \tag{32.6}$$

Solution $\varphi^{(2)}$ is not bounded, because it has a singularity at zero due to properties of Hankel function:

$$\varphi^{(2)}(r) = C_2H_0^{(1)}(ikr) = H_0^{(1)}(ikr)\left(\frac{i\pi r_0}{2}\right)(-f_1 ikJ_1(ikr_0) - f_2J_0(ikr_0)). \tag{32.7}$$

Expression (32.5) provides a general representation of the solution for an internal problem with boundary conditions (32.2), as well as for the corresponding external problem at $r \geq r_0$, which is stated below.

Let us additionally restrict the solution of the boundary-value problem (32.3), (32.2). For the internal problem, this restriction leads to $\varphi^{(2)}(r) \equiv 0 \leftrightarrow C_2 \equiv 0$, which is equal to the condition

$$\frac{f_1}{f_2} = \frac{i}{k}\frac{J_0(ikr_0)}{J_1(ikr_0)}. \tag{32.8}$$

Whereas $\lim_{kr_0\to\infty}\frac{J_0(ikr_0)}{J_1(ikr_0)} = -i$, then, at $k >> 1$

$$\frac{f_1}{f_2} \approx \frac{1}{k}, \quad 0 \le r \le r_0, \tag{32.9}$$

which is equivalent to the boundary conditions:

$$\varphi|_{r=r_0} = f, \left.\frac{\partial\varphi}{\partial r}\right|_{r=r_0} = kf, \quad f = \text{const} \ne 0. \tag{32.10}$$

Similarly, in order to restrict the external problem, the condition $\varphi^{(1)}(r) \equiv 0 \leftrightarrow C_1 \equiv 0$ should be satisfied, then:

$$\frac{f_1}{f_2} = \frac{i}{k}\frac{H_0^{(1)}(ikr_0)}{H_1^{(1)}(ikr_0)}. \tag{32.11}$$

Taking into account that

$$\lim_{kr_0\to\infty}\frac{H_0^{(1)}(ikr_0)}{H_1^{(1)}(ikr_0)} = i, \tag{32.12}$$

then, at $k >> 1$

$$\frac{f_1}{f_2} \approx -\frac{1}{k}, \quad 0 < r_0 \le r, \tag{32.13}$$

Therefore, at $k >> 1$, it is possible to use the expressions (32.10) as approximated boundary conditions for the internal problem, and relations:

$$\varphi|_{r=r_0} = f, \left.\frac{\partial\varphi}{\partial r}\right|_{r=R_0} = -kf, \quad 0 < r_0 \le r, \tag{32.14}$$

for the external problem. The restriction requirement results in the fact that the values f_1, f_2 cannot be arbitrary and must satisfy (32.8) or (32.11). At large k, exact terms (32.8) and (32.11) can be replaced by approximated conditions (32.10) and (32.14), respectively.

32.2 Block Element with a Circular Boundary

According to the general algorithm of the block element method [1] in the case of a two-dimensional domain Ω, we introduce the double forward and inverse Fourier transforms:

$$\begin{aligned}\Phi(\alpha_1,\alpha_2) &= \iint\limits_{\Omega} \varphi(x_1,x_2)\ \exp(i(\alpha_1 x_1+\alpha_2 x_2))\mathrm{d}x_1\mathrm{d}x_2\\ \varphi(x_1,x_2) &= \frac{1}{4\pi^2}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \Phi(\alpha_1,\alpha_2)\exp(-i(\alpha_1 x_1+\alpha_2 x_2))\mathrm{d}x_1\mathrm{d}x_2\end{aligned} \tag{32.15}$$

Let us apply the forward transform (32.15) to the (32.1) and the Stokes formula [3] to the surface integral, and obtain the functional equation:

$$-\left(\alpha_1^2+\alpha_2^2+k^2\right)\Phi(\alpha) = \iint\limits_{\Omega} \mathrm{d}\omega = \int\limits_{\partial\Omega} \omega, \{x_1,x_2\}\in\partial\Omega. \tag{32.16}$$

Here $\partial\Omega$ is the boundary of the domain Ω, ω is the external differential form. As a result, we obtain:

$$\left(\alpha_1^2+\alpha_2^2+k^2\right)\Phi(\alpha_1,\alpha_2) = \int\limits_{\partial\Omega}\left\{\left[\frac{\partial\varphi}{\partial x_2}-i\alpha_2\varphi\right]\mathrm{e}^{(i\alpha_1x_1+i\alpha_2x_2)}\mathrm{d}x_1+\left[\frac{\partial\varphi}{\partial x_1}-i\alpha_1\varphi\right]\mathrm{e}^{(i\alpha_1x_1+i\alpha_2x_2)}\mathrm{d}x_2\right\}. \tag{32.17}$$

