Chapter 1
Introduction

General relativity (GR) is one of the towering achievements of twentieth-century physics. Its predictions have received spectacular experimental confirmation time and time again since its publication over one hundred years ago [1]. However, GR is not the end of the story as far as gravity is concerned. Singularities appearing in the theory provide internal evidence that it is somehow incomplete, and furthermore GR is a classical description of gravity whilst nature at a fundamental level behaves quantum mechanically. At scales approaching the Planck length quantum effects are expected to become important and it is believed that a theory of quantum gravity is needed in order to describe nature at the Planck scale and beyond.[1] Such a theory promises to bring a deeper understanding of fascinating phenomena such as black holes and the big bang, and its discovery remains one of the biggest open challenges in fundamental physics.

Actually, there is nothing preventing us from quantising GR using the standard perturbative techniques that have been successfully applied to nature's other fundamental fields. The resulting quantum field theory (QFT) can be used to make testable predictions, for example in the form of corrections to the Newtonian potential [2].[2] However, if we wish to describe gravity at distances approaching the Planck length predicitivity is lost. It turns out that an infinite number of measurements need to be performed by experiment in order to determine the parameters required to cancel the divergences of the theory i.e. the theory is perturbatively non-renormalizable [3–6].

Perturbative quantisation of GR therefore only provides an effective description of the graviton. Still, effective field theories are commonplace in physics and some of the most successful field theories of the last century come under this umbrella.

[1] Even though probing Planck-scale physics may require energies far above those accessible at current particle accelerators, there are ways to study quantum gravitational effects e.g. from the finger prints of the very early universe left on the CMB. See Chap. 5 for more discussions on experimental searches for quantum gravity.

[2] Although these effects are very small and therefore not likely to be measured any time soon.

Z. H. Slade, *Fundamental Aspects of Asymptotic Safety in Quantum Gravity*,
Springer Theses, https://doi.org/10.1007/978-3-030-19507-6_1

The Standard Model for example can be considered an effective description of the interactions of fundamental particles. Likewise, Newton's theory of gravity is a low energy approximation to Einstein's GR, which in turn must be an effective description of some higher-energy theory of the gravitational field (whether this be a QFT or something more exotic).

The shortcomings of perturbative approaches[3] do not mean that QFT and gravity are incompatible however. A well-behaved quantum theory of gravity might be recovered by taking the dynamics of the non-perturbative regime into account. One such non-perturbative route, which retains the fields and symmetries of GR, is asymptotic safety. Asymptotic safety posits the existence of a non-Gaussian UV fixed point of the gravitational renormalization group flow to control the behaviour of the theory at high energies and thereby keep physical quantities safe from unphysical divergences. This idea was first put forward by Weinberg [8] and has since been the focus of many searches for quantum gravity, the majority of which offer encouraging signs that an appropriate high-energy fixed point could indeed exist.

It may well turn out that we have to go beyond conventional QFT in order to describe gravity at the Planck scale and in the process introduce additional degrees of freedom and symmetries like those of string theory or additional spacetime structure as in loop quantum gravity, or perhaps something else is required altogether. However, whether or not the asymptotic safety hypothesis turns out to be ultimately correct, it is important to make progress with fundamental aspects of the approach, a collection of which provide the focus of this thesis.

In the sections that follow we give the necessary background for the research presented in Chaps. 2–4. We begin with a review of the renormalization group as understood by Kenneth Wilson in Sect. 1.1, before introducing the theory space on which the renormalization group flows play out in Sect. 1.2. In the following Sect. 1.3, we review the specific application of the renormalization group in the asymptotic safety approach to quantum gravity. Section 1.4 contains a discussion on popular approximation schemes employed in asymptotic safety, many of which are then used in the chapters that follow. Finally, we conclude this introductory chapter with an outline of the rest of the thesis.

1.1 The Wilsonian Renormalization Group

Naturally the scale at which we observe the world determines how we describe it. We construct theories in terms of variables appropriate for the viewing scale and in fact we need not worry about what goes on at shorter distances (or equivalently, higher energies) in order to make successful predictions. For example, to describe

[3] Another example comes from [7] in which adding higher derivative operators to the Einstein–Hilbert action leads to a perturbatively renormalizable quantum theory of gravity, but which does not respect unitary.

water flowing in a stream we do not need an understanding of water at the molecular level, instead the physics of fluid mechanics is enough.

However, by the very nature of their construction our theories are often blind to UV dynamics; they are effective theories with limited descriptive power and a finite realm of validity. The scale at which a theory ceases to be applicable is aptly named the cutoff scale. It indicates the point at which our knowledge breaks down and beyond which new physics lies. As we have already mentioned, in the case of perturbative quantum gravity this is the Planck scale. How then are we able to gain access to a high-energy (short-distance) description of nature?

An answer comes from the understanding of the renormalization group (RG) owed to Wilson [9]. The RG is the mathematical formalism that enables us to systematically generate and relate descriptions of a system befitting different viewing scales, and for this reason is often said to be analogous to a microscope with varying magnification. The basic idea is that a system's microscopic degrees of freedom can be replaced by effective ones, together with appropriate rescaled interaction strengths, to give a different description of the system but which produces the same predictions for physical observables. RG methods are at the heart of the asymptotic safety approach to quantum gravity and as such provide the focus of this section.

Wilson's RG has its origins in the study of condensed matter systems and so we begin this section by introducing key RG concepts through a discussion on Kadanoff blocking. We then move on to review the continuum description of the RG due to Wilson and visit Polchinski's flow equation. We end this section with a comparison between the renormalization of perturbation theory and the modern view of renormalizability that Wilson's ideas brought about.

1.1.1 Kadanoff Blocking

Consider a two-dimensional lattice of atoms each possessing two spin degrees of freedom, up or down, and with nearest-neighbour interactions, as shown in Fig. 1.1a. In this example the cutoff scale is given by the lattice spacing δ. Now suppose we average over a group of neighbouring spins and replace them by a single "blocked" spin at the centre. For example, a 3 by 3 block of spins containing mostly up spins is replaced by a single spin-up degree of freedom, and vice versa for down spins. The resulting picture is one with fewer degrees of freedom at an increased separation, see Fig. 1.1b. This procedure is known as blocking or more generally as coarse graining.

In order to compare the coarse-grained description of the system to the original microscopic Fig. 1.1a, a second step is performed—a rescaling—to shrink the lattice spacing back to its original size, see Fig. 1.1c. This two-step process of coarse graining and rescaling is known as block-spin renormalization and was introduced by Leo Kadanoff in 1966 [10]. Together the two steps make up a renormalization group transformation.[4]

[4] Note that these transformations do not form a group in the formal sense as the coarse-graining procedure is not invertible.

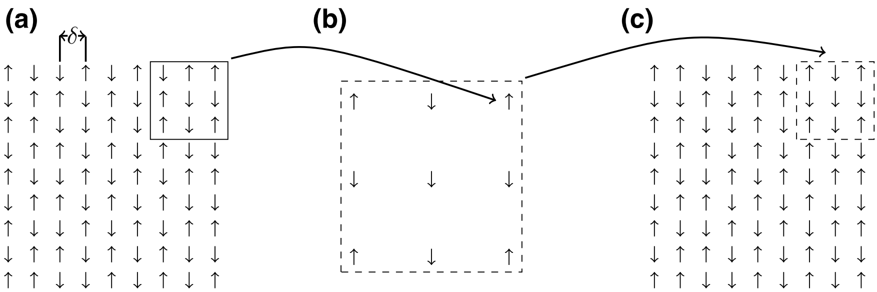

Fig. 1.1 Illustration of a block-spin RG transformation in a 2-dimensional lattice of spins. Coarse graining proceeds from (**a**) to (**b**), followed by rescaling from (**b**) to (**c**). In **c** lattice sites previously outside the picture have been pulled in

Block-spin renormalization gives us an alternative way of describing the lattice of spins, i.e. in terms of coarse-grained variables with appropriately scaled interaction strengths (as opposed to in terms of the original microscopic degrees of freedom). In this sense, an RG transformation is like a reorganization of what we already know. In fact, not only does the block-spin procedure modify the spin-spin interactions, but it also gives rise to new ones. In the original lattice there are only nearest-neighbour interactions, but the block-spin transformation generates next-to-nearest neighbour interactions, next-to-next-to-nearest neighbour interactions and so on.

Crucially though, these different pictures of the system still predict the same values for physical observables, so long as we consider physics at length scales much greater than the cutoff. In other words, performing an RG transformation changes the couplings in such a way so as to leave observables unchanged. Indeed it seems reasonable to expect that when describing some long-distance phenomena, far away from the cutoff scale, predictions for observables should be insensitive to changes in it. In the case of a lattice of spins such an observable would be the resistivity of a metal, which is independent of the precise inter-atomic spacing.

1.1.2 Wilsonian Renormalization

The RG transformations of Kadanoff's blocking procedure are concerned with discrete changes in the cutoff scale. In 1971 Kenneth Wilson introduced a version of the RG adapted to continuous changes in the cutoff which could be implemented through the path integral formulation of quantum field theory [9].

