Exact Solutions of Nonlinear Micropolar Elastic Theory for Compressible Solids

L. M. Zubov, A. M. Kolesnikov and O. V. Rudenko

Abstract In this article we obtain exact solutions of finite inhomogeneous deformations of three-dimensional micropolar elastic bodies. We consider a model of the physically linear isotropic compressible material with six material parameters. The obtained solutions describe following types of finite deformations: cylindrical bending of a rectangular plate, straightening of a cylindrical sector, double cylindrical bending, pure bending of a circular cylinder sector, inflation and reversing of a hollow sphere. The results can be used to verify two-dimensional models of micropolar elastic shells.

1 Introduction

In this paper we derive exact solutions for problems of large inhomogeneous deformations of compressible isotropic micropolar elastic bodies. By a micropolar body we mean a continuous medium with couple stresses and rotational interactions of material particles. This model is also called the Cosserat continuum. The basics of the nonlinear theory of the Cosserat elastic continuum had been given in [5, 8–11, 14, 16]. The model of a micropolar medium is used to describe granular polycrystalline bodies, polymers, composites, suspensions, liquid crystals, geophysical structures, biological tissues, metamaterials, nanostructured materials, etc. The exact solutions can be used to a experimental determination of material parameters in the constitutive relations of the medium. Also they can be used to control the accuracy of calculations in numerical solution of nonlinear equilibrium equations for micropolar bodies.

The derived solutions are special cases of one-dimensional deformations of micropolar elastic bodies. Those are such deformations for which the system of partial

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differential equations of equilibrium reduces to a system of ordinary differential equations. A general method of constructing a class of one-dimensional deformations of a non-linear elastic Cosserat continuum is presented in [12]. In this paper the exact solutions describe the following types of one-dimensional deformations of a micropolar elastic medium: cylindrical bending of a rectangular plate, straightening of a circular cylinder sector, inflation and reversing of a cylindrical tube, pure bending of a circular cylinder sector, double cylindrical bending, inflation and reversing of a hollow sphere. These types of deformations correspond to universal solutions of equilibrium equations for the incompressible isotropic micropolar bodies which were obtained in the earlier paper [18]. In the case of compressed bodies, exact solutions in explicit analytic form for large deformations can be obtain only for some specific constitutive relations of an elastic material. In the present paper we use a model of the physically linear isotropic compressible micropolar body with six material parameters.

The micropolar shell model is two-dimensional analogue of the Cosserat continuum [1, 3, 4, 6, 13]. It is also called the Cosserat surface. In this model, by a shell we mean a material surface or a two-dimensional material continuum. Each point (particle) of this continuum has six degrees of freedom of an absolutely rigid body. The rotational degrees of freedom of a surface particle are kinetically independent of its displacement field. Solutions of problems of stretching, bending, inflation and reversing of cylindrical and spherical Cosserat surfaces, and bending of flat plates were obtained in the paper [17] for finite deformations. The solutions [17] describe nonlinear deformations of the shells, similar to the deformations corresponding to represented here solutions of three-dimensional micropolar theory of elasticity. Comparison of solutions obtained within the theory of shells and solutions obtained within the three-dimensional theory can be used to verify relations of the theory of micropolar shells.

2 Initial Relations of Nonlinear Micropolar Elasticity

The deformation of the elastic medium is described by a mapping of the reference configuration to the current configuration. In the case of a micropolar continuum, it is determined by two kinematically independent fields of displacement and rotation

$$\mathbf{R} = \mathbf{R}(\mathbf{r}) = \mathbf{r} + \mathbf{u}(\mathbf{r}), \quad \mathbf{H} = \mathbf{H}(\mathbf{r}),$$

where $\mathbf{r} = x_s \mathbf{i}_s$, $\mathbf{R} = X_k \mathbf{i}_k$, $(s, k = 1, 2, 3)$, x_s and X_k are Cartesian coordinates of the reference and current configurations, respectively, $\mathbf{i}_k$ are the unit vectors of the Cartesian coordinates, $\mathbf{u}$ is a displacement vector field, $\mathbf{H}$ is a proper orthogonal tensor, which describes the rotational degrees of freedom of the micropolar medium. It is called the microrotation tensor (or turntensor).

Below we use the following gradient, divergence, and rotor operators in the reference configuration coordinates

$$\mathrm{grad}\boldsymbol{\Psi} = \mathbf{r}^n \otimes \frac{\partial \boldsymbol{\Psi}}{\partial q^n}, \quad \mathrm{div}\boldsymbol{\Psi} = \mathbf{r}^n \cdot \frac{\partial \boldsymbol{\Psi}}{\partial q^n},$$
$$\mathrm{rot}\boldsymbol{\Psi} = \mathbf{r}^n \times \frac{\partial \boldsymbol{\Psi}}{\partial q^n}, \quad \mathbf{r}^n = \mathbf{i}_k \frac{\partial q^n}{\partial x_k},$$

where $\boldsymbol{\Psi}$ is an arbitrary differentiable tensor field of any order, $q^n = q^n(x_1, x_2, x_3)$ are curvilinear coordinates (the Lagrangian coordinates).

The system of equations of a micropolar elastic medium in the absence of mass forces and moments includes the following equations [5, 8–11, 14, 16]:

Equilibrium equations

$$\mathrm{div}\mathbf{D} = 0, \quad \mathrm{div}\mathbf{G} + \left(\mathbf{F}^{\mathrm{T}} \cdot \mathbf{D}\right)_\times = 0. \tag{1}$$

Constitutive relations

$$\mathbf{D} = \mathbf{P} \cdot \mathbf{H}, \quad \mathbf{G} = \mathbf{K} \cdot \mathbf{H},$$
$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{E}}, \quad \mathbf{K} = \frac{\partial W}{\partial \mathbf{L}}, \quad W = W(\mathbf{E}, \mathbf{L}). \tag{2}$$

Geometric relations

$$\mathbf{F} = \mathrm{grad}\mathbf{R}, \quad \mathbf{E} = \mathbf{F} \cdot \mathbf{H}^{\mathrm{T}},$$
$$\mathbf{L} = \frac{1}{2}\mathbf{r}^n \otimes \left(\frac{\partial \mathbf{H}}{\partial q^n} \cdot \mathbf{H}^{\mathrm{T}}\right)_\times = \frac{1}{2}\mathbf{I}\mathrm{tr}\left[\mathbf{H} \cdot (\mathrm{rot}\mathbf{H})^{\mathrm{T}}\right] - \mathbf{H} \cdot (\mathrm{rot}\mathbf{H})^{\mathrm{T}}. \tag{3}$$

Here **D**, **G** are the stress and couple stress tensors of the first Piola-Kirchhoff type, **P**, **K** are a stress and a couple stress tensors of the second Piola-Kirchhoff type, **E**, **L** are deformations tensors of a nonlinear micropolar continuum called stretch and wryness tensors, respectively [5, 8–11, 14], **I** is a unit tensor, W is a strain energy density. Symbol $\boldsymbol{\Phi}_\times$ means the vector invariant of a second-order tensor $\boldsymbol{\Phi}$:

$$\boldsymbol{\Phi}_\times = (\Phi_{mn}\mathbf{r}_m \otimes \mathbf{r}_n)_\times = \Phi_{mn}\mathbf{r}_m \times \mathbf{r}_n.$$

Below we will use the model of the compressible isotropic physically linear micropolar continuum [10]. This model is determined by the quadratic function of the strain energy density

$$2W = \lambda\mathrm{tr}^2(\mathbf{E} - \mathbf{I}) + (\mu + \beta)\,\mathrm{tr}\left[(\mathbf{E} - \mathbf{I}) \cdot (\mathbf{E} - \mathbf{I})^{\mathrm{T}}\right]$$
$$+ (\mu - \beta)\,\mathrm{tr}(\mathbf{E} - \mathbf{I})^2 + \delta\mathrm{tr}^2\mathbf{L} + (\gamma + \eta)\,\mathrm{tr}\left(\mathbf{L} \cdot \mathbf{L}^{\mathrm{T}}\right) + (\gamma - \eta)\,\mathrm{tr}\mathbf{L}^2. \tag{4}$$

where λ, μ, β, δ, γ, η are material constants. Also we will use the Poisson's ratio, which is expressed as

$$\nu = \frac{\lambda}{2\,(\lambda + \mu)}.$$

For this material model the stress tensor **P** is a linear function of the tensor ($\mathbf{E} - \mathbf{I}$), and the couple stress tensor **K** is a linear function of the wryness tensor **L**:

$$\begin{aligned} \mathbf{P} = \lambda \mathbf{I} \left(\mathrm{tr} \mathbf{E} - 3 \right) + (\mu + \beta) \left(\mathbf{E} - \mathbf{I} \right) + (\mu - \beta) \left(\mathbf{E}^{\mathrm{T}} - \mathbf{I} \right), \\ \mathbf{K} = \delta \mathbf{I} \mathrm{tr} \mathbf{L} + (\gamma + \eta) \mathbf{L} + (\gamma - \eta) \mathbf{L}^{\mathrm{T}}. \end{aligned} \quad (5)$$

