Stable Coalition Structures in Dynamic Competitive Environment

Elena Parilina and Artem Sedakov

Abstract We consider a finite horizon dynamic competition model in discrete time in which firms are not restricted from cooperation with each other and can form coalitions of any size. For every coalition of firms, we determine profits of its members by two approaches: without the redistribution of profits inside the coalition and with such redistribution using a solution from cooperative game theory. Next, for each approach we examine the stability of a coalition structure in the game. When we find a stable coalition structure, we then verify whether it is dynamically stable, that is, stable over time with respect to the same profit distribution method chosen in the initial time period.

Keywords Dynamic competition · Coalition structure · Stability

1 Introduction

In the chapter we consider a dynamic competition model, in which firms choose their outputs in each time period. The market price is formed based on the decision of firms and on the price in the previous time period. We assume that the level of influence of the previous period price depends on the market state. Having this competitive model, we make an assumption that firms may cooperate in coalitions of any size forming a coalition structure. If the coalition structure is formed, each firm acts to maximize the profit of the coalition it belongs to. If the firms are supposed to have non-transferable profits, they are paid by initially given payoff functions. But if firms' profits are transferable, a cooperative point solution which redistributes the profits between firms is calculated. In both cases, a firm may have an interest in deviating from a coalition it belongs by joining another coalition or becoming a singleton. If no firm has a profitable deviation from its coalition, the coalition

E. Parilina (✉) · A. Sedakov
Saint Petersburg State University, Saint Petersburg, Russia
e-mail: e.parilina@spbu.ru; a.sedakov@spbu.ru

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structure is called stable (Parilina and Sedakov 2014; Sedakov et al. 2013). The stability of a coalition structure is determined with respect to a profit distribution method. In the definition of a stable coalition structure one may find similarity with the Nash equilibrium concept. The existence of a stable coalition structure with respect to the Shapley value in three-person games is proved in Sedakov et al. (2013). In four-person games the existence of a stable coalition structure is proved for special classes of transferable utility games (TU games). It is shown that a stable coalition structure may not exist in general (Sun and Parilina 2018).

The problem of stability of a coalition structure is actual in many applied problems. When the coalition structure is unstable, it is difficult to keep it the same over time and realize the game without changing the structure. There also exist other approaches to determine the stability of a coalition structure (e.g., see Carraro 1999). In the abovementioned paper, to be stable the coalition structure should be (i) internally stable, i.e., each player loses if he leaves his coalition becoming a singleton, (ii) externally stable, i.e., each player-singleton loses if he joins any coalition or another singleton, and, finally, (iii) intracoalitionally stable, i.e., each player from a coalition loses if he leaves his coalition and joins another one. In Parilina and Sedakov (2015) a process of changing coalition structures over time is considered. The authors introduce the concept of d-stability of a coalition structure, which players would never change once it is reached.

Even if the coalition structure is stable in the whole game, i.e., the stability conditions are satisfied in the initial time period, it may become unstable on the corresponding equilibrium state trajectory in some intermediate time period. Therefore, we define a *dynamically stable* coalition structure which is stable not only in the game but in any subgame starting from any intermediate time period and the corresponding state.

In this chapter we examine a competition model with finite time horizon and linear-quadratic profit functions of firms-competitors (see Carlson and Leitmann 2005). The firms are allowed to cooperate by forming a coalition structure. We determine the conditions for firms' strategies to form an open-loop coalition Nash equilibrium. By a coalition Nash equilibrium we mean a Nash equilibrium among players-coalitions in the given coalition structure. There are two options to determine firms' profits in the game. If the profits are non-transferable, the firms are paid according to their initially given payoff functions. If they are transferable, we determine the characteristic function according to the concepts in Chander and Tulkens (1997) and Rajan (1989). Based on the characteristic function, a cooperative point solution is defined using the Shapley value adopted for the games with a given coalition structure (Aumann and Dreze 1974; Shapley 1953). We determine a stable coalition structure for the initial time period and a dynamically stable coalition structure. As an example, a game with three firms is considered which admits five possible coalition structures. Interestingly, in the case of non-transferable payoffs, there are no stable coalition structures, but when firms redistribute their profits according to the Shapley value, there exists a unique stable coalition structure which is not the grand coalition. We then verify that this structure is also dynamically stable.

The chapter is organized as follows. Section 2 presents the theoretical model of the dynamic game and the conditions of a coalition equilibrium. In Sect. 3, we formulate the concept of stability for a coalition structure when profits are both non-transferable or transferable. In the latter case, the Shapley value is chosen as a cooperative point solution. We provide an illustrative example in Sect. 4, and briefly conclude in Sect. 5.

2 The Model

We consider a market of firms composing a finite set N with $|N| = n \geqslant 2$. Producing and selling a product, firms compete in quantities over a finite set of periods $\mathscr{T} = \{0, 1, \ldots, T\}$ with the initial market price p_0 for the product. In each period $t \in \mathscr{T} \setminus T$, a firm $i \in N$ selects its quantity $q_i(p_0, t) \in \mathbb{R}_+$ to be produced for this period. A market price $p(t) \in \mathbb{R}_+$ satisfies the *state equation*

$$p(t+1) = sp(t) + (1-s)\left(a - b\sum_{i \in N} q_i(p_0, t)\right), \quad t \in \mathscr{T} \setminus T, \tag{1}$$

with the initial state $p(0) = p_0$. For a given $s \in [0, 1]$, the first summand in the r.h.s. of (1) represents the inertia in the market price while the second one reflects the price change as a reaction on produced output for some positive constants a and b. Under an open-loop information structure (Haurie et al. 2012), an *open-loop strategy* of firm i is a profile of quantities $q_i(p_0) = (q_i(p_0, 0), \ldots, q_i(p_0, T-1))$ which i decides to produce during the planning horizon. Denote a *strategy profile* by $q(p_0) = (q_1(p_0), \ldots, q_n(p_0))$. Each firm i aims to maximize its total discounted profit of the form

$$\pi_i(p_0, q(p_0)) = \sum_{t=0}^{T-1} \varrho^t \left[p(t)q_i(p_0, t) - \frac{c_i}{2} q_i^2(p_0, t)\right]$$

adopting its strategy $q_i(p_0)$, where $p(t)$ satisfies state equation (1) with initial state $p(0) = p_0$. A parameter $c_i > 0$ reflects firm i's unit cots and $\varrho \in (0, 1]$ is a common discount factor. In period T players have zero payoffs.

From now, we assume that firms are not restricted in cooperating with each other and can form any *coalition*, which is a nonempty subset of N. A partition $\mathscr{B} = \{B_1, \ldots, B_m\}$ of set N is called a *coalition structure*. A strategy profile under the structure $\mathscr{B}$ will be denoted by $q^{\mathscr{B}}(p_0)$. A strategy of a coalition $B \in \mathscr{B}$ is a profile $q_B^{\mathscr{B}}(p_0) = \{q_i^B(p_0), i \in B\}$. Given a coalition structure $\mathscr{B}$, a strategy profile can then be written in terms of the structure, i.e., $q^{\mathscr{B}}(p_0) = \{q_B^{\mathscr{B}}(p_0), B \in \mathscr{B}\}$. Under the coalition structure, the aim of each firm is to maximize the profit of the coalition to which it belongs. More formally, jointly selecting a profile $q_B^{\mathscr{B}}(p_0)$, all firms from coalition $B \in \mathscr{B}$ maximize the sum $\pi_B^{\mathscr{B}}(p_0, q^{\mathscr{B}}(p_0)) = \sum_{i \in B} \pi_i(p_0, q^{\mathscr{B}}(p_0))$ subject to the state Eq. (1) with $p(0) = p_0$.

