Identically Self-blocking Clutters

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Abstract. A clutter is *identically self-blocking* if it is equal to its blocker. We prove that every identically self-blocking clutter different from $\{\{a\}\}$ is nonideal. Our proofs borrow tools from Gauge Duality and Quadratic Programming. Along the way we provide a new lower bound for the packing number of an arbitrary clutter.

1 The Main Result

All sets considered in this paper are finite. Let V be a set of *elements*, and let $\mathcal{C}$ be a family of subsets of V called *members*. If no member contains another, then $\mathcal{C}$ is said to be a *clutter* over *ground set* V [12]. All clutters considered in this paper are different from $\{\}, \{\emptyset\}$. Let $\mathcal{C}$ be a clutter over ground set V. A *cover* is a subset of V that intersects every member. The *covering number*, denoted $\tau(\mathcal{C})$, is the minimum cardinality of a cover. A *packing* is a collection of pairwise disjoint members. The *packing number*, denoted $\nu(\mathcal{C})$, is the maximum cardinality of a packing. Observe that $\tau(\mathcal{C}) \geq \nu(\mathcal{C})$. A cover is *minimal* if it does not contain another cover. The family of minimal covers forms another clutter over ground set V; this clutter is called the *blocker of* $\mathcal{C}$ and is denoted $b(\mathcal{C})$ [12]. It is well-known that $b(b(\mathcal{C})) = \mathcal{C}$ [12,17]. We say that $\mathcal{C}$ is an *identically self-blocking* clutter if $\mathcal{C} = b(\mathcal{C})$. (This terminology was coined in [4].) Observe that $\{a\}$ is the only identically self-blocking clutter with a member of cardinality one.

Theorem 1 ([6]). *A clutter $\mathcal{C}$ is identically self-blocking if, and only if, $\nu(\mathcal{C}) = \nu(b(\mathcal{C})) = 1$.*

Consider for $w \in \mathbb{Z}_+^V$ the dual pair of linear programs

$$\begin{array}{llll} \min & w^\top x & \max & \mathbf{1}^\top y \\ \text{s.t.} & \sum (x_u : u \in C) \geq 1 \ \forall C \in \mathcal{C} & \text{s.t.} & \sum (y_C : u \in C \in \mathcal{C}) \leq w_u \ \forall u \in V \\ & x \geq \mathbf{0} & & y \geq \mathbf{0}, \end{array}$$

labeled $(P), (D)$, respectively. Denote by $\tau^\star(\mathcal{C}, w), \nu^\star(\mathcal{C}, w)$ the optimal values of $(P), (D)$, respectively, and by $\tau(\mathcal{C}, w), \nu(\mathcal{C}, w)$ the optimal values of $(P), (D)$ subject to the additional integrality constraints $x \in \mathbb{Z}^V, y \in \mathbb{Z}^{\mathcal{C}}$, respectively. Observe that by Strong Linear Programming Duality, $\tau(\mathcal{C}, w) \geq \tau^\star(\mathcal{C}, w) = \nu^\star(\mathcal{C}, w) \geq \nu(\mathcal{C}, w)$.

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Notice the correspondence between the 0–1 feasible solutions of (P) and the covers of $\mathcal{C}$, as well as the correspondence between the integer feasible solutions of (D) for $w = \mathbf{1}$ and the packings of $\mathcal{C}$. In particular, $\tau(\mathcal{C}, \mathbf{1}) = \tau(\mathcal{C})$ and $\nu(\mathcal{C}, \mathbf{1}) = \nu(\mathcal{C})$. We will refer to the feasible solutions of (P) as *fractional covers*, and to the feasible solutions of (D) for $w = \mathbf{1}$ as *fractional packings*.

$\mathcal{C}$ has the *max-flow min-cut property* if $\tau(\mathcal{C}, w) = \nu(\mathcal{C}, w)$ for all $w \in \mathbb{Z}_+^V$ [10]. $\mathcal{C}$ is *ideal* if $\tau(\mathcal{C}, w) = \nu^\star(\mathcal{C}, w)$ for all $w \in \mathbb{Z}_+^V$ [11]. Clearly clutters with the max-flow min-cut property are ideal. The max-flow min-cut property is not closed under taking blockers, but

Theorem 2 ([18]). *A clutter is ideal if, and only if, its blocker is ideal.*

If $\mathcal{C}$ is an identically self-blocking clutter different from $\{\{a\}\}$, then $\tau(\mathcal{C}) \geq 2 > 1 = \nu(\mathcal{C})$ by Theorem 1, so $\mathcal{C}$ does not have the max-flow min-cut property. In this paper, we prove the following stronger statement:

Theorem 3. *An identically self-blocking clutter different from* $\{\{a\}\}$ *is non-ideal.*

