Cecilia Kilhamn · Roger Säljö *Editors*

Encountering Algebra

A Comparative Study of Classrooms in Finland, Norway, Sweden, and the USA

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Editors
Cecilia Kilhamn
Department of Pedagogical, Curricular and Professional Studies
University of Gothenburg
Gothenburg, Sweden

Roger Säljö
Department of Education, Communication and Learning
University of Gothenburg
Gothenburg, Sweden

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Preface

This volume builds on a collaborative research project about algebra learning in four countries, Finland, Norway, Sweden, and the USA (California). The idea of the project was to document and analyze the first lessons when students are introduced to algebra. In concrete terms, each country team contacted teachers and asked them for permission to video record the lessons when they introduced the curricular unit of algebra. The choice of the topic of algebra was motivated not only by our joint interest in teaching and learning in this domain, but also by the observations reported in much of the literature that algebra is a hurdle for many students. Concepts such as variable, unknowns, and equivalence and the solving of equations represent activities that many find challenging, and the aim of the project has been to see how instruction is organized, and how students approach, struggle with, and appropriate basic algebraic symbols and modes of thinking. This implies that the spirit of the project has been to explore the perspectives and rationalities of the participants, students, and teachers, as they engage in algebra in regular instructional settings.

As the reader of this volume, and other reports that have been produced in the project, will see, there are both similarities in and variations between the ways in which algebra is introduced in the classrooms we have documented. We are well aware that our empirical materials do now allow us to make strong generalizations in a statistical sense about how algebra is introduced in different countries. The specific background of this volume is that during the extensive discussions over several years in the distributed project group about our data and analyses, we decided to try to pick out elements of introducing algebra that we found characteristic of our respective materials, i.e., elements that stood out as characteristic when compared to what we saw in the materials from the other countries. Thus, each team was encouraged to select one topic or feature of the teaching and learning documented that they perceived as typical of their own educational traditions and of how algebra is introduced in the lessons documented. The empirical chapters (4–7) present these case studies from the respective countries. In the case of Sweden, the focus is on how children participate in teaching and learning when algebra is introduced. In the case of Norway, the focus is on the nature of tasks and examples that teacher use and produce to help clarifying basic algebraic concepts and modes of reasoning. In the chapter from Finland, the ways in which students approach equation solving are explored, and in the US chapter consistencies between teachers' conceptions of what it means to learn mathematics/algebra, on the one hand, and the instruction they engage in, on the other hand, are analyzed. Chapter 8, the final empirical section, reports a comparative analysis of how students in the four different countries solve a patterning task given after the first four lessons of algebra. A patterning task, thus, potentially involves modeling and using algebraic reasoning in order to make mathematical sense of a situation where students have to oscillate between, on the one hand, concrete observations of a pattern that they can see in front of them and, on the other hand, attempts to represent this pattern in mathematical terms. The task itself is taken from an international comparative study of mathematics achievement, and it proved to be an interesting test bed for studying learning trajectories.

The project itself has been an exciting and truly international collaboration between the four teams. Representatives of teams have met physically on a few occasions, but most of the collaboration has taken place through videoconferencing where we have taken decisions on how to proceed with the research at various stages and where we have discussed our data, analyses, and findings. This work has been conducted in real time, which has meant that the members of the team in California had to be ready for academic exercises very early in the morning, while the Finnish team members had to stay in their offices after regular working hours. The fact that it is possible to conduct research seminars under these conditions, including activities such as projecting data to be discussed on a shared screen and scrutinizing analyses suggested, has been both rewarding and inspirational. And we are very satisfied that this idea, built into the design of the project, turned out to work very well technically as well as academically.

The project was originally initiated as a Nordic research project by Roger Säljö, Sweden, Maria Luiza Cestari, Norway, and Ole Björkqvist, Finland, in collaboration with Jim Stiegler, USA. The practical work of collecting empirical data, transcribing, coding, and analyzing was conducted in each country by a team of researchers. In addition to all those participating as authors in this book, we would like to acknowledge the contributions of Rimma Nyman, Anna Lundberg, and Elisabeth Rystedt, three PhD students who joined the project at different times, providing input and new perspectives. Our special thanks go to all the teachers who opened their classrooms for us, allowing us to document their practices on video, and who also willingly took part in individual and focus group interviews.

The project has been funded by the Joint Committee for Nordic Research Councils for the Humanities and Social Sciences (grant 2135-08-210321), and we are very grateful for this possibility to do comparative research in algebra learning, while at the same time testing the possibilities for academic work across time and space. The research teams would also like to express their thanks to the administrative staff at the four universities for their support. The core person in the complicated sub-project of handling resources, overheads, currency fluctuations, and other

facts of academic life has been Mrs. Doris Gustafson at the University of Gothenburg. Doris Gustafson has an incredible experience in the administrative sides of research collaborations, national as well as international, and without her skilled support and problem-solving capacities we would not even have been able to put together a coherent application.

Gothenburg, Sweden

Cecilia Kilhamn
Roger Säljö

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Cecilia Kilhamn, Ann-Sofi Röj-Lindberg, and Ole Björkqvist

Introduction

> There is no single program for learning algebra through the expression of generality. It is a matter of awakening and sharpening sensitivity to the presence and potential for algebraic thinking. (Mason, 1996, p. 65)

The role of algebra within school mathematics reflects very well its central position within mathematics itself. While the historical development of algebra was a major achievement as such (see Varadarajan, 1998), it is also easily appreciated as the starting point for many more advanced topics. In particular, the shift from rhetorical to symbolic algebra has had a great influence on the development of mathematics. In most parts of the world, a significant proportion of lower secondary school curricula in mathematics is devoted to algebra (Leung, Park, Holton, & Clarke, 2014), and mastery of the foundations of algebra, especially the awareness of generality, is in general argued to be a necessary prerequisite for successful study of mathematics at upper school levels.

A common feature of mathematics is its tendency to give precedence to mathematical objects over mathematical actions (nouns over verbs), e.g. the *act of dividing* one number with another is preferably called *a division*, affording the possibility to discuss properties of *divisions* as a mathematical object. Many mathematical topics similarly involve the introduction of object-like concepts based on procedures, and there may be stages of importance in this kind of development (Sfard, 1991). It seems appropriate to analyze school algebra in terms of mathematical concepts and

C. Kilhamn (✉)
Department of Pedagogical, Curricular and Professional Studies,
University of Gothenburg, Gothenburg, Sweden
e-mail: cecilia.kilhamn@gu.se

A.-S. Röj-Lindberg · O. Björkqvist
Faculty of Education and Welfare Studies, Åbo Akademi University, Vasa, Finland
e-mail: aroj@abo.fi

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procedures that may be deemed as characteristic for it, which is a standard way of looking at school mathematics (Lesh & Landau, 1983). Thus, school algebra can be seen as a showcase for school mathematics.

The explicit introduction of algebra within the study of mathematics has traditionally been postponed until the study of arithmetic has been brought to an assumed conclusion, and it has been customary in many countries to begin formal algebra at lower secondary school. For some decades, however, this type of artificial division in time between arithmetic and algebra has been rejected, and it is now quite generally accepted that supporting algebraic thinking and the use of algebraic tools as early as in the first grades is beneficial to both the learning of arithmetic and the learning of algebra (Britt & Irwin, 2011; Cai & Moyer, 2008; Hewitt, 2014; Kaput, Carraher, & Blanton, 2008; Kieran, 2018; Kieran, Pang, Schifter, & Ng, 2016). Although algebraic reasoning without the use of written symbols constitutes the core of what is called "early algebra", developing fluency in written representations is ultimately an essential part of algebra. Early algebra builds on contexts of problems, interweaves algebraic reasoning with existing topics of early mathematics and gradually introduces and extends students' own representations into formal symbolic representations (Carraher, Schliemann, & Schwartz, 2008; Kieran, 2018; Radford, 2011, 2018). In Davydov's curriculum, extensive work is done already in the first years at school to represent part-whole relationships visually and symbolically, before introducing specific numbers (Davydov, Gorbov, Mikulina, & Savaleva, 1999; Schmittau, 2011).

The introduction of algebra has thus been spread out in time, and the approaches to teaching and learning algebra in the early grades vary significantly. Hence, algebra can be approached from different perspectives using different points of departure (Bednarz, Kieran, & Lee, 1996). Furthermore, the teachers' interpretations of curricula and textbooks are certain to introduce even more variability. At the heart of it all is the fact that algebra has many faces, and whatever the teacher momentarily attends to may either support or obscure another aspect of algebra. What is dealt with during a single lesson can reflect momentary learning goals, while it at the same time has a bearing on the long-term learning of mathematics. Both the order and the depth of the treatment of the topics are important, and so is the character of the particular mathematical tools and representations in use during the lessons.

In the VIDEOMAT research project, which is the basis for this book, an overview of the national curricular documents of the four participating countries/states was initiated, i.e. Sweden, Norway, Finland and California, USA. All of these documents describe a similar shift in the teaching of school algebra. In the early years, algebraic content is described in terms of algebraic thinking: dealing with relationships, regularities and patterns without any formalized symbolic notation apart from an emphasis on the equal sign. At around the age of 12, students are expected to start solving equations using variables, expressions and formulas. This can be interpreted as an indication that, although algebraic thinking may have been an issue at an early age and in various ways, a more formalized symbolic language is introduced to students at around the same age in the four countries. Hence, this phase of school algebra was identified as the "introduction of variables in algebra", or the introduction to letter-symbolic algebra. The five empirical chapters in this book document how this introduction is organized in classroom practices in the four contexts.

School Algebra: What Is It?

Against the background of the several mathematical faces of the basic concepts of algebra, it is hardly surprising that many practitioners engaged in algebra teaching and learning find it quite hard to describe what school algebra really is (Kendal & Stacey, 2004). An important feature of algebra is the possibility to represent abstract ideas so that they are easier to access and to use in a routine-like manner. A central idea in algebra is *generality*. Generality presupposes the existence of a great number of specific instances that can be summarized in a concise way, e.g. utilizing the letter n to serve as an index of the instances, while it also appears as a variable in a mathematical expression based on any of the instances. Generality may be established through recognition of patterns, which means identification of that which is constant from instance to instance, as well as that which constitutes variation between instances. Established generalities regarding mathematical structures are starting points for new explorations in mathematics, reliable tools for mathematical problem solving and the formal basis for deductive reasoning.

In an attempt to cover the many faces of algebra, Usiskin (1988) identifies four main categories of school algebra: (1) generalized arithmetic; (2) a way to solve certain types of problems; (3) a study of relationships among quantities; and (4) a study of structures. In recent research about algebraic thinking among 5–12-year-olds, there is a strong emphasis on the structural aspects of algebra (Kieran, 2018). Focusing more on activities related to the use of variables and the art of generalization, Mason (1996) describes four principal roots of algebra as: (1) generalized arithmetic, where letters are used to express the rules of arithmetic; (2) expressing generality; (3) possibilities and constraints, to support awareness of variables; and (4) rearrangement and manipulation, to support understanding why multiple expressions can represent the same thing. When planning classroom activities, an algebra teacher may choose to focus more on some aspects of algebra and less on others, ultimately taking different approaches to the topic. Activities that students commonly engage in during algebra lessons have been categorized by Kieran (1992, 1996) on the basis of many years of research and a thorough overview of the research field. Kieran describes algebraic activities as comprising three core activities: generational activity, transformational activity and global/meta-level activity. Generational activities of algebra involve the forming of expressions and equations that include variables and unknowns, representing problem situations, geometric patterns, numerical sequences and relationships. Transformational activities are rule-based activities, often related to equation solving and changing the form of an expression while maintaining equivalence. Global/meta-level activities refer to those where algebra is used as a tool for mathematical activities such as “problem solving, modeling, noticing structure, studying change, generalizing, analyzing relationships, justifying, proving, and predicting" (Kieran, 2004, p. 142).

While transformational activities have often been the focus of algebra courses in Western countries, Japanese teaching has emphasized more generational and global/meta-level activities. As an example, Watanabe (2011) describes the concept of *Shiki* in the Japanese curriculum. Shiki stands for all mathematical expressions such as: $3 + 5$ or $x - 4$ or $\square \div 3$, as well as mathematical sentences, equalities and inequalities, such as: $3 + 5 = 8$ or $x - 4 = 7$ or $\square \div 3 = 7$ or $x + 5 > 2$. Students are engaged in activities that enhance the ability to construct, interpret and compare mathematical expressions, with less focus on computational proficiency. According to the Japanese curriculum, ideas related to Shiki and the study of functional relationships are the two pillars of elementary school mathematics.

The distinction between arithmetic and algebra is not clear-cut or distinct, perhaps not even possible to make in an obvious way. It may not be evident in a task itself whether or not a student will be engaged in arithmetic or algebraic thinking when working with it. The mathematician Keith Devlin has discussed at length the distinction between arithmetic and algebra on his blog,[1] where he separates arithmetic, where you calculate, from algebra, where you reason logically. He writes:

- Arithmetic involves *quantitative* reasoning *with* numbers.
- Algebra involves *qualitative* reasoning *about* numbers.

Arithmetic involves working with numbers or quantities using clearly defined operations with certain properties. In basic arithmetic, the operations are addition, subtraction, multiplication and division. An arithmetic expression includes numbers and operations, whether or not these numbers are known. Reasoning about unknown or generalized numbers in expressions and equalities is labeled "generalized arithmetic" and is included as one aspect of algebra (e.g. Blanton et al., 2015; Kieran, 2004; Mason, 1996; Usiskin, 1988). The term "arithmetic expression" is also used in programming where it often includes unknown numbers, describing what is to happen when the input is a specific number in a pre-defined set. In this book, we use the term *algebraic* expression for an expression involving at least one variable, in contrast to a *numerical* expression, where only specific numbers are included.

In an attempt to define algebraic thinking in the early grades, Kieran describes it as involving "the development of ways of thinking within activities for which letter-symbolic algebra can be used as a tool but which are not exclusive to algebra and which could be engaged in without using any letter-symbolic algebra at all" (Kieran, 2004, p. 149). In recent years, studies of algebraic thinking in the early years have produced varying results concerning the introduction of letter-symbolic notation (Kieran et al., 2016). A discussion has therefore evolved around when and how symbolic notation is to be introduced in classrooms (Radford, 2018). Against this background, the research focus of the project reported in this book is the point in school algebra when letter-symbolic algebra is explicitly introduced, not separated from, but preceded and accompanied by algebraic thinking.

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[1] https://profkeithdevlin.org/2011/11/20/what-is-algebra/ (retrieved 15 Dec. 2017).

The Use of Variables

Originally, the term variable was introduced by Leibnitz (1646–1716) to represent a varying quantity linked to the notion of function (Philipp, 1992). However, in curricular texts of today the term variable is used both on a meta level to mean the use of letters in algebra in general, and on a more specific level representing quantities that vary in a functional manner (Cai & Knuth, 2011). The use of variables is quite different within different conceptions of algebra. According to Usiskin (1988), it is only in the study of relationships that letters, i.e. alphanumerical symbols, are used to actually represent quantities that vary. In addition to this varying aspect, algebraic letters can also take on the roles of labels, constants, unknowns, generalized numbers, parameters and abstracts symbols (Bush & Karp, 2013).

Students' understanding and use of algebraic letters have been the subject of research in numerous studies (e.g. Carraher & Schliemann, 2007; Kaput, Blanton, & Moreno, 2008; Küchemann, 1981; MacGregor & Stacey, 1997; Philipp, 1992). In a classic study using a written algebra test involving 51 items completed by 3000 students aged 13–15, Küchemann (1978, 1981) found that students interpreted and used letters in algebra at six different levels that are progressively related. These were, in hierarchical order starting with the least sophisticated:

- letters evaluated, i.e. interpreted as numerical values through trial and error
- letters not used, i.e. interpreted as irrelevant
- letters used as objects or labels
- letters used as specific unknowns
- letters used as generalized numbers
- letters used as variables, i.e. interpreted with awareness of functional relationships

Different items invoked different meanings, and the students produced a high percentage of correct results on items where a less sophisticated interpretation of letters was sufficient. For example, an item interpreted as "letter evaluated" is the question "What can you say about a if $a + 5 = 8$", where 92% of the students correctly responded that $a = 3$ (Küchemann, 1981, p. 105). The most difficult items required algebraic reasoning using letters as variables. For example, only 6% of the students responded correctly to the question "Which is larger, $2n$ or $n + 2$? Explain" (p. 111).

Many later studies have built on, or referred to, Küchemann's categories, for example MacGregor and Stacey (1997) who investigated 2000 students of age 11–12, before their first introduction to algebra in school. Although that study also drew on written answers, it differed from Küchemann's study in the sense that the questions were posed in real-world contexts, like "Sue weighs 1 kg less than Chris. Chris weighs y kg. What can you write for Sue's weight?" (MacGregor & Stacey, 1997, p. 5). Students' answers were categorized in relation to the interpretation of the letter, as inferred by the researchers. One student may have ignored the letter, another may have assigned it a specific numeric value. MacGregor and Stacy identified six different interpretations, where the first five are problematic in a mathematical context and often produce incorrect answers. The answer "Sue's weight is ($y - 1$) kg" was categorized as the most advanced interpretation of the letter as an unknown quantity. In contrast to what was the case in Küchemann's study, these were interpretations made by different students dealing with the same item, rather than different items invoking different interpretations. Where Küchemann showed that an unsophisticated interpretation of an algebraic letter was sufficient and gave a correct answer on many items, MacGregor and Stacey only used items where such interpretations did not suffice. In a study by Rystedt, Kilhamn, and Helenius (2016) most of the previously identified interpretations emerged in a discussion among three 12-year-old students when they tried to understand an algebraic expression in an item similar to the one above from MacGregor and Stacey's (1997) study. Their results imply that students will employ all resources they have available, mathematical and non-mathematical, to make sense of letters in algebra, and that many of the less sophisticated interpretations will be tried and discarded when they fail to make sense.