To illustrate this approach let us consider a single-component scalar field $\phi(x)$ with bare action $\hat{\mathcal{S}}[\phi]$. In the language of path integrals, physical observables of the field are then given by derivatives of the generating functional

$$Z[J] = \int^{\Lambda} \mathcal{D}\phi \, \mathrm{e}^{-\hat{\mathcal{S}}[\phi]+J\cdot\phi}\,, \tag{1.1.1}$$

with respect to the external current $J(x)$. We will use a dot notation to denote integration over position or momentum space:

$$J \cdot \phi \equiv J_x \phi_x \equiv \int d^d x J(x)\phi(x) = \int \frac{d^d p}{(2\pi)^d} J(p)\phi(-p)\,. \tag{1.1.2}$$

For bilinear terms we regard the kernel as a matrix, thus the following forms are equivalent:

$$\phi \cdot \Delta^{-1} \cdot \phi \equiv \phi_x \Delta^{-1}_{xy} \phi_y \equiv \int d^d x d^d y\, \phi(x)\Delta^{-1}(x, y)\phi(y) = \int \frac{d^d p}{(2\pi)^d} \phi(p)\Delta^{-1}(p^2)\phi(-p)\,. \tag{1.1.3}$$

Note that when transforming to momentum space, Green's functions $G(p_1, \ldots, p_n)$ come with momentum conserving delta functions such that they are only defined for $p_1 + \cdots + p_n = 0$. Thus two-point functions are functions of just a single momentum $p = p_1 = -p_2$. The integral (1.1.1) is endowed with a sharp UV cutoff Λ such that only those modes propagating with momentum $|p| \equiv \sqrt{p^2} \leq \Lambda$ are integrated over. Here and throughout the rest of this thesis we will now deal with energy scale cutoffs as opposed to length scale cutoffs $\delta = 1/\Lambda$. Note the Euclidean signature of the functional integral needed in order to take proper account of modes with nearly light-like four momenta. (In gravitational theories the Euclidean signature gives rise to the well-known conformal factor problem which has profound consequences for the RG properties of the theory in question. We discuss this in more detail in Sect. 1.4.4.) Finally, the requirement for physics to be independent of the cutoff in the context of the path integral means for the generator of Green's functions $Z[J]$ to be independent of Λ:

$$\Lambda \frac{dZ[J]}{d\Lambda} = 0\,. \tag{1.1.4}$$

Here coarse graining corresponds to lowering the cutoff by integrating out the high-energy degrees of freedom between Λ and some lower energy scale k, cf. Fig. 1.2. For simplicity, let us only consider observables with momenta less than the

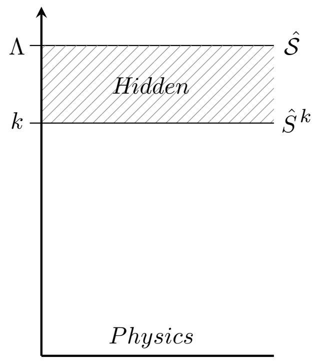

Fig. 1.2 Energy spectrum over which modes are integrated out

lower cutoff k so that $J(p) = 0$ for $|p| > k$. Splitting the modes into two sets, those with momenta $|p| > k$ denoted by $\phi_>$ and those with $|p| \le k$ denoted $\phi_<$, we can rewrite (1.1.1) as

$$Z[J] = \int^{0<|p|\le k} \mathcal{D}\phi_< \int^{k<|p|\le\Lambda} \mathcal{D}\phi_> \, \mathrm{e}^{-\hat{\mathcal{S}}[\phi_< + \phi_>] + J\cdot\phi_<} \tag{1.1.5}$$

$$= \int^{0<|p|\le k} \mathcal{D}\phi_< \, \mathrm{e}^{-\hat{S}^k[\phi_<] + J\cdot\phi_<} , \tag{1.1.6}$$

where the result of integrating over a shell of momenta $k < |p| \le \Lambda$ has been re-expressed in terms of a new, effective action[5]

$$\hat{S}^k[\phi] = -\ln \int^{k<|p|\le\Lambda} \mathcal{D}\phi_> \, \mathrm{e}^{-\hat{\mathcal{S}}[\phi_< + \phi_>]} \ . \tag{1.1.7}$$

This effective action predicts exactly the same low-energy ($E \ll k$) physics as the original bare action $\hat{\mathcal{S}}$. It contains new interactions arising from the coarse-graining procedure (just like we saw in Kadanoff blocking). As the cutoff is lowered, modes are removed from the propagator and "hidden away" in the effective action, manifesting themselves as changes in the couplings, cf. Fig. 1.2. These changes compensate for the change in the cutoff, meaning that $Z[J]$ and its functional derivatives remain unchanged i.e. they obey (1.1.4). It follows that a simple Lagrangian at the cutoff scale Λ will become more complicated as the the cutoff is lowered, growing new interactions, including contributions from irrelevant operators.[6]

We still need to perform the rescaling step. This can be most easily achieved by making all quantities (fields and their couplings) dimensionless by dividing by the effective scale k raised to the power of their scaling dimension. This is equivalent to rescaling distances and momenta, and sends the cutoff back to its original size. Thus writing everything in terms of dimensionless quantities, in addition to the coarse-graining step as described above, completes an RG transformation in the Wilsonian approach. Applying successive RG transformations gives a series of effective actions:

$$\hat{S} \quad \rightarrow \quad \hat{S}' \quad \rightarrow \quad \hat{S}'' \quad \rightarrow \quad \cdots \tag{1.1.8}$$

describing a system up to successively decreasing cutoff scales.

Joseph Polchinski adapted Wilson's RG by introducing a smooth momentum scale cutoff in a more direct way [11]. This was achieved by incorporating a smoothly varying cutoff-dependent function f into the propagator like so[7]

---

[5] See Chap. 2 for a more comprehensive example and further discussions.

[6] With this in mind, it no longer makes sense to insist that Lagrangians only contain relevant operators. Indeed, in the application of the Wilsonian RG to asymptotic safety we allow for all possible operators consistent with symmetry constraints.

[7] Where the mass term is contained within the interactions.

$$\frac{1}{p^2} \to \frac{f(p^2/k^2)}{p^2} \equiv \Delta^k \,. \tag{1.1.9}$$

The function f has the property that for $|p| < k$, $f \approx 1$ and mostly leaves modes unaffected whilst for $|p| > k$, f suppresses modes, vanishing rapidly at infinity. Using the modified propagator, (1.1.6) instead becomes

$$Z[J] = \int \mathcal{D}\phi \, \mathrm{e}^{-\frac{1}{2}\phi\cdot(\Delta^k)^{-1}\cdot\phi - S^k[\phi] + J\cdot\phi} \,, \tag{1.1.10}$$

where for the sake of neatness we have made the replacement $\phi_< \to \phi$ and where the effective action $\hat{S}^k[\phi]$ has been split into a kinetic part and interactions $S^k[\phi]$. The path integral is smoothly regulated in the UV by the cutoff function f. Polchinski showed that if the effective interactions $S^k[\phi]$ satisfy the following integro-differential equation [11]

$$\frac{\partial}{\partial k} S^k[\phi] = \frac{1}{2}\frac{\delta S^k}{\delta\phi} \cdot \frac{\partial \Delta^k}{\partial k} \cdot \frac{\delta S^k}{\delta \phi} - \frac{1}{2}\mathrm{Tr}\left(\frac{\partial \Delta^k}{\partial k} \cdot \frac{\delta^2 S^k}{\delta\phi\delta\phi}\right) \tag{1.1.11}$$

then (1.1.4) (with the replacement $\Lambda \to k$ for the case at hand) follows. This is Polchinski's version of Wilson's flow equation [12, 13][8] which we will see again shortly in Chap. 2. It expresses how the effective interactions must change as the cutoff is lowered in order to keep $Z[J]$ constant. It is commonly referred to as an exact RG equation (ERGE) as no approximation is used in its derivation; in particular, it does not rely on small couplings.

1.1.3 The Wilsonian Perspective

Wilson's approach brought about a new understanding of renormalizability in quantum field theory. In the old view of renormalization a cutoff is introduced to loop integrals to enable their computation on the way to calculating scattering amplitudes and is nothing more than a mathematical trick. Physical quantities are then made independent of the cutoff (they are "renormalized") such that its value can be safely taken to infinity at the end of the calculation with physical quantities remaining finite. This is the familiar renormalization of perturbation theory, implemented for example by redefining bare couplings in terms of renormalized ones or subtracting divergences with a finite number of counter terms.

From the modern Wilsonian perspective the cutoff should be viewed as physically meaningful and all quantum field theories in possession of one should be treated as effective theories only valid up to the cutoff scale. As already mentioned, the cutoff represents the scale at which our knowledge breaks down and therefore we cannot

---

[8] For a more careful comparison between Wilson's and Polchinski's versions see Ref. [14].

justify taking the limit $\Lambda \to \infty$, at least not before knowing the high-energy behaviour of a theory.[9] For a theory to be renormalizable from the Wilsonian perspective means that it is truly free from divergences at *all* scales: no divergences appear, no matter how high we take the cutoff. Technically, this is achieved by arranging for the theory to emanate from a UV fixed point, the subject of the next section.

Unlike perturbation theory, the Wilsonian RG does not rely on couplings being small and therefore represents a non-perturbative approach to renormalization. This is one of its chief advantages as it opens the door to exploring the non-perturbative regime of quantum theories such as gravity.

In summary, in both the perturbative and nonperturbative regimes, the word "renormalization" refers to a way of dealing with divergences, but the methods by which this is done are conceptually and technically different. From the Wilsonian viewpoint, theories such as QED which appear renormalizable where perturbation theory is valid, are not truly renormalizable in the full non-perturbative sense of the word. Wherever we use the term renormalization we will mean it in the sense of the Wilsonian renormalization group, also known in the continuum as the exact renormalization group (ERG), the functional renormalization group (FRG) and the continuous renormalization group.

1.2 Fixed Points and Theory Space

Now that we have reviewed the Wilsonian RG and seen an example of an exact RG equation, we are ready to examine the space on which its solutions live: theory space. In this section we introduce the concept of theory space and discuss its key features, namely fixed points, as well as highlighting the properties they must exhibit in order for asymptotic safety to be realised. We continue to use the scalar field throughout for illustrative purposes.