For an integer $n \geq 3$, denote by Δ_n the clutter over ground set $\{1, \dots, n\}$ whose members are $\{1,2\}$, $\{1,3\}, \dots, \{1,n\}, \{2,3,\dots,n\}$. Notice that the elements and members of Δ_n correspond to the points and lines of a degenerate projective plane. Denote by $\mathbb{L}_7$ the clutter over ground set $\{1,\dots,7\}$ whose members are $\{1,2,3\}, \{1,4,5\}, \{1,6,7\}, \{2,4,7\}, \{2,5,6\}, \{3,4,6\}, \{3,5,7\}$. Notice that the elements and members of $\mathbb{L}_7$ correspond to the points and lines of the Fano plane. It can be readily checked that $\{\Delta_n : n \geq 3\} \cup \{\mathbb{L}_7\}$ are identically self-blocking clutters. There are many other examples of identically self-blocking clutters, and in fact there is one for every pair of blocking clutters ([4], Remark 3.4 and Corollary 3.6). Another example, for instance, is the clutter over ground set $\{1,\dots,6\}$ whose members are $\{6,1,2\}, \{6,2,3\}, \{6,3,4\}, \{6,4,5\}$, $\{6,5,1\}, \{1,2,4\}, \{2,3,5\}, \{3,4,1\}, \{4,5,2\}, \{5,1,3\}$.

Conjecture 4. *An identically self-blocking clutter different from* $\{\{a\}\}$ *has one of* $\{\Delta_n : n \geq 3\}, \mathbb{L}_7, \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}\}$ *as minor.*

(Notice that the last clutter above is *not* identically self-blocking.) For disjoint $X, Y \subseteq V$, the *minor of* $\mathcal{C}$ obtained after *deleting* X and *contracting* Y is the clutter over ground set $V - (X \cup Y)$ whose members are

$$\mathcal{C} \setminus X/Y := \text{the inclusionwise minimal sets of } \{C - Y : C \in \mathcal{C}, C \cap X = \emptyset\}.$$

It is well-known that $b(\mathcal{C} \setminus X/Y) = b(\mathcal{C})/X \setminus Y$ [21], and that if a clutter is ideal, then so is every minor of it [22]. It can be readily checked that the clutters in Conjecture 4 are nonideal. Thus Conjecture 4 – if true – would be a strengthening of Theorem 3.

The rest of the paper is organized as follows: We will present two proofs of Theorem 3, one will be short and indirect (Sect. 2) while the other will be a longer and direct proof that essentially unravels the first proof (Sect. 5). In Sect. 3, by using our techniques, we will provide a new lower bound for the packing number of an arbitrary clutter, and in Sect. 4, we will see a surprising emergence of *cuboids*, a special class of clutters. In Sect. 6 we will address the relevance of studying identically self-blocking clutters, a relatively narrow problem, and why it may be of interest to the community.

2 Gauge Duality

Here we present a short and indirect proof of Theorem 3. Take an integer $n \geq 1$ and let M be a matrix with n columns and nonnegative entries and without a row of all zeros. Consider the polyhedron $P := \left\{x \in \mathbb{R}^n_+ : Mx \geq \mathbf{1}\right\}$. The *blocker of* P is the polyhedron $Q := \{z \in \mathbb{R}^n_+ : z^\top x \geq 1 \; \forall x \in P\}$. Fulkerson showed that there exists a matrix N with n columns and nonnegative entries and without a row of all zeros such that $Q = \left\{z \in \mathbb{R}^n_+ : Nz \geq \mathbf{1}\right\}$, and that the blocker of Q is P [14,15]. In 1987 Chaiken proved the following fascinating result:

Theorem 5 **([8]).** *Take an integer* $n \geq 1$*, let* P, Q *be a blocking pair of polyhedra in* $\mathbb{R}^n$*, and let* R *be a positive definite* n *by* n *matrix. Then* $\min\{x^\top Rx : x \in P\}$ *and* $\min\{z^\top R^{-1} z : z \in Q\}$ *have reciprocal optimal values.*

Theorem 5 exhibits an instance of *gauge duality*, a general framework introduced by Freund later the same year [13]. Theorem 5 in the special case of diagonal R's was also proved by Lovász in 2001 [19]. Both Freund and Lovász seem to have been unaware of Chaiken's result.

Let $\mathcal{C}$ be a clutter over ground set V. Define the *incidence matrix* of $\mathcal{C}$ as the matrix M whose columns are indexed by the elements and whose rows are the incidence vectors of the members, and define $Q(\mathcal{C}) := \left\{x \in \mathbb{R}^V_+ : Mx \geq \mathbf{1}\right\}$. Fulkerson showed that if $\mathcal{C}, \mathcal{B}$ are blocking *ideal* clutters then $Q(\mathcal{C}), Q(\mathcal{B})$ give an instance of blocking polyhedra [14,15]. Therefore Theorem 5 has the following consequence:

Theorem 6. *Let* $\mathcal{C}, \mathcal{B}$ *be blocking ideal clutters. Then* $\min\{x^\top x : x \in Q(\mathcal{C})\}$ *and* $\min\{z^\top z : z \in Q(\mathcal{B})\}$ *have reciprocal optimal values.*

We will need the following lemma whose proof makes use of concepts such as the *Lagrangian* and the *Karush-Kuhn-Tucker* conditions (see [7], Chapter 5):