Let us introduce the following coordinates:

$$\begin{aligned}&\xi_1 = r_0\cos\psi,\quad \xi_2 = r_0\sin\psi,\quad r_0=\sqrt{\xi_1^2+\xi_2^2},\quad \psi = arctg(\xi_2/\xi_1),\\ &x_1 = r\cos\theta,\quad x_2 = r\sin\theta,\quad r=\sqrt{x_1^2+x_2^2},\quad \theta = arctg(x_2/x_1),\\ &\alpha_1 = u\cos\gamma,\quad \alpha_2 = u\sin\gamma,\quad u=\sqrt{\alpha_1^2+\alpha_2^2},\quad \gamma = arctg(\alpha_2/\alpha_1).\end{aligned} \tag{32.18}$$

After some uncomplicated transformations, we obtain

$$\begin{aligned}\left(u^2+k^2\right)\Phi(u,\gamma) = &\int\limits_{\partial\Omega} \exp(iur_0\cos(\psi-\gamma))\times\\ &\times\left(\frac{\partial\varphi}{\partial r}r_0\mathrm{d}\psi-\frac{1}{r_0}\frac{\partial\varphi}{\partial\psi}\mathrm{d}r - iur_0\cos(\gamma-\psi)\varphi\mathrm{d}\psi+iu\sin(\gamma-\psi)\varphi\mathrm{d}r\right).\end{aligned} \tag{32.19}$$

Integrals over dr in (32.19) are equal to zero, while when the circle d$r = 0$, only integrals remain over dψ. As a result, we obtain:

$$\begin{aligned}&\left(u^2+k^2\right)\Phi(u,\gamma)\\&=\int\limits_{\partial\Omega}\exp(ir_0u\cos(\gamma-\psi))\left[r_0\frac{\partial\varphi(r_0,\psi)}{\partial r}-iur_0\cos(\gamma-\psi)\varphi(r_0,\psi)\right]\mathrm{d}\psi.\end{aligned}\tag{32.20}$$

Let us write the formula (32.20) as follows:

$$\Phi(\alpha_1,\alpha_2)=\int\limits_{\partial\Omega}\frac{\exp(i(\alpha_1\xi_1+\alpha_2\xi_2))}{(\alpha_1^2+\alpha_2^2+k^2)}\left[r_0\frac{\partial\varphi(r_0,\psi)}{\partial r}-iur_0\cos(\gamma-\psi)\varphi(r_0,\psi)\right]\mathrm{d}\xi_1\mathrm{d}\xi_2.\tag{32.21}$$

Next, let us apply the inverse Fourier transform (32.15) to the expression (32.21) and obtain

$$\begin{aligned}\varphi(r,\theta)&=\frac{1}{4\pi^2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{\partial\Omega}\frac{\omega}{(u^2+k^2)}\exp(-i(\alpha_1x_1+\alpha_2x_2))\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2\\&=\frac{1}{4\pi^2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{\partial\Omega}\frac{\exp(-i(\alpha_1(x_1-\xi_1)+\alpha_2(x_2-\xi_2)))}{(u^2+k^2)}\\&\quad\times\left[r_0\frac{\partial\varphi(r_0,\psi)}{\partial r}-iur_0\cos(\gamma-\psi)\varphi(r_0,\psi)\right]\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2.\end{aligned}\tag{32.22}$$

The transformations of the integral (32.22), which are described below in Appendix, result in

$$\varphi(r)=r_0\int\limits_0^{\infty}\frac{J_0(ur)u}{(u^2+k^2)}(J_0(ur_0)f_2+J_1(ur_0)uf_1)\mathrm{d}u.\tag{32.23}$$

The expression (32.23) is an integral representation of the restricted solution of the problem (32.3), (32.2), which can be used for calculations, when f_j are known. Using the contour unfolding operation [4], we can calculate this integral accurately applying the residues theory. Let us write the integral (32.23) as follows:

$$\varphi(r) = \int_0^\infty (J_0(ur_0)F_0(u) + J_1(ur_0)F_1(u))u\mathrm{d}u$$
$$= \int_0^\infty J_0(ur_0)uF_0(u)\mathrm{d}u + \int_0^\infty J_1(ur_0)uF_1(u)\mathrm{d}u,$$

where $F_0(u) = f_2 \frac{J_0(ur)r_0}{(u^2+k^2)}, F_1(u) = f_1 \frac{J_1(ur)ur_0}{(u^2+k^2)}$. Functions F_0, F_1 have the property: $F_n(-u) = (-1)^n F_n(u), n = 0, 1$. When we put down the Bessel functions $J_0(z), J_1(z)$ as a sum of the Hankel functions $J_n(z) = \frac{1}{2}\left(H_n^{(1)}(z) + H_n^{(2)}(z)\right)$ and apply the contour unfolding operation, we obtain:

$$\varphi(r) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{H_0^{(1)}(ur_0)}{(u^2+k^2)} uJ_0(ur)f_2 r_0 \mathrm{d}u + \frac{1}{2}\int_{-\infty}^{\infty} \frac{H_1^{(1)}(ur_0)}{(u^2+k^2)} uJ_0(ur)uf_1 r_0 \mathrm{d}u. \quad (32.24)$$