It is well known that students struggle with algebraic expressions and the meaning of variables in expressions (Bush & Karp, 2013). In particular, students seem reluctant to accept a "lack of closure" (Collis, 1975), e.g. they are unwilling to accept an expression such as ($y - 1$) as representing a quantity and as being the answer to a question. Instead of dealing solely with expressions, it could be argued that expressions, and the variables in them, make better sense to students in the context of equations or functions. Since equations build on the important concept of equality, dealing with equations requires a structural understanding of the equal sign (Bush & Karp, 2013; Kieran, 1992, 1996, 2004). When the equal sign is read as "gives" or "yields", students may understand it as a command to calculate. This might lead the student to accept sequences of operations like, e.g. $12 - 5 = 7 + 8 = 15$, but reject an arithmetical decomposition of a number into an operation, such as $7 = 12 - 5$ (Herscovics & Linchevski, 1994, p. 65).

Another aspect of letters in equations concerns the ability to operate on them without evaluating them. Through a historical-epistemological analysis of thirteenth- to fifteenth-century pre-symbolic algebra textbooks, Gallardo (2001) describes what she calls a *didactic cut* between arithmetic and algebra. The cut describes a change; FROM working with an unknown on only one side of the equal sign when it is enough to "undo" the indicated operation (1.1); TO dealing with equations where the unknown appears on both sides and therefore has to be operated on (1.2), as shown below (ibid., p. 127).

(*a, b* and *c* are constants and *x* is a variable)

$$\text{FROM}: \quad ax \pm b = c; \quad a\left(bx \pm c\right) = d; \quad x / a = b; \quad x / a = b / c \tag{1.1}$$

$$\text{TO}: \quad ax \pm b = cx; \quad ax \pm b = cx \pm d \tag{1.2}$$

The same distinct cut was also found among 12–13-year-olds learning algebra (Gallardo, 2001). To separate these two types of equations, the terms arithmetic equation (1.1) and algebraic equation (1.2) have been suggested by Filloy and Rojano (1989). From a student's point of view, following Balacheff (2001), the shift from an arithmetical to an algebraic interpretation of equality corresponds to a needed shift of emphasis related to the validation of the problem solution: from a *pragmatic control* where the solution is validated arithmetically with reference to the initial context of the problem; to a *theoretical control* where the solution is validated with reference to mathematical principles (ibid., p. 256). In a classroom situation, this means that a student may learn to follow procedures for manipulating the operations of solving an equation. But as long as an arithmetically validated solution is more economical from the student's point of view, the reasoning of the student may stay within the arithmetical domain. Chapter 6 in this volume, analyzing Finnish classrooms, documents classroom situations where this supposedly is the case.

School algebra includes illuminating examples of both vertical and horizontal mathematization, as theoretically developed within the Dutch Wiskobas Project (Treffers, 1987). The representation of numbers by letters implies an increasing level of abstraction within mathematics itself (vertical mathematization). It encompasses the possibility to interpret a number as: "any number" (generality) or a number that may take different values (variability), without loss of the possibility to interpret it as a particular number (specificity) or a number that does not change (constancy), even if we do not know its value. The integration of the different interpretations of algebraic letters is a considerable didactical challenge when teaching school algebra at the introductory level. A critical aspect of this challenge is to make the activities where letters are introduced into an algebraic activity. One way forward is to deliberately support students in noticing regularities, articulating generalizations, explaining and proving conjectures (Kieran et al., 2016) and to choose proper tools and representations for such activities.

Tools and Representations

When solving mathematical problems choices need to be made with respect to concepts, procedures and tools. It is common to speak of the problem as *represented* in one way or another, and to work within the affordances and limitations of a specific representation. The standard example is to be found in algebra, with the letter x representing, for example, an unknown quantity to be determined in a problem situation. There are also different modes of representing mathematical concepts and procedures. Lesh, Post, and Behr (1987) distinguish between *experience-based situations*, *manipulative models*, *pictures or diagrams*, *spoken language* and *written symbols.* When Vergnaud (1998) developed what he called A Comprehensive Theory of Representation for Mathematics Education, he emphasized the importance of language and symbols, but pointed out that mediation also occurs without spoken language. Teachers are mediators, Vergnaud argues, whose role is to provide students with fruitful situations and help them develop their repertoire of representations.

When introducing and working with school algebra, a teacher will use a mixture of the available standard modes of representation, adding less formal representations as needed. Radford (2011) highlighted the fruitful interplay between different modes of representation such as *speech, visualization, gestures* and *tactility*. Studying young children as they engaged in patterning activities in primary school, he noticed how new relationships between embodiment, perception and symbol-use emerged. All the teachers in the project reported in this book made frequent use of different modes of representation alongside alphanumerical symbols. Specific examples of the use of embodied representations appear in several of the chapters in this book. For example, in Chap. 5, with data from a Norwegian classroom, the concept of variable is represented by walking, using different strides of different lengths. Chapter 4, reporting on Swedish material, includes data from a lesson when hands-on manipulations of boxes and beans are used to represent equations and equation solving (see also Chap. 8 for a more in-depth discussion of Radford's approach to the role of representations and modalities in learning about patterns).

In addition to embodied representations and manipulatives, different types of symbolic representations and models of concepts and procedures play an important role from the point of view of algebra and higher mathematics. In addition to language, and alphanumerical symbols, there is a wide range of pictorial models commonly used in school algebra. One such example is the balance model to represent the solving of linear equations (e.g. Da Rocha Falcão, 1995; Vlassis, 2002, cf. Chap. 6 in this volume for further discussion). Another example is the bar model method used in Singapore (Ng, 2004), where relations within and between rectangular bars represent algebraic expression. Ng describes the model, as "a structure comprised of rectangles and numerical values that represent all the information and relationships presented in a given problem. The rectangles replace the unknown represented by letters in equations." (Ng, 2004, p. 42). This model is said to be especially useful for the development of proportional reasoning. Through the model method, students with no knowledge of formal algebra are provided with a tool to construct pictorial equations. The assumption is that students, when given such means to visualize problems, will come to see their structural underpinnings. However, the conscious link from solving particular problems with manipulative and pictorial models to "the theoretical solving scheme of algebra" (Balacheff, 2001, p. 250) must still be established.

Since the early digital days of the Logo Maths Project (Sutherland, 1987, 1993), there has been an impressive development of digital environments. Many of these still align with the theoretical assumption of the Logo project that students can learn mathematics through active construction of their own knowledge, facilitated by an iterative process of conjecture and feedback in a computer environment. However, when reanalyzing data from three such research projects, Sutherland found that students' unassisted use of variables was strongly related to their first assisted use of the idea (Sutherland, 1993, p. 110), thus highlighting the role of the teacher also when digital interactive representations are used. In the footsteps of the development of computers and the Internet, much work today is put into creating interactive environments combining algebraic symbolic representations with pictorial geometric representations using a variety of dynamic tools (e.g.Hewitt, 2014; Hohenwarter & Jones, 2007). However, the incorporation of such environments in school practices is a slow process, and, as an illustration of this, there was no trace of dynamic digital tools used in any of the classrooms in the four countries represented in the VIDEOMAT project.

The idea of representing concepts and procedures, thus, must be seen as a very general one, allowing the use of a wide range of tools in teaching and learning mathematics. Representations involve a correspondence between a represented world (the domain of mathematical ideas) and a representing world (the domain of spoken or written language, or some sort of visible objects/entities). They also involve decisions regarding what aspects of the represented world that are represented, and what aspects of the representing world that are utilized (Kaput, 1987). The noun "representation", however, also has a more restricted interpretation as something existing in the representing world, a structure that is the result of a process of representing. Mathematical representations very often have a two-way nature (Goldin & Shteingold, 2001) making it possible to exchange the interpretations. It does not necessarily matter if one of the two worlds lies outside formal mathematics—representing an outside world with mathematical concepts and procedures, or representing mathematical concepts and procedures with objects and actions belonging to the outside world. Both these perspectives are quite acceptable in an educational context. In fact, they relate to abstractions as part of development in mathematics and applications of mathematics, respectively. Representations are thus central to both the internal structure of mathematics, and to the relationships between mathematics and the outside world. In the educational context both of these aspects are developed and used as support for each other, and the possibilities they offer, as well as the difficulties that may be involved, are essential features of school mathematics.

A particular problem, recurring in different areas of school mathematics, has to do with the order in which different representations are introduced and used to promote learning. When several approaches are available, the outcomes in terms of knowledge of representations and flexibility in translations between representations may be quite different (Even, 1998). In the case of school algebra, there exists a variety of approaches that are significantly different in terms of the emphasis they put on different representations and on the order in which they are to be used (Bednarz et al., 1996). Another aspect has to do with the quality and suitability of the representations that are used in school mathematics. In a classroom practice, efforts to make representations accessible to students may or may not preserve important features of the corresponding mathematical world, or may relate to other representations that the students have not yet met. This does not exclude the possibility that it may be advantageous to develop different representations in parallel. In fact, to the extent that different parallel representations exist as part of formal mathematics, achieving flexibility in translation between them can be seen as a learning goal in itself.

School Algebra as Classroom Practice

In an educational context the learning of algebra is highly dependent on the interactive processes in which the teacher(s) and the students take part, and more generally, on the kinds of established institutional practices that exist for teaching and learning mathematics in a particular classroom. This involves a number of conventions that have emerged over long time, and which are only partially determined by mathematics itself or by curriculum materials such as the textbook (see Chaps. 2 and 3). Other determining factors are for example what conceptions of mathematics and mathematical concepts dominate among the students, the knowledge and curriculum orientation of the teacher, and the school culture. Yackel and Cobb (1996) coined the term sociomathematical norms to describe the specific norms that are established within a particular mathematics classroom relating to the core ideas of mathematics. Such norms include what is considered a valid argument when drawing conclusions and making claims. If, for instance, a student is asked to verify a solution, different types of arguments produced in support of a claim could be: (a) to reach the same answers as everyone else, (b) to get the teacher's approval, (c) to show the specific situation using manipulatives, (d) to show it using a picture or a diagram, (e) to reason logically starting with clear assumptions, or (f) to construct an algebraic proof. The sociomathematical norms of a classroom may vary depending on the mathematical topic, the age of the students and on the beliefs about mathematics held by the teacher and the students. In Chap. 7, a detailed comparison between the teaching of two teachers is made, concluding that the teachers' approaches to instruction reflect their different conceptions of what it means to learn and understand algebra.

In relation to teaching and learning of school algebra, classroom practices may be seen as a context for apprenticeship in how to handle conventional mathematical tools and representations. Mason (1996) highlights the teacher's way of acting mathematically, when he describes the "cultural shift" that needs to happen in a classroom when students become "enculturated into mathematical thinking and expression just as naturally as they are into listening to and speaking their native tongue" (p. 66). What is meant by mathematical concepts and mathematical procedures in a particular classroom is something that is shaped only gradually as that enculturation proceeds. Some of the classroom communication going on in a particular lesson may very well be described as involving "pseudo-mathematical" concepts and procedures (without any derogatory connotations). Tools and representations, and especially the role of natural language to support the process of generalizing, are paramount (Kieran et al., 2016), and hence the approach to algebra chosen by a particular teacher may include communicative tools that are meant to evolve over time into a more rigorous set of mathematical/algebraic tools.

The empirical chapters of this book invite us into classrooms in four different countries. They illustrate both how teachers introduce students to basic algebraic ideas and modes of reasoning through a range of procedures and resources, and how students, using their previous experiences and insights into mathematics and problem solving, engage with tasks that are intended to open up the field of algebra. As is evident from the above, there are many ways into algebra and many goals of learning that are central, all the way from understanding how to use concepts, conventions and procedures in productive ways, to grasping how algebraic resources can be used for modeling the world and for understanding patterns and relationships of abstract or concrete entities. As a background to the empirical chapters, we would like to emphasize that what we are studying is the very beginning of a long journey where the students get the opportunity to taste, as it were, algebra. The teachers we have followed have made choices that they consider productive to instruct and guide the students, and our analytical point of departure is that they are rational within the curricular and other constraints they are operating under. In a similar vein, the students are rational in the sense that they are trying to contribute and fulfill their obligations, even when they sometimes seem to lose focus and/or have difficulties understanding what they are supposed to do, or temporarily fail to realize what the goal of an activity is. The point of analyzing the difficulties that both parties occasionally run into is not to suggest short-cuts, but rather to document and reflect on some of the obstacles and hurdles that appear during the first few lessons of algebra learning. Hopefully, insights into what these hurdles are will facilitate the understanding of how they may be overcome.

Chapter 2
Researching Classrooms in Search of Learning: Theoretical and Methodological Considerations

Roger Säljö and Maria Luiza Cestari

Introduction

Classrooms have existed some 5000 years. The particular communicative format in which one teacher lectures to and interacts with a group of pupils, thus, is one of the oldest and most established modes of institutional communication in society, perhaps almost as old as the sermon. Today, classrooms are well established communicative frameworks in most parts of the world, and a growing proportion of children spend an increasing number of years in such environments (Roser & Ortiz-Ospina, 2017; UNESCO, 2015). The first schools appeared in the so-called city states of Mesopotamia in present-day Iraq (Kramer, 1981, p. 3ff.). The major motive behind the establishment of schools was the profound social transformation taking place in this part of the world through the emergence of the so-called city states. Here a more diverse economy developed, people were buying and selling goods, they lived in houses and they were dependent on roads, a defense of the city, a legal system and many other social arrangements that had not existed in previous, pre-urban life forms. For trade to function successfully, contracts had to be written and receipts issued. In this new environment, taxes had to be collected, which, in turn, presupposes records of people, their houses and other assets.

What we see is the emergence of "document societies" (Thomas, 1992), where social life was coordinated through the technology of writing. Literacy (and here we include early forms of numeracy and uses of other symbolic systems) was a new, and quite abstract, intellectual technology, and it had to be taught to young people in order to cover the needs of skilled labor in a society where trade and commerce

R. Säljö (✉)
Department of Education, Communication and Learning, University of Gothenburg, Gothenburg, Sweden
e-mail: roger.saljo@gu.se

M. L. Cestari
Department of Mathematical Sciences, University of Agder, Oslo, Norway

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played a significant role. And this is where scribal schools with classrooms, teachers, headmasters, academic subjects, breaks, homework, tests and many of the other elements that we still recognize as constitutive of schooling emerged (Lundgren & Säljö, 2017; Kramer, 1963, 1981).

This is not the place to expand on the history of schooling, but it is worth considering that the specific institutional arrangements that classrooms represent imply that we are in a communicative environment with particular entitlements and obligations about who does what and when, and for what purposes. In the words of Edwards and Mercer (1989), there are "ground rules" about how to "do" schooling, and mastering those ground rules is an important prerequisite for being successful. Teaching and learning are predominantly linguistic activities; in class, children learn mostly through talking, reading (in a broad sense of this term, including attention to a broad range of representations) and writing about the world. This heavy reliance on language inevitably implies that some of the activities that are central to classroom life sometimes will appear abstract to children. But this, in one sense, is very much the point of schooling. Children should be exposed to knowledge and experiences that are different from what they encounter in everyday life. They are to familiarize themselves with what Vygotsky (1987) refers to as "scientific (or academic) concepts", i.e. insights and knowledge that have been generated in society, sometimes over centuries or even millennia, and that make up important parts of our cultural memory (Donald, 2018). Algebra is a very good example of this move from a world of "spontaneous concepts", acquired through everyday interaction, to academic concepts that are seldom developed spontaneously, but which presuppose systematic instruction and guidance in order to be mastered. We will come back to this.

There is also a range of specific assumptions and ground rules about how to behave and contribute to classroom activities: how to ask and respond to questions, how to solve various kinds of tasks, when to call for help and how to collaborate with fellow students. As we have already alluded to in the review in Chap. 1, there are also specific norms regarding what qualifies as an answer to a question as illustrated in our empirical material by students' unwillingness to consider an expression such as $(y - 1)$ as an adequate answer to a question posed. Children also have to learn how to sit and follow activities that go on for a long time, which may be a challenge for many. Most likely, many of the problems that are referred to as learning difficulties in modern society have their origin in difficulties of living up to such institutional norms. Thus, the role expectations that apply to being a "pupil" or a "student" are different from those that apply to being a "child", and, in some sense, it is easier to be a child than a pupil. Of course, the role expectations of what it implies to be a student will differ in different pedagogical traditions and arrangements and between schools and classrooms across the world, but they will always be there.

Researching Classrooms

The amount of research on classroom activities, teaching and learning, and interactional practices is, as may be expected, enormous, especially during the last three decades. Methodologically this research is very diverse, and most of the methods available in the social science tool kit have been used to document and analyze what goes on in such settings: from ethnographies and micro-sociological studies of the activities of single children in classrooms, to large-scale comparative research building on surveys and/or observations using various kinds of classification systems and, in recent decades, even video recordings of naturally occurring interaction. In some research, the focus is on teaching and teacher practices, sometimes student activities constitute the research object, and some research attempts to connect these objects of inquiry by analyzing how teaching styles or teacher practices may be related to learning outcomes, student motivation or other relevant outcome measures.

The foci of research interests also vary. Some research follows a classical social science approach by observing, analyzing and explaining patterns of success and failure, while other studies, especially in recent decades, are interventions or action research where specific ideas and arrangements are implemented and analyzed. The latter kind of research has grown in importance since teachers came to be more actively involved in research in the late twentieth century (cf. e.g. Hopkins, 2014), and since the interest in various kinds of design-based research emerged in the 1990s (Brown, 1992).

A classroom, in many respects, may be seen as a microcosm of society. The richness in research approaches utilized and issues addressed testifies to this complexity and the number of perspectives that may be applied. The ethnographic approach to analyzing classroom activities has a long history that goes back at least to the so-called Chicago school of sociology, which took an interest in life in urban societies and in the organization of social institutions in such complex surroundings (Waller, 1932). Somewhat later representatives of this tradition, such as Becker, Geer, and Hughes (1968) studying teaching and learning at college level, showed how students adapt to college life and to the expectations they perceive. Through their experiences of classroom practices and tests, they learn what it means to be a student and to learn in this context. Thus, teaching and learning are not given entities, nor are they defined by the official curricular documents or course presentations. Rather, the approaches that students develop to learning are—to paraphrase the authors—responses to how the educational institutions "do business."