Lemma 7 **([8]).** *Let* $\mathcal{C}$ *be a clutter over ground set* V*, and let* M *be its incidence matrix. Then* $\min\{x^\top x : Mx \geq \mathbf{1}, x \geq \mathbf{0}\}$ *has a unique optimal solution* $x^\star \in \mathbb{R}^V_+$*. Moreover, there exists* $y \in \mathbb{R}^{\mathcal{C}}_+$ *such that* $M^\top y = x^\star$*,* $\mathbf{1}^\top y = {x^\star}^\top x^\star$ *and* $y^\top (Mx^\star - \mathbf{1}) = \mathbf{0}$*.*

Proof. Notice that $\min\{x^\top x : Mx \geq \mathbf{1}, x \geq \mathbf{0}\}$ satisfies Slater's condition, that there is a feasible solution satisfying all the inequalities strictly. As $x^\top x$ is a strictly convex function, our quadratic program has a unique optimal solution $x^\star \in \mathbb{R}^V_+$. Denote by $L(x; \mu, \sigma) := x^\top x - \mu^\top (Mx - \mathbf{1}) - \sigma^\top x$ the Lagrangian of the program. Since Slater's condition is satisfied, there exist $\mu^\star \in \mathbb{R}_+^{\mathcal{C}}$ and $\sigma^\star \in \mathbb{R}_+^V$ satisfying the Karush-Kuhn-Tucker conditions:

$$\begin{aligned}\mathbf{0} &= \nabla_x L(x^\star; \mu^\star, \sigma^\star) = 2x^\star - M^\top \mu^\star - \sigma^\star \\ 0 &= {\mu^\star}^\top (Mx^\star - \mathbf{1}) \\ 0 &= {\sigma^\star}^\top x^\star.\end{aligned}$$

Let $y := \frac{1}{2}\mu^\star$. Since $\sigma^\star$ and M have nonnegative entries, and the third equation holds, the first equation implies that $M^\top y = x^\star$. Multiplying the first equation by ${x^\star}^\top$ from the left, and taking the next two equations into account, we get that $\mathbf{1}^\top y = {x^\star}^\top x^\star$. As $y^\top(Mx^\star - \mathbf{1}) = \mathbf{0}$ clearly holds, y is the desired vector. □

We are now ready for the first, short and indirect proof of Theorem 3, stating that an identically self-blocking clutter different from $\{\{a\}\}$ is nonideal:

Proof (of Theorem 3). Let $\mathcal{C}$ be an identically self-blocking clutter over ground set V that is different from $\{\{a\}\}$, and let M be its incidence matrix. Suppose for a contradiction that $\mathcal{C}$ is ideal. Then Theorem 6 applies and tells us that $\min\{x^\top x : Mx \geq \mathbf{1}, x \geq \mathbf{0}\} = 1$. Let $x^\star, y$ be as in Lemma 7; so $x^\star = M^\top y$ and $1 = {x^\star}^\top x^\star = \mathbf{1}^\top y$. As $\mathcal{C}$ is an identically self-blocking clutter different from $\{\{a\}\}$, every member has cardinality at least two, and by Theorem 1 every two members intersect, implying in turn that $MM^\top \geq J + I$.[1] As a result,

$$1 = {x^\star}^\top x^\star = y^\top MM^\top y \geq y^\top (J+I) y = y^\top \mathbf{1}\mathbf{1}^\top y + y^\top y = 1 + y^\top y,$$

implying in turn that $y = \mathbf{0}$, a contradiction. □

3 Lower Bounding the Packing Number

Here we present a lower bound on the packing number of an arbitrary clutter. We need the following lemma from 1965 proved by Motzkin and Straus:

Lemma 8 ([20]). *Let $G = (V, E)$ be a simple graph, and let L be its V by V adjacency matrix: for all $u, v \in V$, $L_{uv} = 1$ if $\{u, v\} \in E$ and $L_{uv} = 0$ otherwise. Then*

$$\max\left\{y^\top L y : \mathbf{1}^\top y = 1, y \geq \mathbf{0}\right\} = 1 - \frac{1}{\omega(G)}$$

where $\omega(G)$ is the maximum cardinality of a clique of G.

Let $\mathcal{C}$ be a clutter over ground set V. Finding $\nu(\mathcal{C})$ can be cast as finding the maximum cardinality of a clique of a graph. This observation, combined with Lemma 8, has the following consequence:

[1] Throughout the paper, J is a square all ones matrix of appropriate dimension, and I is the identity matrix of appropriate dimension.

Lemma 9. *Let $\mathcal{C}$ be a clutter over ground set V, and let M be its incidence matrix. Then*

$$\min\left\{y^\top MM^\top y - \sum_{C\in\mathcal{C}}(|C|-1)y_C^2 : \mathbf{1}^\top y = 1, y \geq \mathbf{0}\right\} = \frac{1}{\nu(\mathcal{C})}.$$