The exponential decrease of the Hankel functions $H_j^{(1)}$ makes it possible to close the contour of integration upwards in the upper half-plane and obtain the integral value (32.24) through the residue in the pole $u = ik$ as

$$\varphi(r) = J_0(ikR)\left(\frac{i\pi r_0}{2}\right)\left(f_1 ikH_1^{(1)}(ikr_0) + f_2 H_0^{(1)}(ikr_0)\right) = \varphi^{(1)}(r), \quad 0 \le r \le r_0 \quad (32.25)$$

It is obvious that the latter expression agrees closely with the bounded solution of the inner problem (32.6). Let us show that at $0 < r_0 < r$, the integral (32.23) is equal to zero. In fact, it is possible to put down the function $J_0(ur)$ as a sum of the Hankel functions and turn the contour as shown above. By closing the contour at $0 < r_0 < r$ in the upper half-plane and by calculating the integral through residue in the pole $u = ik$, we obtain:

$$\varphi(r) = \frac{r_0}{2}\int_{-\infty}^{+\infty} \frac{H_0^{(1)}(ur)u}{(u^2+k^2)}(J_0(ur_0)f_2 + J_1(ur_0)uf_1)\mathrm{d}u =$$
$$= H_0^{(1)}(ikr)\left(\frac{i\pi r_0}{2}\right)\left(-f_1 ikJ_1(ikr_0) - f_2 J_0(ikr_0)\right). \quad (32.26)$$

The factor at the function $H_0^{(1)}$ in (32.26) is equal to zero according to the condition (32.8). In the case of an approximate solution (32.9), the right-hand side of (32.26) will be also equal to zero, approximately. Since $H_0^{(1)}(ikr)$ decreases exponentially with the growth of r, then $\varphi(r) \approx \exp(-k(r - r_0))$ at large r. Moreover, it is obvious that integral (32.26) describes the restricted solution

(32.7) of the corresponding external problem, where f_1, f_2 and k are bounded by the relations (32.11) or (32.13). The estimations for the internal problem, which are shown below, are identical with the estimations for the external problem, therefore, we do not consider them separately.

The integrals (32.23), (32.24) represent the block element for the boundary-value problem (32.3), (32.2). After the values f_1, f_2 are found accurately or approximately, the integral (32.23) can be calculated numerically with a high accuracy, e.g. by means of the integrating algorithms of strongly oscillating functions [5]. In this case, the integral (32.23) using the contour unfolding procedure, which is described above, can be calculated accurately using the theory of residues.

32.3 Approximate Solutions

In the case under consideration, we obtain boundary values f_1, f_2 by applying an analytical solution (32.8), which is not, understandably, always possible. When the second value is given, the block-element method makes is possible to find (accurately or approximately) one of the boundary values $\left(\varphi \text{ or } \frac{\partial \varphi}{\partial r}\right)$ from the solutions of corresponding equations, which are described further.

Let us decompose the unity and introduce local coordinate systems α^μ, x^μ [6]. As a result, the integrals over $\mathrm{d}x_2$ will retreat, and only integrals over $\mathrm{d}x_1$ will remain in the integral (32.17). We then obtain

$$\int\limits_{\partial\Omega} \left[\frac{\partial \varphi}{\partial x_2} - i\alpha_2 \varphi\right] \mathrm{e}^{(i\alpha_1 x_1 + i\alpha_2 x_2)} \mathrm{d}x_1 = \sum_{\mu} \varepsilon_\mu \left[\frac{\partial \varphi_\mu}{\partial x_2^\mu} - i\alpha_2^\mu \varphi_\mu\right] \exp\left(i\alpha_1^\mu x_1^\mu + i\alpha_2^\mu x_2^\mu\right) \Delta x_1^\mu$$

The characteristic equation in Fourier terms takes the form $(\alpha_1^2 + \alpha_2^2 + k^2) = 0$. In any new coordinates, we obtain $\left(\alpha_1^\mu\right)^2 + \left(\alpha_2^\mu\right)^2 + k^2 = 0$. Let us indicate the roots of the characteristic polynomial $\alpha_{2\pm}^\mu$, where «+» stands for the upper half-plane and «−» stands for the lower one. Then

$$\alpha_{2+}^\mu = +i\sqrt{\left(\alpha_1^\mu\right)^2 + k^2}, \quad \alpha_{2-}^\mu = -i\sqrt{\left(\alpha_1^\mu\right)^2 + k^2}.$$