Definition 1 A profile $\bar{q}^{\mathscr{B}}(p_0)$ is an *open-loop coalition Nash equilibrium* (or simply coalition Nash equilibrium) if

$$\pi_B^{\mathscr{B}}(p_0, \bar{q}^{\mathscr{B}}(p_0)) \geqslant \pi_B^{\mathscr{B}}(p_0, (q_B^{\mathscr{B}}(p_0), \bar{q}_{N\setminus B}^{\mathscr{B}}(p_0)))$$

for any coalition $B \in \mathscr{B}$ and its strategy $q_B^{\mathscr{B}}(p_0)$. Alternatively, $\bar{q}^{\mathscr{B}}(p_0)$ satisfies

$$\bar{q}_B^{\mathscr{B}}(p_0) = \arg\max_{q_B^{\mathscr{B}}(p_0)} \pi_B^{\mathscr{B}}(p_0, (q_B^{\mathscr{B}}(p_0), \bar{q}_{N\setminus B}^{\mathscr{B}}(p_0)))$$

for any $B \in \mathscr{B}$.

In particular, when $\mathscr{B} = \{\{1\}, \dots, \{n\}\}$, that is, all coalitions in coalition structure $\mathscr{B}$ are singletons, the coalition Nash equilibrium is a *Nash equilibrium*, while when $\mathscr{B} = \{N\}$, that is, all firms cooperate in one coalition, the coalition equilibrium is a *cooperative optimum*. A sequence of market prices $\bar{p}^{\mathscr{B}} = \{\bar{p}^{\mathscr{B}}(0) \equiv p_0, \bar{p}^{\mathscr{B}}(1), \dots, \bar{p}^{\mathscr{B}}(T)\}$ uniquely determined by coalition equilibrium $\bar{q}^{\mathscr{B}}(p_0)$ and state equation (1) is a *coalition equilibrium trajectory*. A coalition equilibrium trajectory determined by a cooperative optimum $\bar{q}^{\{N\}}(p_0)$ is a *cooperative trajectory* denoted by $\bar{p}^{\{N\}}$. Next, we can define the profit of firm i under a coalition equilibrium $\bar{q}^{\mathscr{B}}(p_0)$, which is $\pi_i(p_0, \bar{q}^{\mathscr{B}}(p_0))$. Similarly, we define firm i's *cooperative profit* $\pi_i(p_0, \bar{q}^{\{N\}}(p_0))$, i.e., its profit under a cooperative optimum $\bar{q}^{\{N\}}(p_0)$.

Let $\Gamma^{\mathscr{B}}(p_0)$ denote the dynamic game over the set of periods $\mathscr{T}$ with coalition structure $\mathscr{B}$ starting in state p_0. We now characterize an open-loop coalition Nash equilibrium in this game. A similar infinite-horizon two-person non-cooperative model is examined in Carlson and Leitmann (2005) for open-loop strategies. One can study this problem also by assuming a feedback information structure. However, to find the corresponding feedback coalition Nash equilibrium, one needs to assume the form of value functions.

Theorem 1 *Under a coalition structure $\mathscr{B}$, an open-loop coalition Nash equilibrium $\bar{q}^{\mathscr{B}}$ is composed of the following strategies:*

$$\bar{q}_i^{\mathscr{B}}(p_0, t) = \frac{1}{c_i}\left[\bar{p}^{\mathscr{B}}(t) - \varrho b(1-s)\mu_B^{\mathscr{B}}(t+1)\right], \quad i \in B, \quad t \in \mathscr{T} \setminus T, \tag{2}$$

where $\bar{p}^{\mathscr{B}}(t)$ and $\mu_B^{\mathscr{B}}(t)$, $B \in \mathscr{B}$, satisfy the recursive relations:

$$\bar{p}^{\mathscr{B}}(t) = s\bar{p}^{\mathscr{B}}(t-1) + (1-s)\left(a - b\sum_{i \in N} \bar{q}_i^{\mathscr{B}}(p_0, t-1)\right), \quad t \in \mathscr{T} \setminus 0,$$

$$\mu_B^{\mathscr{B}}(t) = \sum_{i \in B} \bar{q}_i^{\mathscr{B}}(p_0, t) + \varrho s \mu_B^{\mathscr{B}}(t+1), \quad t \in \mathscr{T} \setminus \{0, T\},$$

with $\bar{p}^{\mathscr{B}}(0) = p_0$ *and* $\mu_B^{\mathscr{B}}(T) = 0$ *for any* $B \in \mathscr{B}$.

Proof For a coalition $B \in \mathscr{B}$, we define the Hamiltonian $\mathscr{H}_B^{\mathscr{B}}$:

$$\mathscr{H}_B^{\mathscr{B}} = \sum_{i \in B} \varrho^t \left[p^{\mathscr{B}}(t) q_i^{\mathscr{B}}(p_0, t) - \frac{c_i}{2} (q_i^{\mathscr{B}}(p_0, t))^2 \right]$$
$$+ \lambda_B^{\mathscr{B}}(t+1) \left[s p^{\mathscr{B}}(t) + (1-s) \left(a - b \sum_{i \in N} q_i^{\mathscr{B}}(p_0, t) \right) \right],$$

where $\lambda_B^{\mathscr{B}}(t+1)$ is a costate variable. From the maximum principle, for any coalition $B \in \mathscr{B}$, the following is true:

$$\frac{\partial \mathscr{H}_B^{\mathscr{B}}}{\partial q_i^{\mathscr{B}}(p_0, t)} = \varrho^t \left[p^{\mathscr{B}}(t) - c_i q_i^{\mathscr{B}}(p_0, t) \right] - (1-s) b \lambda_B^{\mathscr{B}}(t+1)$$
$$= 0, \; i \in B, \; t \in \mathscr{T} \setminus T,$$
$$\frac{\partial \mathscr{H}_B^{\mathscr{B}}}{\partial p^{\mathscr{B}}(t)} = \varrho^t \sum_{i \in B} q_i^{\mathscr{B}}(p_0, t) + s \lambda_B^{\mathscr{B}}(t+1) = \lambda_B^{\mathscr{B}}(t), \quad t \in \mathscr{T} \setminus \{0, T\},$$
$$\lambda_B^{\mathscr{B}}(T) = 0.$$

Replacing costate variables $\lambda_B^{\mathscr{B}}(t)$ with scaled ones $\mu_B^{\mathscr{B}}(t)$ by $\mu_B^{\mathscr{B}}(t) = \varrho^{-t} \lambda_B^{\mathscr{B}}(t)$, $t \in \mathscr{T} \setminus 0$, and rewriting condition $\partial \mathscr{H}_B^{\mathscr{B}} / \partial q_i^{\mathscr{B}}(p_0, t) = 0$, we obtain the expressions from the statement of the theorem. □