One of the most famous, and most discussed, ethnographies of education in history is Philip Jackson's *Life in Classrooms* (1968). Following the logic of an ethnographic approach, Jackson attempted to describe and understand classroom activities in their own right without accepting the institutional definitions of teaching and learning as premises for the analyses. Jackson focused on how teachers developed skills in "crowd control" as they sought to manage the classroom and "as many as 1000 interpersonal exchanges" during a school day. He also showed how students learned one of the most important skills in this particular social setting: how to wait patiently. They waited for their turn, for assignments to be handed out, for teacher attention, for the lesson to be over and so on. Jackson, allegedly, was the first to coin the term "the hidden curriculum" (p. 33), a concept that was to have a considerable influence on research and on the public discussion of schooling. The concept of a hidden curriculum refers not to the official curriculum or the academic subjects students struggle with, but, rather, it points to the wider socialization that takes place as students encounter and adapt to the values, norms and expectations of classrooms and schooling. These values and expectations thus are not taught explicitly, but rather transmitted as an invisible element of classroom practices. Through participation, students learn subtle skills about how to interact with others and how to comply with (or, sometimes, oppose) the tacit institutional expectations. They also learn about themselves, their performance and how it is perceived by teachers and by the institution.

The idea of the hidden curriculum is a good example of how a particular concept may contribute to revising our understanding of a seemingly well-known and highly recognizable activity. An intense discussion and a large number of studies was carried out to further inquire into the nature of such implicit socialization patterns and their implications for what students learn and how successful they are at school (cf. e.g. Giroux & Purpel, 1983; Kentli, 2009; Snyder, 1971). Several authors also argued that such implicit socialization serves as important mechanisms for reproducing social privileges and the class structure of a society (cf. Giroux, 2001; Willis, 1977). Thus, what happens in interactions at the micro-level reflects institutional and societal priorities and constraints.

The motives for engaging in classroom ethnography, as we have done in the project reported here, is to achieve "thick descriptions" (Geertz, 1973) of classroom life and culture, of the routines and daily practices that participants engage in (Hammersley, 1990). A contextual understanding of how such settings are organized, and an in-depth exploration of the nature of the implicit and explicit rules that regulate the activities, are central to such approaches, as is the attention to the perspectives of the participants themselves (Watson-Gegeo, 1999). Beyond these general features of ethnographies, the theoretical orientations of scholars are far from uniform and represent as diverse traditions as symbolic interactionism, marxist/neomarxist perspectives, sociocultural approaches, conversation analysis and several others. Many of these studies have also been influenced by the "ethnography of communication" tradition that followed the pioneering work in sociolinguistics on language norms and language use by Dell Hymes and others (Gumperz & Hymes, 1972).

A rather different tradition of classroom research emerged in the 1950s and 1960s. Here the idea was one of attempting to classify interactional patterns in classrooms in order to search for regularities and to discern the modes of communication that resulted in effective learning. These traditions relied on classification schemes, where communicative "moves" such as questioning, responding, explaining and so on characterizing teacher and student behaviors, were used (cf., e. g., Bellack, 1969; Flanders, 1964). A firm conclusion from this line of research, later confirmed in studies across the world (Lundgren, 1972), is the systematic dominance by teachers of classroom communication. The so-called “rule of two-thirds” applies at several levels (Bellack, Kliebard, Hyman, & Smith, 1966; Flanders, 1970; Westbury & Bellack, 1971). In classrooms, talk takes up two-thirds of the time. Of this time, the teacher talks two-thirds of the time, and two-thirds of this talk, in turn, is used for presenting information, giving instructions, asking questions and for controlling student attention. This mode of communicating is typical of the one-to-many format of classroom communication, and the possibilities for individual students to actively participate and contribute are generally small and, furthermore, unevenly distributed. In a video-based, longitudinal study in science (physics) teaching by Sahlström and Lindblad (1998), for instance, it was shown that the uneven distribution of participation in the shared discourse in the science classrooms observed implied that some students contributed very little to the public activities. These research findings clearly illustrate that we are dealing with institutional traditions for communication, which, in many respects, are special to schooling.

But it is not only these quantitative patterns that testify to the existence of a specific institutional tradition of communicating. Also, the manner in which communication proceeds reflects institutional traditions. The research shows that predominantly teachers ask questions that they know the answers to themselves. The pedagogical roles represent a pattern where “[t]he teacher explains content and asks questions, the student answers and the teacher reacts” (Lundgren, 1977, p. 149). This dominance in traditional teaching of the so-called I-R-E (Initiative-Response-Evaluation) or I-R-F (Feedback) structure has been documented and explored in research using different analytical and theoretical approaches (cf. Cazden, 1988; Sinclair & Coulthard, 1975). Perhaps the most profound study of this specific mode of interaction is Hugh Mehan’s (1979) *Learning lessons*. Mehan followed an ethnomethodological and conversational analytic approach to classroom interaction inspired by the work of Harvey Sacks (Sacks & Jefferson, 1995) and Harold Garfinkel (1967). The point of ethnomethodology is to understand the “methods” that participants use as they engage in social activities and as they produce social order. Consequently, these traditions emphasize detailed description of social practices in order to access the methods people use to contribute to and reproduce such practices. Mehan’s study was thus an explicit reaction against what he held to be theoretically and methodologically vague ethnographies and research that utilized complex, but highly ambiguous and a-theoretical, classification schemes of communication of the kind mentioned above. In his analyses, Mehan gives a detailed presentation and analysis of the regulative function of such interactive IRE-patterns and how the orderliness of classrooms is maintained through the predominance of this pattern of communication. Inspired by this methodological approach, several scholars have continued to explore the nature and consequences of this type of communicative patterns for student learning and participation (cf. e. g. Cestari, 1998; Clarke, Howley, Resnick, & Penstein Rosé, 2016; Wells & Mejia Arauz, 2006).

There is thus a rich repertoire of theoretical perspectives and methodological procedures that has been applied for studying classrooms. And there is also considerable knowledge about central features of the communicative practices that characterize education both in more traditional teacher-led instruction and in more student-centered settings (Edwards & Mercer, 1989). However, there is one technical development that has had a significant impact on research on learning and instruction in recent decades, both in a methodological and theoretical sense, and this is the emerging use of video documentation.

Video in Classroom Research: Analytical Challenges and Potentials

Since the 1950s, recording technologies have developed rapidly. New technologies of audio and video recording have been invented, and these technologies have become increasingly user-friendly, portable and cheap. These developments have had implications for research in the human and social sciences, and many scholars in a range of disciplines have begun to make use of such methods of documentation. Hand-held video cameras can now be used for documenting social practices at work-places, in science laboratories, and in fieldwork in classrooms and other settings, and the recordings may be carried out in fairly unobtrusive manners. In a relatively short time, large amounts of data that document not just verbal interaction but also gestures, posture and bodily movements and other features of situations may be collected (see Goodwin & LeBaron, 2011; Heath, Hindmarsh, & Luff, 2010). In order to cope with such large data sets, there is also an intense development of software that may be used to organize materials and facilitate analysis.

These developments have had a significant impact in the learning sciences and neighboring fields, and they have also triggered methodological and theoretical debates and advances (see Goldman, Pea, Barron, & Derry, 2007). It is now possible to follow learning (or any other) practices in detail, and to discern elements of collective and individual action. Empirical observations of naturally occurring talk and interaction may be inspected repeatedly, even frame by frame if relevant, and it is also possible for researchers to work collectively when analyzing materials. So-called data sessions have become a frequent practice in research units and at conferences. Such sessions in themselves also constitute productive sites for learning among young scholars engaged in developing a theoretically relevant analytical gaze.

Video data are often relatively easy to collect, at least in the technical sense, but call for theoretical and methodological awareness when analyzed. The researcher has to be aware of the "unit of analysis" (Säljö, 2009) that is relevant when studying teaching and learning in a specific theoretical perspective. Cognitive phenomena generally are not visible, one cannot see learning, conceptual change or development happening. Such central features of human life remain covert. We have to work with indicators—performance on a test, contributions to a problem-solving situation or the use of a specific form of argumentation in a discussion—and frame these observations in theoretical terms where the significance of the observations as signs of learning may be clarified and argued for. For research on situated practices, and the interaction and learning that take place, video data play an increasingly important role. Of course, video documentation may also be complemented with other types of data such as interviews, written tests or observations depending on the nature of the research questions.

A rich array of methodological approaches has emerged to cope with the analytical challenges presented by video documentation. Many, if not most, of these approaches imply some form of discourse or interaction analysis (Jordan & Henderson, 1995), where the focus is on how people interact and engage in meaning-making in joint practices. But the theoretical perspectives in which discourse analyses are carried out differ widely. The units of analysis in conversation analysis (CA) and ethnomethodology are different from those that characterize cognitive, pragmatic, dialogical, social-semiotic or sociocultural traditions, to mention just a few contrasts. Of course, this theoretical variation in approaches to analysis is to be expected, since the different traditions build on different theoretical premises, and, consequently, the video data will be explored from these conceptual frameworks. Perhaps a common feature of much video research, though, is an interest in articulating participant perspectives, i.e. analyzing how participants contribute to social interaction as members of a community or when involved in some kind of joint activity.

In research on learning and instruction in mathematics, video analyses have been used successfully in a broad variety of settings, all the way from case studies of individual learners, via explorations of small group learning and classroom practices up to international comparisons of instructional traditions of societies. Perhaps the most well-known example of the latter type of studies is the work carried out within the TIMSS[1] video studies where mathematics instruction and learning across countries have been analyzed (Stevenson & Stigler, 1992; Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999). In these comparative analyses, several interesting differences in organizing classroom activities have been observed. For instance, Stigler and Hiebert (1999) showed how the expectations regarding how to organize mathematics teaching differ between societies. In the US, and in many other countries, the expectation is that there is a teaching cycle where the teachers introduce concepts and procedures and show how to solve a particular class of problems. The role of the student, then, is to work on the problems assigned in order to master this particular type of mathematical procedure. In Japan, on the other hand, it is not unusual that "teachers give students problems to work on that they have not seen before", and they do this under the assumption that "it is good for students to struggle with something they have not been taught" since this will "develop their thinking skills" (Stigler, Gallimore, & Hiebert, 2000, p. 88). In a US context, however, such an initiative by teachers may be seen as a violation of a rather fundamental pillar of the established "didactic contract" (Brosseau, 1997) and the tacit taken-for-granted assumption that students should not encounter problems they

[1] TIMSS: Trends in International Mathematics and Science Study.

have not been explicitly taught. The TIMSS video studies corpus has been used in a range of further analyses (cf. edited volume by Kaiser, Luna, & Huntley, 1999) scrutinizing differences between countries in terms of instructional practices, textbook design, curricula and other features of mathematics instruction. At the methodological level, the findings illustrate the power of video analyses when it comes to understanding cultural and institutional patterns of activity. Mathematics as a school subject may have considerable commonalities between countries, and the classrooms may also look alike in many respects, but the teaching practices reflect assumptions about learning, classrooms and children that differ. An important corollary of this observation is that instructional practices do not follow directly from the subject matter as such. Rather, they are mediated through cultural and institutional assumptions regarding how to teach, the role expectations on teachers and students and other factors that may well be invisible to the participants themselves. This analytical premise has guided the work that will be reported in this volume.

There have been several other analyses of classroom practices and learning patterns specifically in mathematics education. One of the most ambitious and interesting ones is the Learner's Perspective Study (LPS) comparing mathematics instruction in eighth grade in 12 countries and conducted by researchers from the respective countries (cf. e.g. Clarke, Keitel, & Shimizu, 2006; Shimizu, Kaur, Huang, & Clarke, 2010; Kaur, Anthony, Ohtani, & Clarke, 2013). The overall aim of the LPS project has been to pursue "a practice-oriented approach" to mathematics teaching and learning "which situates mathematical activity in relation to the social settings with which the project is fundamentally concerned", and, therefore, "it allows us to interrogate those settings with respect to the practices they afford and constrain" (Clarke et al., 2006, p. 3). LPS has used a range of methods such as video documentation, interviews with teachers and students, data on performance and questionnaires. And, as we have done in the project to be reported here, sequences of lessons have been recorded. A feature of the LPS project is also that it involves "insiders" in the analytical practices in the sense that researchers with an in-depth understanding of the sociocultural traditions of a country and its educational system are involved as analysts. But, in addition, the students are also considered as insiders in these practices that they know well, and the documentation of student activities is much more in-depth than in the TIMSS studies. This was achieved by using a student camera that followed focus students throughout their work (in TIMSS only one camera was used to document activities). This arrangement illustrates how a methodological approach is important in giving voice to a group—the students—who also tend to be very diverse in their orientations, an observation which is confirmed by the data of the VIDEOMAT project.

Again, the impression in these extensive studies of classroom practices, teacher and student contributions, textbooks, normative assumptions, tasks presented and other features of mathematics instruction is the obvious diversity in conditions for teaching and learning. In some settings, the mathematics classrooms are heavily dominated by teacher control of the agenda and progress of instruction (Mook, 2006), while in other settings students are given greater room and responsibility for implementing tasks, receiving less explicit guidance about mathematical content

(Emanuelsson & Sahlström, 2008). However, and as is emphasized in much of the reporting from LPS, this must not be read in a simplistic manner as one approach being better or more efficient than the other. Both students and teachers are accustomed to certain practices, and their criteria for engaging in work reflect their expectations and previous experiences. Also, as in Mook (2006) and Hoon, Kaur, and Kiam (2006), teacher dominance can still incorporate attention to the needs of individual students. Thus, "teacher dominance need not be equated with student passiveness" (Hoon et al., 2006, p. 163). In the study by Emanuelsson and Sahlström (2008), it is shown how active participation in classroom dialog by students to some extent is achieved at a cost: the mathematics content tends to disappear in the interaction or be at the periphery of the argumentation. In this sense, there is "a price of participation" as the researchers put it in the title of their article. And, as yet another example of the embeddedness of teaching practices in traditions and taken-for-granteds, a study of instruction in a German setting within the LPS project (Begehr, 2006, p. 180) concludes that "teachers "outtalk" the students, seemingly without being conscious of the fact." The "verbal guidelines set by the teacher impeded students in their efforts to come to grips with the mathematical content", "since the learners only expressed themselves in disjointed fragments" that had little, if anything, to do with the ability to understand mathematical reasoning. And the conclusion by the author is that teachers of "mathematics classes must evidently learn to 'let go', that is, not to guide their students along a narrow, predefined path, but to grant them the space, including the verbal space, to develop and express their own thoughts." Thus, the difficulties in learning mathematics observed in student-centered learning cultures of Emanuelsson and Sahlström (2008) seem to show similarities, at least at a general level, with those observed in instructional traditions that are characterized by teachers "outtalking" their students when lecturing.

In our opinion, these observations of the complexities of understanding teaching and learning pointed to by the LPS project are extremely interesting. For the development of instructional practices, it is pointless, perhaps even dangerous, to argue for more teacher centeredness or more student centeredness, as if these were opposing poles. The more interesting agenda to pursue in research is to look at complementarities between the parties involved in what they do and try to achieve during mathematics instruction. It is also important to realize that the fact that the instructional patterns and normative assumptions differ between countries is not surprising given how different countries are in other respects. The road ahead clearly is not attempting to achieve uniformity but rather to systematically develop instructional practices in ways recognized as relevant by teachers, students, the curriculum and the wider public.

Chapter 3
The VIDEOMAT Project: Theoretical Considerations and Methodological Procedures

Roger Säljö

Introduction

As we have already pointed out, the ambition of the research reported here has been to shed light on how students encounter introductory algebra in the classroom, and how they understand and begin to appropriate mathematical tools and modes of reasoning. Algebra is a distinctive activity that we learn in school. There are very few, if any, contexts outside classrooms where children develop these kinds of skills, at least in a systematic sense. It has also been well documented in the literature, as we have alluded to in the previous two chapters, that algebra is a hurdle for many students who never seem to grasp the potentials of algebraic modes of thinking and reasoning.

Thus, the basic idea of the project has been to analyze lessons and lesson activities where students are first introduced to algebra. The approach we chose was empirical in the sense that we contacted teachers and asked if we could record the first lessons of school algebra, more precisely the lessons when they first introduce variables in algebra. The teachers, after having decided to be a part of the project, were asked to contact the research teams when they planned to start algebra teaching so that we could document the first lessons. This implies that our documentation of lessons represents the first phases of school algebra teaching and learning as the teachers in the participating schools interpret this curricular unit. There is a number of ways in which one could understand the first introduction to algebra (cf. Chap. 1). For instance, there are various kinds of pre-algebra curricular units where students are asked to fill in missing numbers (2 + _ = 5) and to engage in solving other tasks that may be seen as preparatory for algebra. In textbooks in some of the countries involved in this study, such tasks are included as a part of pre-algebra activities (Reinhardtsen, 2012). It is, of course, also a challenging philosophical question to

R. Säljö (✉)
Department of Education, Communication and Learning, University of Gothenburg, Gothenburg, Sweden
e-mail: roger.saljo@gu.se

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establish exactly when a pedagogical activity may be seen as an introduction to algebra learning. However, in the project we intentionally took a pragmatic approach and decided to let the teachers define when they saw themselves starting with algebra teaching. Some additional information on the specific procedures of how teachers were contacted and enrolled will be given in the empirical chapters from the respective countries.