Let us consider a special case of deformation of the micropolar medium such that $\mathbf{H} = \mathbf{A}$, where **A** a proper orthogonal macrorotation tensor. I.e. it is the orthogonal multiplier in the polar expansion of the strain gradient [7]

$$\mathbf{F} = \mathbf{U} \cdot \mathbf{A}. \quad (6)$$

Here **U** is a symmetric positive definite stretch tensor [7]. In this case, according to (3) and (6), we have

$$\mathbf{E} = \mathbf{E}^{\mathrm{T}} = \mathbf{U}.$$

Let us suppose that material constants δ, γ and η are zero. Then it follows from (5) that the couple stresses are absence: $\mathbf{K} = \mathbf{0}$, and the stress tensor **P** is symmetric and is expressed by the formula

$$\mathbf{P} = \lambda \mathbf{I} \mathrm{tr}(\mathbf{U} - \mathbf{I}) + 2\mu \left(\mathbf{U} - \mathbf{I} \right), \quad (7)$$

and the strain energy density has the form

$$W = \frac{1}{2} \lambda \mathrm{tr}^2 (\mathbf{U} - \mathbf{E}) + \mu \mathrm{tr} \mathbf{U}^2. \quad (8)$$

The relations (7), (8) correspond to the model of the harmonic or the semi-linear material which is well-known in the theory of elasticity of simple materials [7]. Thus the model of the physically linear micropolar body reduces to the model of the simple harmonic material when $\delta = \gamma = \eta = 0$ and $\mathbf{H} = \mathbf{A}$. In other words, the physically linear micropolar material (4) can be considered as a generalization of the harmonic material model to the moment elastic medium.

In the nonlinear theory of elasticity of simple materials, a number of exact solutions are known for finite deformations [7]. These solutions belong to the class of isotropic incompressible bodies. The list of models of compressible nonlinear elastic bodies, that allow explicit exact solutions, is quite small. Most of these exact solutions was found for semi-linear material [7]. As it shows below, the model of the physically linear micropolar body also allows to obtain several exact solutions about inhomogeneous finite deformations.

When the equilibrium problems are solving by the semi-inverse method, we give the finite deformations of a medium by the mapping $Q^M = Q^M(q^s)$, where q^s $(s = 1, 2, 3)$ are curvilinear coordinates in the reference configuration (the Lagrangian coordinates), and Q^M $(M = 1, 2, 3)$ are curvilinear coordinates in the current configuration (the Eulerian coordinates). Further, we use the following coordinate systems:

Cartesian:

$$q^1 = x_1, \quad q^2 = x_2, \quad q^3 = x_3,$$
$$Q^1 = X_1, \quad Q^2 = X_2, \quad Q^3 = X_3.$$

Cylindrical:

$$q^1 = r, \quad q^2 = \varphi, \quad q^3 = z,$$
$$Q^1 = R, \quad Q^2 = \Phi, \quad Q^3 = Z,$$
$$x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi, \quad x_3 = z,$$
$$X_1 = R\cos\Phi, \quad X_2 = R\sin\Phi, \quad X_3 = Z.$$

Spherical:

$$q^1 = r, \quad q^2 = \varphi, \quad q^3 = \theta,$$
$$Q^1 = R, \quad Q^2 = \Phi, \quad Q^3 = \Theta,$$
$$x_1 = r\cos\varphi\cos\theta, \quad x_2 = r\sin\varphi\cos\theta, \quad x_3 = r\sin\theta,$$
$$X_1 = R\cos\Phi\cos\Theta, \quad X_2 = R\sin\Phi\cos\Theta, \quad X_3 = R\sin\Theta.$$

Note, $\theta = \pm\frac{\pi}{2}$ at the sphere poles.

For these orthogonal coordinates we use the orthonormalized base vectors tangent to curvilinear coordinate curves. As above, $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ are the unit vectors of the Cartesian coordinates. The basis vectors $\mathbf{e}_r$, $\mathbf{e}_\varphi$, $\mathbf{e}_z$ and $\mathbf{e}_R$, $\mathbf{e}_\Phi$, $\mathbf{e}_Z$ associated with cylindrical coordinates are expressed as

$$\mathbf{e}_r = \mathbf{i}_1\cos\phi + \mathbf{i}_2\sin\phi, \quad \mathbf{e}_\phi = -\mathbf{i}_1\sin\phi + \mathbf{i}_2\cos\phi, \quad \mathbf{e}_z = \mathbf{i}_3,$$
$$\mathbf{e}_R = \mathbf{i}_1\cos\Phi + \mathbf{i}_2\sin\Phi, \quad \mathbf{e}_\Phi = -\mathbf{i}_1\sin\Phi + \mathbf{i}_2\cos\Phi, \quad \mathbf{e}_Z = \mathbf{i}_3.$$

The basis vectors associated with the spherical coordinates are presented as

$$\mathbf{e}_r = (\mathbf{i}_1\cos\varphi + \mathbf{i}_2\sin\varphi)\cos\theta + \mathbf{i}_3\sin\theta,$$
$$\mathbf{e}_\varphi = -\mathbf{i}_1\sin\varphi + \mathbf{i}_2\cos\varphi,$$
$$\mathbf{e}_\theta = -(\mathbf{i}_1\cos\varphi + \mathbf{i}_2\sin\varphi)\sin\theta + \mathbf{i}_3\cos\theta,$$
$$\mathbf{e}_R = (\mathbf{i}_1\cos\Phi + \mathbf{i}_2\sin\Phi)\cos\Theta + \mathbf{i}_3\sin\Theta,$$
$$\mathbf{e}_\Phi = -\mathbf{i}_1\sin\Phi + \mathbf{i}_2\cos\Phi,$$
$$\mathbf{e}_\Theta = -(\mathbf{i}_1\cos\Phi + \mathbf{i}_2\sin\Phi)\sin\Theta + \mathbf{i}_3\cos\Theta.$$

In this paper we present the exact solution for six families of finite deformations of the micropolar theory of elasticity. Each family is characterized by a mapping $Q^M = Q^M(q^s)$. This mapping gives the field of displacement of the medium and contains one unknown function of one variable. Microrotation in a micropolar medium is kinematically independent of displacements. Thus the mapping $Q^M(q^s)$ have to supplemented by a set of orthogonal tensor fields $\mathbf{H}(q^s)$ to complete the description of the deformation. Each of the six families of the inhomogeneous deformations contains several subfamilies that differ in microrotation fields. For each subfamily, the expressions for the tensors $\mathbf{E}$ and $\mathbf{L}$ are obtained using the formula (3). The expression for the deformation gradient $\mathbf{F}$ is identical for all solutions from one family.

3 Cylindrical Bending of Rectangular Plate

A mapping $Q^M = Q^M(q^s)$ has the form

$$R = R(x_1), \quad \Phi = kx_2, \quad Z = \alpha x_3. \tag{9}$$

where k, α are constants. Considering that $\mathbf{R} = R\mathbf{e}_R + Z\mathbf{e}_Z$, we obtain the deformation gradient

$$\mathbf{F} = R'\mathbf{i}_1 \otimes \mathbf{e}_R + kR\mathbf{i}_2 \otimes \mathbf{e}_\Phi + \alpha\mathbf{i}_3 \otimes \mathbf{e}_Z, \quad R' = \frac{dR}{dx_1}.$$

The mapping (9) is supplemented by four subfamilies of microrotation.

3.1 Subfamily 1A

$$\begin{aligned} \mathbf{H} &= \mathbf{i}_1 \otimes \mathbf{e}_R + \mathbf{i}_2 \otimes \mathbf{e}_\Phi + \mathbf{i}_3 \otimes \mathbf{e}_Z, \\ \mathbf{E} &= R'\mathbf{i}_1 \otimes \mathbf{i}_1 + kR\mathbf{i}_2 \otimes \mathbf{i}_2 + \alpha\mathbf{i}_3 \otimes \mathbf{i}_3, \\ \mathbf{L} &= k\mathbf{i}_2 \otimes \mathbf{i}_3. \end{aligned} \tag{10}$$

According to (2), (5), (10) the stress and couple stress Piola type tensors are defined as

$$\begin{aligned} \mathbf{D} &= D_{1R}\mathbf{i}_1 \otimes \mathbf{e}_R + D_{2\Phi}\mathbf{i}_2 \otimes \mathbf{e}_\Phi + D_{3Z}\mathbf{i}_3 \otimes \mathbf{e}_Z, \\ D_{1R} &= (\lambda + 2\mu)\, R' + \lambda kR + \lambda\alpha - (2\mu + 3\lambda)\,, \\ D_{2\Phi} &= \lambda R' + (\lambda + 2\mu)\, kR + \lambda\alpha - (2\mu + 3\lambda)\,, \\ D_{3Z} &= \lambda R' + \lambda kR + (\lambda + 2\mu)\,\alpha - (2\mu + 3\lambda)\,. \end{aligned} \tag{11}$$

$$\begin{aligned} \mathbf{G} &= G_{2Z}\mathbf{i}_2 \otimes \mathbf{e}_Z + G_{3\Phi}\mathbf{i}_3 \otimes \mathbf{e}_\Phi, \\ G_{2Z} &= k\,(\gamma + \eta)\,, \quad G_{3\Phi} = k\,(\gamma - \eta)\,. \end{aligned}$$