3 Stability of a Coalition Structure

Assuming the firms are exogenously organized in a coalition structure $\mathscr{B}$, Theorem 1 provides equilibrium outputs $\bar{q}_i^{\mathscr{B}}(p_0)$ for each firm $i \in N$ under a coalition Nash equilibrium. Thus following the equilibrium profile $\bar{q}^{\mathscr{B}}(p_0)$, a firm i can determine its profit $\pi_i(p_0, \bar{q}^{\mathscr{B}}(p_0))$ in the game. However under a different coalition structure $\mathscr{B}'$ resulting in a different coalition Nash equilibrium $\bar{q}^{\mathscr{B}'}(p_0)$, firm i's profit $\pi_i(p_0, \bar{q}^{\mathscr{B}'}(p_0))$ will not necessarily coincide with $\pi_i(p_0, \bar{q}^{\mathscr{B}}(p_0))$. If firms were to create a coalition structure themselves, they would do it in a way that each firm would select the coalition which it does not want to leave, thus coming to a *stable* coalition structure. This approach is quite natural, and of course we are aware that there might be other reasons why firms should form a particular coalition structure. Although firms in a coalition focus on the total profit of this coalition, at the same time each firm also takes into account its individual profit in this coalition to measure its "satisfaction" from being a member.

In this section we consider two cases for determining a stable coalition structure: when firms' profits are either non-transferable or transferable.

For a given coalition structure $\mathscr{B}$, let $B(i)$ denote the coalition from $\mathscr{B}$ which contains firm i. Let also for some $B \in \mathscr{B}$ denote $\mathscr{B}_{-B} = \mathscr{B} \setminus B$.

3.1 Non-transferable Profits

We start with a case of non-transferable profits. This means that for a coalition structure $\mathscr{B}$, under the corresponding coalition Nash equilibrium $\bar{q}^{\mathscr{B}}$, a coalition $B \in \mathscr{B}$ receives its profit of $\pi_B^{\mathscr{B}}(p_0, \bar{q}^{\mathscr{B}}(p_0))$ while its member $i \in B$ gets $\pi_i(p_0, \bar{q}^{\mathscr{B}}(p_0))$.

Definition 2 A coalition structure $\mathscr{B}$ is *stable* if for any firm $i \in N$ it holds that

$$\pi_i(p_0, \bar{q}^{\mathscr{B}}(p_0)) \geqslant \pi_i(p_0, \bar{q}^{\mathscr{B}'}(p_0)), \tag{3}$$

where $\mathscr{B}' = \{B(i) \setminus \{i\}, B \cup \{i\}, \mathscr{B}_{-B(i)\cup B}\}$ for any $B \in \mathscr{B} \cup \varnothing$ and $B \neq B(i)$. Otherwise, the coalition structure is *unstable*. Here we recall that $\bar{q}^{\mathscr{B}}(p_0)$ and $\bar{q}^{\mathscr{B}'}(p_0)$ are coalition Nash equilibria for coalition structures $\mathscr{B}$ and $\mathscr{B}'$, respectively.

The definition of the stable coalition structure assumes that a firm may leave a coalition and become a singleton; it may also join any other coalition in the structure. Moreover, if a firm i leaves $B(i)$, the coalition $B(i) \setminus \{i\}$ does not break up and remains a part of the coalition structure. If firm i decides to leave $B(i)$ in favor of some other coalition B, then the members of B allow it to enter and form a coalition $B \cup \{i\}$ not blocking B from the new member.

When a coalition structure, say $\mathscr{B}$, is stable, no firm wishes to change a coalition, i.e., each firm $i \in N$ prefers to be a member of $B(i) \in \mathscr{B}$. Here we stress the reader's attention that the proposed stability concept is related only to the initial game period $t = 0$ when firms are supposed to follow a prescribed equilibrium profile $\bar{q}^{\mathscr{B}}(p_0)$ in the whole game under $\mathscr{B}$. Indeed, for this coalition structure inequality (3) holds true. However in some game period $t \in \mathscr{T} \setminus 0$ under profile $\bar{q}^{\mathscr{B}}(p_0)$ on the coalition equilibrium trajectory $\bar{p}^{\mathscr{B}}$ in state $\bar{p}^{\mathscr{B}}(t)$, coalition structure $\mathscr{B}$ may become unstable. Let $\bar{q}^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))$ denote a coalition Nash equilibrium in the subgame of game $\Gamma^{\mathscr{B}}(p_0)$ starting in period t in state $\bar{p}^{\mathscr{B}}(t)$. We denote this subgame by $\Gamma^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))$. Thus firms' profits in $\Gamma^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))$ are of the form:

$$\pi_i(\bar{p}^{\mathscr{B}}(t), \bar{q}^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))) = \sum_{\tau=t}^{T-1} \varrho^{\tau-t} \left[\bar{p}^{\mathscr{B}}(\tau)\bar{q}_i^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t), \tau) - \frac{c_i}{2}\left(\bar{q}_i^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t), \tau)\right)^2\right]$$

with

$$\bar{p}^{\mathscr{B}}(\tau+1) = s\bar{p}^{\mathscr{B}}(\tau) + (1-s)\left(a - b\sum_{i\in N}\bar{q}_i^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t), \tau)\right), \quad \tau \in \{t, \ldots, T-1\}.$$

If firm i in this state leaves $B(i)$, that is, changes current coalition structure $\mathscr{B}$ to some other $\mathscr{B}'$, this will lead to another coalition Nash equilibrium $\bar{q}^{\mathscr{B}'}(\bar{p}^{\mathscr{B}}(t))$. We notice that the equilibrium in the subgame with new coalition structure $\mathscr{B}'$ will depend upon the state $\bar{p}^{\mathscr{B}}(t)$ in which $\mathscr{B}$ has been changed. Let further

$$\pi_i(\bar{p}^{\mathscr{B}}(t), \bar{q}^{\mathscr{B}'}(\bar{p}^{\mathscr{B}}(t)))$$
$$= \sum_{\tau=t}^{T-1} \varrho^{\tau-t}\left[\bar{p}^{\mathscr{B}'}(\tau)\bar{q}_i^{\mathscr{B}'}(\bar{p}^{\mathscr{B}}(t), \tau) - \frac{c_i}{2}\left(\bar{q}_i^{\mathscr{B}'}(\bar{p}^{\mathscr{B}}(t), \tau)\right)^2\right]$$

with

$$\bar{p}^{\mathscr{B}'}(\tau+1) = s\bar{p}^{\mathscr{B}'}(\tau) + (1-s)\left(a - b\sum_{i\in N}\bar{q}_i^{\mathscr{B}'}(\bar{p}^{\mathscr{B}}(t), \tau)\right), \quad \tau \in \{t, \ldots, T-1\},$$

and $\bar{p}^{\mathscr{B}'}(t) \equiv \bar{p}^{\mathscr{B}}(t)$ denote firm i's profit in the subgame starting in period t in state $\bar{p}^{\mathscr{B}}(t)$ in the new coalition structure $\mathscr{B}'$ under coalition Nash equilibrium $\bar{q}^{\mathscr{B}'}(\bar{p}^{\mathscr{B}}(t))$. Thus we come to the definition.