Theory space by definition is the space containing all possible actions that can be built from a given set of fields obeying certain symmetry constraints. An action in the space is assumed to have the form:

$$S^k[\phi] = \sum g_i(k)\mathcal{O}_i(\phi)\,, \tag{1.2.1}$$

where g_i are the dimensionless, k-dependent couplings and $\mathcal{O}_i$ are operators made up of products of the dimensionless fields and their derivatives. Furthermore, the g_is do not include redundant (a.k.a. inessential) couplings i.e. those which can be eliminated from the action by a field redefinition. The operators form the basis of the theory space whilst the couplings play the role of coordinates. In this way, each

[9]Indeed from this point of view the action of sending $\Lambda \to \infty$ in perturbation theory is misleading. For example, QED can be renormalized perturbatively—at low energy when the couplings are small—but at high enough energies ($\approx 10^{300}$ GeV) it still develops divergences in spite of the limit $\Lambda \to \infty$ having already been taken.

point in the space represents a different possible action. A priori the sum (1.2.1) is infinite as we allow for all possible couplings and therefore so is the dimension of the theory space. In a later Sect. 1.4, we will discuss reducing the dimension by making approximations.

Performing an RG transformation corresponds to moving between effective actions in theory space along an RG trajectory or flow line. In geometrical terms, these RG trajectories are the induced integral curves of the vector field defined by an RG equation, such as Polchinski's in (1.1.11). Thus the trajectory gives a way of visualizing the evolution of a theory with changes in the cutoff scale as described by the RG. By convention we flow from high to low energy, in the direction of increasing coarse graining[10] as indicated by the arrows in Fig. 1.3. It is important to point out here that it is the trajectory itself that we identify with a theory, not the individual actions.

Features of theory space of particular interest are fixed points. These are sources and sinks of RG flows and are home to scale-invariant theories S^*, i.e.

$$k\frac{\partial}{\partial k}S^*[\phi] = 0\,. \tag{1.2.2}$$

Recall that all variables have been made dimensionless using k and so independence of k implies that S^* depends on no scale at all. It follows that fixed point theories are massless. This scale independence also makes fixed point theories trivially renormalizable as we can trivially send $k \to \infty$. This limit is referred to as the continuum limit and theories which have one are said to be UV complete. A fixed point action therefore describes physics at the Planck scale and beyond.

For a given UV fixed point, there exists a submanifold called the critical surface $\mathcal{S}_{UV}$, as shown in Fig. 1.3. By definition, any point in theory space—i.e. any action—on this surface is pulled towards the fixed point under the reverse RG flow (against the directions of the arrows). The portion of the critical surface local to the fixed point, is spanned by so-called relevant operators[11]—those whose coefficients in the action increase as we move out from the fixed point i.e. as $k \to 0$. Perturbing the fixed point action along the relevant directions gives rise to a "renormalized trajectory", indicated by the purple lines in the figure. The trajectory represents a renormalizable theory as its high-energy behaviour is controlled by a fixed point, i.e. as we take the limit $k \to \infty$ and approach the UV fixed point, the couplings of the theory tend to fixed finite values and are protected from blowing up. Since observable quantities can be expressed as functions of the couplings, this means that they will also remain finite when the continuum limit is taken.

The effective actions sitting on a renormalized trajectory are called "perfect actions" [15]. All their scale dependence is carried through the couplings and the anomalous dimension $\eta(k)$: $S^k[\phi] = S[\phi]\,(g_1(k), \ldots, g_n(k), \eta(k))$. This means that

---

[10] Again, coarse graining can only be performed in one direction—we can only integrate out modes, we cannot "integrate them in"—but once the trajectory is defined, we can flow in either direction.

[11] These also include marginally relevant operators.

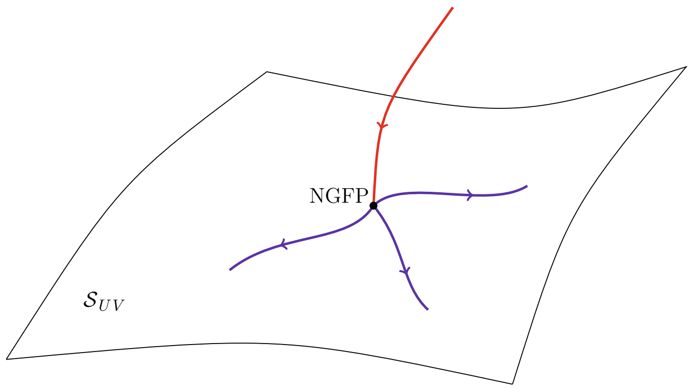

Fig. 1.3 Schematic picture of UV critical surface $\mathcal{S}_{UV}$ in theory space and RG trajectories flowing from high to low k in the direction of coarse graining. The surface contains a non-Gaussian fixed point (NGFP) supporting renormalized trajectories (purple lines). There is also a trajectory coming from outside the surface and flowing into the fixed point (red line); for this trajectory, the fixed point is IR

the actions undergo a self-similar evolution under RG transformations. We return to these perfect actions in the next chapter.

The number of relevant operators spanning the fixed point (a.k.a. eigenperturbations or eigenoperators) gives the dimension d_{UV} of $\mathcal{S}_{UV}$, which will therefore contain a d_{UV}-parameter set of trajectories. Which trajectory is realized in nature will be decided by experiment. For asymptotic safety we require d_{UV} to be finite otherwise we lose predictivity (as we would have to take an infinite number of measurements to fix the infinite number of couplings). Consequently, the smaller the dimension of $\mathcal{S}_{UV}$, the more predictive the theory will be. In the asymptotic safety literature, usually fixed points with a finite number of relevant directions (typically three) are found (see e.g. reviews [16–20] and textbooks [21, 22]), however there are also examples of fixed points which support a continuous spectra of eigenperturbations, see e.g. [23].

Whether a fixed point is classified as UV or IR will depend on the trajectory under consideration. If instead as we flow in the direction of coarse graining, we are pulled *into* a fixed point then, as far as this trajectory is concerned, it is an IR fixed point. Furthermore, what is a UV fixed point for one trajectory may be an IR fixed point for another. Hence, in addition to the renormalized trajectory flowing out of the fixed point in Fig. 1.3, there may also be trajectories flowing into the fixed point, one such trajectory being indicated by the red line. Interestingly, this implies that very different physical systems described by very distinct theories can exhibit the same low-energy behaviour. The observation that the macroscopic description of a phenomenon is independent of the microscopic details is known as universality. Indeed this situation could be realised in a UV complete theory of quantum gravity if it supported more than one high-energy fixed point.

The UV fixed points required for non-perturbative renormalizability can be Gaussian or non-Gaussian, home to either free or interacting theories respectively. For example, the theory of QCD possesses a Gaussian fixed point in the UV supporting interacting relevant directions: a free theory at the fixed point grows into a theory of interacting quarks as we flow into the infrared. Theories exhibiting such fixed points are said to be asymptotically free and are naturally renormalizable since again the UV dynamics are controlled by a fixed point. Of course the UV fixed points of most interest to quantum gravity searches are non-Gaussian. (Asymptotic safety at a Gaussian fixed point would be equivalent to perturbative renormalizability plus asymptotic freedom, but as noted at the start of the chapter, perturbative quantisation of gravity fails.) Theories emanating from such fixed points exhibit asymptotic safety. For this reason renormalized trajectories are also called asymptotically safe.

Preferably we want the theory space to support only one non-Gaussian fixed point (NGFP), or at least a finite number, otherwise again we lose predictivity. However, there are examples in the literature in which lines and planes of fixed points have been uncovered, see [23–26]. On the contrary, it might turn out that the theory space contains no fixed points. One reason found for this in gravitational theories is that background independence has not been properly taken care of [27, 28], as discussed in Chap. 3. The number of fixed points supported by the theory space is determined by counting up the number of independent parameters and constraints coming from the RG equation at the fixed point and its asymptotic solutions. This is the subject of Chap. 4.

In summary, we have seen that for the asymptotic safety scenario to be realised, a theory space must contain NGFPs (and preferably only one) with a finite number of relevant directions. Further to this, fixed points must support a renormalized trajectory that reproduces the behaviour of classical gravity at low energies.

1.3 The Effective Average Action and Its Flow

Having introduced the concept of the RG and the space on which its flows play out, in this section we review the specific application of the RG to asymptotic safety. In the first part we introduce the central tools of the field—namely the effective average action and its flow equation—whilst continuing to work within the setting of scalar field theory so as to illustrate the key concepts in the simplest way possible. The purpose of the proceeding subsection is then to review the necessary modifications when applying these ideas to gravity. The final subsection contains a discussion on background independence, an important requirement for any theory of gravitation, which will be of particular relevance to Chap. 3.

Historically the first hints of asymptotically safe gravity came from applying Wilson's ideas in $2+\epsilon$ dimensions [8]. Nowadays proponents of the field use a reformulation of Wilson's exact RG given in terms of the effective average action Γ_k, a scale dependent version of the usual effective action Γ i.e. the generator of one-particle irreducible Green's functions. For a scalar field the effective average action is defined via the Legendre transform of a functional integral of the following form

$$Z[J] = \int^{\Lambda} \mathcal{D}\phi \, \mathrm{e}^{-\hat{\mathcal{S}}[\phi]-\Delta S_k[\phi]+J\cdot\phi} \equiv \mathrm{e}^{W_k^{\Lambda}[J]}\,, \tag{1.3.1}$$

which is related to the generator of connected Green's functions $W_k^{\Lambda}[J]$ in the usual way. Just as in Sect. 1.1.2, the integral is subject to an overall UV cutoff that is required to make sense of the integration. Here it is implemented by a sharp cutoff at Λ, but it could equally well be of a different type (see Chap. 2 for examples). The functional integral also depends on another cutoff scale, k. When working with the Wilsonian action in the previous chapter, k denoted the effective UV cutoff scale, whereas here it represents an IR cutoff. This might seem like an unnecessary complication, however the reason for this choice becomes clear in Chap. 2 where a relationship between the effective average action and the Wilsonian effective action is derived. To avoid confusion, we will denote any UV cutoff parameter with a superscript and any IR cutoff parameter with a subscript and use this pictorial guide throughout.