Proof. ($\leq$) Let $y \in \mathbb{R}_+^{\mathcal{C}}$ be the incidence vector of a maximum packing of $\mathcal{C}$. Then $\frac{1}{\nu(\mathcal{C})}\cdot y$ is a feasible solution whose objective value is $\frac{1}{\nu(\mathcal{C})}$, implying in turn that $\leq$ holds. ($\geq$) Let G be the graph whose vertices correspond to the members of $\mathcal{C}$, where two vertices are adjacent if the corresponding members are disjoint. Let L be the adjacency matrix of G. Then $MM^\top \geq \mathrm{Diag}(|C|-1 : C \in \mathcal{C}) + J - L$. Notice that there is a bijection between the packings in $\mathcal{C}$ and the cliques in G, and in particular that $\nu(\mathcal{C}) = \omega(G)$. Thus by Lemma 8, for any $y \in \mathbb{R}_+^{\mathcal{C}}$ such that $\mathbf{1}^\top y = 1$,

$$\begin{aligned}1 - \frac{1}{\nu(\mathcal{C})} \geq y^\top Ly &\geq \sum_{C\in\mathcal{C}}(|C|-1)y_C^2 + y^\top Jy - y^\top MM^\top y\\ &= \sum_{C\in\mathcal{C}}(|C|-1)y_C^2 + 1 - y^\top MM^\top y,\end{aligned}$$

implying in turn that $\geq$ holds. □

As a consequence, after employing Carathéodory's theorem (see [9], §3.14) and the Cauchy-Schwarz inequality (see [23]), we get the following lower bound on the packing number of a clutter:

Theorem 10 ([5]). *Let $\mathcal{C}$ be a clutter over ground set V, and let M be its incidence matrix. Then*

$$\nu(\mathcal{C}) \geq \left(\frac{y^\top MM^\top y}{y^\top Jy} - \frac{\min\{|C|-1 : C\in\mathcal{C}\}}{\min\{|V|,|\mathcal{C}|\}}\right)^{-1} \quad \forall y \in \mathbb{R}_+^{\mathcal{C}}, y \neq \mathbf{0}.$$

Proof. Pick a nonzero $y \in \mathbb{R}_+^{\mathcal{C}}$. By Carathéodory's theorem there is a $y' \in \mathbb{R}_+^{\mathcal{C}}$ such that $M^\top y' \leq M^\top y$, $\mathbf{1}^\top y' = \mathbf{1}^\top y$ and $|\mathrm{supp}(y')| \leq |V|$. Lemma 9 applied to $\frac{1}{\mathbf{1}^\top y'}\cdot y'$ implies that

$$\begin{aligned}\nu(\mathcal{C}) &\geq \left(\frac{y'^\top MM^\top y' - \sum_{C\in\mathcal{C}}(|C|-1){y'_C}^2}{y'^\top Jy'}\right)^{-1}\\ &\geq \left(\frac{y^\top MM^\top y}{y^\top Jy} - \frac{\sum_{C\in\mathcal{C}}(|C|-1){y'_C}^2}{y'^\top Jy'}\right)^{-1}.\end{aligned}$$

By the Cauchy-Schwarz inequality applied to the nonzero entries of y',

$$\frac{\sum_{C\in\mathcal{C}}(|C|-1){y'_C}^2}{y'^\top Jy'} \geq \frac{\left(\sum_{C\in\mathcal{C}}\sqrt{|C|-1}\cdot y'_C\right)^2}{|\mathrm{supp}(y')|\cdot y'^\top Jy'} \geq \frac{\min\{|C|-1 : C\in\mathcal{C}\}}{\min\{|V|,|\mathcal{C}|\}}.$$

Combining the last two inequalities proves the theorem. □

This theorem was proved implicitly by Aharoni, Erdős and Linial in 1988. Given that M is the incidence matrix of $\mathcal{C}$, the authors explicitly proved Theorem 10 for y a maximum fractional packing of $\mathcal{C}$:

$$\nu(\mathcal{C}) \geq \left(\frac{\mathbf{1}^\top \mathbf{1}}{\nu^{\star 2}(\mathcal{C})} - \frac{\min\{|C|-1 : C \in \mathcal{C}\}}{\min\{|V|, |\mathcal{C}|\}}\right)^{-1} \geq \frac{\nu^{\star 2}(\mathcal{C})}{|V|}.$$

But one can do better:

Theorem 11. *Let $\mathcal{C}$ be a clutter over ground set V, and let $\alpha := \min\left\{x^\top x : x \in Q(\mathcal{C})\right\}$. Then*

$$\nu(\mathcal{C}) \geq \left(\frac{1}{\alpha} - \frac{\min\{|C|-1 : C \in \mathcal{C}\}}{\min\{|V|, |\mathcal{C}|\}}\right)^{-1}.$$

Proof. Let M be the incidence matrix of $\mathcal{C}$, and let $x \in \mathbb{R}^V_+$ be the point in $Q(\mathcal{C})$ such that $x^\top x = \alpha$. By Lemma 7, there exists $y \in \mathbb{R}^{\mathcal{C}}_+$ such that $x = M^\top y$ and $\mathbf{1}^\top y = \alpha$. By Theorem 10,

$$\begin{aligned}\nu(\mathcal{C}) &\geq \left(\frac{x^\top x}{y^\top \mathbf{1}\mathbf{1}^\top y} - \frac{\min\{|C|-1 : C \in \mathcal{C}\}}{\min\{|V|, |\mathcal{C}|\}}\right)^{-1} \\ &= \left(\frac{\alpha}{\alpha^2} - \frac{\min\{|C|-1 : C \in \mathcal{C}\}}{\min\{|V|, |\mathcal{C}|\}}\right)^{-1},\end{aligned}$$

as required. □

Let

$$\beta := \min\left\{\frac{y^\top M M^\top y}{y^\top J y} : y \geq \mathbf{0}, y \neq \mathbf{0}\right\} \quad \text{and} \quad \alpha := \min\{x^\top x : Mx \geq \mathbf{1}, x \geq \mathbf{0}\}.$$