The equilibrium equation (1)$_2$ is satisfied identically for the considered deformation. The equilibrium equation (1)$_1$ is reduced to a scalar equation

$$\frac{\partial D_{1R}}{\partial x_1} - kD_{2\Phi} = 0. \tag{12}$$

From (11) the Eq. (12) is reduced to a ordinary differential equation in the unknown function $R(x_1)$:

$$R'' - k^2 R = -\frac{k\,(1-\nu\,(\alpha-1))}{1-\nu}. \tag{13}$$

The solution of this differential equation is

$$R(x_1) = c_1 e^{-kx_1} + c_2 e^{kx_1} + \frac{(1-\nu\,(\alpha-1))}{k\,(1-\nu)}. \tag{14}$$

In view of $0 \le x_1 \le h$, where h is a plate thickness, we can write the boundary conditions of the absence of external loads on the faces of the plate

$$D_{1R}\Big|_{x_1=0,h} = 0. \tag{15}$$

The constants of integration are determined from (11), (13), (14) and (15) as

$$c_1 = -\frac{(1-\nu\,(\alpha-1))\,\mathrm{e}^{kh}}{k\,(1-\nu)\left(1+\mathrm{e}^{kh}\right)},$$
$$c_2 = \frac{(1-\nu\,(\alpha-1))\,(1-2\nu)}{k\,(1-\nu)\left(1+\mathrm{e}^{kh}\right)}.$$

3.2 Subfamily 1B

$$\mathbf{H} = -\mathbf{i}_1 \otimes \mathbf{e}_R + \mathbf{i}_2 \otimes \mathbf{e}_\Phi - \mathbf{i}_3 \otimes \mathbf{e}_Z,$$
$$\mathbf{E} = -R'\mathbf{i}_1 \otimes \mathbf{i}_1 + kR\mathbf{i}_2 \otimes \mathbf{i}_2 - \alpha\mathbf{i}_3 \otimes \mathbf{i}_3,$$
$$\mathbf{L} = -k\mathbf{i}_2 \otimes \mathbf{i}_3.$$

So we have

$$D_{1R} = (\lambda + 2\mu)\,R' - \lambda kR + \lambda\alpha + (2\mu + 3\lambda)\,,$$
$$D_{2\Phi} = -\lambda R' + (\lambda + 2\mu)\,kR - \lambda\alpha - (2\mu + 3\lambda)\,,$$
$$D_{3Z} = \lambda R' - \lambda kR + (\lambda + 2\mu)\,\alpha + (2\mu + 3\lambda)\,,$$
$$G_{2Z} = k\,(\gamma + \eta)\,, \quad G_{3\Phi} = -k\,(\gamma - \eta)\,.$$

The equation for the unknown function $R(x_1)$ differs from the case of 1A:

$$R'' - k^2 R = \frac{k\,(1+\nu\,(1+\alpha))}{1-\nu}.$$

The solution of this differential equation is

$$R(x_1) = c_1 e^{-kx_1} + c_2 e^{kx_1} + \frac{(1+\nu\,(\alpha+1))}{k\,(1-\nu)}.$$

The integrations constants are determined from (15) as

$$c_1 = \frac{(1-2\nu)\,(1+\nu\,(\alpha+1))\,\mathrm{e}^{kh}}{k\,(1-\nu)\left(1+\mathrm{e}^{kh}\right)},$$
$$c_2 = -\frac{1+\nu\,(\alpha+1)}{k\,(1-\nu)\left(1+\mathrm{e}^{kh}\right)}.$$

3.3 Subfamily 1C

$$\mathbf{H} = \mathbf{i}_1 \otimes \mathbf{e}_R - \mathbf{i}_2 \otimes \mathbf{e}_\Phi - \mathbf{i}_3 \otimes \mathbf{e}_Z,$$
$$\mathbf{E} = R'\mathbf{i}_1 \otimes \mathbf{i}_1 - kR\mathbf{i}_2 \otimes \mathbf{i}_2 - \alpha\mathbf{i}_3 \otimes \mathbf{i}_3,$$
$$\mathbf{L} = -k\mathbf{i}_2 \otimes \mathbf{i}_3.$$

The stress and couple stress tensors have components

$$D_{1R} = (\lambda + 2\mu)\,R' - \lambda kR - \lambda\alpha - (2\mu + 3\lambda)\,,$$
$$D_{2\Phi} = -\lambda R' + (\lambda + 2\mu)\,kR + \lambda\alpha + (2\mu + 3\lambda)\,,$$
$$D_{3Z} = -\lambda R' + \lambda kR + (\lambda + 2\mu)\,\alpha + (2\mu + 3\lambda)\,,$$
$$G_{2Z} = k\,(\gamma + \eta)\,, \quad G_{3\Phi} = k\,(\gamma - \eta)\,.$$

The equation for the unknown function $R(x_1)$ is written as

$$R'' - k^2 R = \frac{k\,(1+\nu\,(\alpha+1))}{1-\nu}.$$

The solution of this differential equation is

$$R(x_1) = c_1 e^{-kx_1} + c_2 e^{kx_1} - \frac{1+\nu\,(\alpha+1)}{k\,(1-\nu)}.$$

From (15) we obtain

$$c_1 = -\frac{(1-2\nu)\,(1+\nu\,(\alpha+1))\,\mathrm{e}^{kh}}{k\,(1-\nu)\left(1+\mathrm{e}^{kh}\right)},$$
$$c_2 = \frac{1+\nu\,(\alpha+1)}{k\,(1-\nu)\left(1+\mathrm{e}^{kh}\right)}.$$

3.4 Subfamily 1D

$$\mathbf{H} = -\mathbf{i}_1 \otimes \mathbf{e}_R - \mathbf{i}_2 \otimes \mathbf{e}_\Phi + \mathbf{i}_3 \otimes \mathbf{e}_Z,$$
$$\mathbf{E} = -R'\mathbf{i}_1 \otimes \mathbf{i}_1 - kR\mathbf{i}_2 \otimes \mathbf{i}_2 + \alpha\mathbf{i}_3 \otimes \mathbf{i}_3,$$
$$\mathbf{L} = k\mathbf{i}_2 \otimes \mathbf{i}_3.$$

We have

$$D_{1R} = (\lambda+2\mu)\,R' + \lambda kR - \lambda\alpha + (2\mu+3\lambda)\,,$$
$$D_{2\Phi} = \lambda R' + (\lambda+2\mu)\,kR - \lambda\alpha + (2\mu+3\lambda)\,,$$
$$D_{3Z} = -\lambda R' - \lambda kR + (\lambda+2\mu)\,\alpha - (2\mu+3\lambda)\,,$$
$$G_{2Z} = k\,(\gamma+\eta)\,,\quad G_{3\Phi} = -k\,(\gamma-\eta)\,.$$

The equation for the unknown function $R(x_1)$ has the form

$$R'' - k^2 R = \frac{k\,(1-\nu\,(\alpha-1))}{1-\nu}.$$

The solution of this differential equation is

$$R(x_1) = c_1 e^{-kx_1} + c_2 e^{kx_1} - \frac{(1-\nu\,(\alpha-1))}{1-\nu}.$$

Using (15) we have

$$c_1 = \frac{(1-\nu\,(\alpha-1))\,\mathrm{e}^{kh}}{k\,(1-\nu)\left(1+\mathrm{e}^{kh}\right)},$$
$$c_2 = -\frac{(1-2\nu)\,(1-\nu\,(\alpha-1))}{k\,(1-\nu)\left(1+\mathrm{e}^{kh}\right)}.$$

It can be shown that resultant force vector acting in sections of a deformable body $\Phi = \mathrm{const}$ is equal to zero for all subfamilies 3A – 3D. The resultant moment has direction of the vector $\mathbf{e}_Z$ and value

$$M = l \int_0^h \left(R D_{\phi\Phi} + G_{\phi Z} \right) dr,$$

where l is length of the sector of the cylinder along the coordinate z $(0 \leq z \leq l)$.

The constant κ can be computed from given bending moment M. And constant α can be calculated from given longitudinal force acting in sections $Z = \mathrm{const}$

$$F = \phi_1 \int_{r_0}^{r_1} D_{zZ} r dr.$$

Here ϕ_1 is a sector angle $(0 \leq \phi \leq \phi_1)$.