The dependence on k is introduced via the IR cutoff operator R_k which lives inside the cutoff action:

$$\Delta S_k[\phi] = \frac{1}{2}\phi \cdot R_k \cdot \phi\,. \tag{1.3.2}$$

The cutoff operator is a function of the Laplacian: $R_k = R_k(-\nabla^2)$, and acts on the field ϕ to turn ΔS_k into a mass-like term. Roughly speaking, R_k suppresses modes propagating with momentum $p^2 < k^2$, otherwise leaving them unaffected. The precise way in which it does this is unimportant but it must satisfy the two limits

$$\lim_{p^2/k^2\to 0} R_k(p^2) = k^2 \quad \text{and} \quad \lim_{p^2/k^2\to\infty} R_k(p^2) = 0\,. \tag{1.3.3}$$

Popular choices for R_k include the optimized cutoff $R_k(p^2) = (k^2 - p^2)\Theta(k^2 - p^2)$ [29–31] and the exponential cutoff $R_k(p^2) = (p^2/k^2)[\exp(p^2/k^2) - 1]^{-1}$.

The effective average action Γ_k^{Λ} is obtained by subtracting the cutoff action $\Delta_k S$ (as a functional of the classical fields) from the Legendre transform of (1.3.1):

$$\Gamma_k^{\Lambda}[\varphi] \equiv \tilde{\Gamma}_k^{\Lambda}[\varphi] - \frac{1}{2}\varphi \cdot R_k \cdot \varphi\,, \tag{1.3.4}$$

where $\tilde{\Gamma}_k^{\Lambda} = -W_k^{\Lambda}[J] + J\cdot\varphi$ is the Legendre transform and $\varphi(x) \equiv \langle\phi(x)\rangle$ is the expectation value a.k.a. classical field.

The flow equation for the effective average action is obtained by taking the derivative of (1.3.4) with respect to k[12]:

$$\partial_k \Gamma_k^{\Lambda}[\varphi] = \frac{1}{2}\mathrm{Tr}^{\Lambda}\left[\left(\frac{\delta^2\Gamma_k^{\Lambda}}{\delta\varphi\delta\varphi} + R_k\right)^{-1} \partial_k R_k\right]. \tag{1.3.5}$$

[12] The steps are given in Chap. 3.

The trace is taken over position (or momentum) space coordinates and here is restricted to only those modes propagating with momentum $|p| \leq \Lambda$. Note that at this point both the effective action Γ_k^Λ and the flow equation depend on *two* scales: the IR cutoff scale k and the UV cutoff scale Λ. However, the derivative $\partial_k R_k$ is sharply peaked around $p^2 = k^2$, dying off rapidly for $p^2 \gg k^2$, and so the left-hand side of the flow equation only receives contributions from modes near (or below) k. This means that the trace is prevented from blowing up in the limit $\Lambda \to \infty$ and the UV cutoff can be safely removed. Doing this yields the following RG equation [12, 32][13]

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left[\left(\frac{\delta^2 \Gamma_k}{\delta\varphi\delta\varphi} + R_k\right)^{-1} \partial_k R_k\right]. \tag{1.3.6}$$

It is this "Λ-free" flow equation which is employed in current investigations into asymptotically safe gravity. In contrast to (1.3.5), its solutions Γ_k depend only on a single scale, k. This is crucial to its use as it allows us to express everything in terms of dimensionless couplings $g_i(k)$ with respect to the single dimensionful parameter k, i.e. to recover the power of the Wilsonian RG. From now on when referring to the flow equation and its solutions we will mean the Λ-free versions.

Now let us comment on some key features of the flow equation (1.3.6). First of all, a solution Γ_k of (1.3.6) represents an action for a system in which the high-energy modes (with respect to k) have been integrated out and provides a natural effective action for processes occurring at energies $E \approx k$. A complete set of well-behaved solutions to the flow equation $\{\Gamma_k, 0 \leq k < \infty\}$ corresponds to a complete RG trajectory, free from divergences in both the IR and UV. As we saw in the previous section, the latter condition is realized by arranging the trajectory to originate from a high-energy fixed point.[14]

Secondly, just like Polchinski's, (1.3.6) is an exact RG equation suitable for the non-perturbative regime. However, despite the flow equation itself being exact, in practice it is not possible to solve it exactly and an approximation to the effective average action has to be made. These approximations are the subject of Sect. 1.4.

Thirdly, since the infrared cutoff k is introduced by hand, it is an artificial quantity that must not feature in physical observables. The physical part of the effective action is therefore only recovered when the cutoff is removed. This is done by taking the limit $k \to 0$ whilst holding all physical, i.e. unscaled, quantities fixed. It is in this limit that we recover the information contained in the full path integral.

There is a further property of this set up which is important to recognise: given a solution Γ_k of the flow equation, it is not possible to exactly recover the path integral (1.3.1) from which it was derived. Or more specifically, there is no exact way to reconstruct the bare action $\hat{S}$ from an effective average action Γ_k. In short, the reason for this is that a UV regulated path integral cannot, through the Legendre

---

[13]Dimensionless RG time $t = \ln(k/\mu)$ where μ is a fixed reference scale is also commonly used instead of k.

[14]Note that all theory space concepts described in the previous section apply equally well to the effective average action.

transform procedure, return an effective action Γ_k but instead necessarily gives rise to an effective action Γ_k^Λ which depends explicitly on two cutoffs. At the point of defining Γ_k, all reference to the UV cutoff is lost and so there is no way to gain access to the bare action in the original UV regularized functional integral from the solutions of the Λ-free flow equation.

Conceptually there is nothing wrong with simply working with the flow equation (1.3.6) and forgoing defining a path integral representation of the theory. In this way, we dispense with the need to define a bare action at the overall cutoff scale and concomitant tuning required to reach the continuum limit. One of the main advantages of working with the effective average action over the path integral is that it lends itself to more powerful approximation techniques. Being able to work with approximations is crucial as solving the flow equation is equivalent to, and practically as difficult as, solving the original path integral from which it came. Furthermore, since Γ_k is the k-dependent generator of one-particle irreducible Green's functions, it is directly related to scattering amplitudes which means that once we have found a complete trajectory, taking consecutive functional derivatives of Γ_k give us all the Green's functions of the theory and in the limit $k \to 0$ they coincide with those of the standard effective action $\Gamma \equiv \Gamma_0$.

Despite the advantages of using the effective average action, there are still reasons for wanting a path integral formulation of the theory. For example, to more easily understand certain properties of the QFT such as constraints and symmetries and to compare with other approaches to quantum gravity. The challenge of obtaining a path integral representation is called the reconstruction problem and is the subject of Chap. 2.

Even though we cannot directly obtain the bare action from the effective average action as emphaszied above, a simple and exact relationship between Γ_k and the Wilsonian effective action $\hat{S}^k$ (introduced in (1.1.7)) does exist [12, 33]. Referring back to Fig. 1.2, it need not seem so surprising that there is such a relationship [13]. In the discussions on the Wilsonian RG in Sect. 1.1.2, we saw that integrating out degrees of freedom between Λ and some lower cutoff scale k resulted in a Wilsonian effective action S^k with the scale k acting as a UV cutoff for the unintegrated modes. On the other hand, k can also be regarded as an infrared cutoff for the modes which have already been integrated out (those which reside in the shaded area of Fig. 1.2). From this perspective we see that the Wilsonian effective action is almost equivalent to the original functional integral, but modified by an infrared cutoff k, which in turn is straightforwardly related to Γ_k in the continuum limit (cf. Eq. (2.6.3) in Chap. 2). In Chap. 2 we derive this relationship and show how $\hat{S}^k$ can play the role of a perfect bare action which lives inside a fully UV regularised functional integral.

1.3.1 *The Effective Average Action for Gravity*

Up to this point we have been using a scalar field to introduce key concepts in functional RG methods, but of course we need to go beyond scalar theory to study quantum gravity. Instead of quantizing some field living on some predetermined spacetime background, in quantum gravity spacetime itself becomes the dynamical variable we wish to quantise, and with this give meaning to the path integral over all metrics

$$\int \mathcal{D}\tilde{g}_{\mu\nu}\, \mathrm{e}^{-\hat{S}[\tilde{g}_{\mu\nu}]} \tag{1.3.7}$$

and its associated effective average action. This brings with it new challenges, both conceptual and technical in nature. In the following we give an overview of the construction of the effective average action for gravity and its flow equation. The derivation is more involved than for the case of the scalar field but the procedure follows the same general pattern.

To deal with the obstacles arising when applying the functional RG to gravity, a technique called the background field method is used. It consists of decomposing the full metric $\tilde{g}_{\mu\nu}$ into a background metric $\bar{g}_{\mu\nu}$ and a fluctuation field $\tilde{h}_{\mu\nu}$ like so

$$\tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \tilde{h}_{\mu\nu}\,. \tag{1.3.8}$$

The background metric is fixed but left completely arbitrary. The split shifts the integration (1.3.7) over the total metric to one over the fluctuation field $\tilde{h}_{\mu\nu}$ i.e. it is the fluctuation field that is quantised in the path integral. Note that the fluctuation $\tilde{h}_{\mu\nu}$ is not restricted to being small here like in perturbation theory.