Definition 3 A coalition structure $\mathscr{B}$ is *dynamically stable* if for any firm $i \in N$ and any game period $t \in \mathscr{T}$ it holds that

$$\pi_i(\bar{p}^{\mathscr{B}}(t), \bar{q}^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))) \geqslant \pi_i(\bar{p}^{\mathscr{B}}(t), \bar{q}^{\mathscr{B}'}(\bar{p}^{\mathscr{B}}(t))), \tag{4}$$

where $\mathscr{B}' = \{B(i) \setminus \{i\}, B \cup \{i\}, \mathscr{B}_{-B(i)\cup B}\}$ for any $B \in \mathscr{B} \cup \varnothing$ and $B \neq B(i)$.

The dynamic stability of $\mathscr{B}$ means its stability in *any* game period along the coalition equilibrium trajectory $\bar{p}^{\mathscr{B}}$.

3.2 Transferable Profits

Now we move to the case of transferable profits. Here we assume that for a coalition structure $\mathscr{B}$ under the corresponding coalition Nash equilibrium $\bar{q}^{\mathscr{B}}(p_0)$ a coalition $B \in \mathscr{B}$ receives its profit of $\pi_B^{\mathscr{B}}(p_0, \bar{q}^{\mathscr{B}}(p_0))$ while the profit of its members from cooperation has to be determined by redistributing $\pi_B^{\mathscr{B}}(p_0, \bar{q}^{\mathscr{B}}(p_0))$ among them. We will determine the profits of firms by a *cooperative solution* of a corresponding TU game with a coalition structure. A TU game with a coalition structure is a triple $(N, v_0, \mathscr{B})$, where N is a player set (the set of firms), v_0 is a characteristic function measuring a *worth* of any coalition, and finally $\mathscr{B}$ is a coalition structure. When transiting from a normal-form game to the corresponding TU game, there is no unique way in determining the characteristic function. We define it in two steps. Given a coalition structure $\mathscr{B}$, at the first step we define the value $v_0(B, \mathscr{B})$ as the profit of the coalition of firms $B \in \mathscr{B}$ under a coalition Nash equilibrium $\bar{q}^{\mathscr{B}}(p_0)$ in the dynamic game between players-coalitions from $\mathscr{B}$. Thus $v_0(B, \mathscr{B}) = \pi_B^{\mathscr{B}}(p_0, \bar{q}^{\mathscr{B}}(p_0))$. Next, for a coalition $S \subset B$, we define the value $v_0(S, \mathscr{B})$ as the total profit of its members under a coalition Nash equilibrium $\hat{q}^{S,\mathscr{B}}(p_0)$ in a dynamic game between players-firms from $\mathscr{B}$ when (i) firms from coalition S jointly maximize the total profit of this coalition, (ii) each firm from $B \setminus S$ maximizes its own profit, and (iii) each firm $i \in N \setminus B$ maximizes the total profit of coalitions they belong to, given the state equation (1). In other words, $\hat{q}^{S,\mathscr{B}}(p_0)$ is of the form:

$$\hat{q}_i^{S,\mathscr{B}}(p_0) = \begin{cases} \arg\max\limits_{q_S^{S,\mathscr{B}}(p_0)} \pi_S^{\mathscr{B}}(p_0, (q_S^{S,\mathscr{B}}(p_0), \hat{q}_{N\setminus S}^{S,\mathscr{B}}(p_0))), & i = S, \\ \arg\max\limits_{q_i^{S,\mathscr{B}}(p_0)} \pi_i^{\mathscr{B}}(p_0, (q_i^{S,\mathscr{B}}(p_0), \hat{q}_{N\setminus\{i\}}^{S,\mathscr{B}}(p_0))), & i \in B \setminus S, \\ \arg\max\limits_{q_{B'}^{S,\mathscr{B}}(p_0)} \pi_{B'}^{\mathscr{B}}(p_0, (q_{B'}^{S,\mathscr{B}}(p_0), \hat{q}_{N\setminus B'}^{S,\mathscr{B}}(p_0))), & i = B',\ B' \in \mathscr{B}_{-B}. \end{cases}$$

Therefore, the characteristic function is given by

$$v_0(S, \mathscr{B}) = \begin{cases} \pi_B^{\mathscr{B}}(p_0, \bar{q}^{\mathscr{B}}(p_0)), & S = B,\ B \in \mathscr{B}, \\ \pi_S^{\mathscr{B}}(p_0, \hat{q}^{S,\mathscr{B}}(p_0)), & S \subset B,\ B \in \mathscr{B}, \\ 0, & S = \varnothing, \\ \sum\limits_{\substack{B \in \mathscr{B}, \\ B \subseteq S}} \pi_B^{\mathscr{B}}(p_0, \bar{q}^{\mathscr{B}}(p_0)) + \sum\limits_{\substack{B \in \mathscr{B}, \\ B \not\subseteq S, B \cap S \neq \varnothing}} \pi_{B\cap S}^{\mathscr{B}}(p_0, \hat{q}^{B\cap S,\mathscr{B}}(p_0)), & \text{otherwise}. \end{cases}$$

Theorem 2 *Under a coalition structure $\mathscr{B}$, for any $S \subset B$, an open-loop coalition Nash equilibrium $\hat{q}^{S,\mathscr{B}}(p_0)$ is composed of the following strategies:*

$$\hat{q}_i^{S,\mathscr{B}}(p_0, t) = \begin{cases} \frac{1}{c_i}\Big[\hat{p}^{S,\mathscr{B}}(t) - \varrho b(1-s)\mu_S^{S,\mathscr{B}}(t+1)\Big], & i \in S, \\ \frac{1}{c_i}\Big[\hat{p}^{S,\mathscr{B}}(t) - \varrho b(1-s)\mu_i^{S,\mathscr{B}}(t+1)\Big], & i \in B \setminus S, \quad t \in \mathscr{T} \setminus T, \\ \frac{1}{c_i}\Big[\hat{p}^{S,\mathscr{B}}(t) - \varrho b(1-s)\mu_{B'}^{S,\mathscr{B}}(t+1)\Big], & i \in B',\ B' \in \mathscr{B}_{-B}, \end{cases}$$

where $\hat{p}^{S,\mathscr{B}}(t)$ and $\mu_S^{S,\mathscr{B}}(t)$, $\mu_i^{S,\mathscr{B}}(t)$, $i \in B \setminus S$, $\mu_{B'}^{S,\mathscr{B}}(t)$, $B' \in \mathscr{B}_{-B}$, satisfy the recursive relations:

$$\hat{p}^{S,\mathscr{B}}(t) = s\hat{p}^{S,\mathscr{B}}(t-1) + (1-s)\left(a - b\sum_{i\in N}\hat{q}_i^{S,\mathscr{B}}(p_0, t-1)\right), \quad t \in \mathscr{T} \setminus 0,$$

$$\mu_S^{S,\mathscr{B}}(t) = \sum_{i \in S} \hat{q}_i^{S,\mathscr{B}}(p_0, t) + \varrho s \mu_S^{S,\mathscr{B}}(t+1), \quad t \in \mathscr{T} \setminus \{0, T\},$$

$$\mu_i^{S,\mathscr{B}}(t) = \hat{q}_i^{S,\mathscr{B}}(t) + \varrho s \mu_i^{S,\mathscr{B}}(t+1), \quad i \in B \setminus S, \quad t \in \mathscr{T} \setminus \{0, T\},$$

$$\mu_{B'}^{S,\mathscr{B}}(t) = \sum_{i \in B'} \hat{q}_i^{S,\mathscr{B}}(p_0, t) + \varrho s \mu_{B'}^{S,\mathscr{B}}(t+1), \quad B' \in \mathscr{B}_{-B}, \quad t \in \mathscr{T} \setminus \{0, T\},$$

with $\hat{p}^{S,\mathscr{B}}(0) = p_0$, $\mu_S^{S,\mathscr{B}}(T) = 0$, $\mu_i^{S,\mathscr{B}}(T) = 0$, $i \in B \setminus S$, *and* $\mu_{B'}^{S,\mathscr{B}}(T) = 0$, $B' \in \mathscr{B}_{-B}$.