By Strong Conic Programming Duality (see [7], Chapter 5),

$$\frac{1}{\sqrt{\beta}} = \max\left\{\mathbf{1}^\top y : \|M^\top y\| \leq 1, y \geq \mathbf{0}\right\} = \min\{\|x\| : Mx \geq \mathbf{1}, x \geq \mathbf{0}\} = \sqrt{\alpha},$$

so $\beta = \frac{1}{\alpha}$. As a result, the inequality given by Theorem 11 is the best lower bound derived from Theorem 10.

As an immediate consequence of Theorem 11, we get another new lower bound on the packing number of a clutter:

Theorem 12. *Let $\mathcal{C}$ be a clutter. Then $\nu(\mathcal{C}) \geq \min\left\{x^\top x : x \in Q(\mathcal{C})\right\}$.*

See ([1], Chapter 3, Theorem 3.2) for an alternative proof of this theorem.

4 Cuboids

Take an even integer $n \geq 2$. A *cuboid* is a clutter over ground set $\{1, \dots, n\}$ where every member C satisfies $|C \cap \{1,2\}| = |C \cap \{3,4\}| = \cdots = |C \cap \{n-1, n\}| = 1$. Introduced in [2], cuboids form a very special class of clutters, to the point that some of the main conjectures in the field can be phrased equivalently in terms of cuboids [3]. Cuboids play a special role here also:

Theorem 13. *Let $\mathcal{C}$ be an ideal clutter over n elements whose members do not have a common element, and let $\alpha := \min\{x^\top x : x \in Q(\mathcal{C})\}$. Then $\alpha \geq \frac{4}{n}$. Moreover, the following statements are equivalent:*

(i) $\alpha = \frac{4}{n}$,
(ii) n *is even, after a possible relabeling of the ground set the sets* $\{1,2\}, \{3,4\}, \dots, \{n-1, n\}$ *are minimal covers, and the members of minimum cardinality form an ideal cuboid over ground set* $\{1, \dots, n\}$ *whose members do not have a common element.*

To prove this theorem we need a few preliminary results. Given a simple graph $G = (V, E)$, a *fractional perfect matching* is a $y \in \mathbb{R}^E_+$ such that $\mathbf{1}^\top y = \frac{|V|}{2}$ and for each vertex $u \in V$, $\sum (y_e : u \in e) = 1$. We need the following classic result:

Lemma 14 (folklore). *If a bipartite graph has a fractional perfect matching, then it has a perfect matching.*

Lemma 15. *Take an integer $n \geq 2$, and let $\mathcal{C}$ be a clutter over ground set $\{1, \dots, n\}$. Then the following statements are equivalent:*

(i) $\mathcal{C}$ *has a fractional packing of value* 2 *and* $b(\mathcal{C})$ *has a fractional packing of value* $\frac{n}{2}$,
(ii) n *is even, and in* $\mathcal{C}$, *after a possible relabeling of the ground set the sets* $\{1,2\}, \{3,4\}, \dots, \{n-1, n\}$ *are minimal covers, and the members of minimum cardinality form a cuboid over ground set* $\{1, \dots, n\}$ *with a fractional packing of value* 2.

Proof. **(ii)** $\Rightarrow$ **(i)** is immediate. **(i)** $\Rightarrow$ **(ii)**: Let M, N be the incidence matrices of $\mathcal{C}, b(\mathcal{C})$, respectively. Let $y \in \mathbb{R}^{\mathcal{C}}_+$ be a fractional packing of $\mathcal{C}$ of value 2; so $M^\top y \leq \mathbf{1}$ and $\mathbf{1}^\top y = 2$. Let $t \in \mathbb{R}^{b(\mathcal{C})}_+$ be a fractional packing of $b(\mathcal{C})$ of value $\frac{n}{2}$; so $N^\top t \leq \mathbf{1}$ and $\mathbf{1}^\top t = \frac{n}{2}$. Then $n = \mathbf{1}^\top \mathbf{1} \geq t^\top N M^\top y \geq t^\top J y = t^\top \mathbf{1}\mathbf{1}^\top y = \frac{n}{2} \cdot 2 = n$. Thus equality holds throughout, implying in turn that **(1)** $M^\top y = \mathbf{1}$ and $\mathbf{1}^\top y = 2$, **(2)** $N^\top t = \mathbf{1}$ and $\mathbf{1}^\top t = \frac{n}{2}$, **(3)** if $y_C > 0$ and $t_B > 0$ for some $C \in \mathcal{C}$ and $B \in b(\mathcal{C})$, then $|C \cap B| = 1$. Notice that $\tau(\mathcal{C}) \geq 2$ and $\tau(b(\mathcal{C})) \geq \frac{n}{2}$, so every member of $\mathcal{C}$ has cardinality at least $\frac{n}{2}$ while every member of $b(\mathcal{C})$ has cardinality at least 2. Together with (1) and (2), these observations imply that n is even, and **(4)** if $y_C > 0$ for some $C \in \mathcal{C}$, then $|C| = \frac{n}{2}$, **(5)** if $t_B > 0$ for some $B \in b(\mathcal{C})$, then $|B| = 2$. Let G be the graph over vertices $\{1, \dots, n\}$ whose edges correspond to $\{B \in b(\mathcal{C}) : t_B > 0\}$. Pick $C \in \mathcal{C}$ such that $y_C > 0$. Then by (3) the vertex subset C intersects every edge of G exactly once, implying in turn that G is a bipartite graph. By (2) G has a fractional perfect matching, and as the graph is bipartite, there must be a perfect matching by Lemma 14, labeled as $\{1,2\},\{3,4\},\ldots,\{n-1,n\}$ after a possible relabeling of the ground set. As a consequence, the members of $\mathcal{C}$ of minimum cardinality form a cuboid over ground set $\{1,\ldots,n\}$ which by (1) and (4) has a fractional packing of value 2. Thus (ii) holds. $\square$