The bare action $\hat{S}[\tilde{g}_{\mu\nu}]$ is invariant under diffeomorphisms,

$$\delta\tilde{g}_{\mu\nu} = \mathcal{L}_v\tilde{g}_{\mu\nu} \equiv v^\rho\partial_\rho\tilde{g}_{\mu\nu} + \partial_\mu v^\rho\tilde{g}_{\rho\nu} + \partial_\nu v^\rho\tilde{g}_{\rho\mu}\,, \tag{1.3.9}$$

which after performing the background split can be written as

$$\delta\tilde{h}_{\mu\nu} = \mathcal{L}_v\tilde{g}_{\mu\nu} \quad \text{and} \quad \delta\bar{g}_{\mu\nu} = 0\,. \tag{1.3.10}$$

Here $\mathcal{L}_v$ is the Lie derivative along the vector field $v^\mu\partial_\mu$. These gauge transformations must be gauge-fixed to avoid over-counting seemingly distinct but physically indistinguishable metric configurations. A gauge-fixing condition $F_\mu[\tilde{h};\bar{g}] = 0$ is introduced into the path integral via the Fadeev-Popov procedure. This results in a ghost action which then appears alongside the bare action. The broken gauge symmetry of the path integral will eventually be communicated to the effective action via the generator of connected Green's functions, however we can restore diffeomorphism invariance to the effective action if we insist that it is invariant under the so-called background gauge transformations:

$$\delta\bar{g}_{\mu\nu} = \mathcal{L}_v\bar{g}_{\mu\nu} \quad \text{and} \quad \delta\tilde{h}_{\mu\nu} = \mathcal{L}_v\tilde{h}_{\mu\nu}\,. \tag{1.3.11}$$

These extra gauge choices are made possible thanks to the background field method.

Another key advantage of the this method is that it allows the construction of a covariant IR cutoff. In this gravitational context, the IR cutoff operator becomes a function of the covariant Laplacian associated with the background field: $R_k(-\bar{\nabla}^2) = R_k(-\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu)$. It is then with respect to the spectrum of $-\bar{\nabla}^2$ that fluctuation modes are compared to the cutoff scale k and are either integrated out or suppressed. A Laplacian of the total metric cannot be used as it would not preserve the structure of the flow equation as represented in (1.3.6). This fact actually turns out to be of key significance in the quest for background independence, an important issue which we shall return to shortly. Note that the ghost fields also come with their own IR cutoff.

Once the gauge fixing, ghost and cutoff terms have all been included in the functional integral alongside the bare action and source terms for all the fields, the effective average action is obtained by following the analogous steps described in Sect. 1.3 and which are explicitly laid out in [34]. The result is the effective average action for gravity [34]:

$$\Gamma_k[h, \bar{g}, \xi, \bar{\xi}], \tag{1.3.12}$$

where h is the classical fluctuation field, $\bar{g}$ is the background metric as before and $\xi, \bar{\xi}$ are the classical ghost fields. The crucial observation here is that the effective action depends *separately* on the background metric $\bar{g}$. This is due to the extra background field dependence of the ghost, gauge fixing and cutoff terms, which is in turn a consequence of using the background field method. As mentioned above, as long as the background gauge transformations (1.3.11) are obeyed, i.e. the background metric transforms as an ordinary tensor field $\delta\bar{g}_{\mu\nu} = \mathcal{L}_\nu \bar{g}_{\mu\nu}$, the effective action is a diffeomorphism invariant functional of its fields: $\Gamma_k[\Phi + \mathcal{L}_\nu \Phi] = \Gamma_k[\Phi]$ where $\Phi = \{h_{\mu\nu}, \bar{g}_{\mu\nu}, \xi^\mu, \bar{\xi}_\mu\}$.

The derivation of the flow equation for gravity goes through in much the same way as in the case of the scalar field (the explicit steps can be found in [34]). The result has the same general structure as (1.3.6) but with the right-hand side featuring a trace for both the fluctuation field h and ghosts $\xi, \bar{\xi}$ (with an additional minus sign for the anti-commuting ghost term). The functional derivatives in the traces are taken at fixed $\bar{g}$. Again, the UV cutoff on the functional integral drops out at the level of the flow equation due to the protective properties on the cutoff function R_k.

1.3.2 Background Independence

As pointed out already, an essential ingredient for any theory of gravity is background independence. Background independence is the requirement that a theory be free from any prior geometry; instead, the properties of the spacetime should emerge as a prediction of the theory. With this in mind, it might seem like a misstep to introduce dependence on a background metric through the background field method. However, by leaving the background metric completely unspecified, no background configuration plays a distinguished role in the construction of the flow equation. This means that the flow equation does not rely on the properties of any particular background field, implying that quantisation of the fluctuation $\tilde{h}$ occurs on all backgrounds simultaneously.[15] Nevertheless, the *solutions* of the flow equation do depend on the background. They are forced to carry separate dependence on the background metric $\bar{g}_{\mu\nu}$ through the cutoff operator $R_k(-\bar{\nabla}^2)$ as previously emphasized. Physics should depend only on the full metric, and not also on a background metric that was introduced by hand through the background field technique. This separate background dependence means that in general each background configuration would lead to different results for physical observables.

Not only do these solutions live in an appropriately enlarged theory space, spanned by operators of both the total metric and background metric, but the separate background field dependence makes further artificial enlargement of the theory space possible. A solution of the flow equation can be modified by an arbitrary scale-independent functional of the background field $\mathcal{F}[\bar{g}]$ such that the result $\Gamma_k[h, \bar{g}, \xi, \bar{\xi}] + \mathcal{F}[\bar{g}]$ is also a solution to the flow equation. This additional freedom, also introduced by hand through the background field method, needs to be controlled as well.

It is thus necessary to go beyond simply making sure the formalism does not depend on any particular background and to also somehow manage the separate background field dependence of the effective action. In most of the literature, the requirement of background independence refers only to the construction of the flow equation about an arbitrary background, whereas background independence in the sense that we mean it here is much more than this, and is in fact a strong extra constraint.

One way of circumventing these issues is to use the single field approximation.[16] This approximation consists of neglecting the evolution of the gauge-fixing and ghost sectors and setting $\bar{g} = g$ (equivalently, $h = 0$) in $\Gamma_k[h, \bar{g}]$ such that the effective action becomes a functional of only one field, namely the total metric g. Note that this can only be done once the functional derivatives in the trace have been performed as they are taken at fixed $\bar{g}$. With the solutions of the flow equation then just depending on the total metric, the aforementioned issues are bypassed.

The single field approximation has been employed in the majority of works in asymptotic safety to date. A severe drawback of this approximation however is that it cannot be used to explore the effects of background dependence as of course dependence on the background metric becomes invisible. This can lead to unphysical results as has been seen in the Local Potential Approximation (LPA) for scalar field theory [35] and obscures the significance of fixed point solutions at large field in the $f(R)$ approximation as emphasized in [23]. Instead, background dependence can only be investigated in bi-metric truncations in which dependence on both the full metric and the background metric is retained. For studies going beyond the single field approximation in different ways see [36–45].

Working within bi-metric truncations, and therefore being able to take full account of the effects of background dependence, requires us to find an alternative way to

---

[15] Even then, background independence of the formalism is not guaranteed due to the inherent background dependence of the RG scale k. See end of section for further discussion.

[16] Spoken about in more detail in Sect. 1.4.

manage the separate background dependence of the effective action. This can be achieved by imposing an additional constraint alongside the flow equation known as a modified split Ward Identity (msWI). (See Eq. (3.2.6) for an example of what the msWI looks like in the context of conformally reduced gravity.) Even though for all $k > 0$ background independence will inevitably be lost due to the cutoff, imposing the msWI in addition to the flow equation ensures that exact[17] background independence is recovered in the limit $k \to 0$ (the limit in which R_k drops out) after going "on-shell". This is imperative for the attainment of background independent physical observables as it is in this limit that the physical part of the effective action is recovered, as already explained at the start of this section. Furthermore, solutions of the flow equation do not automatically satisfy the msWI and in this way the msWI also controls the arbitrary enlargement of the theory space manufactured by the background split.

It is important to note that the msWI constraint is not an optional extra. It is derived from the same functional integral as the flow equation and therefore any set of (exact) solutions to the flow equation must also satisfy the msWI. In other words, the flow equation and msWI must be *compatible*. In Chap. 3 we prove that this is indeed true at the exact level, before any approximation to the effective action has been made. For approximate solutions, compatibility is not automatically guaranteed. We show that in the case of approximation, namely a derivative expansion up to $\mathcal{O}(\partial^2)$ for conformally reduced gravity, extra conditions must be placed on the form of the cutoff or the anomalous dimension in order to achieve compatibility.

An unsettling conclusion from the research reported in [27] was that fixed points with respect to the RG scale k are in general forbidden by the msWIs that are enforcing background independence. With hindsight, this can be seen as a useful signal that a background dependent description of quantum gravity does not make sense and a hint that there might be some deeper understanding of the meaning of RG in quantum gravity to be unearthed. For scalar field theory at the level of the LPA in [35] and later in the setting of conformally reduced gravity in [27], it was discovered that it is possible to combine the msWI and flow equation to uncover a background independent description of the entire flow, written in terms of background independent variables, including a background independent notion of the RG scale.

The employment of the msWI thus also remedies the issue of the ambiguity in the meaning of the scale k in a gravitational setting. Since it is the metric that provides us with the definition of length, the RG scale k (which can be equally thought of as some inverse length $1/k$) is inherently dependent on it. But moreover, in a quantum gravity theory, length scales fluctuate and so it is not clear what meaning should be ascribed to k or indeed scale dependence as expressed through the RG. Using the background field method alone does not resolve this issue since then k is defined with respect to modes of the covariant background field Laplacian $-\bar{\nabla}^2$ and becomes inherently dependent on the background metric instead.

[17] By exact we mean background independence in the strict sense defined previously.

1.4 Approximations

As emphasized in Sect. 1.3, it is usually impossible to solve the flow equation exactly and in order to actually make any progress we need to make an approximation for the effective average action. Making an approximation corresponds to truncating the theory space to some lower dimensional subspace and evaluating the flow equation there.[18] The subspace should be chosen in such a way that it is small enough to make calculations feasible but yet still big enough to capture the essential physics. Despite not retaining all the information within the full effective action (or equivalently, the path integral), approximations make computations manageable and prove a fruitful way to gain insights into important foundational issues in asymptotic safety. The purpose of this section is to introduce well-known and much-used approximation schemes, the majority of which are employed in the chapters to come.