4 Straightening of a Circular Hollow–Cylinder Sector

A mapping $Q^M = Q^M(q^s)$ is described by

$$X_1 = X_1(r), \quad X_2 = \xi\phi, \quad X_3 = \alpha z, \tag{16}$$

where ξ, α are constants. In view of

$$\mathbf{R} = X_1 \mathbf{i}_1 + X_2 \mathbf{i}_2 + X_3 \mathbf{i}_3,$$

we find the deformation gradient

$$\mathbf{F} = X_1' \mathbf{e}_r \otimes \mathbf{i}_1 + \frac{\xi}{r} \mathbf{e}_\phi \otimes \mathbf{i}_2 + \alpha \mathbf{e}_z \otimes \mathbf{i}_3, \quad X_1' = \frac{dX_1}{dr}.$$

The mapping (16) is supplemented by four subfamilies of microrotations.

4.1 Subfamily 2A

$$\begin{aligned} \mathbf{H} &= \mathbf{e}_r \otimes \mathbf{i}_1 + \mathbf{e}_\phi \otimes \mathbf{i}_2 + \mathbf{e}_z \otimes \mathbf{i}_3, \\ \mathbf{E} &= X_1' \mathbf{e}_r \otimes \mathbf{e}_r + \frac{\xi}{r} \mathbf{e}_\phi \otimes \mathbf{e}_\phi + \alpha \mathbf{e}_z \otimes \mathbf{e}_z, \\ \mathbf{L} &= -\frac{1}{r} \mathbf{e}_\phi \otimes \mathbf{e}_z. \end{aligned} \tag{17}$$

From (2), (5) and (17) the stress and couple stress tensors have the form

$$\begin{gathered}
\mathbf{D} = D_{r1}\mathbf{e}_r \otimes \mathbf{i}_1 + D_{\phi 2}\mathbf{e}_\phi \otimes \mathbf{i}_2 + D_{z3}\mathbf{e}_z \otimes \mathbf{i}_3, \\
D_{r1} = (\lambda + 2\mu)\, X_1' + \lambda\frac{\xi}{r} + \lambda\alpha - (2\mu + 3\lambda)\,, \\
D_{\phi 2} = \lambda X_1' + (\lambda + 2\mu)\,\frac{\xi}{r} + \lambda\alpha - (2\mu + 3\lambda)\,, \\
D_{z3} = \lambda X_1' + \lambda\frac{\xi}{r} + (\lambda + 2\mu)\,\alpha - (2\mu + 3\lambda)\,, \\
\mathbf{G} = G_{\phi 3}\mathbf{e}_\phi \otimes \mathbf{i}_3 + G_{z2}\mathbf{e}_z \otimes \mathbf{i}_2, \\
G_{\phi 3} = -\,(\gamma + \eta)\,\frac{1}{r}, \quad G_{z2} = -\,(\gamma - \eta)\,\frac{1}{r}.
\end{gathered} \tag{18}$$

The equilibrium equation $(1)_2$ is satisfied identically for the considered deformation. The equilibrium equation $(1)_1$ is reduced to a scalar ordinary differential equation

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\,D_{r1}\Big) = 0. \tag{19}$$

In view of $r_0 \le r \le r_1$, where r_0, r_1 are inner and outer radii, respectively, the boundary conditions for the absence of external loads on the lateral surfaces of the cylinder sector are written as

$$D_{r1}\Big|_{r=r_0,r_1} = 0. \tag{20}$$

In view of (19), (20), we obtain

$$D_{r1} \equiv 0. \tag{21}$$

The identity (21) is reduced to an ordinary differential equation in the unknown function $X_1(r)$, namely

$$X_1' + \frac{\xi\nu}{(1-\nu)\,r} = \frac{1 + (1-\alpha)\,\nu}{1-\nu}.$$

The solution of this differential equation has the form

$$X_1(r) = \frac{1 + (1-\alpha)\,\nu}{1-\nu} r - \frac{\xi\nu}{1-\nu}\ln r + \text{const}.$$

The integration constant corresponds to a rigid displacement and could assume any value, including zero.

4.2 Subfamily 2B

$$\mathbf{H} = -\mathbf{e}_r \otimes \mathbf{i}_1 + \mathbf{e}_\phi \otimes \mathbf{i}_2 - \mathbf{e}_z \otimes \mathbf{i}_3,$$
$$\mathbf{E} = k_1 X_1' \mathbf{e}_r \otimes \mathbf{e}_r + k_2 \frac{\xi}{r} \mathbf{e}_\phi \otimes \mathbf{e}_\phi + k_3 \alpha \mathbf{e}_z \otimes \mathbf{e}_z,$$
$$\mathbf{L} = -\frac{1}{r} \mathbf{e}_\phi \otimes \mathbf{e}_z.$$

We have

$$D_{r1} = (\lambda + 2\mu)\, X_1' - \lambda \frac{\xi}{r} + \lambda\alpha + (2\mu + 3\lambda)\,,$$
$$D_{\phi 2} = -\lambda X_1' + (\lambda + 2\mu)\, \frac{\xi}{r} - \lambda\alpha - (2\mu + 3\lambda)\,,$$
$$D_{z3} = \lambda X_1' - \lambda \frac{\xi}{r} + (\lambda + 2\mu)\, \alpha + (2\mu + 3\lambda)\,.$$
$$G_{\phi 3} = (\gamma + \eta)\, \frac{1}{r}, \quad G_{z2} = -\,(\gamma - \eta)\, \frac{1}{r}.$$

The equation for the unknown function $X_1(r)$ differs from the case 2A:

$$X_1' - \frac{\xi\nu}{(1-\nu)\, r} = -\frac{1 + (1+\alpha)\, \nu}{1 - \nu}.$$

The solution of this differential equation is

$$X_1(r) = -\frac{1 + (1+\alpha)\, \nu}{1 - \nu} r + \frac{\xi\nu}{1-\nu} \ln r + \text{const}.$$

4.3 Subfamily 2C

$$\mathbf{H} = \mathbf{e}_r \otimes \mathbf{i}_1 - \mathbf{e}_\phi \otimes \mathbf{i}_2 - \mathbf{e}_z \otimes \mathbf{i}_3,$$
$$\mathbf{E} = X_1' \mathbf{e}_r \otimes \mathbf{e}_r - \frac{\xi}{r} \mathbf{e}_\phi \otimes \mathbf{e}_\phi - \alpha \mathbf{e}_z \otimes \mathbf{e}_z,$$
$$\mathbf{L} = -\frac{1}{r} \mathbf{e}_\phi \otimes \mathbf{e}_z.$$

The components of the stress and couple stress tensors are expressed as

$$D_{r1} = (\lambda + 2\mu)\, X_1' - \lambda \frac{\xi}{r} - \lambda\alpha - (2\mu + 3\lambda)\,,$$
$$D_{\phi 2} = -\lambda X_1' + (\lambda + 2\mu)\, \frac{\xi}{r} + \lambda\alpha + (2\mu + 3\lambda)\,,$$
$$D_{z3} = -\lambda X_1' + \lambda \frac{\xi}{r} + (\lambda + 2\mu)\, \alpha + (2\mu + 3\lambda)\,,$$
$$G_{\phi 3} = (\gamma + \eta)\, \frac{1}{r}, \quad G_{z2} = (\gamma - \eta)\, \frac{1}{r}.$$

The equation in the unknown function $X_1(r)$ is

$$X_1' - \frac{\xi\nu}{(1-\nu)\, r} = \frac{1 + (1+\alpha)\,\nu}{1-\nu}.$$

The solution of this differential equation is given by

$$X_1(r) = \frac{1 + (1+\alpha)\,\nu}{1-\nu} r + \frac{\xi\nu}{1-\nu} \ln r + \text{const.}$$

4.4 Subfamily 2D

$$\mathbf{H} = -\mathbf{e}_r \otimes \mathbf{i}_1 - \mathbf{e}_\phi \otimes \mathbf{i}_2 + \mathbf{e}_z \otimes \mathbf{i}_3,$$
$$\mathbf{E} = -X_1' \mathbf{e}_r \otimes \mathbf{e}_r - \frac{\xi}{r} \mathbf{e}_\phi \otimes \mathbf{e}_\phi + \alpha \mathbf{e}_z \otimes \mathbf{e}_z,$$
$$\mathbf{L} = -\frac{1}{r} \mathbf{e}_\phi \otimes \mathbf{e}_z.$$

We have

$$D_{r1} = (\lambda + 2\mu)\, X_1' + \lambda \frac{\xi}{r} - \lambda\alpha + (2\mu + 3\lambda)\,,$$
$$D_{\phi 2} = \lambda X_1' + (\lambda + 2\mu)\, \frac{\xi}{r} - \lambda\alpha + (2\mu + 3\lambda)\,,$$
$$D_{z3} = -\lambda X_1' - \lambda \frac{\xi}{r} + (\lambda + 2\mu)\, \alpha - (2\mu + 3\lambda)\,,$$
$$G_{\phi 3} = -\,(\gamma + \eta)\, \frac{1}{r}, \quad G_{z2} = (\gamma - \eta)\, \frac{1}{r}.$$

The equation for the unknown function $X_1(r)$ has the form

$$X_1' + \frac{\xi\nu}{(1-\nu)\, r} = -\frac{1 + (1-\alpha)\,\nu}{1-\nu}.$$

The solution of this differential equation is

$$X_1(r) = -\frac{1+(1-\alpha)\,\nu}{1-\nu} r - \frac{\xi\nu}{1-\nu} \ln r + \text{const}.$$

It can be shown that resultant force vector acting in sections of a deformable body $X_2 = \text{const}$ has the direction of the vector $\mathbf{i}_2$ and value

$$F_2 = l \int_{r_0}^{r_1} D_{\phi 2} dr,$$

where l is a cylinder sector length ($0 \leq z \leq l$).