The integral equation set to determine the unknown quantities $\frac{\partial \varphi_\mu}{\partial x_2^\mu}, \varphi_\mu$, which are constant in any coordinate system, can be written as follows:

$$\int\limits_{-\varepsilon}^{\varepsilon} \left(\frac{\partial \varphi_\mu}{\partial x_2^\mu} - i\alpha_{2-}^\mu \varphi_\mu\right) \exp\left(i\alpha_1^\mu x_1^\mu\right) \mathrm{d}x_1^\mu = 0$$

When $\varphi_\mu = f$, the degenerate solution of the integral equation can be written as

$$\frac{\partial \varphi_\mu}{\partial x_2^\mu} - \sqrt{\left(\alpha_1^\mu\right)^2 + k^2} \cdot f = 0, \quad \alpha_1^\mu = 0.$$

It follows that

$$\frac{\partial \varphi_\mu}{\partial x_2^\mu} = f\sqrt{k^2} = f \cdot k. \tag{32.27}$$

Thus, we obtain approximate solutions of the function $\varphi_\mu = f$ and the derivative $\frac{\partial \varphi_\mu}{\partial x_2^\mu} = kf$. It is possible to obtain a more accurate solution of the integral equation by using the Fourier transform, i.e.

$$\int_{-\varepsilon}^{\varepsilon} \frac{\partial \varphi_\mu}{\partial x_2^\mu} \exp\left(i\alpha_1^\mu x_1^\mu\right) \mathrm{d}x_1^\mu = i\alpha_{2-}^\mu \cdot f \cdot \frac{\mathrm{e}^{i\alpha_1^\mu \varepsilon} - \mathrm{e}^{-i\alpha_1^\mu \varepsilon}}{i\alpha_1^\mu}.$$

We therefore obtain approximately:

$$\frac{\partial \varphi_\mu}{\partial x_2^\mu} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} i\alpha_{2-}^\mu \cdot f \cdot \frac{\mathrm{e}^{i\alpha_1^\mu \varepsilon} - \mathrm{e}^{-i\alpha_1^\mu \varepsilon}}{i\alpha_1^\mu} \exp\left(-i\alpha_1^\mu x_1^\mu\right) \mathrm{d}\alpha_1^\mu. \tag{32.28}$$

In Table 32.1, the solutions of (32.8) $f_2^{(1)}$, approximated solution of the integral equation (32.28) $f_2^{(2)}$, and the degenerate solution (32.27) $f_2^{(3)}$ are compared depending on $1 \le k \le 10$ at $r_0 = 1, f_1 = 1$. It should be noted that solutions of (32.27) and (32.28) are obtained without using analytical solution.

Table 32.1 Values $f_2^{(j)}(k)$ at $1 \le k \le 10$

k	Solution of (32.8) $f_2^{(1)}$	Solution of (32.28) $f_2^{(2)}$	Solution of (32.27) $f_2^{(3)}$
1	0.4463900	0.7176216	1.0
2	1.395549	1.689416	2.0
3	2.429956	2.686028	3.0
4	3.454090	3.685591	4.0
5	4.466916	4.685533	5.0
6	5.474156	5.685527	6.0
7	6.478725	6.685524	7.0
8	7.481884	7.685526	8.0
9	8.484209	8.685524	9.0
10	9.485998	9.685521	10.0

Figure 32.1a shows an example of the bounded solution of the internal boundary-value problem (32.3), (32.2): $\varphi^{(1)}(r,k)$ depending on r and k, $(f_1 = 1, r_0 = 1)$. The difference between the exact solution (32.6) and the numerical calculation of the integral (32.23) are unobservable within the scale of the picture.

Let us show that the integral representation (32.23) is numerically stable. Let f_1 be set exactly, while f_2 is set approximately and $\tilde{f}_2 = f_2 - \varepsilon_2$, where ε_2 is the absolute error of the solution f_2. Then, the absolute error is as follows:

$$\varepsilon(r,k) = \varphi(r) - \tilde{\varphi}(r) = \int\limits_0^\infty \frac{J_0(ur)u}{(u^2+k^2)}\varepsilon_2 J_0(ur_0)\mathrm{d}u = \varepsilon_2 J_0(ikr)\left(\frac{i\pi r_0}{2}\right)H_0^{(1)}(ikr_0).$$

At large $k >> 1$

$$\varepsilon(r,k) \approx \varepsilon_2 \exp[-k(r_0-r)], \quad 0<r<r_0. \tag{32.29}$$

The relative error δ

$$\delta(k) = \left(1 - \frac{\tilde{\varphi}(r)}{\varphi(r)}\right) = \frac{H_0^{(1)}(ikr_0)\varepsilon_2}{\left(f_1 ikH_1^{(1)}(ikr_0) + f_2 H_0^{(1)}(ikr_0)\right)} = \frac{\varepsilon_2}{f_1 ik\frac{H_1^{(1)}(ikr_0)}{H_0^{(1)}(ikr_0)} + f_2}.$$