The empirical materials consist of four consecutive introductory lessons in each country. Four or five classes/teachers in each country were included. This implies that the empirical materials consist of at least 16 recorded lessons from each country, 69 lessons in all. In the analysis of data presented in this volume, we focus on specific aspects of algebra teaching and learning, activities such as using letters, formulating and using mathematical expressions and solving equations. In addition to this material, we recorded a fifth lesson in each classroom in all countries. In this lesson, the students engaged in group work. The idea behind including this fifth lesson was that we wanted to make sure that the project would include data where the students were active and engaged in tasks where their understanding of early forms of algebraic reasoning would surface. In this fifth lesson, the students in all the classes worked with three selected problems that are interesting from the point of view of learning algebra. One of the tasks in this lesson—a pattern task taken from an international comparative study—turned out to be particularly rich when it comes to understanding student meaning-making and the problems they struggled with. The problem-solving activities generated by this task in the four countries will be analyzed in Chap. 7. This implies that the total classroom material involves recordings of 80 lessons, and this is a very extensive corpus of recordings. In addition, there are recorded interviews with teachers and students, which makes the recorded material even more extensive.

Recording Lessons

For the recordings we used three different cameras documenting different types of activities. One camera recorded the whole classroom from a set position, one followed the teacher to capture whole class instruction as well as teacher-student interaction with groups or individuals. The third camera followed a student group—referred to as a focus group—capturing the peer-to-peer interaction when the students worked individually or with group assignments handed out by the teacher (see Fig. 3.1). The extent to which teachers mixed lecturing styles of teaching with group (and individual) work varied in the different classrooms, as would be expected. The fifth lesson, however, was specifically set up to document student activities in groups, designating two of the cameras as focus group cameras and one as a teacher camera, with a few local deviations from the general camera set-up. Additional information regarding the specifics of the recordings will be given in the introduction of each of the empirical chapters.

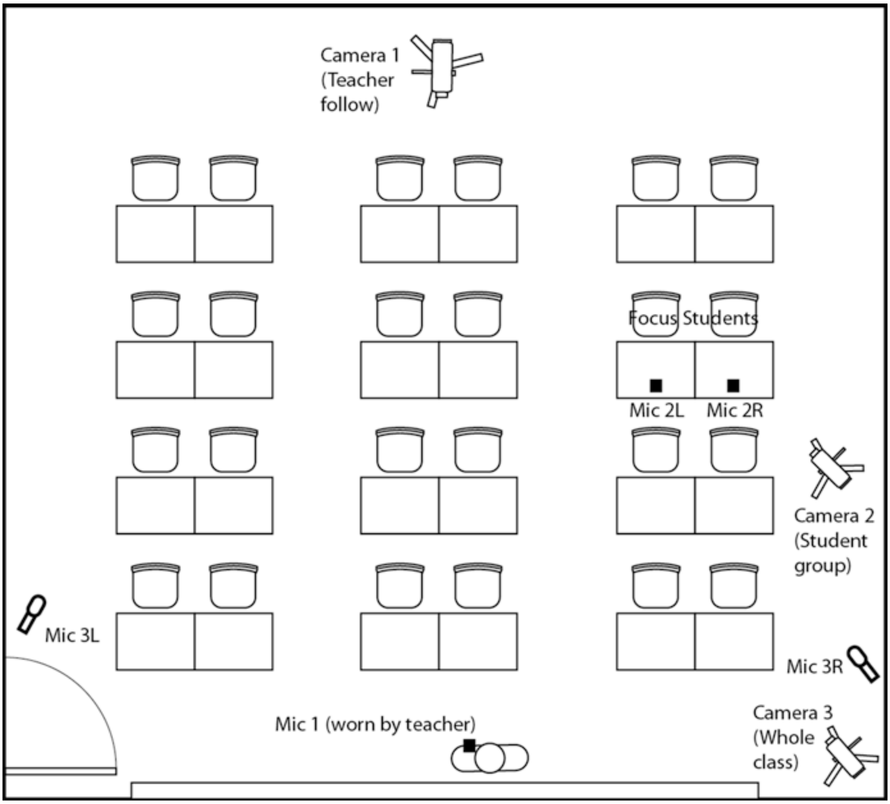

Fig. 3.1 Camera set-up for data collection (picture: Thomas Hillman)

Research Ethics

When contacts with teachers and schools had been established and their collaboration secured, the project and its basic interests were introduced to the schools and to the parties concerned: headmasters, fellow teachers, parents and students. In all four countries, ethical clearance of the project activities took place. The details of these procedures differ somewhat between countries but the general principles are similar, for instance informed consent by parents had to be secured. The research ethics has been evaluated by local ethics committees in the four countries.

In the contacts with the teachers, we emphasized that the purpose was not to evaluate the teaching. Our interest was in a comparison of how algebra is introduced in different countries. Names of schools or participants would not be mentioned, and only the researchers in the teams would be given access to the data. We have used fictive names for participants and the images and photos have been stylized through the use of software.

A Note on Theory: Learning Through Interaction

In Chap. 1, we presented a review of some of the literature on algebra teaching and learning. Research in this field represents a variety of research approaches in terms of the theoretical orientations and methodological procedures. In the present project, and as we pointed out above, the empirical material is extensive and the analyses require an explicit theoretical framework. The analyses are data driven and focus on meaning-making and interaction in a sociocultural and dialogical perspective (Bakhtin, 1986; Vygotsky, 1978, 1987). This implies that the focus is on the situated activities that participants engage in; their verbal interaction, gestures (for instance, hand-raising, and other bodily contributions such as pointing and writing in books or on the whiteboard), and the use of artifacts present in the situation (matchsticks, paper and pencil, etc.). In other words, we treat cognition as embodied and as "stretched out" (Lave, 1988) between participants, and between participants and the artifacts that serve as resources for meaning-making.

In sociocultural perspectives, learning is understood in terms of the appropriation of cultural tools (Wertsch, 2002; Wertsch & Addison Stone, 1985). Appropriation, which is a concept taken from Bakhtin (1981), implies that a person encounters a concept or a symbol and begins to use it as a tool for thinking and reasoning. Even a cursory glance at our videos documenting early algebra teaching and learning testifies to an environment that is rich in thinking tools that may be appropriated: concepts (variable, unknown number) and symbols (letters representing numbers or variables, the equal sign) as well as procedures (how to solve simple equations or recognize patterns). Furthermore, the tools appear both in talk and in written form, and there are also images, drawings and artifacts that play a vital role in the meaning-making. Several of these tools and procedures appear unknown to the children, while some of them—for instance the idea that *x* may be used as a representation of a number—the children, or at least some of them, report having encountered previously.

Appropriation is a gradual process that generally presupposes exposure to, and use of, a concept or a cultural tool over a number of situations where the tool is put to use in different contexts and for various purposes. Such prolonged and varied exposure, most likely, would be the rule for successful appropriation in this particular setting where the tools are quite dense. To appropriate something—i.e. to "absorb" it and make it "one's own", to borrow Bakhtin's (1981) expressions—is an activity that requires conscious effort by the individual, it is not a passive or abstract process of conceptual change. The process of appropriation also has no end, cultural tools are never completely mastered as Wertsch (1998) points out. There is always space for re-specification and novel modes of using them.

A central assumption of this Vygotskian-Bakhtinian framework for understanding learning is that we often use cultural tools before we fully understand them (cf. Daniels, 2008, pp. 66–67, for a discussion). In this sense, we may say more than we know when we put a particular cultural tool to use, as Wertsch and Kazak (2011) show. In the algebra lessons, this perspective is important to keep in mind. Students are engaging in problem-solving activities where they are operating at the boundary of their competences, and where they are expected to use cultural tools that they are not completely familiar with. But it is precisely through participation in such practices that they eventually may make the specific mathematical tools "their own."

This view of learning, in turn, builds on Vygotsky's concept of semiotic mediation and processes he referred to as externalization and internalization (1978, pp. 54–57). In this perspective, learning takes place through social interaction where individuals adapt to and mentally reconstruct tools and operations that they encounter in joint activities. Thus, in socially shared activities individuals will be exposed to concepts and categories that are transformed into tools that the individual later will use when thinking. This is part of the famous Vygotskian (1981) "genetic law of cultural development" where cultural tools appear on "two planes", first between people in social interaction (inter-psychological tools), and then within people as tools for thinking (intra-psychological categories). Language is the most significant means of mediation; we mediate the world for each other through the categories that we use in social interaction. But mediation also takes place through many other resources such as texts and artifacts of various kinds. Materiality, thus, is an important aspect of mediation. Artifacts "are simultaneously ideal (conceptual) and material" as Cole (1996, p. 117) puts it, and using artifacts is a way of coming into contact with and utilizing concepts for solving problems (Säljö, 2019).

In the dialogical version of sociocultural theory inspired by Bakhtin, the emphasis is on how people in communicative practices share knowledge and experiences through joint activity. In this epistemology, and in this approach to studying learning and development, situated communicative practices make up the most significant setting in which knowledge is displayed, produced, understood and recycled. As Linell (1998, p. 277) puts it, dialogism "construe[s] practices and interactions as primary entities, and seeks to explicate how language, traditions, routines, roles, knowledge, theories etc. are embodied as aspects pertaining to the continuity of praxis", and, we might add, to innovation and learning. Thus, learning is an emergent phenomenon which may be studied by focusing on how participants in interaction make concepts, ideas and other features of life understandable and learnable for others.

A corollary of this Vygotskian-Bakhtinian position, thus, is that we do not consider the individual as the privileged unit of analysis for understanding learning. Of course, whether individuals learn early algebra or not, is a primary knowledge interest of the work presented here. However, the focus on interactional practices seeks to explore the opportunities for learning that children are offered, and that they are able to make use of. The ability to contribute to classroom practices, to ask questions and to offer solutions, tentative as they may be, in themselves represent knowledge that paves the way for appropriation of concepts and procedures. Thus, our prime interest is in the processes that make up the contexts for learning early algebra, and to see what students encounter, and how far they get toward realizing some of the potentials of algebra.

Instruction in the context of schooling is, as has been mentioned, an institutional activity with specific obligations and entitlements for those who participate.

Lessons, the prototypical form in which instruction is organized around the world, may be thought of as a continuous reproduction of this institution and its roles. Thus, and as we have pointed out, students have to know both how to learn and what to learn. The reflexive elements of learning in institutional settings may be understood in terms of the "double dialogicality" (Linell, 1998, p. 132 *et passim*) that pertains to communication in such settings. Thus, the communication, and the contributions by the participants, must be relevant to the local setting in which the interaction takes place, and they must be understood in this specific context. The students, for instance, have to have some understanding of what are relevant manners of making meaning in this particular setting when encountering artifacts (matchsticks to count or figures to interpret) and letters (x) and mathematical expressions ($y - 3$). And the teachers have to realize in situ the problems that students may have in this respect and provide assistance so that they can continue their work. At this level, meaning-making is situated and local, subject to a range of factors that evolve in the specific context.

However, meaning-making in this context also takes place in the framework of a set of established communicative routines and practices that are anchored in a wider cultural understanding of what a classroom is, what it implies to be a teacher or a student and what it means to learn. There are, to use Linell's (1998) terminology, situation transcending elements of these instructional projects which suggest ways of talking and behaving that make these situations into what we perceive as instruction or schooling. An interesting example of such situation transcending features of the classroom interaction that will be explored and commented on in some of the empirical sections concerns the nature of the sociomathematical norms (Yackel & Cobb, 1996) that are established and adhered to by the participants. Thus, to what extent are the classroom activities, and the meaning-making that goes on, conducive to learning specific mathematical modes of reasoning and arguing? To what extent do the students learn to argue from within mathematics and its conceptual frameworks? These are questions that concern the culture of learning that is established and the extent to which it supports specific modes of meaning-making that eventually will be productive for doing mathematics. These are some of the questions addressed in the empirical analyses that follow in the chapters to come.

The First Encounters with Algebra: A Guide to the Empirical Sections

As we have pointed out, the VIDEOMAT project has been an international collaboration between research groups in four countries/settings. The planning of the activity of following and analyzing the first encounters with algebra in the different settings was done in collaboration between all groups and resulted in an application to the funding agency. In the preparatory phases, the project groups agreed on a common approach to the collection of data and the general design of the project.

The decision was to have parallel data sets from classrooms and, as we have already mentioned, the data consist of recordings of the initial lessons of algebra in the schools contacted in each country.

The analyses of this material have been done partially within each of the research teams, and partially in collaboration between the teams. The collaboration between the teams is rather interesting in its own right since it was carried out largely by means of videoconferencing in principle once a month during term time for the duration of the project. The groups were also able to meet physically on two occasions during the project period, but most of the interaction took place through video link. During the videoconferences and the physical meetings, the various groups presented their work and their analyses, and together we took decisions on how to deal with the various practical, methodological and substantive problems that inevitably occur in such a comparative undertaking. Among the issues that we discussed were: When does algebra teaching begin? Is pre-algebra algebra? We also had to consider the fact that algebra teaching starts at different grades in different school systems, and, in addition, children also start school at different ages, which, in turn, means that the ages at which they start studying algebra will be different. Also, the curricula and text books differ in various respects.

As we pointed out above, our solution to these, and other, problems was that we contacted teachers teaching at the middle-school level and asked them if they were willing to let us record their first lessons of algebra. This pragmatic approach implies that we did not engage in any philosophical debate about exactly what algebra is and when teaching of this particular area starts. Or, to be more accurate, we did discuss such issues at some length, but after consulting the literature and reviewing the curricula in the four countries (cf. Reinhardtsen, 2012), we decided rather quickly that it would not be possible to find one unequivocal definition that could guide how data should be generated. Instead, we turned to the teachers, and they defined when they would start with the topic of algebra in their class. However, in all countries there is a change in school systems at about this time, where teachers in the lower grades have a generalist training, teaching many school subjects to the same group of children, and teachers in the higher grades have a specialist training as mathematics teachers, teaching mathematics to different groups of children. We decided therefore to approach teachers on both sides of this point of change. A consequence of this approach, and the differences between the settings in terms of when algebra teaching (as the teachers see it) begins, is that the data sets come from classrooms in different grades: grades 7 and 8 in Norway, and grades 6 and 7, in Finland, Sweden and the USA. In this context, it is interesting to observe that in spite of these differences, the outcome of this procedure is that the mathematical contents of the introductory lessons are fairly similar. The teachers introduce students to unknown numbers (and/or remind them that they had encountered such numbers, represented by an x, previously), they introduce a static interpretation of the equal sign and equations (through various pedagogical arrangements), and they show how simple linear equations can be solved. They also touch upon, although more or less clearly, the idea of what constitutes a mathematical expression. Thus, the mathematical substance, i.e. the "what" of instruction and learning, is fairly similar across the settings, while the approaches to communicating early algebra show both similarities and differences, as will be evident below. In addition to the video documentation, we also generated other material such as interviews with teachers and students, lesson plans, work material that the students used during class, and we scrutinized the curricula.

All four teams had access to the video documentation (and other materials) from all classes. The languages of instruction are English, Norwegian and Swedish. Thus, in Finland the recordings were made in Swedish speaking classes in a Swedish speaking part of Finland. Swedish is a national language in Finland and there are textbooks and curricula in Swedish. The teachers have been trained in teacher training at a university where Swedish is the language of instruction. These decisions resulted in a situation where the teams from Finland, Norway and Sweden were able to read everything that was collected and understand all the recordings. For the benefit of the U.S. team, the first lesson from all countries was translated into English for the preliminary analyses. As the analyses proceeded, sections of the materials that were interesting from an analytical point of view were also translated for purposes of comparison.

The Idea Behind This Volume

As the analysis of the empirical materials continued, the idea of putting together this volume emerged. Thus, the point of the book is to present significant illustrations of how students in contemporary Western societies with different school systems and, perhaps also different instructional cultures, are introduced to algebra. Given the design of our project, we cannot argue that the observations we report are typical of the various countries in a statistical sense, i.e. that our classrooms demonstrate how algebra is taught in the four settings. Indeed, the very idea that there are commonalities within countries that do not overlap with differences between countries at the level of detail that we study meaning-making and conceptual understanding is not credible. However, as the work progressed it became clear that teaching and learning is organized differently in some respects; in the materials there seem to be slightly different instructional cultures where, for instance, the role expectations on teachers and students are slightly different. Thus, in the classrooms teachers and students “do” mathematics teaching and learning slightly differently. Again, we do not want to claim that these different patterns characterize the educational systems in the respective countries. But the variations we observed are interesting and reflect different traditions and assumptions that most likely exist within the countries as well.

In concrete terms, the preparation of this volume has been guided by an interest in singling out features that the four teams found characteristic of their own classes/schools/teachers/students in the light of what they could see in the other materials from the other three countries. An initial overview analysis was done of all the teacher-planned lessons by coding each 30 s of each lesson using a coding system of mutually exclusive coverage codes (described in more detail in Kilhamn & Röj-Lindberg, 2013). The codes were then used to produce lesson graphs in which each lesson was summarized and visualized on one page. (See Chap. 5 for some examples of lesson graphs.) Observations of candidate features of differences in the organization of teaching and learning algebra were discussed during the videoconferences and physical meetings, and the groups then outlined to the other teams what they wanted to focus on in this volume. The result of these deliberations were that the teams would focus on slightly different themes, but still try to capture some of the elements of cultures of learning that seem familiar in the educational system they know very well. An important, and joint, premise of the analytical ambitions throughout the project has been to include and give recognition to the perspectives of the participants, i.e. the students and teachers who together produce the instructional activities that are consequential for learning. The focus is on how they work and what they are trying to accomplish during the lessons.

The joint decision was that the Swedish team would focus on issues of mathematization and participation in classroom discourse during algebra teaching with a particular focus on what is referred to as the participation frameworks and the opportunities for learning algebra they afford (Chap. 4). In the Norwegian material, the design and use of instructional examples and illustrations by teachers used to help students realize what algebra is all about stood out as interesting from a more general point of view (Chap. 5). In the Finnish material, the focus is on the introduction of how to solve equations and to understand the principles of equality (Chap. 6). In the material from California, case studies of the relationships between teachers' beliefs about what algebra (and mathematics more generally) is, and how it should be taught, and the instructional activities that follow from these beliefs are in focus (Chap. 7). In the final empirical chapter, there is a comparative analysis of how students during the fifth lesson solve the matchstick task mentioned above. Thus, in this chapter we tried to trace elements of the teaching that students had just been involved in in order to see to what extent it would be consequential for how they engage in a patterning task that invites algebraic reasoning. The materials used in this chapter come from all four countries (Chap. 8).