Remark 16. *Let $\mathcal{C}$ be a clutter over n elements, and let $\alpha, x^\star$ be the optimal value and solution of $\min\{x^\top x : x \in Q(\mathcal{C})\}$, respectively. Then the following statements hold:*

(i) $\alpha \geq \frac{\nu^{\star 2}(\mathcal{C})}{n}$. Moreover, equality holds if and only if $x^\star = \frac{\nu^\star(\mathcal{C})}{n}\cdot \mathbf{1}$.
(ii) Assume that every member has cardinality at least two. Then $\alpha \leq \frac{n}{4}$. Moreover, equality holds if and only if $x^\star = \frac{1}{2}\cdot\mathbf{1}$.

Proof. **(i)** The Cauchy-Schwarz inequality implies that $\alpha = x^{\star\top}x^\star \geq \frac{(\mathbf{1}^\top x^\star)^2}{n} \geq \frac{\tau^{\star 2}(\mathcal{C})}{n} = \frac{\nu^{\star 2}(\mathcal{C})}{n}$. Moreover, equality holds throughout if and only if the entries of $x^\star$ are equal and $\mathbf{1}^\top x^\star = \nu^\star(\mathcal{C})$, i.e. $x^\star = \frac{\nu^\star(\mathcal{C})}{n}\cdot\mathbf{1}$. **(ii)** is immediate. $\square$

We also need the following result proved implicitly in [2] (its proof can be found in the proof of Theorem 1.6, Claim 3 on p. 543):

Lemma 17 ([2]). *Take an even integer $n \geq 2$, and let $\mathcal{C}$ be an ideal clutter over ground set $\{1,\ldots,n\}$ where $\{1,2\},\{3,4\},\ldots,\{n-1,n\}$ are minimal covers. Then $\left\{C \in \mathcal{C} : |C| = \frac{n}{2}\right\}$ is an ideal cuboid.*

We are now ready to prove Theorem 13:

Proof (of Theorem 13). By Remark 16 (i), $\alpha \geq \frac{\nu^{\star 2}(\mathcal{C})}{n} = \frac{\tau^2(\mathcal{C})}{n} \geq \frac{4}{n}$, where the equality follows from the fact that $\mathcal{C}$ is ideal, and the last inequality holds because the members have no common element. **(i)** $\Rightarrow$ **(ii)**: Assume that $\alpha = \frac{4}{n}$. Let $x^\star$ be the optimal solution of $\min\{x^\top x : x \in Q(\mathcal{C})\}$. Then by Remark 16 (i), $\tau(\mathcal{C}) = 2$ and $x^\star = \left(\frac{2}{n},\frac{2}{n},\ldots,\frac{2}{n}\right)$. Let M be the incidence matrix of $\mathcal{C}$. By Lemma 7, there is a $y \in \mathbb{R}^{\mathcal{C}}_{+}$ such that $M^\top y = x^\star$ and $\mathbf{1}^\top y = \frac{4}{n}$, that is,

$\frac{n}{2}\cdot y$ is a fractional packing of $\mathcal{C}$ of value 2.

Let $\beta, z^\star$ be the optimal value and solution of $\min\{z^\top z : z \in Q(b(\mathcal{C}))\}$. As $\mathcal{C}$ is an ideal clutter, it follows from Theorem 6 that $\beta = \frac{1}{\alpha} = \frac{n}{4}$. Thus by Remark 16 (ii), $z^\star = \left(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}\right)$. Let N be the incidence matrix of $b(\mathcal{C})$. By Lemma 7, there is a $t \in \mathbb{R}^{b(\mathcal{C})}_{+}$ such that $N^\top t = z^\star$ and $\mathbf{1}^\top z^\star = \frac{n}{4}$, that is,

$2t$ is a fractional packing of $b(\mathcal{C})$ of value $\frac{n}{2}$.