Since the properties of (32.27), (32.12) we obtain at $k\;\; >> 1$:

$$\delta(k) \approx \frac{\varepsilon_2}{f_1 k + f_1 k} \approx \frac{\varepsilon_2}{2fk}. \tag{32.30}$$

The obtained estimations of the absolute and relative errors (32.29), (32.30) mean the stability of the solution (32.23) at non-zero errors of derivative f_2, which was obtained by formula (32.28) or (32.27). These estimations are obtained by assuming that the integral (32.23) can be calculated accurately. In practice, when we calculate the integral (32.23) numerically, the mean absolute and relative errors will be unavoidably higher, since any computing algorithm leads to additional errors. Let us introduce relative errors $\delta^{(n)}$ as

$$\delta^{(n)}(k) = \frac{1}{r_0}\int\limits_0^{r_0} |1 - \varphi_n(r,k)/\varphi(r,k)|\mathrm{d}r. \tag{32.31}$$

Here, φ_1 corresponds to the accurate solution (32.6) with a mean absolute error $\varepsilon_2^{(1)} \approx 10^{-6}$, φ_2 corresponds to the approximate solution of the integral equation (32.28) with $\varepsilon_2^{(2)} \approx 0.23$, φ_3 corresponds to the degenerate solution of the integral equation (32.27) with $\varepsilon_2^{(3)} \approx 0.54$.

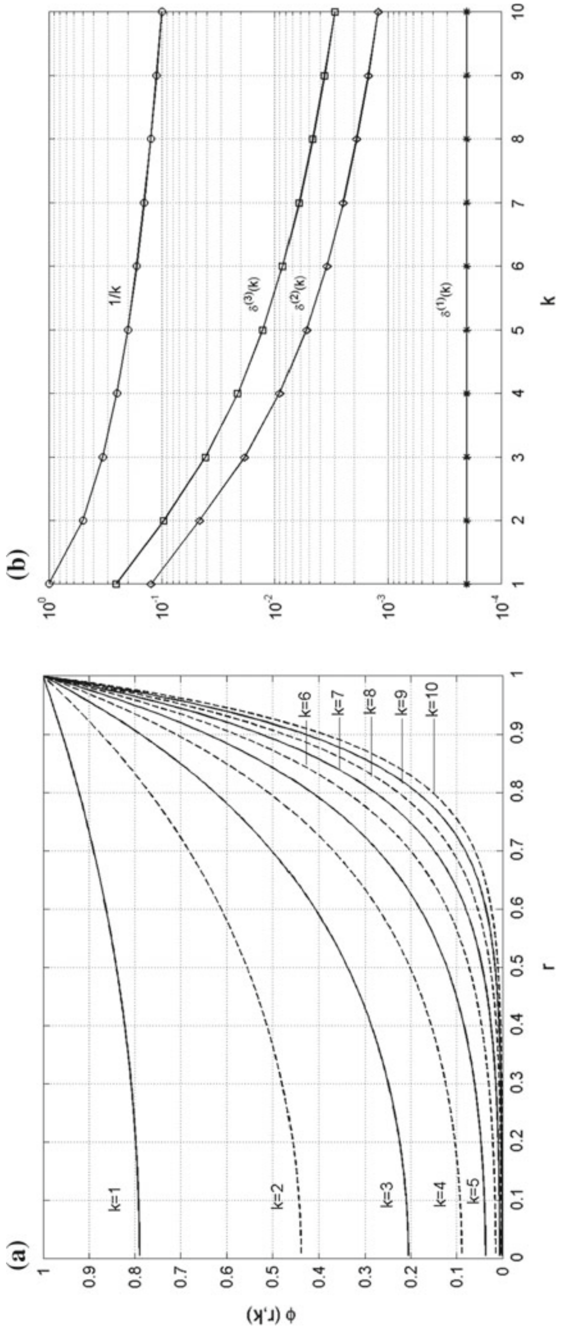

Fig. 32.1 **a** Solutions $\varphi^{(1)}(k,r), (f_1 = 1, r_0 = 1)$; **b** average relative errors $\delta^{(n)}(k)$ from (32.31)

In Fig. 32.1b, the values $\delta^{(n)}$ are shown on the logarithmical scale in comparison with the value $1/k$. As is obvious, at small $k \leq 10$ the average relative errors (32.31) decrease with the growth of k even more rapidly than the asymptotic estimation (32.30).

32.4 Conclusion

The work describes the algorithm for solving a model boundary-value problem, which corresponds to the Bessel equation for the circular domain, using the block element method. The boundary-value problem has a simple analytical solution, which makes it possible to compare the accurate and approximate solution, which is obtained using the block element method. The solution is represented as an improper integral of the Bessel functions, which can be easily calculated in quadratures and can be calculated exactly using the residues theory. The unknown coefficients of the exterior form can be found by solving the integral equation. The results of the numerical solution of the integral equation are demonstrated in the work, as well as practical and theoretical estimations of absolute and ratio solution errors obtained using the block element method and their solution stability.