Format of the Empirical Chapters and Transcription Conventions

The general approach of the project work, in terms of theory and empirical work, has been described above. The empirical chapters, however, refer to slightly different issues, and for each of these issues there is a research literature that is relevant in the vast field of research that concerns itself with algebra learning. Therefore, the chapters contain a short introduction to research that situates the issues and the analyses in the international literature. Also, some information on when and how algebra is introduced in the curriculum is included. Furthermore, an analysis of the differences between textbooks in the four countries has been carried out within the project and reported by Reinhardtsen (2012).

The presentation of data in the chapters through excerpts from the classrooms follows a pattern where all data are written in `this specific typeface`. Uses of `this typeface` thus consistently signal that the utterances are from the participants. In some cases `this typeface` also indicates that the text is a description of what happens in the classroom during a specific sequence, for instance signaling non-verbal behavior, `[teacher points at the drawing on the whiteboard]`, or that some passage is `[inaudible]`. In some cases, the typeface is used in a summary of what happens immediately before or after an excerpt. This will be evident as the excerpts will be read. In addition to spoken language, a number of mathematical symbols are used in the classrooms. In order to make use of authentic teacher and student writing we have stayed within the symbolic conventions of each country. As a consequence, there are a few differences in notation in the different chapters. Most obvious is the symbol for multiplication, which is · in the Nordic countries and × in the US.

As for the transcriptions, we are well aware of the fact that transcribing per se is a theoretical enterprise, and that there are many alternatives when it comes to representing spoken language in writing. This insight, that transcription is a theoretical enterprise, was formulated in a distinctive way a long time ago by Ochs (1979) in a seminal text. Subsequently, we have seen a whole literature appearing about how to transcribe interactional practices in accordance with the expectations of various research traditions, such as ethnomethodology, conversation analysis and a range of ethnographic approaches.

Realizing that transcription implies abstraction and a partial representation of spoken interaction, and furthermore that the procedures adopted are relative to research interests as Linell (2011, p. 129ff) points out, we have chosen not to make very detailed transcriptions marking overlaps, prosody and other linguistic details. An additional issue here is that in three of the empirical materials, the excerpts are translations from Norwegian and Swedish, respectively. Rather, the transcriptions are at an intermediate level where it is possible for readers interested in learning and instruction in mathematics to follow the contents that the conversations and activities are about. Some significant elements of the interaction such as pausing and uses of non-verbal communication, for instance, using fingers when counting, pointing at the blackboard or to representations in textbooks, etc., are indicated. In the activities we study, symbolic and material artifacts, such as drawings on blackboards, objects to count, etc., play an important role, and such elements of the communication are represented both in pictures and by comments/explanations inserted in the transcripts. At a general level, our data testify to the fact that teaching and learning mathematics are very much multimodal activities. References to physical objects, images, formulas and pieces of texts are constitutive elements of the meaning-making in almost all the conversations we have documented.

Chapter 4
Participation and Mathematization in Introductory Algebra Classrooms: The Case of Sweden

Cecilia Kilhamn, Thomas Hillman, and Roger Säljö

Introduction

During an algebra lesson, the teacher picks up a box in which an unknown number of beans is hidden. Shaking the box, she says:

Teacher: `It can be any number, as long as they can fit in the box. But it can't be zero.`
Student: `No.`
Teacher: `Because then we wouldn't, have heard when it rattled.`
Student: `Here it's five and here it's five. And here, it can be anything, but you say it has to be something particular…`

This dialog is a snapshot from a Swedish Grade 6 algebra lesson on the use of variables and equation solving. The situation engages students in active hands-on investigations and conversations about mathematical ideas, in this case based on handling boxes with beans representing unknown numbers. But, in spite of the high level of student participation, mathematical ideas seem at the same time evasive, and the whole discussion is strongly tied to examples and materials present in the situation. One aspect of the Swedish data that stood out in comparison to the other countries in the project was the relation between participation and mathematization, where a strong emphasis on participation and attention to student thinking seemed not to support or enhance student's opportunities for mathematizing.

C. Kilhamn (✉)
Department of Pedagogical, Curricular and Professional Studies, University of Gothenburg, Gothenburg, Sweden
e-mail: cecilia.kilhamn@gu.se

T. Hillman · R. Säljö
Department of Education, Communication and Learning, University of Gothenburg, Gothenburg, Sweden

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In the Swedish classrooms observed as part of this study, teachers valued student participation and student engagement in mathematical activities, giving students many opportunities to express their thinking. Students were often asked to share their work with the whole class, which they seemed happy to do, and they were praised by the teachers for their good work and participation. Mathematical tasks and activities offered students opportunities to discuss in small groups, to investigate using manipulatives and to engage in creative processes. However, we saw little evidence of the process of mathematizing in terms of making connections and conjectures, and there were few occasions where the participants attempted to see the general in the specific.

In this chapter, the reader will be given insights into activities in three Grade 6 classrooms where the focus was the introduction of variables. Each activity will be described and analyzed in terms of (a) the nature of mathematizing that takes place, and (b) the kinds of participation that could be observed.

Mathematizing

When we look at excerpts from the Swedish classrooms, we use the term mathematizing to describe mathematics as an activity in line with Freudenthal (1991). Mathematizing, in the sense that Freudenthal uses the term, is an inclusive term describing mathematics as an activity rather than as a closed system of already preformulated and reified forms of knowledge. Systematization, for example, is one of the fundamental aspects of mathematizing, and what students need to learn is the activity of systematizing (Freudenthal, 1968). Mathematizing includes attending to form, such as axiomatizing and formalizing, as well as to aspects of content; what Freudenthal called schematizing, but what is now often referred to as modeling. Mathematizing particularly refers to the activity of looking for essentials within and across contexts to discover similarities, analogies and isomorphisms as a way of generalizing through progressive formalizing and symbolizing. In addition, Freudenthal highlights that mathematization is a reflective activity, claiming that “A particularly important aspect of mathematizing is that of reflecting on one’s own activities, which may instigate a change of perspective.” (Freudenthal, 1991, p. 36).

The terms horizontal and vertical mathematization (Freudenthal, 1991; Treffers, 1987) distinguish between two slightly different mathematical activities. Horizontal mathematization describes activities that lead from the world of life to the world of symbols, where “real world” objects and events are modeled using the language of mathematics. Although real world situations are a starting point in horizontal mathematization, it is the activity of mathematizing and the mathematics involved that are in focus. Systematizations and generalizations across real world contexts are possible as a result of this mathematization. Vertical mathematization involves activities within the world of symbols: shaping, reshaping and manipulating different sets of symbols mechanically, comprehendingly and reflectingly (Freudenthal, 1991).

Algebra, and particularly the use of variables, is very much a literate practice where inscriptions and the use of symbols play an important role both as a means to learning and as end products. This literate nature of algebra directs our attention to the uses of documentation, and to the correspondence between what is said and what is written in instructional practices. Written signs (inscriptions) provide significant anchor points for understanding algebra at the same time as they force learners to externalize their thinking with the precision required when using symbols in writing. Inscriptions, furthermore, have a permanent character that verbal communication does not have, providing rich opportunities in instructional settings for questions, reflections and meta-level discussions as the work progresses.

We look at the learning of mathematics as the use of one's powers of imagining, generalizing, abstracting, specializing, conjecturing and convincing (Mason & Johnston-Wilder, 2004). Algebra is a field of mathematics where generalizing is the most important feature. Algebraic thinking is to a large extent the recognition and articulation of generality, of seeing the general through the particular and of seeing the particular in the general (Mason, Graham, & Johnston-Wilder, 2005; cf. Chap. 1). When we look at the episodes to be presented in this chapter, we look to see in what ways learners have the opportunity to discern similarities and differences, to systematize and reflect, and to recognize and articulate generality. In short, we look to see in what ways they have the opportunity to mathematize.

Participation

A core element of learning analytical skills is participation in collective practices, where concepts and modes of reasoning are encountered and put to use for specific purposes. Participation is a broad concept that can be understood in many different ways, but for the purposes of this study we draw on three key theoretical constructs that help to interpret the collective practices in the Swedish classrooms. To identify and describe forms and shifts in participation we draw on the notion of a "participation framework" (Goffman, 1981). This notion describes the interactional roles held by the circle of participants within a particular situation. It acknowledges that participation frameworks are dynamic, and that participants' roles are defined by their relationship to the utterances and actions that take place within a particular interactional frame as the collaboration evolves. These roles change as new frames are established. In the shifts, participation frameworks become especially visible, and thus available for analysis.

Within the participation frameworks established in the classrooms, we draw upon the idea of sociomathematical norms (Yackel & Cobb, 1996) to examine the mathematical character of the collective practices. This idea brings into focus the differences between the general social norms established through the practices of teachers and students, and those that are mathematical in nature, such as norms that govern what constitutes acceptable mathematical explanations and justifications. To illuminate this distinction, Yackel and Cobb (1996, p. 461) offer the following examples:

> The understanding that students are expected to explain their solutions and their ways of thinking is a social norm, whereas the understanding of what counts as an acceptable mathematical explanation is a sociomathematical norm. Likewise, the understanding that when discussing a problem students should offer solutions different from those already contributed is a social norm, whereas the understanding of what contributes mathematical difference is a sociomathematical norm.

Attending to the enactment of sociomathematical norms in relation to the more general social norms negotiated by teachers and students provides a way of identifying, examining and characterizing the specific aspects of the classroom practices conducive to learning mathematics.

To examine the establishment of, and adherence to, sociomathematical norms through the practices of teachers and students in relation to mathematical learning, we draw upon Gresalfi, Martin, Hand, and Greeno's (2008) notion of construction of competence. This notion challenges the idea that competence is a characteristic of an individual and instead proposes that it is an attribute of participation in a particular setting. As teachers and students engage in collective practices, they construct a system of competence that comes to define mathematical competence in that context:

> This system of competence gets constructed as students and the teacher negotiate (1) the kind of mathematical agency that the task and the participation structure afford, (2) what the students are supposed to be accountable *for* doing, and (3) whom they need to be accountable *to* in order to participate successfully in the classroom activity system. (Gresalfi et al., 2008, p. 52)

By examining the practices of teachers and students, the enacted forms of mathematical competence in a classroom can be discerned, and through an analysis of the particular sociomathematical norms and participation frameworks established and endorsed, the character of participation in a mathematics classroom can be understood. Drawing on these theoretical constructs, we unpack the participation in the episodes from the Swedish classrooms in this study in relation to the mathematical activity that takes place.

Background: A Brief Note on Early Algebra Teaching in Sweden

In general terms, the Swedish school system is organized into primary school including grades K-6, lower secondary school for grades 7–9 and upper secondary school for grades 10–12. In primary school the teachers are educated as generalist teachers, often certified to teach all subjects. This implies that teachers have limited university training in mathematics or mathematics education. Most frequently, one teacher stays with a class for 3 years (grades 1–3 or 4–6) teaching all subjects. Hence, a teacher for grades 4–6 revisits Grade 6 mathematics only once every 3 years.

Sweden has a long tradition of compulsory schooling, and since the comprehensive school reform in 1962, all children are obliged to attend school for 9 years (extended to 10 years in 2018 when preschool class, i.e. grade K, became compulsory). Ideals of equity and equal access to high-quality education are high on the political agenda, and schools preferably should have a mix of students in terms of gender and socioeconomic backgrounds. In its time, i.e. in the post-war period, this reform was seen as important in the ambitions of educating democratic citizens within a society with egalitarian ideals. During the last 50 years globalization has affected Sweden, and one of the consequences has been a more diverse student population. All the schools included in this study are public schools with students of mixed backgrounds, gender and ability.

Since 1842 Sweden has had a national curriculum prescribed by the state. Although the details of state regulation have varied over the years, central features of school life such as organization, curriculum, grading and other elements have been centrally controlled. Since the reform of 1962, the national curriculum has changed almost every 10 years (1962, 1969, 1980, 1994 and 2011) as a result of political intervention following rapid social transformations caused by changes in the labor market and increasing globalization. New educational ideals and political ambitions about citizenship in a modern democratic society have also played a role in the frequent revisions of the curriculum. Since 1989, when much of the responsibility for administering public schools was decentralized, shifting from the national to the municipal level, the diversity in schools has increased, and recently equity in the current educational system has been questioned by many parties, including state authorities (Kilhamn & Hillman, 2015; Skolverket, 2012).

Mathematics teaching in Sweden is often described as relying heavily on textbooks (Hansson, 2011; Johansson, 2006; Skolverket, 2012) with extensive use of independent desk work. Many have argued that this implies that the responsibility for learning has been handed over to the students (Carlgren, Klette, Mýrdal, Schnack, & Simola, 2006; Hansson, 2010). Government control of textbooks ceased in 1992, and since then textbooks are sold on a free market, leaving teachers, headmasters and municipalities the responsibility to guarantee quality. As a reaction to this, there is a current movement, mainly in grades 1–6, to increase hands-on material and problem-solving activities, and as a result many schools have established what they call "mathematics workshops." Considerable government and other resources have been invested into these workshops equipping schools with concrete materials suitable for hands-on activities intended to make the subject more attractive and enjoyable and to promote learning (Skolverket, 2003). However, an evaluation of these initiatives has shown that most schools failed to combine purchases of materials with sufficient teacher in-service training about how to make use of the teaching resources offered (Skolverket, 2011b). In two of the three classrooms described in this chapter, the textbook has been replaced by a number of hands-on activities and worksheets for the topic of algebra.

In the current national curriculum (Skolverket, 2011a), the mathematics syllabus for grades one through nine is described on 14 pages. Aligned with research concerning mathematical competences (Niss, 2003), the general aims of the subject are described in terms of specific “abilities” students should be given the opportunity to develop. Such abilities are, for example, the ability to formulate and solve problems, to apply and follow mathematical reasoning, to use and analyze mathematical concepts and relationships, and to use mathematical forms of expression. Core mathematical content is outlined for 3-year periods (i.e. for grades 1–3, 4–6 and 7–9, respectively). In connection to this, knowledge requirements that relate to the abilities are defined. The core algebra content for grades 4–6 is described as follows (Skolverket, 2011a):

- Unknown numbers and their properties and also situations where there is a need to represent an unknown number by a symbol.
- Simple algebraic expressions and equations in situations that are relevant for pupils.
- Methods of solving simple equations.
- How patterns in number sequences and geometrical patterns can be constructed, described and expressed.

Participants

The episodes analyzed in this chapter originate from three Grade 6 classrooms in two schools (A and B) in municipalities just outside a big city. Both schools are public schools following the regular curriculum, and the student groups in the catchment areas are diverse in terms of their backgrounds. All students participating spoke Swedish, although there were some students with Swedish as a second language in every class. The three teachers are certified generalist teachers, who said they planned their teaching on the basis of the national curriculum, and the textbook, which was the same in all classes (Carlsson, Liljegren, & Picetti, 2004). However, school A had recently invested in an “algebra activity box” (the so-called NTA-box designed by a national science agency to reform teaching and learning in mathematics and natural science), containing teaching materials supporting hands-on algebra and patterning activities. This led to the decision by the two teachers in school A not to use the textbook at all for the unit on algebra.

While there was considerable similarity between the three classes, they differed in two important ways. First, class size varied among the classes, with 13 students in class A1, 18 in class A2 and 30 in class B. The large variability in class size is a result of decentralized resource allocation within the Swedish school system, where schools may decide to split classes for specific activities. Second, teaching experience varied considerably among the teachers with 3 years of experience for the teacher in class A1, 22 years of experience for the teacher in class A2 and 10 years for the teacher in class B. Since teaching experience and class size may be important background factors to consider in terms of how teaching is organized, this difference should be kept in mind.

Results

The prime purpose of this section is to present an empirical study of introductory algebra teaching in Swedish schools. Of course, and as may be seen from the lessons we have recorded, there is no unified mode of introducing algebra. The variations in approaches to teaching and learning are considerable, as they are in other subjects. Also, teaching is a performative activity where communicative patterns are never entirely parallel or simply repeated. Rather, such activities are embedded and enacted in an intricate web of assumptions, obligations and situated affordances that will co-determine the nature of participation and the outcomes in terms of learning. The point of this analysis is to present excerpts that document some features of what we see as characteristic manners of introducing new topics in teaching and learning in a Swedish context. The sample is small and cannot be generalized in the statistical sense as we have pointed out earlier, but the aspects of algebra teaching brought up in this chapter stood out as different from what was seen in the algebra classrooms in the other three countries and therefore worthwhile to report. Thus, our ambition is to provide substantive insights into teaching practices that allow for conceptual comparisons, generalizations and analyses.

Using Symbols Instead of Numbers (Class A1)

The first example is an activity in Class A1 (see Fig. 4.1) at the start of the second lesson dealing with the introductory algebra unit. The task is introduced after a repetition of the use of the equal sign from the lesson before, when the students created different numerical equalities. In that lesson the equal sign was referred to through the metaphor of a "balance", the point repeatedly made that each side "weighs even" or "has the same value" as the other.

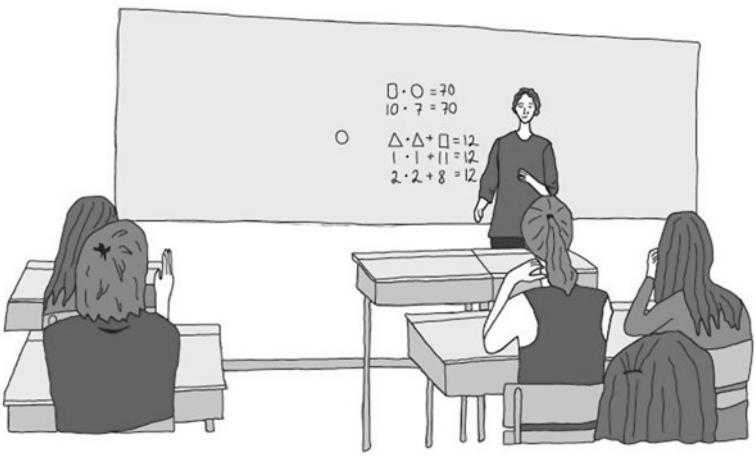

Fig. 4.1 Class A1

Introducing the Task

The teacher starts by introducing three symbols, cut from paper and put on the board (Fig. 4.2).