The resultant force vector acting in sections of a deformable body $X_3 = \text{const}$ has the direction of the vector $\mathbf{i}_3$ and value

$$F_3 = \phi_1 \int_{r_0}^{r_1} D_{z3} r dr,$$

where ϕ_1 is a cylinder sector angle ($0 \leq \phi \leq \phi_1$).

The constants ξ and α can be computed from given forces F_2 and F_3. And we can compute the bending moment which is required for straightening a circular hollow-cylinder sector. The resultant moment acts in sections of a deformable body $X_2 = \text{const}$, and has the direction of the vector $\mathbf{i}_3$ and value

$$M = l \int_{r_0}^{r_1} \left(X_1 D_{\phi 2} + G_{z3} \right) dr.$$

5 Pure Bending and Reversing of a Circular Hollow–Cylinder Sector

We set up a mapping $Q^M = Q^M(q^s)$ by defining

$$R = R(r), \quad \Phi = \kappa\phi, \quad Z = \alpha z. \tag{22}$$

where κ and α are constants. In view of

$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{e}_z, \quad \mathbf{R} = R\mathbf{e}_R + Z\mathbf{e}_Z,$$

we find the deformation gradient

$$\mathbf{F} = R'\mathbf{e}_r \otimes \mathbf{e}_R + \frac{\kappa R}{r}\mathbf{e}_\phi \otimes \mathbf{e}_\Phi + \alpha \mathbf{e}_z \otimes \mathbf{e}_Z, \quad R' = \frac{dR}{dr}.$$

The mapping (22) is supplemented by four subfamilies of microrotations.

5.1 Subfamily 3A

$$\begin{aligned} \mathbf{H} &= \mathbf{e}_r \otimes \mathbf{e}_R + \mathbf{e}_\phi \otimes \mathbf{e}_\Phi + \mathbf{e}_z \otimes \mathbf{e}_Z, \\ \mathbf{E} &= R'\mathbf{e}_r \otimes \mathbf{e}_r + \frac{\kappa R}{r}\mathbf{e}_\phi \otimes \mathbf{e}_\phi + \alpha \mathbf{e}_z \otimes \mathbf{e}_z, \\ \mathbf{L} &= \frac{\kappa - 1}{r}\mathbf{e}_\phi \otimes \mathbf{e}_z. \end{aligned} \tag{23}$$

Using (2), (5), (23) stress and couple stress tensors are given by

$$\begin{aligned} \mathbf{D} &= D_{rR}\mathbf{e}_r \otimes \mathbf{e}_R + D_{\phi\Phi}\mathbf{e}_\phi \otimes \mathbf{e}_\Phi + D_{zZ}\mathbf{e}_z \otimes \mathbf{e}_Z, \\ D_{rR} &= (\lambda + 2\mu)\, R' + \lambda\frac{\kappa R}{r} + \lambda\alpha - (2\mu + 3\lambda)\,, \\ D_{\phi\Phi} &= \lambda R' + (\lambda + 2\mu)\, \frac{\kappa R}{r} + \lambda\alpha - (2\mu + 3\lambda)\,, \\ D_{zZ} &= \lambda R' + \lambda\frac{\kappa R}{r} + (\lambda + 2\mu)\, \alpha - (2\mu + 3\lambda)\,, \\ \mathbf{G} &= G_{\phi Z}\mathbf{e}_\phi \otimes \mathbf{e}_Z + G_{z\Phi}\mathbf{e}_z \otimes \mathbf{e}_\Phi, \\ G_{\phi Z} &= (\gamma + \eta)\, \frac{\kappa - 1}{r}, \quad G_{z\Phi} = (\gamma - \eta)\, \frac{\kappa - 1}{r}. \end{aligned} \tag{24}$$

The equilibrium equation $(1)_2$ is satisfied identically for the considered deformation. The equilibrium equation $(1)_1$ is reduced to a scalar equation

$$\frac{\partial D_{rR}}{\partial r} + \frac{D_{rR} - \kappa D_{\phi\Phi}}{r} = 0. \tag{25}$$

Using (24), the Eq. (25) is reduced to an ordinary differential equation in the unknown function $R(r)$:

$$R'' + \frac{R'}{r} - \kappa^2\frac{R}{r^2} = \frac{(\kappa - 1)\,((\alpha - 1)\,\nu - 1)}{(1 - \nu)\, r}.$$

The solution for this differential equation is

$$R(r) = c_1 r^{\kappa} + c_2 r^{-\kappa} + \frac{(\kappa - 1)\left((\alpha - 1)\nu - 1\right)}{\left(1 - \kappa^2\right)(1 - \nu)} r.$$

In view of $r_0 \leq r \leq r_1$, where r_0, r_1 are inner and outer radii, respectively, the boundary conditions for the absence of external loads on the lateral surfaces of the cylinder sector are written in the form

$$D_{rR}\Big|_{r=r_0,r_1} = 0. \tag{26}$$

The constants of integration c_1 and c_2 can be obtained from the boundary conditions (26). Their expressions are cumbersome enough to write them explicit here.

5.2 Subfamily 3B

$$\mathbf{H} = -\mathbf{e}_r \otimes \mathbf{e}_R + \mathbf{e}_\phi \otimes \mathbf{e}_\Phi - \mathbf{e}_z \otimes \mathbf{e}_Z,$$
$$\mathbf{E} = k_1 R' \mathbf{e}_r \otimes \mathbf{e}_r + k_2 \frac{\kappa R}{r} \mathbf{e}_\phi \otimes \mathbf{e}_\phi + k_3 \alpha \mathbf{e}_z \otimes \mathbf{e}_z,$$
$$\mathbf{L} = -\frac{\kappa + 1}{r} \mathbf{e}_\phi \otimes \mathbf{e}_z.$$

Now we have

$$D_{rR} = (\lambda + 2\mu) R' - \lambda \frac{\kappa R}{r} + \lambda\alpha + (2\mu + 3\lambda),$$
$$D_{\phi\Phi} = -\lambda R' + (\lambda + 2\mu) \frac{\kappa R}{r} - \lambda\alpha - (2\mu + 3\lambda),$$
$$D_{zZ} = \lambda R' - \lambda \frac{\kappa R}{r} + (\lambda + 2\mu)\alpha + (2\mu + 3\lambda).$$
$$G_{\phi Z} = (\gamma + \eta) \frac{\kappa + 1}{r}, \quad G_{z\Phi} = -(\gamma - \eta) \frac{\kappa + 1}{r}.$$

The equation in the unknown $R(r)$ differs from the case 3 :

$$R'' + \frac{R'}{r} - \kappa^2 \frac{R}{r^2} = -\frac{(\kappa + 1)\left((\alpha + 1)\nu + 1\right)}{(1 - \nu) r}.$$

The solution of this differential equation is given by

$$R(r) = c_1 r^{\kappa} + c_2 r^{-\kappa} - \frac{(\kappa + 1)\left((\alpha + 1)\nu + 1\right)}{\left(1 - \kappa^2\right)(1 - \nu)} r.$$

5.3 Subfamily 3C

$$\mathbf{H} = \mathbf{e}_r \otimes \mathbf{e}_R - \mathbf{e}_\phi \otimes \mathbf{e}_\Phi - \mathbf{e}_z \otimes \mathbf{e}_Z,$$
$$\mathbf{E} = R'\mathbf{e}_r \otimes \mathbf{e}_r - \frac{\kappa R}{r}\mathbf{e}_\phi \otimes \mathbf{e}_\phi - \alpha \mathbf{e}_z \otimes \mathbf{e}_z,$$
$$\mathbf{L} = -\frac{\kappa + 1}{r}\mathbf{e}_\phi \otimes \mathbf{e}_z.$$