1.4.1 The Einstein–Hilbert Truncation

The earliest truncation for which RG flows have been found is the Einstein–Hilbert truncation [34]:

$$\Gamma_k[h, \bar{g}, \xi, \bar{\xi}] = \frac{1}{16\pi G_k} \int d^4x \sqrt{g}\,(-R + 2\Lambda_k) + S_{\text{gf}}[h, \bar{g}] + S_{\text{gh}}[h, \bar{g}, \xi, \bar{\xi}]\,, \tag{1.4.1}$$

where the classical gauge fixing S_{gf} and ghost actions S_{gh} are chosen to be independent of k. This ansatz utilizes the single field approximation which, now stated more precisely, means that the evolution of the ghosts is neglected, it features no k-dependent piece for which $\bar{g} \neq g$ and as before, we set $h = 0$ once the Hessians have been computed. Most notably, (1.4.1) contains two parameters which are allowed to run with energy: the cosmological constant Λ_k and Newton's coupling G_k.

By inserting the ansatz into the flow equation, RG flows for the dimensionless Newton's coupling $\tilde{G}_k = k^2 G_k$ and dimensionless cosmological constant $\tilde{\Lambda}_k = k^{-2}\Lambda_k$ can be determined. This requires projecting the flow on to the chosen subspace of theory space. Let us briefly review how this is done in the general case of a theory space comprised of functionals of the form $\Gamma_k[\varphi] = \sum_{i=1} g_i(k)\mathcal{O}_i(\varphi)$. An approximation $\check{\Gamma}_k[\varphi]$ is made up of operators (perhaps infinitely many of them) belonging to the subspace only, for example $\check{\Gamma}_k[\varphi] = \sum_{j=1}^{N} g_j(k)\mathcal{O}_j(\varphi)$. The general idea is to expand the trace on the right-hand side of the flow equation with the ansatz $\check{\Gamma}_k$ inserted on the basis $\{\mathcal{O}_i\}$ of the full theory space i.e.

[18] One option is to do this by expanding the trace with respect to a small coupling, but of course this would only then allow us to explore the perturbative regime.

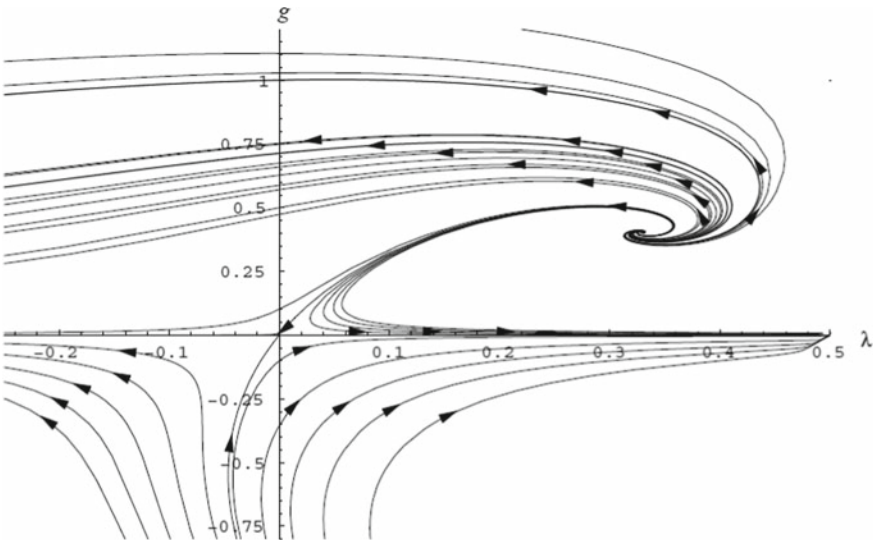

Fig. 1.4 Plot of RG flows from the Einstein–Hilbert truncation (1.4.1) in the $g - \lambda$ plane ($\tilde{G}_k - \tilde{\Lambda}_k$ in our notation). Reprinted figure with permission from [M. Reuter and F. Saueressig, Physical Review D, Vol. 65, 065016, 2002.] Copyright (2002) by the American Physical Society. https://dx.doi.org/10.1103/PhysRevD.65.065016

$$\frac{1}{2}\mathrm{Tr}[\cdots] = \sum_{i=1}^{\infty} \beta_i \mathcal{O}_i(\varphi) = \sum_{j=1}^{N} \beta_j \mathcal{O}_j(\varphi) + \text{rest} \tag{1.4.2}$$

and retaining only those terms contained within the subspace i.e. neglecting the "rest". Here $\beta = \beta(g_1, g_2, \ldots)$ are the beta functions for the couplings which, unlike the beta functions of perturbation theory, are not restricted to be functions of only small couplings. Equating (1.4.2) to the left-hand side of the flow equation, $\partial_k \check{\Gamma}_k = \sum_{j=1}^{N} \beta_j \mathcal{O}_j$, yields a system of N coupled ODEs for the couplings. Once these equations are solved, we say that the RG flow in the space of all couplings has been projected onto the N-dimensional subspace. Here we have used an approximation of the polynomial type as an example, but the same ideas apply to approximations involving full functionals as well; then instead of having coupled differential equations we obtain an evolution equation for the functional.[19]

Carrying out this procedure for the Einstein–Hilbert truncation gives rise to the flows displayed in Fig. 1.4. Notably the figure features two fixed points: a Gaussian one at the origin and a NGFP at positive values for both couplings. Whilst the employment of different cutoff types shifts the position of the NGFP, it continues

[19] In fact this highlights a computational advantage of polynomial truncations over those retaining a full functional: the flow equation for a polynomial truncation is simply an ODE in k yielding a finite number of relations for the couplings, whereas working with a full functional results in a partial differential equation which is technically more involved.

to be present for all cutoffs tested to date [16]. Furthermore, it is always found in the quadrant of positive $\tilde{G}_k$ and $\tilde{\Lambda}_k$ and is UV attractive for both couplings i.e. has two relevant directions. The Einstein–Hilbert truncation (1.4.1) predicts a NGFP with the desired properties for asymptotic safety and has been the subject of many studies within the community [34, 46–52]. However, to be sure that a reputed fixed point is not just an artifact of an insufficient approximation, we must go beyond the Einstein–Hilbert truncation.

1.4.2 Polynomial Truncations

The natural next step is to explore less severe truncations. These so-called polynomial truncations keep successively higher powers of the scalar curvature R and have to date included all powers up to R^{34} [53–55]. In all cases asymptotically safe fixed points have been found. This is encouraging, but it is easy to be misled into thinking fixed points exist as past studies have shown. For example, in the LPA for a single-component scalar field, spurious fixed points have been found to persist in polynomial truncations of the potential to very high order. These fixed points can then be shown to disappear when the full potential is considered [56]. Another example is given by [57] which analysed the RG properties of U(1) theory in three dimensions using the approximation $f(F^2_{\mu\nu})$. There again non-Gaussian fixed points were found for $f(F^2_{\mu\nu})$ truncated to a polynomial, whereas using the full function resulted in no such fixed points.

We find that even though careful treatment of polynomial approximations taken to high order can allow extraction of convergent results, one does not see in this way the singularities at finite field or asymptotic behaviour at diverging field which are actually responsible for determining their high order behaviour. Indeed such large field effects can invalidate deductions from polynomial truncations [56–58] and/or restrict or even exclude the existence of global solutions [13, 59–61]. Another good example is provided by some of the most impressive evidence for asymptotic safety to date: the polynomial expansions up to R^{34}. These are however derived from a differential equation for an $f(R)$ fixed point Lagrangian [51] which was shown in [24] to have no global solutions as a consequence of fixed singularities at finite field.

Furthermore, since fixed points are effectively the solutions of polynomial equations in the couplings, they only allow for discrete solutions. But physical systems exist with lines or even higher dimensional surfaces of fixed points, parameterised by exactly marginal couplings (in supersymmetric theories these are common and called moduli). Moreover, lines and planes of fixed points have been found in other approximations within asymptotic safety [23–25].[20]

[20] And in a perhaps related approximation in scalar-tensor gravity [62].

As well as this, by construction, polynomial truncations only deal with small curvatures, which has to be the case for an expansion in powers of R to make sense. This means that polynomial truncations are insensitive to strong curvature effects and the deep non-perturbative regime of quantum gravity that we are ultimately interested in.

1.4.3 The $f(R)$ Approximation

In order to have confidence that asymptotically safe fixed points exist we must therefore go beyond even polynomial truncations to approximations that keep an infinite number of couplings. Arguably the simplest such approximation is to keep all powers of the scalar curvature, making the ansatz

$$\Gamma_k[g] = \int d^4x \sqrt{g}\, f_k(R)\,. \tag{1.4.3}$$

This is called the $f(R)$ approximation and has been investigated in many works [51, 63–75].[21] Inserting such an approximation into the flow equation results in a non-linear partial differential equation which governs the evolution of $f_k(R)$ with changes in the RG scale k. At fixed points, where the k-dependence drops out, it reduces to an ODE of either second or third order (depending on the cutoff scheme used). See Eq. (4.2.1) in Chap. 4 for an example written in terms of scaled variables, $\varphi(r) := k^4 f(Rk^{-2})$.

In the $f(R)$ approximation we are no longer restricted to small curvatures, however this then raises the question: what significance should we attach to the behaviour of $f_k(R)$ for $R \gg 1$? Since then the size of the spacetime is much smaller than the cutoff $1/k$. This puzzle is addressed and resolved in [73] and also discussed in more detail in the introduction to Chap. 4.