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Appendix

In the appendix, transformations of a multidimensional integral:

$$\begin{aligned}\varphi(r,\theta) = \frac{1}{4\pi^2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{\partial\Omega}\frac{\exp(-i(\alpha_1(x_1-\xi_1)+\alpha_2(x_2-\xi_2)))}{(u^2+k^2)} \\ \times\left[r_0\frac{\partial\varphi(r_0,\psi)}{\partial r} - iur_0\cos(\gamma-\psi)\varphi(r_0,\psi)\right]\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2,\end{aligned} \tag{32.32}$$

into the integral of a single variable are present. Despite the awkwardness of the stated expressions, the transformations are in principle simple. The relations:

$$\exp(-iz\cos\tau) = \sum_{k=-\infty}^{+\infty} \exp\left(-ik\left(-\tau-\frac{\pi}{2}\right)\right)J_{-k}(-z)$$
$$= \sum_{k=-\infty}^{+\infty} \exp\left(ik\left(-\tau-\frac{\pi}{2}\right)\right)J_k(z) = \sum_{k=-\infty}^{+\infty} \exp\left(ik\left(\tau-\frac{\pi}{2}\right)\right)J_k(z), \tag{32.33}$$

are used further. When we transform the exponent in formula 32.32), we obtain a product of series, which can be written down as a two-fold series:

$$\exp(-i[\alpha_1(x_1-\xi_1)+\alpha_2(x_2-\xi_2)]) = \exp(-iur\cos(\gamma-\theta))\exp(iur_0\cos(\gamma-\psi))$$
$$= \sum_{m=-\infty}^{+\infty} \exp(im(\theta-\gamma-\frac{\pi}{2}))J_m(ur)\sum_{n=-\infty}^{+\infty} \exp(in(\psi-\gamma+\frac{\pi}{2}))J_n(ur_0)$$
$$= \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} \exp(im(\theta-\gamma-\frac{\pi}{2}))J_m(ur)\exp(in(\psi-\gamma+\frac{\pi}{2}))J_n(ur_0). \tag{32.34}$$

Then

$$\varphi(r,\theta) = \frac{1}{4\pi^2}\int_0^{\infty}\int_0^{2\pi}\int_0^{2\pi} \frac{\sum\limits_{m=-\infty}^{+\infty}\sum\limits_{n=-\infty}^{+\infty}}{(u^2+k^2)} \exp(im(\theta-\gamma-\tfrac{\pi}{2}))J_m(ur)\exp(in(\psi-\gamma+\tfrac{\pi}{2}))J_n(ur_0)$$
$$\times\left[r_0\frac{\partial\varphi(r_0,\psi)}{\partial r} - \frac{iur_0}{2}(\exp i(\gamma-\psi) + \exp(-i(\gamma-\psi)))\varphi(r_0,\psi)\right]u\mathrm{d}\psi\mathrm{d}\gamma\mathrm{d}u. \tag{32.35}$$

We further assume that:

$$\varphi(r_0,\psi) = \varphi(r_0)\exp(i\lambda\psi). \tag{32.36}$$

We obtain:

$$\varphi(r,\theta) = \frac{1}{4\pi^2}\int_0^{\infty}\int_0^{2\pi}\int_0^{2\pi} \frac{\sum\limits_{m=-\infty}^{+\infty}\sum\limits_{n=-\infty}^{+\infty}}{(u^2+k^2)} \exp\left(im\left(\theta-\gamma-\tfrac{\pi}{2}\right)\right)J_m(ur)\exp(i\psi(n+\lambda))\exp\left(in\left(-\gamma+\tfrac{\pi}{2}\right)\right)$$
$$\times J_n(ur_0)\left[r_0\frac{\partial\varphi(r_0)}{\partial r} - \frac{iur_0}{2}(\exp(-i\psi)\exp(i\gamma) + \exp(i\psi)\exp(-i\gamma))\varphi(r_0)\right]u\mathrm{d}\psi d\gamma\mathrm{d}u. \tag{32.37}$$