The components of the stress and couple stress tensors are expressed as

$$D_{rR} = (\lambda + 2\mu)\, R' - \lambda\frac{\kappa R}{r} - \lambda\alpha - (2\mu + 3\lambda)\,,$$
$$D_{\phi\Phi} = -\lambda R' + (\lambda + 2\mu)\, \frac{\kappa R}{r} + \lambda\alpha + (2\mu + 3\lambda)\,,$$
$$D_{zZ} = -\lambda R' + \lambda\frac{\kappa R}{r} + (\lambda + 2\mu)\, \alpha + (2\mu + 3\lambda)\,,$$
$$G_{\phi Z} = (\gamma + \eta)\, \frac{\kappa + 1}{r}, \quad G_{z\Phi} = (\gamma - \eta)\, \frac{\kappa - 1}{r}.$$

The equation in the unknown $R(r)$ is written as

$$R'' + \frac{R'}{r} - \kappa^2\frac{R}{r^2} = \frac{(\kappa + 1)\,((\alpha + 1)\,\nu + 1)}{(1 - \nu)\, r}.$$

The solution of this differential equation is given by

$$R(r) = c_1 r^\kappa + c_2 r^{-\kappa} + \frac{(\kappa + 1)\,((\alpha + 1)\,\nu + 1)}{\left(1 - \kappa^2\right)(1 - \nu)} r.$$

5.4 Subfamily 3D

$$\mathbf{H} = -\mathbf{e}_r \otimes \mathbf{e}_R - \mathbf{e}_\phi \otimes \mathbf{e}_\Phi + \mathbf{e}_z \otimes \mathbf{e}_Z,$$
$$\mathbf{E} = -R'\mathbf{e}_r \otimes \mathbf{e}_r - \frac{\kappa R}{r}\mathbf{e}_\phi \otimes \mathbf{e}_\phi + \alpha \mathbf{e}_z \otimes \mathbf{e}_z,$$
$$\mathbf{L} = \frac{\kappa - 1}{r}\mathbf{e}_\phi \otimes \mathbf{e}_z.$$

Now we have

$$D_{rR} = (\lambda + 2\mu)\, R' + \lambda\frac{\kappa R}{r} - \lambda\alpha + (2\mu + 3\lambda)\,,$$

$$D_{\phi\Phi} = \lambda R' + (\lambda + 2\mu)\,\frac{\kappa R}{r} - \lambda\alpha + (2\mu + 3\lambda)\,,$$

$$D_{zZ} = -\lambda R' - \lambda\frac{\kappa R}{r} + (\lambda + 2\mu)\,\alpha - (2\mu + 3\lambda)\,,$$

$$G_{\phi Z} = (\gamma + \eta)\,\frac{\kappa - 1}{r}, \quad G_{z\Phi} = -\,(\gamma - \eta)\,\frac{\kappa + 1}{r}.$$

The equation in the unknown $R(r)$ is written as

$$R'' + \frac{R'}{r} - \kappa^2\frac{R}{r^2} = -\frac{(\kappa - 1)\,((\alpha - 1)\,\nu - 1)}{(1 - \nu)\,r}.$$

The solution of this differential equation is

$$R(r) = c_1 r^{\kappa} + c_2 r^{-\kappa} - \frac{(\kappa - 1)\,((\alpha - 1)\,\nu - 1)}{\left(1 - \kappa^2\right)(1 - \nu)} r.$$

It can be shown that the resultant force vector acting in sections of a deformable body $\Phi = \text{const}$ equals zero for all subfamilies 3A – 3D. And the resultant moment has the direction of the vector $\mathbf{e}_Z$ and value

$$M = l \int_0^h \left(R D_{\phi\Phi} + G_{\phi Z}\right) dr,$$

where l is a length of cylinder sector along the coordinate z $(0 \le z \le l)$.

The constant κ can be computed from given bending moment M. The constant α can be calculated from given longitudinal force acting in sections $Z = \text{const}$,

$$F = \phi_1 \int_{r_0}^{r_1} D_{zZ} r dr.$$

Here ϕ_1 is sector angle $(0 \le \phi \le \phi_1)$.

6 Double Cylindrical Bending of a Circular Hollow–Cylinder Sector

A mapping $Q^M = Q^M(q^s)$ is described by

$$R = R(r), \quad \Phi = sz, \quad Z = t\phi, \tag{27}$$

where s, t are constants. Since

$$\mathbf{r} = R\mathbf{e}_r + z\mathbf{e}_z, \quad \mathbf{R} = R\mathbf{e}_R + Z\mathbf{e}_Z,$$

the deformation gradient is found

$$\mathbf{F} = R'\mathbf{e}_r \otimes \mathbf{e}_R + \frac{t}{r}\mathbf{e}_\phi \otimes \mathbf{e}_Z + sR\mathbf{e}_z \otimes \mathbf{e}_\Phi, \quad R' = \frac{dR}{dr}.$$

The mapping (27) is supplemented by four subfamilies of microrotations.

6.1 Subfamily 4A

$$\begin{aligned} \mathbf{H} &= \mathbf{e}_r \otimes \mathbf{e}_R + \mathbf{e}_\phi \otimes \mathbf{e}_Z - \mathbf{e}_z \otimes \mathbf{e}_\Phi, \\ \mathbf{E} &= R'\mathbf{e}_r \otimes \mathbf{e}_r + \frac{t}{r}\mathbf{e}_\phi \otimes \mathbf{e}_\phi + sR\mathbf{e}_z \otimes \mathbf{e}_z, \\ \mathbf{L} &= -\frac{1}{r}\mathbf{e}_\phi \otimes \mathbf{e}_z + s\mathbf{e}_z\mathbf{e}_\phi. \end{aligned} \tag{28}$$

Using (2), (5), (28) the stress and couple stress tensors are determined as

$$\begin{aligned} \mathbf{D} &= D_{rR}\mathbf{e}_r \otimes \mathbf{e}_R + D_{\phi Z}\mathbf{e}_\phi \otimes \mathbf{e}_Z + D_{z\Phi}\mathbf{e}_z \otimes \mathbf{e}_\Phi, \\ D_{rR} &= (\lambda + 2\mu) R' + \lambda\frac{t}{r} - \lambda s R - (2\mu + 3\lambda), \\ D_{\phi Z} &= \lambda R' + (\lambda + 2\mu)\frac{t}{r} - \lambda s R - (2\mu + 3\lambda), \\ D_{z\Phi} &= -\lambda R' - \lambda\frac{t}{r} + (\lambda + 2\mu) s R + (2\mu + 3\lambda), \\ \mathbf{G} &= G_{\phi\Phi}\mathbf{e}_\phi \otimes \mathbf{e}_\Phi + G_{zZ}\mathbf{e}_z \otimes \mathbf{e}_Z, \\ G_{\phi\Phi} &= -s(\gamma - \eta) + \frac{1}{r}(\gamma + \eta), \\ G_{zZ} &= s(\gamma + \eta) - \frac{1}{r}(\gamma - \eta). \end{aligned} \tag{29}$$

The equilibrium equation $(1)_2$ is satisfied identically for the considered deformation. The equilibrium equation $(1)_1$ is reduced to a scalar equation

$$\frac{\partial D_{rR}}{\partial r} + \frac{D_{rR}}{r} - sD_{z\Phi} = 0. \tag{30}$$

From (29), the Eq. (30) is reduced to an ordinary differential equations in the unknown $R(r)$:

$$R'' + \frac{1}{r}R' + \left(a_1 + \frac{a_2}{r}\right)R = a_3 + \frac{a_4}{r}, \tag{31}$$

$$a_1 = -s^2, \quad a_2 = -\frac{s\nu}{1-\nu},$$
$$a_3 = \frac{s\,(1+\nu)}{1-\nu}, \quad a_4 = \frac{1+(1-st)\,\nu}{1-\nu}.$$

The Eq. (31) is a particular case of the extended confluent hypergeometric equation [2]. Its solution can be represented by Kummer's functions.