Excerpt 1

Teacher: `Today I thought that we would exchange some digits or numbers for symbols instead. And it is to get some sort of brain-exercise, to challenge the brain a bit, to think differently.`

Teacher: `This is what we will do, you don't need to care about, these aren't worth any particular numbers, now when we are playing with them. You will get to play too. This is what we will do. [Puts the first example on the board: yellow polygon times yellow polygon equals 36]` (see Fig. 4.3)

Teacher: `What number can I use there, to make it the same on the other side? [Points first to 36 and then to the yellow polygons on the left side of the equal sign]. Tom?`

Tom: `six times six`

Teacher: `good six times six. How about if I do this? [Puts the next example on the board: [yellow polygon times pink smiley equals 70]`

In Excerpt 1, the teacher presents three examples on the board. In the first equation she points out that two identical symbols represent the same number, in this case six. Then she contrasts this to a second equation, where she tells the students to think of two different numbers. A student asks if the yellow polygon can be another number now, and the solution $7 \cdot 10 = 70$ is suggested and written underneath the figures. In the third equation, the teacher repeats that two instances of the same symbol represent the same number. The students suggest two different solutions here: $1 \cdot 1 + 11 = 12$ and $2 \cdot 2 + 8 = 12$, which are both written on the board but not further commented on.

The teacher then hands out note paper, instructing the students to work in pairs making up `challenges` for their peers (see Excerpt 2). She cleans the board leaving only the paper symbols.

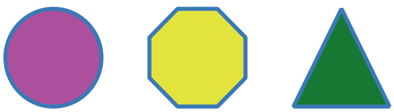

Fig. 4.2 Symbols for numbers—a round pink smiley, a yellow polygon and a green triangle

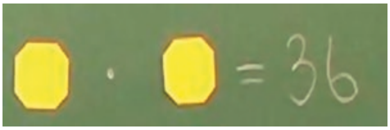

Fig. 4.3 First example on the board

Excerpt 2

Teacher: Write some numbers, with a few triangles, or, with these here figures that I've made today. And then you'll get a chance to come forward yourselves and challenge your friends. You can work two and two. And then you make a small challenge. It shouldn't be too difficult. That I am choosing rather low numbers on purpose is not because you're bad at maths or something like that. Rather, low numbers can, can be difficult, too. Am I right? Don't make too big challenges, so that they get too long, so that it gets too tricky. And we'll let a few of you come forward later, and test them out. Mm. Hans?

Hans: there are some or there are some very low... for instance minus numbers.

Teacher: minus numbers, yes of course you may use anything you can, sort of

Hans: it is quite easy, sort of

Teacher: all four rules of arithmetic that you've learned

Per: are we supposed to use the symbols then?

Teacher: yes, and then you come forward and use the symbols. So, it is good if you draw the symbols here.

In this introduction, we see that the mathematical terms equation, expression and variable are not used. A symbol is treated as something that stands for an unknown number (but known by the person who designed the challenge). The terms used to describe the activity are: `sum`, to mean the value of a numerical expression, or the answer to the calculation; `equality` to mean equality/equation; and `challenge` or `number` to mean the equation created.[1] Formal mathematical terminology is rare, and the activity is introduced and framed as a game-like project by the teacher pointing out that `you will get to play too`. It is also apparent that the students are somewhat uncertain about the grammar of the exercise, i.e. what numbers and symbols they should use and how. For example, when Hans points out that `there are some very low minus numbers`, and when Tom asks if they are `supposed to use the symbols`.

[1] In Swedish mathematics classrooms the term number is commonly used to mean exercise, for example when talking about what exercise (or task) a student is working on in the book.

Student Work

Following Excerpt 2, students spend around 6 min working in pairs with the task, while the teacher circulates in the classroom scaffolding and encouraging them to be creative and to draw and write their `challenges` on paper. The teacher is not very specific about how they should document their work and some student-pairs chose to record their work individually, while others produce shared inscriptions. We can see from the students' work (see Fig. 4.4) that they produce examples of equations with one unknown as well as equations with two or more variables. All the equations have an expression on the left side and a number on the right side of the equal sign.

The teacher prioritizes the students' opportunities to express themselves freely, sometimes at the expense of developing the theme of the lesson. For example, when Anna wants to use two different symbols to represent the same number (see Fig. 4.5), the teacher discusses this with her but lets her decide what to do (see Excerpt 3).

Excerpt 3

Teacher:	`mm, two of the same`
Anna:	`that's 40`
Teacher:	`do you want, okay, we can take. 40. Plus 40 then`
Anna:	`40 plus 40 [the teacher shows Anna how to write the numbers under the symbols and tries to engage Laura as well, who is working with Anna]`
Anna:	`plus eh, 10`
Teacher:	`plus 10. What have you come up to now?`
Anna:	`[mumbles, writing 40+40+10-10=80]`
Teacher:	`but then those two [polygon and triangle] mean the same number. Don't you think that will confuse them a bit?`
Anna:	`but that's the whole point`
Teacher:	`oh. Okay.`
Anna:	`[giggles]`

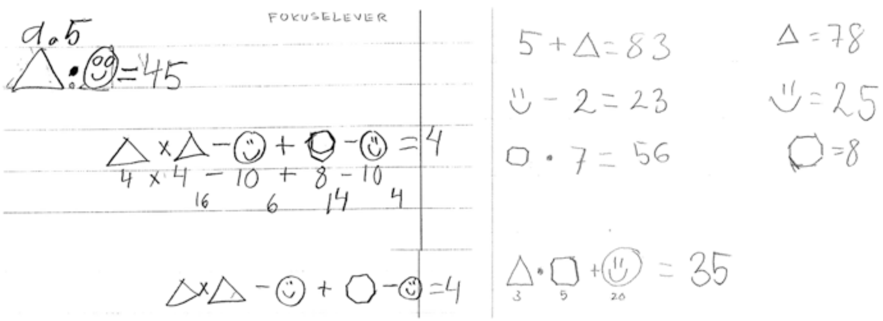

Fig. 4.4 Student work using symbols to represent numbers

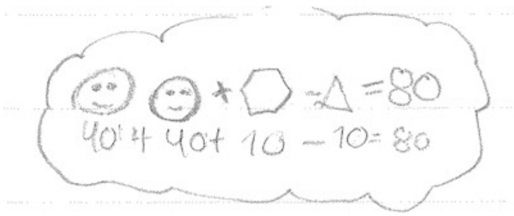

Fig. 4.5 Anna's work using symbols to represent numbers

Teacher: yes well that was a good number in that case. But then you will need to explain, that two can be, that two of your signs are the same.

As can be seen, the complexity Anna introduces when designing a challenge, and that she describes as the whole point of her suggestion, involves allowing two different symbols to represent the same number. From the perspective of the participation framework that Anna adheres to, her proposal may be understood as an instance of a challenge in the sense that it makes the task tricky for her friends, but as an instance of mathematizing and learning about variables or the equal sign, the specific point of this proposal is ambiguous and not further developed in the conversation with the teacher or in the following whole class discussion (see Excerpt 4).

Follow-Up on the Board with Students Showing Their Examples

Following student work, 12 min are spent on whole class interaction where the students show their examples on the board, challenging their peers to find the numbers. The students who have produced a "challenge" also act as teachers when it is shown on the board, the teacher stepping back to leave the floor to the students. Each example is erased when showing the next one, so no comparison of different equations can be made as the work progresses. Most of the challenges involve several symbols and could have many solutions. The first one can be seen in Fig. 4.6a, with the solution $44 + 1 - 2 = 43$ suggested by a student. Another student makes a conjecture that there are many solutions, and the teacher says: yes it can be done in many ways and yes, just change the numbers a little. She asks the students who produced the challenge to give their solution: $40 + 6 - 3 = 43$ and responds yes, that's a solution. There is no comment about the relationship between the different solutions or the variables in the equation. Thus, this opportunity to systematize, make conjectures and express generalities is not made use of. The claims that the tasks can be done in many ways, and that one can change the numbers a little, are not explored as, or converted into, explicit forms of mathematization.

When Anna comes up with her challenge, where two symbols represent the same number (Fig. 4.6b), the teacher intervenes, describing it as something individually connected to that student, rather than as a particular case of the general idea of variable and how values may be represented (see Excerpt 4).

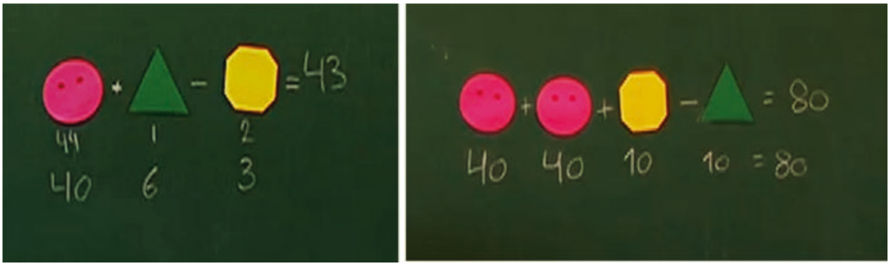

Fig. 4.6 (**a**) Two different solutions to the same equation; (**b**) Two symbols represent the same number

Excerpt 4

Teacher: `Oh, this is going to be hard. And then you have a sum there, we've got to have that. mm [Anna writes = 80]`
Teacher: `80. Does anybody... [points to the polygon and the triangle but then moves the finger to the two circles]`
Teacher: `are these the same, these numbers?`
Anna: `yes`
Teacher: `yes, and are those different? [points to the polygon and the triangle while looking at A]`
Anna: `no`
Teacher: `no [turns to the class] Because this is how it was: this is a small riddle Anna has. She chose to have, they have the same value these two`
Hans: `but then she can put the triangle on that one too`
Teacher: `yes, but she wanted to have it that way. So that's why I said we have to be clear, to explain`

When the teacher here ends by saying, *`we have to be clear, to explain`*, her comment concerns the fact that Anna has used different symbols for the same number, but, again, there is no obvious reference to relevant mathematical concepts, nor to what a variable is and what values it may assume. The example is treated as an isolated case where a particular symbolic representation introduced for instructional purposes is focused, and there is no visible attempt to go further.

Closing of the Activity

After 20 min the teacher rounds up the activity by telling the students that they had been `very clever` during this exercise. Then she tells them that symbols can be letters, and that next time they will be working with the letter `x` to replace numbers.

Excerpt 5

Teacher: `mm, can we do it another way [turns to the class] No, well in any case I think that was the most suitable. Good. You've been very clever.`

Teacher: `Now you have seen, now you have had these... In the past, a long long time ago, there were a lot of professors who played around with numbers. And, Per, when yesterday, you showed instead, instead of making a symbol? [interrupted by a student arriving late]`

Teacher: `you could do an x [gestures x with her finger in the air] Why, why did you come up and write an x?`

Per: `Because, you can do it.`

Teacher: `yes`

Per: `but, eh, 5 times x will... is equal to 25.`

Teacher: `Yes, precisely. It's the same. And, why do you think, well why did you choose x?`

Per: `you can just as well take anything`

Teacher: `yes, you can just as well use any letter. And, that's what we will work with next time we meet here. Then we will work with x. x-values instead.`

After this, the lesson moves on to a different activity solving a magical square, and no reference is made to the first activity as the work continues. The papers where the students had written their equations are disposed of. There is no homework given. There is no explicit closing, summarizing what was expected to be learned through the activity, no conjectures have been made and discussed, no mathematical arguments are given, and no documentation of the work is saved for future use. The activity appears as an exercise about combining numbers and using figures as representations, and this is also the manner in which the students contribute within this participation framework. The point of this activity as a step into the use of mathematical language and relationships is not made explicit.

Teacher Comments and Student Reflections

In a short post lesson interview, the teacher repeats that she thinks the students were very `clever`. When planning for this class she says that they sometimes have a different mathematics teacher, and that this teacher had `already done much of the fun activities in the algebra box`. She has problems finding something the students had not already `done`. When questioned why the activities need to be new, she does not give an answer. The following two lessons are spent working on hands-on activities constructing equations with one variable using "boxes and beans" that will be described in the next section. In the last lesson, the students are asked to describe an equation using boxes and beans and to represent it by also using numbers, words and algebra/equation. The teacher does not produce lesson plans and, thus, it is not possible to see how the exercises relate to her ideas about the progress into algebra.

At the end of the four lessons, the students are encouraged by the teacher to reflect in writing on what they have learned and how they felt about it. Most of the students, in quite general terms, express that the four lessons had been `fun and quite easy`, and that they `learned more about the equal sign` and more about calculations. Only two students mention equations: `I have learned how to work with equations`. One student writes, `don't know`, and another one that he `did not learn anything`.

Analysis of Class A1

The Nature of Mathematizing

The activity engages students in constructing equations to understand the idea of working with symbols for numbers, and to understand the correct way of using the equal sign. The task is a good example of vertical mathematization, where one set of symbols (numbers) is translated into another set of symbols (geometric figures). The new symbols create equations on a more general level. The activity affords possibilities for learning through creative investigation, through seeing similarities and differences, and through systematizing and reflecting in order to articulate general principles. There are many important algebraic ideas buried in the activity that come up in the introduction and in the equations created by the students, for example the difference between equations with one unknown and equations with several variables, and in what cases different symbols represent different numbers. Both internal relationships between variables within equations and relationships between equations could be highlighted. The task obviously is rich in terms of possible insights into the concept of variable, unknown number and equation, and the use of symbols to represent numbers.

However, it is clear that these issues are not exploited, and there are no obvious signs that the activity moves from "doing" to "mathematizing." The lack of a systematic closure of the task, the sparse use of mathematical terminology, the absence of mathematical conjectures, and the fact that all the examples presented and solved on the board are erased straight away so that they cannot be compared and therefore no generalizations are voiced, lead to the conclusion that the lesson is more focused on the activity as such than on the conceptual ideas that the activity was intended to introduce. This interpretation is strengthened by the fact that the teacher does not have a lesson plan other than the decision to do the activities she had chosen, and she talks about the mathematics lessons as a collection of activities that `are done` rather than as a sequence of activities with specific objectives of learning that concern mathematics and mathematization. Communication is mostly about each given example and the calculations done. When referring to what they have learned during four lessons on introductory algebra, the students mostly produce very general statements about learning `more`, mainly mentioning calculations (arithmetic) and the equal sign, with no mention of symbols, unknown numbers or variables. Since each presented equation is treated as unique, the students are left on their own to discover generalities through the particulars.

The written work in this lesson appears to play the role of creating a temporarily shared representation of the activity or task during the process, but it does not seem to be intended for further reflection, nor does it serve such a role. Examples are written on the board so that all can see them while they are worked on. They are then erased directly afterwards. The student pairs collaborate and draw and write on a common piece of paper while they create their `challenges`. At the end of the lesson the paper has played out its role and is disposed of. In other words, the permanent character of inscriptions is not taken advantage of in the teaching and learning activity. The written work serves as *a representation of* their equation, but does not serve as a *model for thinking* mathematically on a more general level or for reflecting on the nature of algebra. We see no evidence of inscriptions serving as tools for mathematization.

The Nature of Participation

The social norms of the classroom allow the students to collaborate and to engage in a creative process, this is obvious. Students work in pairs facilitating mathematical communication. The equations they come up with are all given attention, and students exercise agency in relation to their solutions. The students are given much praise on a general level as being `clever`, and their confidence and motivation are most likely strengthened through this. There is no, or very little, hesitation from the students to show their equations on the board and "play teacher", and students listen respectfully to each other during the whole class interaction. In their reflections after the four algebra lessons, the students write that mathematics is fun and easy. The framing of the activities in this classroom implies that the tasks are presented as `exercises for the brain`. The students are expected to produce `challenges` for their classmates, and they get to `play too`, as the teacher puts it. Inclusive and respectful social norms are established supporting collaboration and sharing of ideas, and the students make use of these opportunities for active participation throughout the lessons.

In terms of the sociomathematical norms endorsed, each task is attended to and discussed as a separate and self-sustained unit. Considerable time is spent on discussing the representations and the numbers, but the potentials of them as tools for mathematization are not explicitly attended to. Concepts such as variable, generalization and others that are essential for learning about algebra are not used to any significant extent. The written work that students engage in is not discussed in terms of its mathematical content but is seen as a report of the work. It is not used as a means for clarifying mathematical relationships and patterns. When equations are written on the board or on paper, there is no evidence that these are seen as instances of mathematical ideas and concepts that are more general. We will return to these observations.

Creating Equations Using Boxes and Beans (Class A2)

The second example is an activity spanning over three lessons as an introduction to the algebra unit in class A2 (see Fig. 4.7). The class is in the same school as class A1, and students have access to the same material in the algebra activity box. The activity utilizes concrete material in the shape of boxes and beans to illustrate equations with one variable. The aim of the activity is to introduce (a) equations as equalities with expressions on both sides, and (b) equation solving as `doing the same on both sides`.

Lesson One

After briefly introducing the topic of algebra as being about using expressions, formulas and letters, particularly *x*, the teacher starts the lesson by focusing on the equal sign, asking the students about it and ending with the definition `the value of one side is equal to the value of the other side`. The teacher emphasizes the two different interpretations of the equal sign as dynamic (gives the answer) and as static (is equal value). After this, the boxes-and-beans activity is introduced. Students are engaged in understanding the activity. At first they engage in tasks presented by the teacher. Then they make up tasks for each other to solve. The lesson ends when time is up, and the activity is continued the next day.