Proof For a coalition $S \subset B$, each firm $i \in B \setminus S$, and each coalition $B' \in \mathscr{B}_{-B}$ we define the Hamiltonians $\mathscr{H}_S^{S,\mathscr{B}}$, $\mathscr{H}_i^{S,\mathscr{B}}$, and $\mathscr{H}_{B'}^{S,\mathscr{B}}$, respectively:

$$\begin{aligned}
\mathscr{H}_S^{S,\mathscr{B}} &= \sum_{i \in S} \varrho^t \left[p^{S,\mathscr{B}}(t) q_i^{S,\mathscr{B}}(p_0, t) - \frac{c_i}{2} (q_i^{S,\mathscr{B}}(p_0, t))^2 \right] \\
&\quad + \lambda_S^{S,\mathscr{B}}(t+1) \left[s p^{S,\mathscr{B}}(t) + (1-s) \left(a - b \sum_{i \in N} q_i^{S,\mathscr{B}}(p_0, t) \right) \right], \\
\mathscr{H}_i^{S,\mathscr{B}} &= \varrho^t \left[p^{S,\mathscr{B}}(t) q_i^{S,\mathscr{B}}(p_0, t) - \frac{c_i}{2} (q_i^{S,\mathscr{B}}(p_0, t))^2 \right] \\
&\quad + \lambda_i^{S,\mathscr{B}}(t+1) \left[s p^{S,\mathscr{B}}(t) + (1-s) \left(a - b \sum_{i \in N} q_i^{S,\mathscr{B}}(p_0, t) \right) \right], \\
\mathscr{H}_{B'}^{S,\mathscr{B}} &= \sum_{i \in B'} \varrho^t \left[p^{S,\mathscr{B}}(t) q_i^{S,\mathscr{B}}(p_0, t) - \frac{c_i}{2} (q_i^{S,\mathscr{B}}(p_0, t))^2 \right] \\
&\quad + \lambda_{B'}^{S,\mathscr{B}}(t+1) \left[s p^{S,\mathscr{B}}(t) + (1-s) \left(a - b \sum_{i \in N} q_i^{S,\mathscr{B}}(p_0, t) \right) \right],
\end{aligned}$$

where $\lambda_S^{S,\mathscr{B}}(t)$, $\lambda_i^{S,\mathscr{B}}(t)$, $i \in B \setminus S$, and $\lambda_{B'}^{S,\mathscr{B}}(t)$, $B' \in \mathscr{B}_{-B}$, are costate variables. From the maximum principle the following is true:

$$\frac{\partial \mathscr{H}_S^{S,\mathscr{B}}}{\partial q_i^{S,\mathscr{B}}(p_0, t)} = \varrho^t \left[p^{S,\mathscr{B}}(t) - c_i q_i^{S,\mathscr{B}}(p_0, t) \right] - (1-s) b \lambda_S^{S,\mathscr{B}}(t+1) = 0,$$
$$i \in S, \quad t \in \mathscr{T} \setminus T,$$

$$\frac{\partial \mathscr{H}_S^{S,\mathscr{B}}}{\partial p^{S,\mathscr{B}}(t)} = \varrho^t \sum_{i \in S} q_i^{S,\mathscr{B}}(p_0, t) + s \lambda_S^{S,\mathscr{B}}(t+1) = \lambda_S^{S,\mathscr{B}}(t), \quad t \in \mathscr{T} \setminus \{0, T\},$$

$$\lambda_S^{S,\mathscr{B}}(T) = 0,$$

$$\frac{\partial \mathscr{H}_i^{S,\mathscr{B}}}{\partial q_i^{S,\mathscr{B}}(p_0,t)} = \varrho^t \left[p^{S,\mathscr{B}}(t) - c_i q_i^{S,\mathscr{B}}(p_0,t) \right] - (1-s) b \lambda_i^{S,\mathscr{B}}(t+1) = 0,$$

$$i \in B \setminus S,\ t \in \mathscr{T} \setminus T,$$

$$\frac{\partial \mathscr{H}_i^{S,\mathscr{B}}}{\partial p^{S,\mathscr{B}}(t)} = \varrho^t q_i^{S,\mathscr{B}}(p_0,t) + s\lambda_i^{S,\mathscr{B}}(t+1) = \lambda_i^{S,\mathscr{B}}(t), \quad t \in \mathscr{T} \setminus \{0,T\},$$

$$\lambda_i^{S,\mathscr{B}}(T) = 0,$$

$$\frac{\partial \mathscr{H}_{B'}^{S,\mathscr{B}}}{\partial q_i^{S,\mathscr{B}}(p_0,t)} = \varrho^t \left[p^{S,\mathscr{B}}(t) - c_i q_i^{S,\mathscr{B}}(p_0,t) \right] - (1-s) b \lambda_{B'}^{S,\mathscr{B}}(t+1) = 0,$$

$$i \in B',\ t \in \mathscr{T} \setminus T,$$

$$\frac{\partial \mathscr{H}_{B'}^{S,\mathscr{B}}}{\partial p^{S,\mathscr{B}}(t)} = \varrho^t \sum_{i \in B'} q_i^{S,\mathscr{B}}(p_0,t) + s\lambda_{B'}^{S,\mathscr{B}}(t+1) = \lambda_{B'}^{S,\mathscr{B}}(t), \quad t \in \mathscr{T} \setminus \{0,T\},$$

$$\lambda_{B'}^{S,\mathscr{B}}(T) = 0.$$

First we replace costate variables $\lambda_S^{S,\mathscr{B}}(t)$, $\lambda_i^{S,\mathscr{B}}(t)$, $i \in B \setminus S$, and $\lambda_{B'}^{S,\mathscr{B}}(t)$, $B' \in \mathscr{B}_{-B}$, with scaled ones by $\mu_S^{S,\mathscr{B}}(t) = \varrho^{-t}\lambda_S^{S,\mathscr{B}}(t)$, $\mu_i^{S,\mathscr{B}}(t) = \varrho^{-t}\lambda_i^{S,\mathscr{B}}(t)$, $i \in B \setminus S$, and $\mu_{B'}^{S,\mathscr{B}}(t) = \varrho^{-t}\lambda_{B'}^{S,\mathscr{B}}(t)$. Next, rewriting conditions $\partial \mathscr{H}_S^{S,\mathscr{B}} / \partial q_i^{S,\mathscr{B}}(p_0,t) = 0$, $i \in S$, $\partial \mathscr{H}_i^{S,\mathscr{B}} / \partial q_i^{S,\mathscr{B}}(p_0,t) = 0$, $i \in B \setminus S$, and $\partial \mathscr{H}_{B'}^{S,\mathscr{B}} / \partial q_i^{S,\mathscr{B}}(p_0,t) = 0$, $i \in B'$, $B' \in \mathscr{B}_{-B}$, we obtain the expressions from the statement of the theorem.