Finally, as already hinted at above in Polynomial truncations, in order to ascertain the true nature of fixed points it is crucial to explore the regime of large scaled curvature: $r \to \infty$, i.e. to develop the asymptotic solutions. We could have already guessed that the behaviour of solutions in this limit is important to understand since for fixed background curvature[22] R it corresponds to the limit in which the physical effective action is recovered, $k \to 0$. These asymptotic solutions are the central topic of Chap. 4.

[21] In fact, to date this is the only such approximation that has been investigated, together with some closely related approximations in scalar-tensor [76, 77] and unimodular [78] gravity, and in three space-time dimensions [79].

[22] Here we commit a slight abuse of notation as, at the level of the projected flow equation, R now represents the background curvature which emerges from employing the single field approximation.

1.4.4 Conformally Reduced Gravity

Conformally reduced gravity is the regime in which only fluctuations of the conformal factor of the metric are quantised. A small number of works have studied it using the exact RG, starting with Ref. [80]. To arrive at conformally reduced gravity we only consider a subset of metrics that are conformally equivalent to some fixed reference metric $\hat{g}_{\mu\nu}$:

$$\tilde{g}_{\mu\nu} = f(\tilde{\phi})\hat{g}_{\mu\nu}\,. \tag{1.4.4}$$

Here $\tilde{\phi}$ is the total conformal factor field and f is some choice of parameterisation. It is then the fluctuation field $\tilde{\phi}$ that is integrated over in the path integral. This leads to a scalar-like theory and a simpler model than say $f(R)$ for investigating the effects of background dependence, and is of particular relevance to Chap. 3.

Recent investigations in conformally reduced gravity have shed light on important foundational issues in asymptotic safety which deserve some comment. Even though conformally reduced gravity and standard 4-dimensional scalar theory are very similar in structure (after all the conformal factor is a single-component scalar field), the flow equation for the former comes with an additional minus sign, a result of the conformal factor problem already mentioned below Eq. (1.1.1). As is well-known, the Euclidean signature functional integral for the Einstein–Hilbert action suffers from this problem [81], which is that the negative sign for the kinetic term of the conformal factor yields a wrong-sign Gaussian destroying convergence of the integral. At first sight, providing the cutoff is adapted, the change in sign seems not to pose any special problem for the exact RG equation [34]. However as is shown in [23], this one sign change has profound consequences for the RG properties of the solutions, broadly resulting in a continuum of fixed points supporting both a discrete and a continuous eigenoperator spectrum.

The conclusions reached in [23] seem to be strongly at variance with the asymptotic safety literature where a single fixed point with a handful of relevant directions (typically three) is found.[23] The great majority of work in the literature however focuses on the single field approximation and/or polynomial truncations which can obscure the effects of the conformal factor problem; whereas, in [23] use of these type of approximations was avoided—the only approximations used were that of conformally reduced gravity itself and the slow field limit for the background field—and furthermore, background independence was incorporated. Further work is needed to understand whether this picture persists when working with the full metric; perhaps this might qualitatively alter the results.

[23] Actually a continuum of fixed points supporting a continuous spectra of eigenoperators has been found for the $f(R)$ approximation already in [24].

1.4.5 *The Derivative Expansion*

The derivative expansion is an approximation originally developed for scalar field theory [59] and as such can be straightforwardly applied to conformally reduced gravity. It consists of expanding an action in powers of derivatives of the field. For standard scalar field theory, an expansion of the effective average action up to the third order looks like

$$\Gamma_k[\varphi] = \int d^dx \Big\{ V(\varphi, t) + \frac{1}{2} K(\varphi, t)(\partial_\mu \varphi)^2 + H_1(\varphi, t)(\partial_\mu \varphi)^4 + H_2(\varphi, t)(\Box\varphi)^2 + H_3(\varphi, t)(\partial_\mu \varphi)^2(\Box\varphi) + \cdots \Big\}, \quad (1.4.5)$$

which in momentum space amounts to an expansion in powers of momenta.

The leading order of the derivative expansion is the LPA, introduced in [82] and since rediscovered by many authors e.g. [56, 83, 84]. This functional truncation keeps a general potential $V(\varphi)$ for the field and therefore incorporates infinitely many operators. When keeping all components of the metric tensor, the $f(R)$-approximation is as close to the LPA as one can get, as emphasized in [64]. We make use of the LPA, and more generally the derivative expansion, in Chap. 3 in the setting of conformally reduced gravity.

Let us close this section by remarking that in practice expanding the trace and extracting the terms belonging to the subspace of an approximation is a rather involved technical process. The background metric is often fixed to be that of a four-sphere to simplify calculations.[24] A transverse-traceless decomposition of the fluctuation field $\tilde{h}_{\mu\nu}$ is performed to facilitate the computation of the inverse Hessian on the right-hand side of the flow equation and this introduces new fields. Also to facilitate computation, different types of cutoffs are used, e.g. a type I cutoff where R_k is just a function of $-\bar{\nabla}^2$ as in Sect. 1.3, or a type II cutoff, $R_k = R_k(-\bar{\nabla}^2 + E)$, where E is a non-trivial endomorphism [51]. Type II cutoffs allow flexibility in how different modes are integrated out and will appear again in Chap. 4. The spacetime trace in the flow equation itself is evaluated by a type of heat kernel expansion. Finally, solving the differential equations resulting from the projection often entails a combination of analytical and numerical methods.

1.5 Thesis Outline

Each of the following three chapters focuses on a different fundamental aspect of asymptotic safety. In Chap. 2 we consider the reconstruction problem. As explained in Sect. 1.3, this is the problem of how to recovery a path integral formulation of a theory from the effective average action. Presenting the research of [33], we provide

[24] But note that there is no conceptual necessity for this and final results should be independent of the choice of background metric.

two exact solutions to this problem and understand how they compare to a one-loop approximate solution in the existing literature. In Chap. 3 we present the work of [28] in which the fundamental requirement of background independence in quantum gravity is addressed. Working within the derivative expansion of conformally reduced gravity, we explore the notion of compatibility (introduced in Sect. 1.3.2) and uncover the underlying reasons for background dependence generically forbidding fixed points in such models, extending the work of [27]. As emphasized in Sect. 1.4.3, in order to understand the true nature of fixed point solutions it is necessary to study their asymptotic behaviour. Chapter 4 presents the work of [26] in which we explain how to find the asymptotic form of fixed point solutions in the $f(R)$ approximation. In the fifth and final chapter we give a brief summary of the research presented in Chaps. 2–4, discussing the significance of the key findings and commenting on useful extensions of the work. We finish by considering the need to incorporate matter into the formalism in a compatible way and touch upon potential opportunities to test asymptotic safety in the future.

References

1. Einstein A (1915) The field equations of gravitation. Sitzungsber Preuss Akad Wiss Berlin (Math Phys) 1915:844–847
2. Donoghue JF (1994) General relativity as an effective field theory: the leading quantum corrections. Phys Rev D 50:3874–3888. arXiv:gr-qc/9405057
3. 't Hooft G, Veltman MJG (1974) One loop divergencies in the theory of gravitation. Ann Inst H Poincare Phys Theor A 20:69–94
4. Goroff MH, Sagnotti A (1985) Quantum gravity at two loops. Phys Lett B 160:81–86
5. Goroff MH, Sagnotti A (1986) The ultraviolet behavior of einstein gravity. Nucl Phys B 266:709–736
6. van de Ven AEM (1992) Two loop quantum gravity. Nucl Phys B 378:309–366
7. Stelle KS (1977) Renormalization of higher derivative quantum gravity. Phys Rev D 16:953–969
8. Weinberg S (1980) Ultraviolet divergences in quantum theories of gravitation. In: Hawking SW, Israel W (eds) General relativity. Cambridge University Press, pp 790–831
9. Wilson KG (1971) Renormalization group and critical phenomena. I. Renormalization group and the Kadanoff scaling picture. Phys Rev B 4:3174–3183
10. Kadanoff LP (1966) Scaling laws for Ising models near T(c). Physics 2:263–272
11. Polchinski J (1984) Renormalization and effective Lagrangians. Nucl Phys B 231:269–295
12. Morris TR (1994) The exact renormalization group and approximate solutions. Int J Mod Phys A 09:2411–2450. arXiv:hep-ph/9308265
13. Morris TR (1998) Elements of the continuous renormalization group. Prog Theor Phys Suppl 131:395–414. arXiv:hep-th/9802039
14. Bervillier C (2013) The Wilson exact renormalization group equation and the anomalous dimension parameter. Condens Matter Phys 16:23003. arXiv:1304.4131
15. Hasenfratz P, Niedermayer F (1994) Perfect lattice action for asymptotically free theories. Nucl Phys B 414:785–814. arXiv:hep-lat/9308004
16. Reuter M, Saueressig F (2012) Quantum Einstein gravity. New J Phys 14:055022. arXiv:1202.2274
17. Percacci R (2011) A short introduction to asymptotic safety. In: Time and matter: proceedings, 3rd international conference, TAM2010, Budva, Montenegro, 4–8 Oct 2010, pp 123–142. arXiv:1110.6389