Let us write down this integral as a sum of three integrals:

$$\begin{aligned}\varphi(r,\theta) = \frac{1}{4\pi^2}\int_0^\infty\int_0^{2\pi}\int_0^{2\pi}\sum_{m=-\infty}^{+\infty}F_m\sum_{n=-\infty}^{+\infty}\exp(i\psi(n+\lambda))\\ \times\exp\Big(in\Big(-\gamma+\frac{\pi}{2}\Big)\Big)J_n(ur_0)\left[R_0\frac{\partial\varphi(r_0)}{\partial r}\right]u\mathrm{d}\psi\mathrm{d}\gamma\mathrm{d}u\\ +\frac{1}{4\pi^2}\int_0^\infty\int_0^{2\pi}\int_0^{2\pi}\sum_{m=-\infty}^{+\infty}F_m\sum_{n=-\infty}^{+\infty}\exp(i\psi(n+\lambda-1))\\ \times\exp\Big(in\Big(-\gamma+\frac{\pi}{2}\Big)\Big)J_n(ur_0)\left[-\frac{iur_0}{2}\exp(i\gamma)\varphi(r_0)\right]u\mathrm{d}\psi\mathrm{d}\gamma\mathrm{d}u\\ +\frac{1}{4\pi^2}\int_0^\infty\int_0^{2\pi}\int_0^{2\pi}\sum_{m=-\infty}^{+\infty}F_m\sum_{n=-\infty}^{+\infty}\exp(i\psi(n+\lambda+1))\\ \times\exp\Big(in\Big(-\gamma+\frac{\pi}{2}\Big)\Big)J_n(ur_0)\left[-\frac{iur_0}{2}\exp(-i\gamma)\varphi(r_0)\right]u\mathrm{d}\psi\mathrm{d}\gamma\mathrm{d}u.\end{aligned} \tag{32.38}$$

The coefficients at ψ in exponents can be set to zero, since the final solution should not depend on ψ. Thus, we obtain for the first integral $n=-\lambda$, for the second integral $n=-\lambda+1$, and for the third one $n=-\lambda-1$. As a result, the integral 32.38) can be written as:

$$\begin{aligned}\varphi(r,\theta) = \frac{1}{4\pi^2}\int_0^\infty\int_0^{2\pi}\int_0^{2\pi}\sum_{m=-\infty}^{+\infty}F_m\exp\Big(i(-\lambda)\Big(-\gamma+\frac{\pi}{2}\Big)\Big)\\ \times J_{-\lambda}(ur_0)\left[r_0\frac{\partial\varphi(r_0)}{\partial r}\right]u\mathrm{d}\psi\mathrm{d}\gamma\mathrm{d}u\\ +\frac{1}{4\pi^2}\int_0^\infty\int_0^{2\pi}\int_0^{2\pi}\sum_{m=-\infty}^{+\infty}F_m\exp\Big(i(-\lambda+1)\Big(-\gamma+\frac{\pi}{2}\Big)\Big)\\ \times J_{-\lambda+1}(ur_0)\left[-\frac{iur_0}{2}\exp(i\gamma)\varphi(r_0)\right]u\mathrm{d}\psi\mathrm{d}\gamma\mathrm{d}u\\ +\frac{1}{4\pi^2}\int_0^\infty\int_0^{2\pi}\int_0^{2\pi}\sum_{m=-\infty}^{+\infty}F_m\exp\Big(i(-\lambda-1)\Big(-\gamma+\frac{\pi}{2}\Big)\Big)\\ \times J_{-\lambda-1}(ur_0)\left[-\frac{iur_0}{2}\exp(-i\gamma)\varphi(r_0)\right]u\mathrm{d}\psi\mathrm{d}\gamma\mathrm{d}u.\end{aligned} \tag{32.39}$$

While developing type F_m in integrals and summing up the terms in exponents, we obtain:

$$\begin{aligned}\varphi(r,\theta) = &\frac{1}{4\pi^2}\int_0^\infty\int_0^{2\pi}\int_0^{2\pi}\frac{\sum_{m=-\infty}^{+\infty}J_m(ur)}{(u^2+k^2)}\exp\Big(i\gamma(\lambda-m)+im\Big(\theta-\frac{\pi}{2}\Big)+i(-\lambda)\frac{\pi}{2}\Big)\\ &\times J_{-\lambda}(ur_0)\left[r_0\frac{\partial\varphi(r_0)}{\partial r}\right]u\mathrm{d}\psi\mathrm{d}\gamma\mathrm{d}u\\ &+\frac{1}{4\pi^2}\int_0^\infty\int_0^{2\pi}\int_0^{2\pi}\frac{\sum_{m=-\infty}^{+\infty}J_m(ur)}{(u^2+k^2)}\exp\Big(i\gamma(\lambda-m)+im\Big(\theta-\frac{\pi}{2}\Big)+i(-\lambda+1)\frac{\pi}{2}\Big)\\ &\times J_{-\lambda+1}(ur_0)\left[-\frac{iur_0}{2}\varphi(r_0)\right]u\mathrm{d}\psi\mathrm{d}\gamma du\\ &+\frac{1}{4\pi^2}\int_0^\infty\int_0^{2\pi}\int_0^{2\pi}\frac{\sum_{m=-\infty}^{+\infty}J_m(ur)}{(u^2+k^2)}\exp\Big(i\gamma(\lambda-m)+im\Big(\theta-\frac{\pi}{2}\Big)+i(-\lambda-1)\frac{\pi}{2}\Big)\\ &\times J_{-\lambda-1}(ur_0)\left[-\frac{iur_0}{2}\varphi(r_0)\right]u\mathrm{d}\psi\mathrm{d}\gamma\mathrm{d}u.\end{aligned} \tag{32.40}$$