□

A *cooperative point solution* to the game $(N, v_0, \mathscr{B})$ with a coalition structure $\mathscr{B}$ is a map that assigns a profile $\xi[v_0, \mathscr{B}] \in \mathbb{R}^n$ to the TU game such that $\sum_{i \in B} \xi_i[v_0, \mathscr{B}] = v_0(B, \mathscr{B})$ for all $B \in \mathscr{B}$. In this definition we relax the individual rationality condition as the characteristic function may not be superadditive by its construction. As cooperative point solutions we may consider different ones, e.g., the Shapley value, the nucleolus, etc. (see Aumann and Dreze (1974) for cooperative solutions of a TU game with a coalition structure).

Definition 4 A coalition structure $\mathscr{B}$ is *stable* with respect to a cooperative point solution if for any firm $i \in N$ it holds that $\xi_i[v_0, \mathscr{B}] \geqslant \xi_i[v_0, \mathscr{B}']$ where $\mathscr{B}' = \{B(i) \setminus \{i\}, B \cup \{i\}, \mathscr{B}_{-B(i) \cup B}\}$ for any $B \in \mathscr{B} \cup \varnothing$ and $B \neq B(i)$. Otherwise the coalition structure is *unstable*.

In a similar way, we can determine a dynamically stable coalition structure. For this reason we have to determine the cooperative point solution in each subgame starting in state $\bar{p}^{\mathscr{B}}(t)$, $t \in \mathscr{T} \setminus 0$ on coalition equilibrium trajectory $\bar{p}^{\mathscr{B}}$. To do this, we first define a TU subgame $(N, v_t, \mathscr{B})$ with coalition structure $\mathscr{B}$ where v_t is the characteristic function in this subgame. We let $v_t(B, \mathscr{B}) = \pi_B^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t), \bar{q}^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t)))$ for any $B \in \mathscr{B}$. And for a coalition $S \subset B$, we define the value $v_t(S, \mathscr{B})$ as the profit of the coalition of firms S under a coalition Nash equilibrium $\hat{q}^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))$ in a dynamic subgame similarly, i.e., when (i) firms from coalition S jointly maximize the total profit of this coalition, (ii) each firm from $B \setminus S$ maximizes its own profit, and (iii) each firm $i \in N \setminus B$ maximizes the total profit of coalitions they belong to. In other words, $\hat{q}^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))$ is of the form:

$$\hat{q}_i^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t)) = \begin{cases} \arg\max\limits_{q_S^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))} \pi_S^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t), (q_S^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t)), \hat{q}_{N\setminus S}^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t)))), & i = S, \\ \arg\max\limits_{q_i^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))} \pi_i^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t), (q_i^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t)), \hat{q}_{N\setminus\{i\}}^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t)))), & i \in B \setminus S, \\ \arg\max\limits_{q_{B'}^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))} \pi_{B'}^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t), (q_{B'}^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t)), \hat{q}_{N\setminus B'}^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t)))), & i = B', \\ & B' \in \mathscr{B}_{-B}. \end{cases}$$

Therefore the characteristic function is given by

$$v_t(S, \mathscr{B}) = \begin{cases} \pi_B^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t), \bar{q}^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))), & S = B,\ B \in \mathscr{B}, \\ \pi_S^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t), \hat{q}^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))), & S \subset B,\ B \in \mathscr{B}, \\ 0, & S = \varnothing, \\ \sum\limits_{B \in \mathscr{B}, B \subseteq S} \pi_B^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t), \bar{q}^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))) \\ \quad + \sum\limits_{B \in \mathscr{B}, B \not\subseteq S, B \cap S \neq \varnothing} \pi_{B\cap S}^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t), \hat{q}^{B\cap S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))), & \text{otherwise.} \end{cases}$$

Using the same cooperative point solution in the subgame $(N, v_t, \mathscr{B})$, we get a profile $\xi[v_t, \mathscr{B}]$ of firms' cooperative profits.

Definition 5 We call a coalition structure $\mathscr{B}$ *dynamically stable* with respect to a cooperative point solution in case of transferable profits if for any firm $i \in N$ and any game period $t \in \mathscr{T}$ it holds that

$$\xi_i[v_t, \mathscr{B}] \geqslant \xi_i[v_t, \mathscr{B}'], \tag{5}$$

where $\mathscr{B}' = \{B(i) \setminus \{i\}, B \cup \{i\}, \mathscr{B}_{-B(i)\cup B}\}$ for any $B \in \mathscr{B} \cup \varnothing$ and $B \neq B(i)$, meaning that $\mathscr{B}$ is stable at *any* time period along the coalition equilibrium trajectory $\bar{p}^{\mathscr{B}}$.

Remark 1 In Rajan (1989), the author proposes an alternative scheme of determining the characteristic function in TU oligopoly games. Following this approach, (i) firms from coalition S jointly maximize the total profit of this coalition, (ii) firms from $B \setminus S$ jointly maximize the total profit of $B \setminus S$, and (iii) each firm $i \in N \setminus B$ maximizes the total profit of coalition $B(i)$. In other words, the coalition Nash equilibrium $\check{q}^{S,\mathscr{B}}(p_0)$ is given by:

$$\check{q}_R^{S,\mathscr{B}}(p_0) = \begin{cases} \arg\max\limits_{q_S^{S,\mathscr{B}}(p_0)} \pi_S^{\mathscr{B}}(p_0, (q_S^{S,\mathscr{B}}(p_0), \hat{q}_{N\setminus S}^{S,\mathscr{B}}(p_0))), & R = S, \\ \arg\max\limits_{q_{B\setminus S}^{S,\mathscr{B}}(p_0)} \pi_{B\setminus S}^{\mathscr{B}}(p_0, (q_{B\setminus S}^{S,\mathscr{B}}(p_0), \hat{q}_{N\setminus(B\setminus S)}^{S,\mathscr{B}}(p_0))), & R = B \setminus S, \\ \arg\max\limits_{q_{B'}^{S,\mathscr{B}}(p_0)} \pi_{B'}^{\mathscr{B}}(p_0, (q_{B'}^{S,\mathscr{B}}(p_0), \hat{q}_{N\setminus B'}^{S,\mathscr{B}}(p_0))), & R = B',\ B' \in \mathscr{B}_{-B}, \end{cases}$$

while the coalition Nash equilibrium $\check{q}^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))$ in subgame, starting in state $\bar{p}^{\mathscr{B}}(t)$, $t \in \mathscr{T} \setminus 0$, on coalition equilibrium trajectory $\bar{p}^{\mathscr{B}}$, takes the following form:

$$\check{q}_R^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t)) \\ = \begin{cases} \arg\max\limits_{q_S^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))} \pi_S^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t), (q_S^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t)), \hat{q}_{N\setminus S}^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t)))), & R = S, \\ \arg\max\limits_{q_{B\setminus S}^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))} \pi_{B\setminus S}^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t), (q_{B\setminus S}^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t)), \hat{q}_{N\setminus(B\setminus S)}^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t)))), & R = B \setminus S, \\ \arg\max\limits_{q_{B'}^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))} \pi_{B'}^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t), (q_{B'}^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t)), \hat{q}_{N\setminus B'}^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t)))), & R = B', \\ & B' \in \mathscr{B}_{-B}. \end{cases}$$