We suppose $r_0 \le r \le r_1$, where r_0, r_1 are inner and outer radii, respectively. The boundary conditions for the absence of external loads on the lateral surfaces of the cylinder sector are written as

$$D_{rR}\Big|_{r=r_0,r_1} = 0. \tag{32}$$

6.2 Subfamily 4B

$$\mathbf{H} = \mathbf{e}_r \otimes \mathbf{e}_R - \mathbf{e}_\phi \otimes \mathbf{e}_Z + \mathbf{e}_z \otimes \mathbf{e}_\Phi,$$
$$\mathbf{E} = R'\mathbf{e}_r \otimes \mathbf{e}_r - \frac{t}{r}\mathbf{e}_\phi \otimes \mathbf{e}_\phi + sR\mathbf{e}_z \otimes \mathbf{e}_z,$$
$$\mathbf{L} = -\frac{1}{r}\mathbf{e}_\phi \otimes \mathbf{e}_z - s\mathbf{e}_z\mathbf{e}_\phi.$$

Now we have

$$D_{rR} = (\lambda + 2\mu)\,R' - \lambda\frac{t}{r} + \lambda sR - (2\mu + 3\lambda)\,,$$
$$D_{\phi Z} = -\lambda R' + (\lambda + 2\mu)\,\frac{t}{r} - \lambda sR + (2\mu + 3\lambda)\,,$$
$$D_{z\Phi} = \lambda R' - \lambda\frac{t}{r} + (\lambda + 2\mu)\,sR - (2\mu + 3\lambda)\,,$$
$$G_{\phi\Phi} = -s\,(\gamma - \eta) - \frac{1}{r}\,(\gamma + \eta)\,,$$
$$G_{zZ} = s\,(\gamma + \eta) + \frac{1}{r}\,(\gamma - \eta)\,.$$

The equation in the unknown $R(r)$ differs from the case 4:

$$R'' + \frac{1}{r}R' + \left(a_1 + \frac{a_2}{r}\right)R = a_3 + \frac{a_4}{r},$$
$$a_1 = -s^2, \quad a_2 = \frac{s\nu}{1-\nu},$$
$$a_3 = -\frac{s\,(1+\nu)}{1-\nu}, \quad a_4 = \frac{k_1\,(1+(1+st)\,\nu)}{1-\nu}.$$

6.3 Subfamily 4C

$$\mathbf{H} = -\mathbf{e}_r \otimes \mathbf{e}_R - \mathbf{e}_\phi \otimes \mathbf{e}_Z - \mathbf{e}_z \otimes \mathbf{e}_\Phi,$$
$$\mathbf{E} = -R'\mathbf{e}_r \otimes \mathbf{e}_r - \frac{t}{r}\mathbf{e}_\phi \otimes \mathbf{e}_\phi - sR\mathbf{e}_z \otimes \mathbf{e}_z,$$
$$\mathbf{L} = -\frac{1}{r}\mathbf{e}_\phi \otimes \mathbf{e}_z - s\mathbf{e}_z\mathbf{e}_\phi.$$

The components of the stress and couple stress tensors are expressed as

$$D_{rR} = (\lambda + 2\mu) R' + \lambda\frac{t}{r} + \lambda s R + (2\mu + 3\lambda),$$
$$D_{\phi Z} = \lambda R' + (\lambda + 2\mu) \frac{t}{r} + \lambda s R + (2\mu + 3\lambda),$$
$$D_{z\Phi} = \lambda R' + \lambda\frac{t}{r} + (\lambda + 2\mu) s R + (2\mu + 3\lambda),$$
$$G_{\phi\Phi} = s(\gamma - \eta) + \frac{1}{r}(\gamma + \eta),$$
$$G_{zZ} = s(\gamma + \eta) + \frac{1}{r}(\gamma - \eta).$$

The equation in the unknown $R(r)$ has the form

$$R'' + \frac{1}{r}R' + \left(a_1 + \frac{a_2}{r}\right) R = a_3 + \frac{a_4}{r},$$
$$a_1 = -s^2, \quad a_2 = \frac{s\nu}{1-\nu},$$
$$a_3 = \frac{s(1+\nu)}{1-\nu}, \quad a_4 = -\frac{(1 + (1 - st)\nu)}{1-\nu}.$$

6.4 Subfamily 4D

$$\mathbf{H} = -\mathbf{e}_r \otimes \mathbf{e}_R + \mathbf{e}_\phi \otimes \mathbf{e}_Z + \mathbf{e}_z \otimes \mathbf{e}_\Phi,$$
$$\mathbf{E} = -R'\mathbf{e}_r \otimes \mathbf{e}_r + \frac{t}{r}\mathbf{e}_\phi \otimes \mathbf{e}_\phi + sR\mathbf{e}_z \otimes \mathbf{e}_z,$$
$$\mathbf{L} = -\frac{1}{r}\mathbf{e}_\phi \otimes \mathbf{e}_z + s\mathbf{e}_z\mathbf{e}_\phi.$$

Now we obtain

$$D_{rR} = (\lambda + 2\mu)\, R' - \lambda \frac{t}{r} - \lambda s R + (2\mu + 3\lambda)\,,$$
$$D_{\phi Z} = -\lambda R' + (\lambda + 2\mu)\, \frac{t}{r} + \lambda s R - (2\mu + 3\lambda)\,,$$
$$D_{z\Phi} = -\lambda R' + \lambda \frac{t}{r} + (\lambda + 2\mu)\, s R - (2\mu + 3\lambda)\,,$$
$$G_{\phi\Phi} = s\,(\gamma - \eta) - \frac{1}{r}\,(\gamma + \eta)\,,$$
$$G_{zZ} = s\,(\gamma + \eta) - \frac{1}{r}\,(\gamma - \eta)\,.$$

The equation in the unknown $R(r)$ is written as

$$R'' + \frac{1}{r} R' + \left(a_1 + \frac{a_2}{r}\right) R = a_3 + \frac{a_4}{r}, \tag{33}$$

$$a_1 = -s^2, \quad a_2 = -\frac{s\nu}{1-\nu},$$
$$a_3 = -\frac{s\,(1+\nu)}{1-\nu}, \quad a_4 = -\frac{(1 + (1 - st)\,\nu)}{1-\nu}.$$

Let us consider the deformation (27) of the cylinder sector such that the ends $z = 0$ and $z = l$ are joined. Here l is a length of the cylinder sector. In this case the deformed body is a circular hollow cylinder in the current configuration. Then we have the conditions

$$\Phi\Big|_{z=0} = 0, \quad \Phi\Big|_{z=l} = 2\pi.$$

And we find that $s = \frac{2\pi}{l}$.

It can be shown that the resultant moment acting in sections of a deformable body $Z =$ const equal zero for all subfamilies 4A – 4D. And the resultant force vector has the direction of the vector $\mathbf{e}_Z$ and its value is

$$F = l \int_{r_0}^{r_1} D_{\phi Z} dr.$$

The deformation parameter t can be computed from given force F.

7 Radially Symmetric Deformation of a Hollow Sphere

We give a mapping $Q^M = Q^M(q^s)$ in the form

$$R = R(r), \quad \Phi = \phi, \quad \Theta = \theta. \tag{34}$$

In view of

$$\mathbf{r} = R\mathbf{e}_r, \quad \mathbf{R} = R\mathbf{e}_R, \tag{35}$$

the deformation gradient has the form

$$\mathbf{F} = R'\mathbf{e}_r \otimes \mathbf{e}_R + \frac{R}{r}\mathbf{e}_\phi \otimes \mathbf{e}_\Phi + \frac{R}{r}\mathbf{e}_\theta \otimes \mathbf{e}_\Theta, \quad R' = \frac{dR}{dr}.$$

The mapping (34) is supplemented by four subfamilies of microrotation.

7.1 Subfamily 5A

$$\begin{aligned} \mathbf{H} &= \mathbf{e}_r \otimes \mathbf{e}_R + \mathbf{e}_\phi \otimes \mathbf{e}_\Phi + \mathbf{e}_\theta \otimes \mathbf{e}_\Theta, \\ \mathbf{E} &= R'\mathbf{e}_r \otimes \mathbf{e}_r + \frac{R}{r}\mathbf{e}_\phi \otimes \mathbf{e}_\phi + \frac{R}{r}\mathbf{e}_\theta \otimes \mathbf{e}_\theta. \end{aligned} \tag{36}$$

From (2), (5) and (36), the stress and couple stress tensors have the form

$$\begin{aligned} \mathbf{D} &= D_{rR}\mathbf{e}_r \otimes \mathbf{e}_R + D_{\phi\Phi}\mathbf{e}_\phi \otimes \mathbf{e}_\Phi + D_{\theta\Theta}\mathbf{e}_\theta \otimes \mathbf{e}_\Theta, \\ D_{rR} &= (\lambda + 2\mu)\, R' + 2\lambda\frac{\kappa R}{r} - (3\lambda + 2\mu)\,, \\ D_{\phi\Phi} = D_{\theta\Theta} &= \lambda R' + 2\,(\lambda + \mu)\,\frac{R}{r} - (3\lambda + 2\mu)\,. \end{aligned} \tag{37}$$

The wryness tensor $\mathbf{L}$ and couple stress tensor $\mathbf{G}$ are zero tensors.

The equilibrium equation $(1)_2$ is satisfied identically for the considered deformation. The equilibrium equation $(1)_1$ is reduced to a scalar equation

$$\frac{\partial D_{rR}}{\partial r} + 2\frac{D_{rR} - D_{\phi\Phi}}{r} = 0. \tag{38}$$

In consequence of (37), the Eq. (38) is reduced to an ordinary differential equation in the unknown function $R(r)$, namely

$$R'' + 2\frac{R'}{r} - 2\frac{R}{r^2} = 0. \tag{39}$$

The solution for this differential equation is

$$R(r) = c_1 r + \frac{c_2}{r}. \tag{40}$$

We give the boundary conditions in the form

$$D_{rR}\Big|_{r=r_0} = -p_0, \quad D_{rR}\Big|_{r=r_1} = -p_1. \tag{41}$$

Here r_0 and r_1 are inner and outer radii, respectively, p_0 and p_1 are pressures per unit area in the reference configuration in the inner and outer surfaces, respectively.