It therefore follows from Lemma 15 that n is even, after a possible relabeling of the ground set the sets $\{1,2\},\{3,4\},\ldots,\{n-1,n\}$ are minimal covers of $\mathcal{C}$, and the members of $\mathcal{C}$ of minimum cardinality form a cuboid $\mathcal{C}_0$ over ground set

$\{1,\ldots,n\}$ with a fractional packing of value 2. In particular, the members of $\mathcal{C}_0$ do not have a common element. Moreover, since $\mathcal{C}$ is an ideal clutter, it follows from Lemma 17 that $\mathcal{C}_0$ is an ideal cuboid, thereby proving (ii). **(ii)** $\Rightarrow$ **(i)**: Observe that every member has cardinality at least $\frac{n}{2}$, so $\left(\frac{2}{n},\frac{2}{n},\ldots,\frac{2}{n}\right)\in Q(\mathcal{C})$, implying in turn that $\alpha\leq\frac{4}{n}$. Since $\alpha\geq\frac{4}{n}$ also, (i) must hold. □

We showed that among ideal clutters $\mathcal{C}$ whose members do not have a common element, it is essentially cuboids that achieve the smallest possible value for $\min\{x^\top x : x\in Q(\mathcal{C})\}$. Our proof relied on Lemma 15, which in itself has another consequence. Viewing clutters as *simple games*, Hof et al. [16] showed that given a clutter $\mathcal{C}$ over n elements, its *critical threshold value* is always at most $\frac{n}{4}$, and this maximum is achieved if, and only if, $\mathcal{C}$ has a fractional packing of value $\frac{n}{2}$ and $b(\mathcal{C})$ has a fractional packing of value 2. Thus by Lemma 15, it is essentially blockers of cuboids that achieve the largest possible critical threshold value.

5 Bypassing Gauge Duality

Here we present a longer and direct proof of the main result of the paper, Theorem 3. This proof will bypass the use of Theorem 6. We will need the following two lemmas:

Lemma 18. *Let $\mathcal{C},\mathcal{B}$ be clutters over ground set V such that $|C\cap B|=1$ for all $C\in\mathcal{C},B\in\mathcal{B}$, for which there exist nonzero $y\in\mathbb{R}^{\mathcal{C}}_+$ and $t\in\mathbb{R}^{\mathcal{B}}_+$ such that $\sum_{C\in\mathcal{C}}y_C\chi_C=\sum_{B\in\mathcal{B}}t_B\chi_B$. Then*

$$\nu(\mathcal{C})\geq\left(\frac{\mathbf{1}^\top t}{\mathbf{1}^\top y}-\frac{\min\{|C|-1 : C\in\mathcal{C}\}}{\min\{|V|,|\mathcal{C}|\}}\right)^{-1}$$
$$\nu(\mathcal{B})\geq\left(\frac{\mathbf{1}^\top y}{\mathbf{1}^\top t}-\frac{\min\{|B|-1 : B\in\mathcal{B}\}}{\min\{|V|,|\mathcal{B}|\}}\right)^{-1}.$$

Proof. Due to the symmetry between $\mathcal{C}$ and $\mathcal{B}$, it suffices to prove the first inequality. After possibly scaling t, we may assume that $\mathbf{1}^\top t=1$. Our hypotheses imply that for each $C'\in\mathcal{C}$,

$$\sum_{C\in\mathcal{C}}y_C|C'\cap C|=\sum_{C\in\mathcal{C}}y_C\chi_{C'}^\top\chi_C=\sum_{B\in\mathcal{B}}t_B\chi_{C'}^\top\chi_B=\sum_{B\in\mathcal{B}}t_B|C'\cap B|=1.$$

Thus, given that M is the incidence matrix of $\mathcal{C}$, the equalities above state that $MM^\top y=\mathbf{1}$. And Theorem 10 applied to y implies that

$$\nu(\mathcal{C})\geq\left(\frac{y^\top\mathbf{1}}{y^\top Jy}-\frac{\min\{|C|-1 : C\in\mathcal{C}\}}{\min\{|V|,|\mathcal{C}|\}}\right)^{-1},$$

therefore implying the first inequality. □

Given a clutter, a fractional cover is *minimal* if it is not greater than or equal to another fractional cover. Given an ideal clutter, it is well-known that every minimal fractional cover can be written as a convex combination of the incidence vectors of minimal covers (see for instance [11]). We will use this fact below:

Lemma 19. *Let $\mathcal{C}, \mathcal{B}$ be blocking ideal clutters. Then there exist nonempty $\mathcal{C}' \subseteq \mathcal{C}$ and $\mathcal{B}' \subseteq \mathcal{B}$ such that $|C \cap B| = 1$ for all $C \in \mathcal{C}', B \in \mathcal{B}'$, for which there exist nonzero $y \in \mathbb{R}_+^{\mathcal{C}'}$ and $t \in \mathbb{R}_+^{\mathcal{B}'}$ such that $\sum_{C \in \mathcal{C}'} y_C \chi_C = \sum_{B \in \mathcal{B}'} t_B \chi_B$.*