Since the Fourier term $\Phi(u,\gamma)$ is axially symmetric, we set the coefficients at γ to zero and obtain $m=\lambda$. Then:

$$\begin{aligned}\varphi(r,\theta) = &\frac{1}{4\pi^2}\int_0^\infty\int_0^{2\pi}\int_0^{2\pi}\frac{J_\lambda(ur)}{(u^2+k^2)}\exp\Big(i\lambda\Big(\theta-\frac{\pi}{2}\Big)+i(-\lambda)\frac{\pi}{2}\Big)\\ &\times J_{-\lambda}(ur_0)\left[-r_0\frac{\partial\varphi(r_0)}{\partial r}\right]u\mathrm{d}\psi\mathrm{d}\gamma\mathrm{d}u\\ &+\frac{1}{4\pi^2}\int_0^\infty\int_0^{2\pi}\int_0^{2\pi}\frac{J_\lambda(ur)}{(u^2+k^2)}\exp\Big(i\lambda\Big(\theta-\frac{\pi}{2}\Big)+i(-\lambda+1)\frac{\pi}{2}\Big)\\ &\times J_{-\lambda+1}(ur_0)\left[-\frac{iur_0}{2}\varphi(r_0)\right]u\mathrm{d}\psi\mathrm{d}\gamma\mathrm{d}u\\ &+\frac{1}{4\pi^2}\int_0^\infty\int_0^{2\pi}\int_0^{2\pi}\frac{J_\lambda(ur)}{(u^2+k^2)}\exp\Big(i\lambda\Big(\theta-\frac{\pi}{2}\Big)+i(-\lambda-1)\frac{\pi}{2}\Big)\\ &\times J_{-\lambda-1}(ur_0)\left[-\frac{iur_0}{2}\varphi(r_0)\right]u\mathrm{d}\psi\mathrm{d}\gamma\mathrm{d}u.\end{aligned} \tag{32.41}$$

Let us now set λ in the expression (32.36) equal to zero $\lambda = 0$. Then

$$\varphi(r) = \frac{1}{4\pi^2}\int_0^\infty\int_0^{2\pi}\int_0^{2\pi}\frac{J_0(ur)}{(u^2+k^2)}J_0(ur_0)\left[r_0\frac{\partial\varphi(r_0)}{\partial r}\right]u\mathrm{d}\psi\mathrm{d}\gamma\mathrm{d}u$$
$$+\frac{1}{4\pi^2}\int_0^\infty\int_0^{2\pi}\int_0^{2\pi}\frac{J_0(ur)}{(u^2+k^2)}J_1(ur_0)\left[\frac{ur_0}{2}\varphi(r_0)\right]u\mathrm{d}\psi\mathrm{d}\gamma\mathrm{d}u \tag{32.42}$$
$$+\frac{1}{4\pi^2}\int_0^\infty\int_0^{2\pi}\int_0^{2\pi}\frac{J_0(ur)}{(u^2+k^2)}J_{-1}(ur_0)\left[-\frac{ur_0}{2}\varphi(r_0)\right]u\mathrm{d}\psi\mathrm{d}\gamma\mathrm{d}u.$$

Let us simplify:

$$\varphi(r) = \frac{1}{4\pi^2}\int_0^\infty\int_0^{2\pi}\int_0^{2\pi}\frac{J_0(ur)u}{(u^2+k^2)}\left\{J_0(ur_0)\left[r_0\frac{\partial\varphi(r_0)}{\partial r}\right]\right.$$
$$\left.+(J_1(ur_0)-J_{-1}(ur_0))\left[\frac{ur_0}{2}\varphi(r_0)\right]\right\}\mathrm{d}\psi\mathrm{d}\gamma\mathrm{d}u. \tag{32.43}$$

Let us integrate over ψ, γ:

$$\varphi(r) = \int_0^\infty\frac{J_0(ur)u}{(u^2+k^2)}\left\{J_0(ur_0)\left[r_0\frac{\partial\varphi(r_0)}{\partial r}\right]+(J_1(ur_0)-J_{-1}(ur_0))\left[\frac{ur_0}{2}\varphi(r_0)\right]\right\}\mathrm{d}u. \tag{32.44}$$

Taking into consideration that:

$$J_{-1}(u) = (-1)^1 J_1(u) = -J_1(u), \tag{32.45}$$

we finally obtain:

$$\varphi(r) = r_0\int_0^\infty\frac{J_0(ur)u}{(u^2+k^2)}(J_0(ur_0)f_2 + J_1(ur_0)uf_1)\mathrm{d}u. \tag{32.46}$$

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