Given the above coalition Nash equilibria, one can determine characteristic functions $v_t(S, \mathscr{B})$, $S \subseteq N$, $t \in \mathscr{T}$, under this approach:

$$\check{v}_t(S, \mathscr{B}) = \begin{cases} \pi_B^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t), \bar{q}^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))), & S = B,\ B \in \mathscr{B}, \\ \pi_S^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t), \check{q}^{S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))), & S \subset B,\ B \in \mathscr{B}, \\ 0, & S = \varnothing, \\ \sum\limits_{B\in\mathscr{B}, B\subseteq S} \pi_B^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t), \bar{q}^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))) \\ \quad + \sum\limits_{B\in\mathscr{B}, B\nsubseteq S, B\cap S\neq\varnothing} \pi_{B\cap S}^{\mathscr{B}}(\bar{p}^{\mathscr{B}}(t), \check{q}^{B\cap S,\mathscr{B}}(\bar{p}^{\mathscr{B}}(t))), & \text{otherwise}, \end{cases}$$

with $\bar{p}^{\mathscr{B}}(0) \equiv p_0$. Then we are able to determine the corresponding cooperative solutions $\xi[\check{v}_t, \mathscr{B}]$, $t \in \mathscr{T}$, and verify whether the coalition structure $\mathscr{B}$ is (dynamically) stable.

4 An Example

We consider a market of three firms, $N = \{1, 2, 3\}$ competing in quantities over a finite set of periods $\mathscr{T} = \{0, 1, \ldots, 10\}$ with parameters: $s = 0.8$, $a = 30$, $b = 1$, $c_1 = 1$, $c_2 = 2$, $c_3 = 3$, a discount factor $\varrho = 0.9$ and the initial market price $p(0) = 10$.

Five coalition structures can be formed by three firms: $\mathscr{B}_1 = \{\{1\}, \{2\}, \{3\}\}$, $\mathscr{B}_2 = \{\{1, 2\}, \{3\}\}$, $\mathscr{B}_3 = \{\{1, 3\}, \{2\}\}$, $\mathscr{B}_4 = \{\{1\}, \{2, 3\}\}$, $\mathscr{B}_5 = \{\{1, 2, 3\}\}$.

First, for each coalition structure we calculate a coalition Nash equilibrium, the corresponding price trajectory, and the profits of the firms under this equilibrium for two cases: when the profits are non-transferable and when they are the components of the Shapley value in the game with a given coalition structure. Non-transferable profits are represented in Table 1. The analysis of these profits shows that all five coalition structures are unstable if the firms are paid by the initially given payoff functions:

- for $\mathscr{B}_1$, firm 1 benefits if it joins firm 2 which results in coalition structure $\mathscr{B}_2$;
- for $\mathscr{B}_2$, firm 2 has an incentive to become a singleton thus forming structure $\mathscr{B}_1$;
- for $\mathscr{B}_3$, firm 1 will benefit by joining firm 2;
- for $\mathscr{B}_4$, firm 1 will benefit by joining coalition {2, 3};
- and finally for $\mathscr{B}_5$, firm 2 has an incentive to deviate becoming a singleton.

Since in the case of non-transferable profits there is no stable coalition structure, then there cannot be any dynamically stable coalition structure.

Now consider the case of transferable profits. We use the Shapley value $\text{Sh}[v_t, \mathscr{B}] = (\text{Sh}_1[v_t, \mathscr{B}], \ldots, \text{Sh}_n[v_t, \mathscr{B}])$, $t \in \mathscr{T}$, as a cooperative point solution in the game and any subgame. Its components are given by

$$\text{Sh}_i[v_t, \mathscr{B}] = \sum_{S \subseteq B(i),\ i \in S} \frac{(|B(i)| - |S|)!(|S| - 1)!}{|B(i)|!} \left(v_t(S, \mathscr{B}) - v_t(S \setminus \{i\}, \mathscr{B}) \right), \quad i \in N.$$

We note that the Shapley value for a TU game with a coalition structure (or the Aumann–Dreze value (Aumann and Dreze 1974)) is defined by a so-called restricted characteristic function. For any coalition $B \in \mathscr{B}$ and subcoalition $S \subseteq B$, the value of the restricted characteristic function coincides with $v_t(S, \mathscr{B})$. The Shapley values for all possible coalition structures and all subgames are represented in Table 2. The analysis of firms' profits in the transferable case shows that $\mathscr{B}_2$ is the only stable coalition structure with respect to the Shapley value at $t = 0$ because there are no profitable deviations for any firm. Other four coalition structures are unstable with respect to the Shapley value. Indeed,

Table 1 Firms' profits

$\mathscr{B}$	$\pi_1^{\mathscr{B}}(p_0, \bar{q}^{\mathscr{B}}(p_0))$	$\pi_2^{\mathscr{B}}(p_0, \bar{q}^{\mathscr{B}}(p_0))$	$\pi_3^{\mathscr{B}}(p_0, \bar{q}^{\mathscr{B}}(p_0))$
$\mathscr{B}_1$	422.344	231.706	158.167
$\mathscr{B}_2$	446.836	223.418	186.940
$\mathscr{B}_3$	442.399	261.002	147.466
$\mathscr{B}_4$	454.596	235.079	156.719
$\mathscr{B}_5$	486.875	243.438	162.292

Non-transferable case

- for $\mathscr{B}_1$, firm 1 will benefit if it joins firm 2;
- for $\mathscr{B}_3$, firm 1 can make a profitable deviation by joining firm 2 and therefore forming a structure $\mathscr{B}_2$;
- for $\mathscr{B}_4$, firm 2 will benefit by joining firm 1;
- and finally for $\mathscr{B}_5$, firm 1 has an incentive to become a singleton.

Coalition equilibrium trajectories (equilibrium prices) for different coalition structures are depicted in Fig. 1. For any $t = 1, \ldots, 10$, the price $\bar{p}^{\mathscr{B}_1}(t)$ for the case when firms do not cooperate is the smallest, and the price $\bar{p}^{\mathscr{B}_5}(t)$ for the case of full cooperation is the largest as expected.

Moreover, the analysis of Table 2 shows that the structure $\mathscr{B}_2$ is also dynamically stable, so it satisfies Definition 5, i.e., there are no profitable deviations of any firm in any time period $t = 0, \ldots, 9$ when the game is realized along the coalition equilibrium trajectory $\bar{p}^{\mathscr{B}_2}$ calculated for the game with coalition structure $\mathscr{B}_2$.