Using (37), (40) and (41) the integration constants are presented as

$$c_1 = 1 - \frac{(1-2\nu)\left(r_1^3 p_1 - r_0^3 p_0\right)}{2\mu\,(1+\nu)\left(r_1^3 - r_0^3\right)}, \quad c_2 = -\frac{(p_1 - p_0)\, r_1^3 r_0^3}{4\mu\left(r_1^3 - r_0^3\right)}.$$

7.2 Subfamily 5B

$$\mathbf{H} = \mathbf{e}_r \otimes \mathbf{e}_R - \mathbf{e}_\phi \otimes \mathbf{e}_\Phi - \mathbf{e}_\theta \otimes \mathbf{e}_\Theta,$$
$$\mathbf{E} = R'\mathbf{e}_r \otimes \mathbf{e}_r - \frac{R}{r}\mathbf{e}_\phi \otimes \mathbf{e}_\phi - \frac{R}{r}\mathbf{e}_\theta \otimes \mathbf{e}_\theta,$$
$$\mathbf{L} = -\frac{2}{r}\mathbf{e}_\phi \otimes \mathbf{e}_\theta + \frac{2}{r}\mathbf{e}_\theta \otimes \mathbf{e}_\phi.$$

Now we obtain

$$D_{rR} = (\lambda + 2\mu)\, R' - 2\lambda \frac{R}{r} - (3\lambda + 2\mu)\,,$$
$$D_{\phi\Phi} = D_{\theta\Theta} = -\lambda R' + 2\,(\lambda + \mu)\,\frac{R}{r} + (3\lambda + 2\mu)\,,$$

$$\mathbf{G} = \frac{4\eta}{r}\mathbf{e}_\phi \otimes \mathbf{e}_\Theta - \frac{4\eta}{r}\mathbf{e}_\theta \otimes \mathbf{e}_\Phi.$$

The equation in the unknown function $R(r)$ differs from the case 5A:

$$R'' + \frac{2R'}{r} - \frac{2aR}{r^2} = \frac{4a}{r}, \quad a = \frac{1+\nu}{1-\nu}.$$

The solution of this equation is given by

$$R(r) = c_1 r^{-\frac{1-b}{2}} + c_2 r^{-\frac{1+b}{2}} - \frac{1+\nu}{\nu} r, \quad b = \sqrt{\frac{9+7\nu}{1-\nu}}.$$

The integration constants c_1 and c_2 are determined by the boundary conditions (41). Their expressions are cumbersome enough to write them explicit here.

8 Reversing of a Hollow Sphere

A mapping $Q^M = Q^M(q^s)$ has the form

$$R = R(r), \quad \Phi = \phi, \quad \Theta = -\theta. \tag{42}$$

In view of (35), the deformation gradient is defined by

$$\mathbf{F} = R'\mathbf{e}_r \otimes \mathbf{e}_R + \frac{R}{r}\mathbf{e}_\phi \otimes \mathbf{e}_\Phi - \frac{R}{r}\mathbf{e}_\theta \otimes \mathbf{e}_\Theta, \quad R' = \frac{dR}{dr}.$$

The mapping (42) is supplemented by four subfamilies of microrotations.

8.1 Subfamily 6A

$$\mathbf{H} = -\mathbf{e}_r \otimes \mathbf{e}_R + \mathbf{e}_\phi \otimes \mathbf{e}_\Phi - \mathbf{e}_\theta \otimes \mathbf{e}_\Theta, \tag{43}$$

$$\mathbf{E} = -R'\mathbf{e}_r \otimes \mathbf{e}_r + \frac{R}{r}\mathbf{e}_\phi \otimes \mathbf{e}_\phi - \frac{R}{r}\mathbf{e}_\theta \otimes \mathbf{e}_\theta, \tag{44}$$

$$\mathbf{L} = -\frac{2}{r}\mathbf{e}_\phi \otimes \mathbf{e}_\theta + \frac{2}{r}\mathbf{e}_\theta \otimes \mathbf{e}_\phi. \tag{45}$$

In view of (2), (5) and (43), the stress and couple stress tensors are defined as

$$\begin{aligned} \mathbf{D} &= D_{rR}\mathbf{e}_r \otimes \mathbf{e}_R + D_{\phi\Phi}\mathbf{e}_\phi \otimes \mathbf{e}_\Phi + D_{\theta\Theta}\mathbf{e}_\theta \otimes \mathbf{e}_\Theta, \\ D_{rR} &= (\lambda + 2\mu)\, R' - 2\lambda\frac{R}{r} + (2\mu + 3\lambda)\,, \\ D_{\phi\Phi} &= -\lambda R' + 2\,(\lambda + \mu)\,\frac{R}{r} - (2\mu + 3\lambda)\,, \\ D_{\theta\Theta} &= \lambda R' - 2\,(\lambda + \mu)\,\frac{R}{r} + (2\mu + 3\lambda)\,, \end{aligned} \tag{46}$$

$$\begin{aligned} \mathbf{G} &= G_{\phi\Theta}\mathbf{e}_\phi \otimes \mathbf{e}_\Theta + G_{\theta\Phi}\mathbf{e}_\theta \otimes \mathbf{e}_\Phi, \\ G_{\phi\Theta} &= \frac{4\eta}{r}, \quad G_{\theta\Phi} = \frac{4\eta}{r}. \end{aligned} \tag{47}$$

The equilibrium equation $(1)_2$ is satisfied identically for the considered deformation. The equilibrium equation $(1)_1$ is written in the form . Using (46) the equilibrium equation is reduced to an ordinary differential equation in the unknown function $R(r)$:

$$R'' + \frac{2R'}{r} - \frac{2aR}{r^2} = -\frac{4a}{r}, \quad a = \frac{1+\nu}{1-\nu}.$$

The solution of this differential equation has the form

$$R(r) = c_1 r^{-\frac{1+b}{2}} + c_2 r^{-\frac{1-b}{2}} + \frac{1+\nu}{\nu} r, \quad b = \sqrt{\frac{9+7\nu}{1-\nu}}.$$

The integration constants c_1 and c_2 are determined by the boundary conditions (41). Their expressions are cumbersome enough to write them explicit here.

8.2 Subfamily 6B

$$\mathbf{H} = -\mathbf{e}_r \otimes \mathbf{e}_R - \mathbf{e}_\phi \otimes \mathbf{e}_\Phi + \mathbf{e}_\theta \otimes \mathbf{e}_\Theta,$$
$$\mathbf{E} = -R'\mathbf{e}_r \otimes \mathbf{e}_r - \frac{R}{r}\mathbf{e}_\phi \otimes \mathbf{e}_\phi - \frac{R}{r}\mathbf{e}_\theta \otimes \mathbf{e}_\theta.$$

Now we have

$$D_{rR} = (\lambda + 2\mu)\, R' - (2\mu + 3\lambda)\,,$$
$$D_{\phi\Phi} = -\lambda R' + 2\mu \frac{R}{r} + (2\mu + 3\lambda)\,,$$
$$D_{\theta\Theta} = -\lambda R' - 2\mu \frac{R}{r} + (2\mu + 3\lambda)\,.$$

The wryness tensor **L** and the couple stress tensor **G** are zero tensors.

The equation in the unknown function $R(r)$ is same as Eq. (39). Its solution has the form (40). From the boundary conditions (41) we determine the constants c_1 and c_2 as

$$c_1 = -1 - \frac{(1-2\nu)\left(r_1^3 p_1 - r_0^3 p_0\right)}{\mu\,(1+\nu)\left(r_1^3 - r_0^3\right)}, \quad c_2 = -\frac{(p_1 - p_0)\, r_1^3 r_0^3}{4\mu\left(r_1^3 - r_0^3\right)}.$$

9 Conclusion

In this paper, several families of finite deformations of a micropolar body had been considered. Following the semi-inverse method, we reduced the original system of differential equilibrium equations with three independent variables to a system of ordinary differential equations. In the present paper we used the model of the physically linear isotropic compressible micropolar body with six material parameters. Its strain energy density is quadratic form of stretch and wryness tensors.

We consider arbitrary large strains and rotations in spite of this, the ordinary differential equations obtained in solving problems are linear. This allowed us to construct the exact solutions for the problems of strong cylindrical bending of a rectangular plate, straightening of a cylindrical layer, inflation and reversing of a cylindrical tube, pure bending of a circular cylinder sector, double cylindrical bending, inflation and reversing of a hollow sphere.

The solutions obtained in this paper within the three-dimensional non-linear micropolar elasticity theory can be used to verify the two-dimensional theory of micropolar shells. Also these solutions can be used for establishing the connection of material constants in the constitutive relations of the two-dimensional shell model with the material constants of the three-dimensional micropolar medium. To do this we can use the formulae obtained in the paper [15]. These formulae express the resultant stresses and the resultant couple stresses in a shell by the stresses and the couple stresses of a three-dimensional medium averaging through a thickness.

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