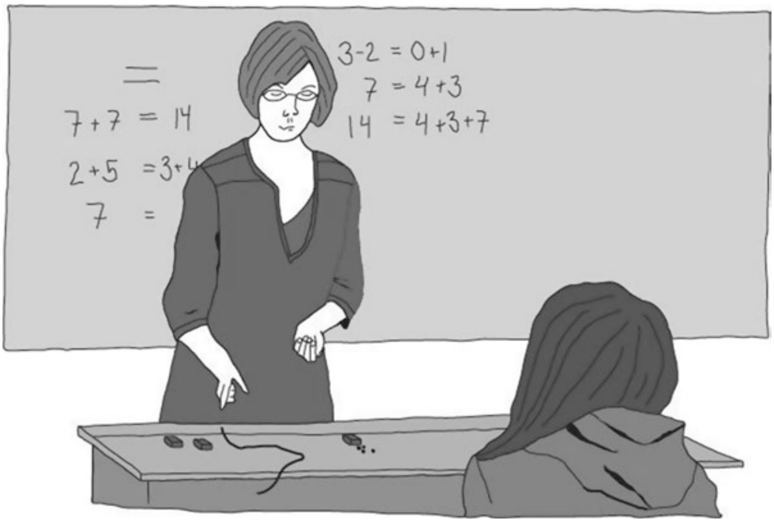

Fig. 4.7 Boxes and beans exercise in Class A2

Introducing the Activity

The teacher brings out beans and matchboxes prepared with the same number of beans in each. She puts a string on the table, and on each side of the string she places boxes with beans inside and/or outside on the table (see Figs. 4.7 and 4.8). Each box holds the same number of beans and the aim is to find out how many beans are in the boxes under the condition that the total number of beans on each side of the string is the same. Each task is prepared in advance so that the boxes can be opened to uncover the correct number of beans.

Excerpt 6

Teacher: And then I'll put some string, what do you think the string is supposed to symbolize? Freddy?

Freddy: Equal sign!

Teacher: Eq- Equal sign yes. And on the other side I'll put, one matchbox, and three spare beans.

Teacher: Now you'll need your pencil, and you'll need your eraser. And I want you to think about: how many beans could there be in each matchbox? And then it's like this that there are as many beans in this one, as there are in this one, as there are in this one. The same amount of beans in all of them. How many beans? Do y- do you see from there? Otherwise you'll have to go up and, check what it looks like.

S1: Should we, draw it although-

Teacher: You can draw it, I think that's a good idea.

S2: Should we write?

Teacher: Write or draw. Then you'll have to stand up if you can't see.

S3: Should we write or draw?

Teacher: You can write or draw.

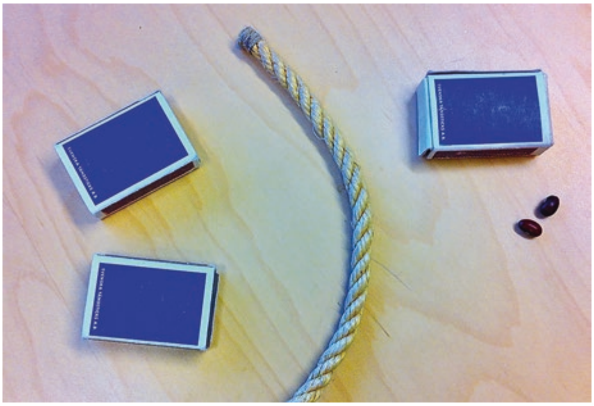

Fig. 4.8 The first boxes-and-beans equation on the table: two boxes on one side and one box and three beans on the other side

In Excerpt 6 we see that the teacher lets the students begin to negotiate the frame of participation by accepting their suggestions of drawing and writing. The teacher emphasizes the importance of looking closely at the arrangement. As the students work individually or in pairs to figure out how many beans there are in the boxes, the teacher circulates the room, probing students for answers and justifications (see Excerpt 7). When uncertain, the students are encouraged to test their conjectures, using the material or their drawing to see if the equality is true given the number of beans they think is in each box. The mathematical work is expected to be mental, and the material and inscriptions are used to check and justify. The teacher does not introduce the word equation nor any algebraic notation, but talks about the specific concrete situation in terms of boxes, beans and that there should be `the same on both sides`. Excerpt 7 shows an interaction between the teacher and two students, and Fig. 4.9 shows the documentation produced by some students. In the students' inscriptions, the beans are sometimes drawn and sometimes represented by a number.

Excerpt 7

Teacher: Have you got it yet?
Daniel: No.
Teacher: No?
S2: I think it's three.
Teacher: Why do you think that?
Daniel: I think it's two.
Teacher: Why do you think it's three?
Daniel: I think it's two.
Daniel: No, I just feel it.
Teacher: You've got a feeling. Well, if you draw it with two beans here in each, can you see if it goes together then? So you draw-
Daniel: Should it be the same amount on each side?
Teacher: Yes. Exactly, this is like-
Sara: It's three.
Teacher: Yes, you did think two.
Daniel: So I should write two?

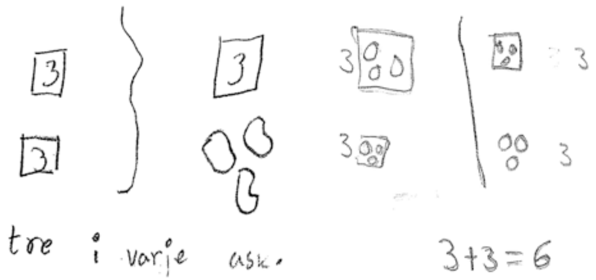

Fig. 4.9 Two students' inscriptions representing the first task

Teacher: Well, if you test it, does it work out with two in each then? You, don't l- don't look that much now, think of you're saying that you think there's two in each. Try it. Does it work out then?

Student Work

After this, several tasks are presented one at a time, and the students work individually, drawing them on their papers and solving them by figuring out the number of beans in the boxes. In Fig. 4.10, we see more of the students' work of representing and solving equations described in terms of boxes and beans on two sides of a string. One student exchanges the string for an equal sign (Fig. 4.10b). Some students make a drawing which does not represent equal numbers of beans on each side (Fig. 4.10c).

During the last 5 min of the lesson the students are asked to create similar situations for each other to solve. Each pair of students is given a tray to place their boxes and beans on. Instead of a string, the boxes and beans are placed on two sides of a drawn line. Initially this is done without documentation, but when time is running out the students with tasks that are yet unsolved are asked to draw them to be able to reconstruct them during the next lesson.

Excerpt 8

Teacher: And then I've done like this, that I took a ma- a tray to use as a mat to make it easier to set up, these, eh, matchsticks and beans, so that you won't drop them on the floor when you move them, and you'll then choose to: well, maybe should have two boxes on this side, and one spare, and then I'll have four spares, and one match- matchstick box on that side. And instead of string, what have I done? Yes, I've drawn lines with the ruler here then. So that it's that that's the equal sign. So your group, your pair, should make a task that someone else should then

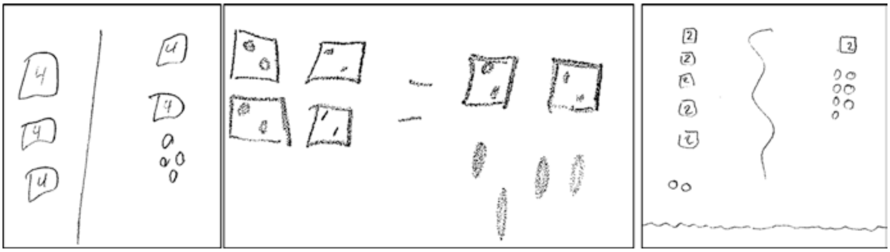

Fig. 4.10 (**a**) Representation of three boxes = two boxes and four beans; (**b**) Representation using an equal sign instead of a string; (**c**) Representation without equality between the sides

solve. And then, the rules are: there's the same amount on each side of the line, there's the same number of beans in each box. Have you understood the task? Good.

At this point some students are quite inventive. One pair of students comes up with an equality that has many solutions by setting up the following situation: two boxes and five beans on each side of the line (Fig. 4.11). When the teacher comes, they discuss possible solutions for this task (Excerpt 9).

Excerpt 9

Linda:	It could be... It could be one in there, there could be two in there, there could be three in there, heh…
Teacher:	mm, Exactly. It can be any number, as long as they can fit in the box.
Teacher:	But it can't be zero.
Linda:	No.
Teacher:	Because then we wouldn't, have heard it rattle.
Linda:	Here it's five and here it's five. And here, it can be anything, but you say it has to be something particular…
Freddy:	It can be anything here
Teacher:	yes, mm
Linda:	... but you can't figure that out
Freddy:	Yes.
Linda:	what in particular it should be
Teacher:	mm
Freddy:	You can figure it out. [shakes the box and looks inside]
Teacher:	Mm. Yes you can look yes, exactly.
Linda:	But we're not allowed to do that. That's just it.
Teacher:	Though, you could write that eh, it's…
Freddy:	Five! Five, five! I'm writing five.
Teacher:	yes, it could be five in there.
Linda:	yes
Freddy:	I'm drawing

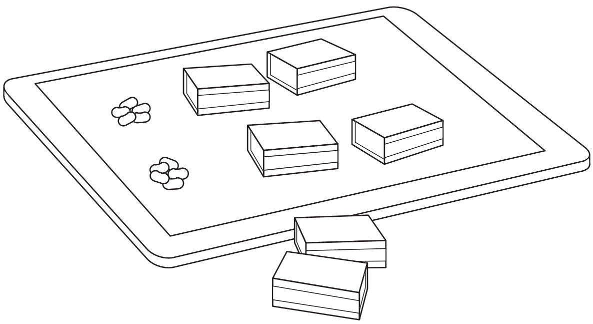

Fig. 4.11 Setting up boxes and beans on a tray

Teacher: `mm, well there could be seven in there, there could be eleven in there…`
Linda: `It could be… It could be one in there, there could be two in there, there could be three in there, heh…`
Teacher: `mm, exactly. It can be any number, as long as they can fit in the box`
Teacher: `But it can't be zero`
Linda: `No`
Teacher: `Because then we wouldn't have heard it rattle.`

In their discussion, the students and the teacher are closely tied to the particular example of the concrete material in front of them on the table. Although they realize that the two sides are equal for any number of beans in the boxes, they agree that the number has to fit into the box, and it cannot be zero because they can hear it rattle when they shake the box. While there are many solutions in theory, the aim of the task is still to find out the exact number in the box, and when in doubt the teacher suggests they look into the box. During this first lesson the students are expected to come up with solution strategies themselves, no set strategy is presented, and the justification for success is found in the concrete situation. The different strategies are not evaluated. The example is the problem, and there is no evidence in the discussion that the more general properties of the task are attended to.

Lesson Two

The activity continues during the second lesson. First some of the student tasks from lesson one are solved in a whole class discussion, where several students are asked to suggest solution strategies. Most strategies are presented through a guess and check process. One student, Pete, solves a task by removing the same number of boxes and beans on each side until he has only beans on one side and only boxes on the other. This strategy is not picked up by anyone else and not developed further by the teacher. The impression is that all suggested solution strategies are accepted as equally relevant. The lesson continues with two new tasks presented by the teacher. This time there are many boxes, and she presents a shorter way of symbolizing the situation, drawing only one box and one bean on each side using numbers to show how many of each there are (see Fig. 4.12).

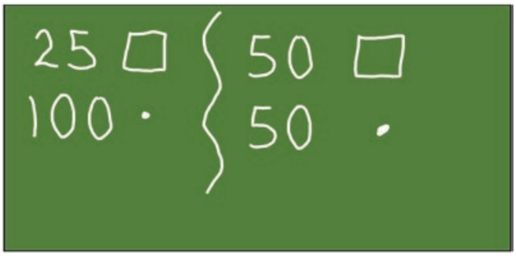

Fig. 4.12 A shorter way of representing large numbers of boxes and beans

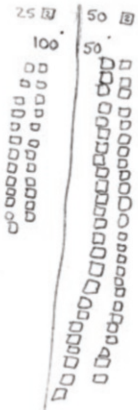

Fig. 4.13 Student representation of the situation 25 boxes and 100 beans = 50 boxes and 50 beans

Despite the large numbers, most of the students make drawings of the boxes, but in most cases the beans are represented only by numbers (see Fig. 4.13).

Towards the end of the lesson, there is a whole class round-up where several students are called to the board to show their solutions. All the presented solutions use a guess and check strategy, looking for relations between the numbers.

One of the solutions that come up on the board shows a misuse of the equal sign when the student writes: $25 \cdot 2 = 50 + 100 = 150$ and $50 \cdot 2 = 100 + 50 = 150$. The teacher points to the mistake but does not explain the nature of it, nor does she show a correct notation. The strategy of solving by subtracting the same number of boxes and the same number of beans on both sides, presented by Pete at the beginning of the lesson, is not brought up again. But after the lesson, Pete goes to the board and shows a few other students who stay behind how to solve the equation using that method.

Lesson Three

At the start of this lesson the teacher introduces the equal sign and the standard solution algorithm of doing the same on both sides. She goes back to drawing all boxes and beans and then, step by step, she removes or crosses out the same number of beans and boxes on each side until she only has one box on one side and beans on the other (Fig. 4.14). When doing so she makes a note of how many are left but does not connect to the inverse operation subtraction.

When the solution is shown on the board, students look away and do not pay much attention. Finally, the teacher introduces an algebraic representation, writing the following equation below her drawing: $4x + 4 = 2x + 8$. At this point one student exclaims `Yes!` as if feeling relieved and/or enlightened. Now the teacher introduces the term `equation`. As a next task the teacher draws a line with two boxes and 10 beans on one side and one box and 13 beans on the other, asking the students to write this as an equation (see Fig. 4.15). Although the algebraic notation is intro-

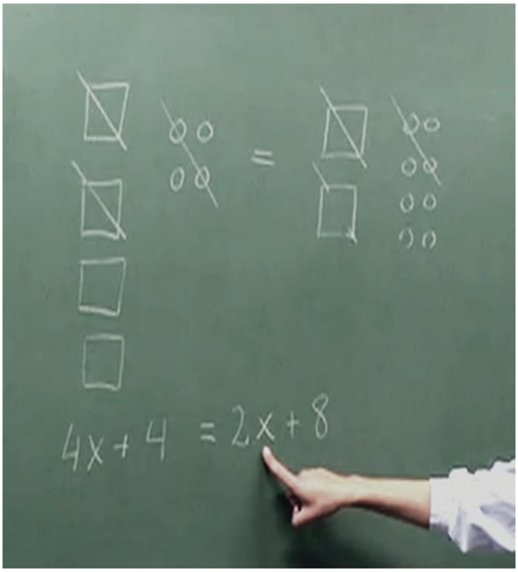

Fig. 4.14 In the third lesson the teacher presents the solution strategy of crossing out the same on both sides, and then writes the corresponding equation

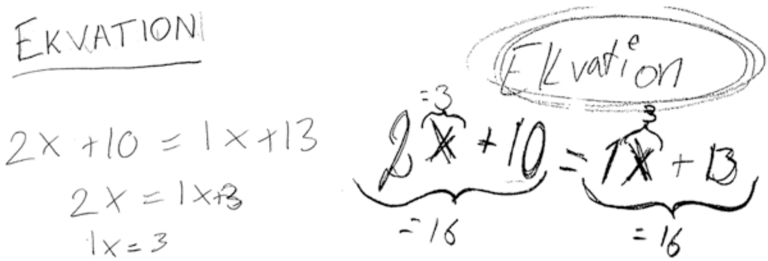

Fig. 4.15 Students' algebraic representation of the situation 2 boxes and 10 beans = 1 box and 13 beans under the heading Equation

duced, the formal notation for the crossing out strategy (inverse operation) is not, and the strategy is presented more as a procedure connected to the boxes and beans than as an algebraic procedure.

Closing of the Activity

A worksheet is handed out and students are asked to solve the following four equations:

(a) $x + 5 = 13$
(b) $9 = 12 - x$
(c) $4x + 6 = 3x + 8$
(d) $5x + 5 = 3x + 15$

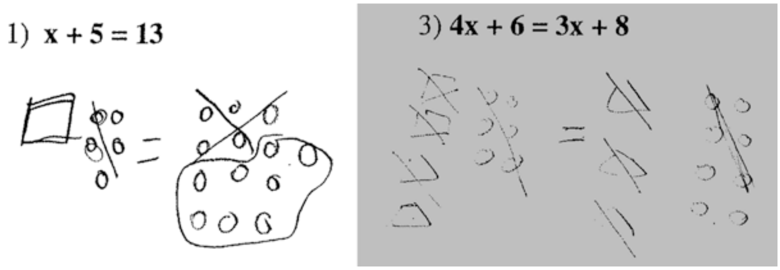

Fig. 4.16 Some students draw the equations as boxes and beans solving them by crossing out the same number on each side

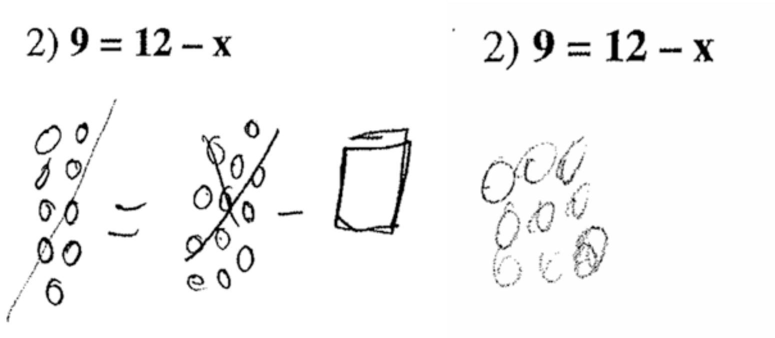

Fig. 4.17 Student's work trying to represent subtraction using boxes and beans

The students are encouraged to draw each equation as boxes and beans if they find it helpful, but most of the students only write answers using trial-and-error solution strategies. Some students make drawings removing equal amounts on both sides, as shown in Fig. 4.16.