Table 2 Firms' profits (the Shapley values)

t	$\mathscr{B}$	$\text{Sh}_1[v_t, \mathscr{B}]$	$\text{Sh}_2[v_t, \mathscr{B}]$	$\text{Sh}_3[v_t, \mathscr{B}]$	t	$\mathscr{B}$	$\text{Sh}_1[v_t, \mathscr{B}]$	$\text{Sh}_2[v_t, \mathscr{B}]$	$\text{Sh}_3[v_t, \mathscr{B}]$
0	$\mathscr{B}_1$	422.344	231.706	158.167	5	$\mathscr{B}_1$	336.968	177.409	119.639
	$\mathscr{B}_2$	430.446	239.808	186.940		$\mathscr{B}_2$	342.448	182.890	133.982
	$\mathscr{B}_3$	427.021	261.002	162.844		$\mathscr{B}_3$	340.182	191.721	122.853
	$\mathscr{B}_4$	454.596	232.669	159.130		$\mathscr{B}_4$	351.803	178.421	120.651
	$\mathscr{B}_5$	452.758	258.405	181.441		$\mathscr{B}_5$	353.934	192.172	132.136
1	$\mathscr{B}_1$	440.274	239.261	162.871	6	$\mathscr{B}_1$	288.484	150.155	100.959
	$\mathscr{B}_2$	448.325	247.312	190.187		$\mathscr{B}_2$	292.709	154.381	110.950
	$\mathscr{B}_3$	444.94	266.987	167.537		$\mathscr{B}_3$	290.968	160.062	103.444
	$\mathscr{B}_4$	470.542	240.335	163.945		$\mathscr{B}_4$	298.540	150.973	101.777
	$\mathscr{B}_5$	469.588	264.982	185.207		$\mathscr{B}_5$	300.822	160.827	109.890
2	$\mathscr{B}_1$	433.687	233.943	158.893	7	$\mathscr{B}_1$	231.874	119.054	79.790
	$\mathscr{B}_2$	441.417	241.673	183.845		$\mathscr{B}_2$	234.562	121.742	85.409
	$\mathscr{B}_3$	438.182	259.186	163.388		$\mathscr{B}_3$	233.457	124.586	81.374
	$\mathscr{B}_4$	460.982	235.080	160.030		$\mathscr{B}_4$	237.343	119.588	80.324
	$\mathscr{B}_5$	460.929	257.827	179.542		$\mathscr{B}_5$	239.214	125.345	84.976
3	$\mathscr{B}_1$	411.162	220.230	149.256	8	$\mathscr{B}_1$	165.154	83.545	55.816
	$\mathscr{B}_2$	418.359	227.426	171.183		$\mathscr{B}_2$	166.215	84.607	57.727
	$\mathscr{B}_3$	415.359	242.328	153.452		$\mathscr{B}_3$	165.780	85.409	56.442
	$\mathscr{B}_4$	434.787	221.384	150.409		$\mathscr{B}_4$	166.932	83.757	56.028
	$\mathscr{B}_5$	435.606	241.631	167.657		$\mathscr{B}_5$	167.842	85.818	57.654
4	$\mathscr{B}_1$	378.076	200.926	135.856	9	$\mathscr{B}_1$	86.073	43.036	28.691
	$\mathscr{B}_2$	384.533	207.383	154.221		$\mathscr{B}_2$	86.073	43.036	28.691
	$\mathscr{B}_3$	381.852	219.350	139.632		$\mathscr{B}_3$	86.073	43.036	28.691
	$\mathscr{B}_4$	397.501	202.044	136.973		$\mathscr{B}_4$	86.073	43.036	28.691
	$\mathscr{B}_5$	399.088	219.280	151.529		$\mathscr{B}_5$	86.073	43.036	28.691

Transferable case

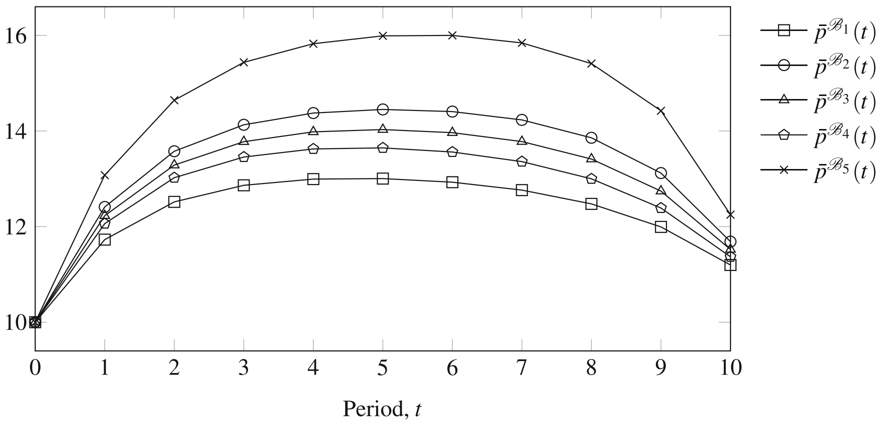

Fig. 1 Price under coalition Nash equilibrium for all possible coalition structures (coalition equilibrium trajectories)

Table 3 The Shapley values based on characteristic functions given in Remark 1 for the game with coalition structure $\mathscr{B}_5$

t	$\mathrm{Sh}_1[\check{v}_t, \mathscr{B}_5]$	$\mathrm{Sh}_2[\check{v}_t, \mathscr{B}_5]$	$\mathrm{Sh}_3[\check{v}_t, \mathscr{B}_5]$
0	453.830	257.999	180.775
1	470.503	264.627	184.647
2	461.662	257.534	179.103
3	436.144	241.405	167.346
4	399.432	219.123	151.343
5	354.103	192.08	132.059
6	300.858	160.788	109.893
7	239.179	125.341	85.0157
8	167.805	85.8249	57.6838
9	86.0727	43.0364	28.6909

This motivates firms to keep this coalition structure the same in the game and not to change it in any intermediate game period once the game has been started.

Following Remark 1, we may calculate the characteristic functions under another approach (see the definition of $\check{v}_t(S, \mathscr{B})$). The values of these functions and the corresponding Shapley values for the three-person game differ only for coalition structure $\mathscr{B}_5$. The Shapley values $\mathrm{Sh}[\check{v}_t, \mathscr{B}_5]$ for $t = 0, \ldots, 9$ are presented in Table 3. Analyzing the values in Tables 2 and 3, we observe that the coalition structure $\mathscr{B}_2$ is also dynamically stable under this approach.

5 Conclusion

We have considered a linear-quadratic dynamic game in which firms, competing in a market, may cooperate and form not only the grand coalition but also smaller coalitions being components of a coalition structure. The firms in the coalitions obtain their profits according to a cooperative point solution (e.g., the Shapley value, the nucleolus). The conditions for coalition Nash equilibrium strategies of firms have been obtained. We examined the stability of the coalition structure meaning its Nash stability according to which no firm has an incentive to individual deviation from the coalition it belongs to. We have considered an example for which the grand coalition is unstable, but there exists another coalition structure which is stable not only for the whole game but also along the state equilibrium trajectory corresponding to this coalition structure, that is, dynamically stable. It is interesting to find the general conditions under which a coalition structure is stable (or dynamically stable) for the class of dynamic games considered in the chapter. One can also develop stronger stability conditions which would protect a coalition structure against deviations of any group of players. These developments are left for future research.

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