18. Niedermaier M, Reuter M (2006) The asymptotic safety scenario in quantum gravity. Living Rev Rel 9:5–173
19. Nagy S (2014) Lectures on renormalization and asymptotic safety. Annals Phys 350:310–346. arXiv:1211.4151
20. Litim DF (2011) Renormalisation group and the Planck scale. Phil Trans Roy Soc Lond A 369:2759–2778. arXiv:1102.4624
21. Percacci R (2017) An introduction to covariant quantum gravity and asymptotic safety. 100 years of general relativity, vol 3 . World Scientific
22. Reuter M, Saueressig F (2019) Quantum gravity and the functional renormalization group. Cambridge University Press
23. Dietz JA, Morris TR, Slade ZH (2016) Fixed point structure of the conformal factor field in quantum gravity. Phys Rev D 94(12):124014. arXiv:1605.0763
24. Dietz JA, Morris TR (2013) Asymptotic safety in the f(R) approximation. JHEP 01:108. arXiv:1211.0955
25. Dietz JA, Morris TR (2013) Redundant operators in the exact renormalisation group and in the f(R) approximation to asymptotic safety. JHEP 07:064. arXiv:1306.1223
26. Gonzalez-Martin S, Morris TR, Slade ZH (2017) Asymptotic solutions in asymptotic safety. Phys Rev D 95(10):106010. arXiv:1704.0887
27. Dietz JA, Morris TR (2015) Background independent exact renormalization group for conformally reduced gravity. JHEP 04:118. arXiv:1502.0739
28. Labus P, Morris TR, Slade ZH (2016) Background independence in a background dependent renormalization group. Phys Rev D 94(2):024007. arXiv:1603.0477
29. Litim DF (2000) Optimization of the exact renormalization group. Phys Lett B 486:92–99. arXiv:hep-th/0005245
30. Litim DF (2001) Optimized renormalization group flows. Phys Rev D 64:105007. arXiv:hep-th/0103195
31. Litim DF (2001) Mind the gap. Int J Mod Phys A 16:2081–2088. arXiv:hep-th/0104221
32. Wetterich C (1993) Exact evolution equation for the effective potential. Phys Lett B 301:90–94
33. Morris TR, Slade ZH (2015) Solutions to the reconstruction problem in asymptotic safety. JHEP 11:094. arXiv:1507.0865
34. Reuter M (1998) Nonperturbative evolution equation for quantum gravity. Phys Rev D 57:971–985. arXiv:hep-th/9605030
35. Bridle IH, Dietz JA, Morris TR (2014) The local potential approximation in the background field formalism. JHEP 03:093. arXiv:1312.2846
36. Manrique E, Reuter M, Saueressig F (2011) Bimetric renormalization group flows in quantum Einstein gravity. Annals Phys 326:463–485. arXiv:1006.0099
37. Manrique E, Reuter M (2010) Bimetric truncations for quantum Einstein gravity and asymptotic safety. Annals Phys 325:785–815. arXiv:0907.2617
38. Becker D, Reuter M (2014) En route to background independence: broken split-symmetry, and how to restore it with bi-metric average actions. Annals Phys 350:225–301. arXiv:1404.4537
39. Manrique E, Reuter M, Saueressig F (2011) Matter induced bimetric actions for gravity. Annals Phys 326:440–462. arXiv:1003.5129
40. Codello A, D'Odorico G, Pagani C, Percacci R (2013) The renormalization group and weyl-invariance. Class Quant Grav 30:115015. arXiv:1210.3284
41. Christiansen N, Knorr B, Pawlowski JM, Rodigast A (2016) Global flows in quantum gravity. Phys Rev D 93(4):044036. arXiv:1403.1232
42. Groh K, Saueressig F (2010) Ghost wave-function renormalization in asymptotically safe quantum gravity. J Phys A 43:365403. arXiv:1001.5032
43. Eichhorn A, Gies H (2010) Ghost anomalous dimension in asymptotically safe quantum gravity. Phys Rev D 81:104010. arXiv:1001.5033
44. Donà P, Eichhorn A, Percacci R (2014) Matter matters in asymptotically safe quantum gravity. Phys Rev D 89(8):084035. arXiv:1311.2898
45. Donà P, Eichhorn A, Percacci R (2015) Consistency of matter models with asymptotically safe quantum gravity. arXiv:1410.4411

46. Lauscher O, Reuter M (2002) Ultraviolet fixed point and generalized flow equation of quantum gravity. Phys Rev D 65:025013. arXiv:hep-th/0108040
47. Reuter M, Saueressig F (2002) Renormalization group flow of quantum gravity in the Einstein–Hilbert truncation. Phys Rev D 65:065016. arXiv:hep-th/0110054
48. Souma W (1999) Nontrivial ultraviolet fixed point in quantum gravity. Prog Theor Phys 102:181–195. arXiv:hep-th/9907027
49. Litim DF (2004) Fixed points of quantum gravity. Phys Rev Lett 92:201301. arXiv:hep-th/0312114
50. Fischer P, Litim DF (2006) Fixed points of quantum gravity in extra dimensions. Phys Lett B 638:497–502. arXiv:hep-th/0602203
51. Codello A, Percacci R, Rahmede C (2009) Investigating the ultraviolet properties of gravity with a Wilsonian renormalization group equation. Annals Phys 324:414–469. arXiv:0805.2909
52. Benedetti D, Groh K, Machado PF, Saueressig F (2011) The universal RG machine. JHEP 06:079. arXiv:1012.3081
53. Falls K, Litim DF, Nikolakopoulos K, Rahmede C (2013) A bootstrap towards asymptotic safety. arXiv:1301.4191
54. Falls K, Litim DF, Nikolakopoulos K, Rahmede C (2016) Further evidence for asymptotic safety of quantum gravity. Phys Rev D 93(10):104022. arXiv:1410.4815
55. Falls K, Litim DF, Nikolakopoulos K, Rahmede C (2018) On de Sitter solutions in asymptotically safe $f(R)$ theories. Class Quant Grav 35(13):135006. arXiv:1607.0496
56. Morris TR (1994) On truncations of the exact renormalization group. Phys Lett B 334:355–362. arXiv:hep-th/9405190
57. Morris TR (1995) Noncompact pure gauge QED in 3-D is free. Phys Lett B 357:225–231. arXiv:hep-th/9503225
58. Morris TR (1996) On the fixed point structure of scalar fields. Phys Rev Lett 77:1658. arXiv:hep-th/9601128
59. Morris TR (1994) Derivative expansion of the exact renormalization group. Phys Lett B 329:241–248. arXiv:hep-ph/9403340
60. Morris TR (1995) The renormalization group and two-dimensional multicritical effective scalar field theory. Phys Lett B 345:139–148. arXiv:hep-th/9410141
61. Morris TR, Turner MD (1998) Derivative expansion of the renormalization group in O(N) scalar field theory. Nucl Phys B 509:637–661. arXiv:hep-th/9704202
62. Benedetti D, Guarnieri F (2014) Brans-Dicke theory in the local potential approximation. New J Phys 16:053051. arXiv:1311.1081
63. Machado PF, Saueressig F (2008) On the renormalization group flow of f(R)-gravity. Phys Rev D 77:124045. arXiv:0712.0445
64. Benedetti D, Caravelli F (2012) The local potential approximation in quantum gravity. JHEP 1206:017. arXiv:1204.3541
65. Demmel M, Saueressig F, Zanusso O (2012) Fixed-functionals of three-dimensional quantum Einstein gravity. JHEP 11:131. arXiv:1208.2038
66. Demmel M, Saueressig F, Zanusso O (2015) Fixed functionals in asymptotically safe gravity. In: Proceedings, 13th Marcel Grossmann meeting on recent developments in theoretical and experimental general relativity, astrophysics, and relativistic field theories (MG13), Stockholm, Sweden, 1–7 July 2012, pp 2227–2229. arXiv:1302.1312
67. Benedetti D (2013) On the number of relevant operators in asymptotically safe gravity. Europhys Lett 102:20007. arXiv:1301.4422
68. Demmel M, Saueressig F, Zanusso O (2015) RG flows of quantum Einstein gravity in the linear-geometric approximation. Annals Phys 359:141–165. arXiv:1412.7207
69. Demmel M, Saueressig F, Zanusso O (2015) A proper fixed functional for four-dimensional quantum Einstein gravity. JHEP 08:113. arXiv:1504.0765
70. Ohta N, Percacci R, Vacca GP (2015) Flow equation for $f(R)$ gravity and some of its exact solutions. Phys Rev D 92(6):061501. arXiv:1507.0096
71. Ohta N, Percacci R, Vacca GP (2016) Renormalization group equation and scaling solutions for f(R) gravity in exponential parametrization. Eur Phys J C 76(2):46. arXiv:1511.0939

72. Percacci R, Vacca GP (2017) The background scale Ward identity in quantum gravity. Eur Phys J C 77(1):52. arXiv:1611.0700
73. Morris TR (2016) Large curvature and background scale independence in single-metric approximations to asymptotic safety. JHEP 11:160. arXiv:1610.0308
74. Falls K, Ohta N (2016) Renormalization group equation for $f(R)$ gravity on hyperbolic spaces. Phys Rev D 94(8):084005. arXiv:1607.0846
75. Ohta N (2017) Background scale independence in quantum gravity. PTEP 2017(3):033E02. arXiv:1701.0150
76. Percacci R, Vacca GP (2015) Search of scaling solutions in scalar-tensor gravity. Eur Phys J C 75(5):188. arXiv:1501.0088
77. Labus P, Percacci R, Vacca GP (2016) Asymptotic safety in $O(N)$ scalar models coupled to gravity. Phys Lett B 753:274–281. arXiv:1505.0539
78. Eichhorn A (2015) The renormalization group flow of unimodular f(R) gravity. JHEP 04:096. arXiv:1501.0584
79. Demmel M, Saueressig F, Zanusso O (2014) RG flows of quantum Einstein gravity on maximally symmetric spaces. JHEP 06:026. arXiv:1401.5495
80. Reuter M, Weyer H (2009) Background independence and asymptotic safety in conformally reduced gravity. Phys Rev D 79:105005. arXiv:0801.3287
81. Gibbons G, Hawking S, Perry M (1978) path integrals and the indefiniteness of the gravitational action. Nucl Phys B 138:141
82. Nicoll JF, Chang TS, Stanley HE (1974) Approximate renormalization group based on the Wegner-Houghton differential generator. Phys Rev Lett 33:540–543
83. Hasenfratz A, Hasenfratz P (1986) Renormalization group study of scalar field theories. Nucl Phys B 270:687–701
84. Morris TR (1996) Momentum scale expansion of sharp cutoff flow equations. Nucl Phys B 458:477–503. arXiv:hep-th/9508017