Proof. Let M, N be the incidence matrices of $\mathcal{C}, \mathcal{B}$, respectively. Let $x^\star$ be the optimal solution of $\min\{x^\top x : x \in Q(\mathcal{C})\}$. Then $x^\star$ is a minimal fractional cover of $\mathcal{C}$. As $\mathcal{C}$ is an ideal clutter, $x^\star = N^\top t$ for some $t \in \mathbb{R}_+^{\mathcal{B}}$ such that $\mathbf{1}^\top t = 1$. Moreover, by Lemma 7, there exists $y \in \mathbb{R}_+^{\mathcal{C}}$ such that $M^\top y = x^\star$ and $y^\top(Mx^\star - \mathbf{1}) = \mathbf{0}$. Thus, $\mathbf{1}^\top y = x^{\star\top} M^\top y = t^\top N M^\top y \geq t^\top J y = t^\top \mathbf{1}\mathbf{1}^\top y = \mathbf{1}^\top y$, implying that $t^\top N M^\top y = t^\top J y$. Therefore, if $\mathcal{C}' := \{C \in \mathcal{C} : y_C > 0\}$ and $\mathcal{B}' := \{B \in \mathcal{B} : t_B > 0\}$, we have that $|C \cap B| = 1$ for all $C \in \mathcal{C}', B \in \mathcal{B}'$. Moreover, the equation $M^\top y = x^\star = N^\top t$ implies that $\sum_{C \in \mathcal{C}'} y_C \chi_C = \sum_{B \in \mathcal{B}'} t_B \chi_B$. As $\mathcal{C}'$ and $\mathcal{B}'$ are clearly nonempty, they are the desired clutters. $\square$

Theorem 20. *Let $\mathcal{C}, \mathcal{B}$ be blocking ideal clutters. If $\tau(\mathcal{C}) \geq 2$ and $\tau(\mathcal{B}) \geq 2$, then $\nu(\mathcal{C}) \geq 2$ or $\nu(\mathcal{B}) \geq 2$.*

Proof. Assume that $\tau(\mathcal{C}) \geq 2$ and $\tau(\mathcal{B}) \geq 2$. By Lemma 19, there exist nonempty $\mathcal{C}' \subseteq \mathcal{C}$ and $\mathcal{B}' \subseteq \mathcal{B}$ such that $|C \cap B| = 1$ for all $C \in \mathcal{C}', B \in \mathcal{B}'$, for which there exist nonzero $y \in \mathbb{R}_+^{\mathcal{C}'}$ and $t \in \mathbb{R}_+^{\mathcal{B}'}$ such that $\sum_{C \in \mathcal{C}'} y_C \chi_C = \sum_{B \in \mathcal{B}'} t_B \chi_B$. As the members of $\mathcal{C}'$ and $\mathcal{B}'$ have cardinality at least two, we get from Lemma 18 that $\nu(\mathcal{C}') > \frac{\mathbf{1}^\top y}{\mathbf{1}^\top t}$ and $\nu(\mathcal{B}') > \frac{\mathbf{1}^\top t}{\mathbf{1}^\top y}$. As $\nu(\mathcal{C}) \geq \nu(\mathcal{C}')$ and $\nu(\mathcal{B}) \geq \nu(\mathcal{B}')$, it follows that $\nu(\mathcal{C}) \geq 2$ or $\nu(\mathcal{B}) \geq 2$, as required. $\square$

We are now ready to prove Theorem 3 again, stating that an identically self-blocking clutter different from $\{\{a\}\}$ is nonideal:

Proof (of Theorem 3). Let $\mathcal{C}$ be an identically self-blocking clutter different from $\{\{a\}\}$. Then $\tau(\mathcal{C}) \geq 2$, and $\nu(\mathcal{C}) = 1$ by Theorem 1. Theorem 20 now applies and tells us that $\mathcal{C}$ cannot be ideal, as required. $\square$

6 Concluding Remarks

Given a general blocking pair $\mathcal{C}, \mathcal{B}$ of ideal clutters, what can be said about them? This is an important research topic in Integer Programming and Combinatorial Optimization. We showed that if $\tau(\mathcal{C}) \geq 2$ and $\tau(\mathcal{B}) \geq 2$, then $\nu(\mathcal{C}) \geq 2$ or $\nu(\mathcal{B}) \geq 2$ (Theorems 3 and 20). Equivalently, if the members of $\mathcal{C}, \mathcal{B}$ have cardinality at least two, then one of the two clutters has a *bicoloring*, i.e. the ground set can be bicolored so that every member receives an element of each color. Next to Lehman's *width-length* characterization [18], this is the only other fact known about the structure of $\mathcal{C}$ and $\mathcal{B}$. As such, we expect the results as well as the tools introduced here to help us address the question in mind.

Our main result led us to two computable lower bounds – one weaker than the other – on the packing number of an arbitrary clutter (Theorems 11 and 12). We believe these lower bounds will have applications beyond the scope of this paper. We also characterized the clutters on which one of the lower bounds is at its weakest; we showed that these clutters are essentially cuboids (Theorem 13). Combined with evidence from [2,3], this only stresses further the central role of cuboids when studying ideal clutters.

Finally, we used techniques from Convex Optimization to prove the main result of the paper. A natural question is whether there is an elementary and discrete approach for proving the result? Conjecture 4 provides a potential approach and leads to an interesting research direction.

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