The equations in Fig. 4.16 include only addition and multiplication. Equations including subtraction turn out to cause some problems for students who attempt to represent them as boxes and beans (see Fig. 4.17). Visualizing and drawing the subtraction of a box is a difficult, if not impossible, task. It is up to the students to make a transfer of the procedure of *taking away the same number of boxes and beans from each side* to the more general idea of *doing the same operation on each side* of an equation. In this case the process has been carried out using concrete materials and drawings, but no real mathematical interpretation of it has been articulated. In the interview after the lesson, the teacher expresses surprise that the subtraction task was so much more difficult to solve. For students not bothering about connecting to the boxes-and-beans activity, the equation was an easy missing value problem. It seems as if the activity made an easy problem complicated, and the teacher had not thought about how to show a subtraction using the concrete material.

Analysis of Class A2

The Nature of Mathematizing

As in the first example, the gist of this activity is based on the idea that students themselves should construct equations. Several aspects of algebraic equations are built into the activity (see Table 4.1). However, the activity also constrains the learning of equation solving to equations involving addition and multiplication (not subtraction or division), limiting the value of the variable to a whole number between one and what fits in the box, and always gives the variable in the algebraic expression $ax + b$ the role of being the multiplicand. The number of boxes is known, and the unknown number in each box is the variable x. When later confronted with a problem of solving the equation $150 = 4x + 30$ in a context where x was the multiplier, some students from this group immediately found $x = 30$, but could not interpret the meaning of x in the question posed because they kept thinking of it as “the number of beans in one box” (Rystedt, Kilhamn, & Helenius, 2016).

Another feature of the activity is that it could serve as an introduction to the standard algorithm for solving equations through a series of simplifications producing equivalent equations by means of doing the same operation on each side of the equal sign. By starting the activity in situations using concrete material, it provides a potential for horizontal mathematization. The object of our analysis is the extent to which mathematics and mathematization are in focus during the activities.

The analysis shows that the mathematical interpretations of the issues described in Table 4.1 are not made explicit in the lessons. During the first two lessons, mathematical terminology is rare and mathematical notation is absent. Time is spent manipulating the materials and making drawings, and, throughout, the particular concrete real world situations are focused and discussed rather than more general features of the equality. Possibilities afforded by the activity are not taken up, such as comparing equations and solutions to make conjectures of a general kind based on the specific examples. Formalizing, symbolizing and mathematical modeling are scarce and not a joint, public concern. The teacher waits for the students to come up with ways of symbolizing. Using mathematical notation such as plus sign and equal sign is done by a few students but not expanded by the teacher. The teacher leaves

Table 4.1 Aspects of algebraic equations implicitly present for horizontal mathematization

Situation	Mathematical interpretation
Equal number of beans on both sides	Equal value on both sides of the equal sign
Equal number of beans in each box if there is more than one box	A variable has the same value if it appears several times in one equation
Boxes and beans can appear on both sides	Expressions including both numbers and variables can be on both sides of the equal sign
The unknown number of beans in the boxes is a specific number although for some equations this specific number can be any number	A variable represents a number; sometimes this number can be an arbitrary number or assume a range of values

the responsibility for mathematizing to the students. Instead of encouraging mathematical reasoning the teachers suggest they use the concrete material or drawings to justify and check their results within the context of the objects and drawings themselves. During the whole boxes-and-beans activity only straightforward addition problems are posed and solved. When the teacher finally introduces the word equation and algebraic notation in the third lesson, we can see from the students' solutions of the worksheet tasks that they have problems generalizing the ideas from the boxes-and-beans activity, for instance, they do not manage to apply them to a simple subtraction task. There is no evidence of transfer here, and those who solve the problems treat them as missing value tasks.

In this classroom the teacher expects each student to make individual documentation of his/her work, but the nature of that documentation is negotiable. The written work mostly consists of drawings and serves the purpose of symbolizing the concrete situation, helping students to visualize the beans hidden in the boxes. From an analytical point of view, the inscriptions can be seen as an intermediate step, an alternative form of mediation (Vygotsky, 1978) that bridges between the concrete situations and mathematical notation. In this activity, the drawings are a *representation of* concrete equality situations and a *representation of* the process of eliminating equal amounts on both sides. They could very well serve as a tool for mathematizing if embedded in a richer mathematical terminology and notation.

The Nature of Participation

In this classroom, much of the mathematical work is expected to be carried out by the students. The students are asked to be inventive and they are encouraged to come up with solution strategies themselves. Only during the third lesson does the teacher suggest a strategy compatible with the standard algorithm. All student solution strategies are commented on as valuable. As was the case in class A1, the social norms of the classroom encourage the students to collaborate, to take initiatives and to engage in a creative process constructing tasks for each other. Students are given a lot of time for the activity, there is no hurry and a few times we see students who are well ahead of the teacher, patiently waiting for more substantial things to work with.

The framing of the activities during these lessons is similar to what we found in the previous example, with the possible exception that the game-like quality of the tasks is not present to the same extent. Inclusive and respectful social norms prevail, and they clearly encourage collaboration and sharing of ideas and proposals for how to work with the tasks. The approach to learning is inductive, students are supposed to generate understandings of the tasks that make it possible for them to solve the problems they are assigned. However, again, there is little overt and public mathematization where the conceptual potentials of the materials, for instance the use of the boxes and the beans, are exploited. The concept of variable—what Vygotsky (1978) would refer to as a typical instance of a scientific concept—is not introduced as a means for communicating about the objects in situ. The idea that dominates is that the boxes contain a certain number of beans, and this number is to be found in each and every instance. No discussion that provides continuity between the examples and their mathematical interpretation is carried out, and there is also no summary of the activities where it is made clear what was to be learned.

Introducing Variables Through Group Discussions (Class B)

The third example is drawn from the first of five consecutive days of observation and filming in class B (see Fig. 4.18). This lesson explicitly introduced the concept of variable and the use of letters. During the first part of the lesson, the teacher focuses on algebraic notation as a textual practice by giving students several tasks involving the translation of expressions from spoken language to written algebraic notation. Later, the teacher introduces the concept of variable and the notion "to vary", and the students work on problem-solving tasks that involve interpreting algebraic expressions. Throughout this work, the teacher makes extensive use of an interactive whiteboard (IWB) with a document camera attached. She uses this arrangement to display the tasks in the textbook along with solutions suggested by the students.

The first task the teacher assigns to the class is one in which students are asked to calculate the age of two people in relation to a third whose age is given (Carlsson et al., 2004, p. 100: Task 27). When presenting the task, the teacher uses the docu-

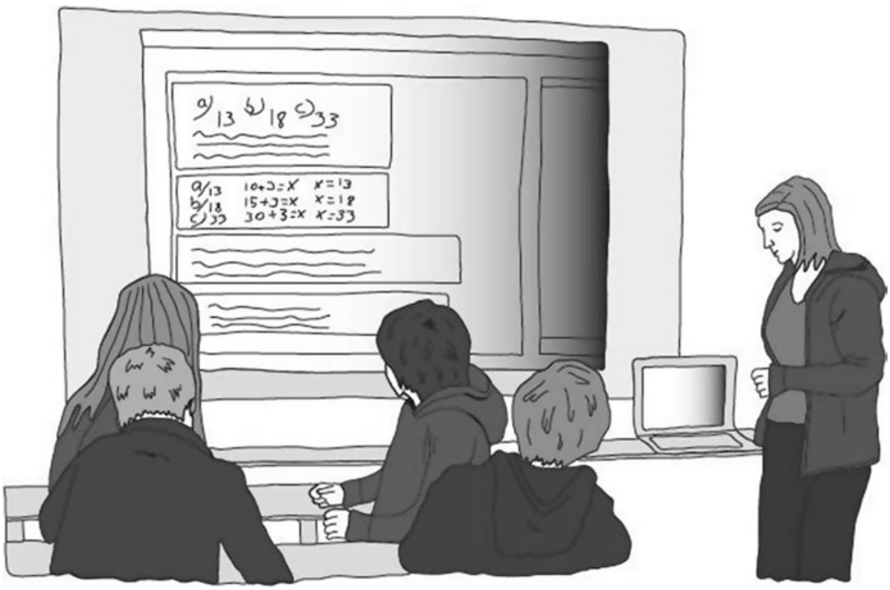

Fig. 4.18 Interactive whiteboard activity in Class B

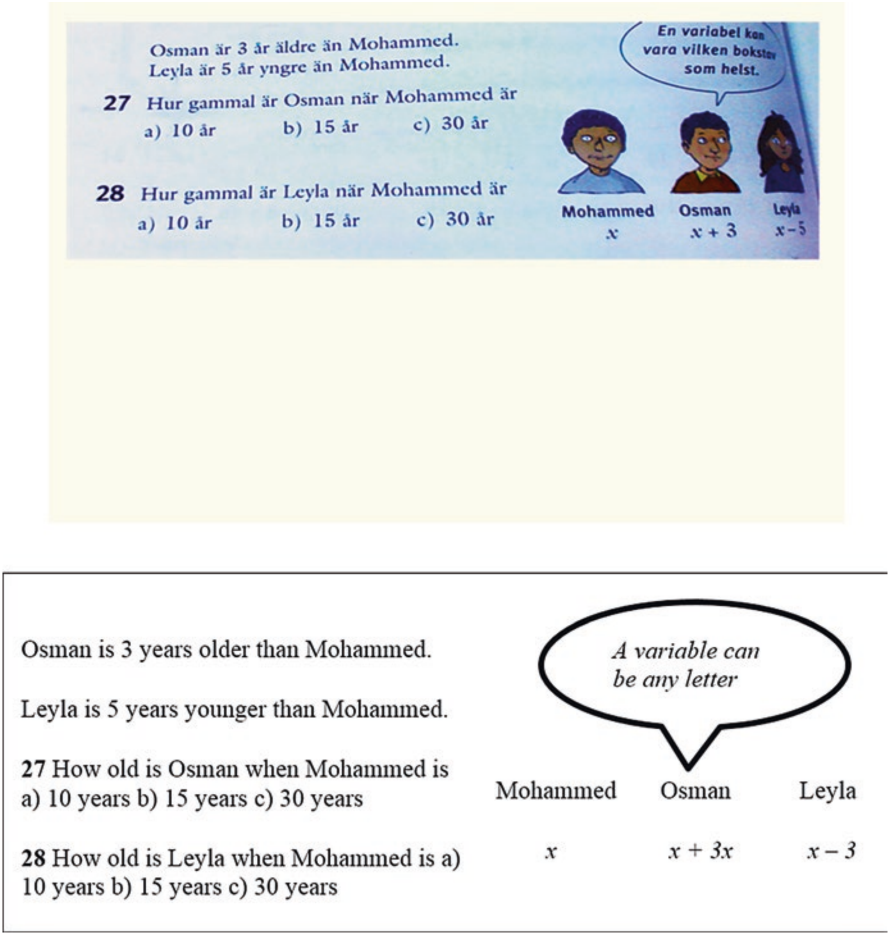

Fig. 4.19 Age task displayed in (**a**) the IWB, and (**b**) in stylized version (translated by the authors)

ment camera to photograph the textbook page and chooses to show only the relevant section on the IWB (see Fig. 4.19).

In the task, students are given the relationship between the age of Mohammed and that of Osman who is 3 years older and Leyla who is 5 years younger. For each question in the task, students are given Mohammed's age and asked to determine the age of one of the other two people. After reading the task aloud to the class, the teacher asks the students to work in groups of four or five and to produce a joint document showing their work.

Collecting and Discussing Student Work on the IWB

After a few minutes of working on the task, one group raise their hands and tell the teacher that they `are done`. The teacher listens to their explanation of the solution and reminds the class that she would like them to document their work, not just the answers. She says `I want you to fill in how you have reasoned using a variable. You don't need to erase, rather you add that, your reasoning. And keep in mind that I want not just one, I want a twofold result, one could say. I want to know` *`how old`* `Osman is, but I also want to find out` *`how you got hold of`* `Osman's age`. Thus, the teacher explicitly points out that she wants the product as well as a documentation of the process of producing a mathematical expression. Following several more minutes of group work, the teacher asks the students to present their documents so that she can photograph them with the document camera and arrange them across several pages on the interactive whiteboard. As it is not possible to find space to display all the group work at the same time, the teacher organizes those solutions she wants to be visible and starts a whole class discussion:

Excerpt 10

Teacher:	`we couldn't find room for all the solutions you have come up with on one single page, but I will try to collect it if we start, since on this here first page all think the same thing for both a b and c do they agree?`
Anders:	`ehh`
Teacher:	`here it says 13 right?`
Anders:	`13`
Bea:	`15`
Carl:	`15 it says 15 there`
Teacher:	`10 15 30 right? Or was it Osman you were finding out?`
Bea:	`oh but [inaudible] where does it start`
Carl:	`you must, now it's wrong`
Anders:	`no it says there`
Teacher:	`but what did you say?`
Bea:	`just look at the task [laughing] it was all written there`
Anders:	`no but it says when Mohamed is 15 years Osman is`
Bea:	`it says 13 years underneath`
Dana:	`it says 18 there`
Anders:	`it's wrong`
Teacher:	`ah okay ah and here 13 18`
Anders:	`but in, here in the calculation it's wrong, but in the answer it says 33 no 38 oh then we took that, it was wrong`

Teacher: ah then you had it, what information did you use to solve this task, if we look at the last there? nothing? So you also meet 18 and 33 if we return to the task itself what information do you use to work out what Osman was? Edvin
Edvin: how old the others were
Teacher: and how did you find that out then? Frans
Frans: it's in the text
Teacher: it's in the text

In Excerpt 10, we can see that many of the students solve the task as an arithmetic task, using only the information in the text where the relationship between the children's ages was expressed in words. They do not find any need to introduce the idea of a variable into their own work to find the answers. One group uses an *x* in the place of the sum (the answer), which is not consistent with what *x* represents in the textbook picture of the task shown in Fig. 4.19.

One Student Group Presenting Their Work

Following this initial discussion, the teacher switches pages on the IWB back to the image of the textbook task and asks one group to come up to the front of the classroom to explain their solutions. The students in this particular group have chosen to use algebraic notation when documenting their work:

Excerpt 11
Teacher: there was a group who discussed something else on this page can anybody see-, did anyone use anything else on this page than just what it says right here? Frans [pointing at the text block in textbook task]
Frans: the three pictures
Teacher: was there any group who explained the ages in this way? [pointing at expression under the picture of Osman] when you tried to work out how old Osman was?
Carl: no
Teacher: you did it [pointing at a student group] why did you do it?
Gunnar: that's how we wanted to do it
Hilda: we just did it
Teacher: you just did it?
Gunnar: yeah
Hilda: we just worked it out
Teacher: here it's like this [pointing to the picture of Osman] that for each task you do the same [touches IWB at

task c] but one thing varies [touches IWB at task a] that's why they have been a bit nice there helping you [pointing at the expressions under the pictures] because once you've worked this out it is quite quick [pointing at the tasks] and those of you who decided that we will use- how how did you work from there [pointing at the expression under the picture of Mohammed] if we look at that group, those who use what it says there, can one of you explain a bit how you did, can't Gunnar, can you come here and just show us here when you discussed

Gunnar: but Hilda can

At this point, the teacher switches to a page on the IWB that displays the solution documented by the group she has asked to come up to the front of the room and explain their work (see Fig. 4.20).

With their work visible on the IWB, the group of students, while being prompted by the teacher, explain how they arrived at a solution. The teacher sees and alludes to a connection between the students' work and the algebraic expressions in the textbook. The students claim that they first checked what the *x* was and then did the calculation, but the teacher points out that first there has to be an expression involving *x*. The dialog is characterized by everyday language referring to `something`, `this`, and pointing at the IWB. The students speak of the variable as `the sign`.

Excerpt 12

Teacher: ahh come up the whole group

Teacher: what was the first thing you did?

Hilda: mm wrote who they were or something or we wrote something

Teacher: you wrote what was in the pictures

Dana: I can't hear

a) $x + 3 = 13$

Osman is 3 years older and if Mohammed is 10 years old then Osman is 13 year old

b) $x + 3 = 18$ Osman is 3 years older and if

$x = 15$ Mohammed is 15 years old then

Osman is 18

c) $x + 3 = 33$

$x = 30$

Osman is 3 years older and if Mohammed is 30 years old then Osman is 33 years old.

Fig. 4.20 Page of student solutions that use variables displayed (**a**) on the IWB, and (**b**) in stylized version (translated by the authors). The solution is discussed in Excerpt 12

Teacher: we have to speak up, we can turn facing out so you don't speak into the board then you need to speak even louder than necessary
Inge: I need to check that
Hilda: umm we checked what the x:s mean and calculated it
Dana: can't hear
Teacher: what you want to say is that x had, it was the same on those different, when you looked at the different figures, was that what you did or did you make your own first?
Gunnar: we didn't look at the figures before we worked this out this
Teacher: mmm
Gunnar: that we really were supposed to use the sign, no
Teacher: no you found out that it said there what you had worked out if we return to the picture, what really happened was that you didn't look at the pictures but [changes the IWB screen to show the textbook task] the first thing you did was to decide this [pointing at x under the picture of Osman] because that's really what it says Osman is 3 years younger than Mohammed, Leila what was she?
Hilda: 5 years younger
Gunnar: 5 years younger
Teacher: Inge
Inge: 5 years younger

As can be seen in the discussion in these two excerpts, the interpretive work the students engage in involves identifying the value of *x* (Mohammed's age) and adding or subtracting from that. They thus show that they have understood the picture and the operations they have to perform to find the answer to the specific questions asked. This is also recognized by the teacher when she pushes them to begin to use the concept of variable.

Finishing the Activity

After guiding the student group in discussing how they had decided to use a variable to describe the relative ages of the children in the task, the teacher then connects their approach to the approach she wants to see in solutions to the next task:

Excerpt 13

Teacher: good so the first thing you did was you simply wanted to find out [points at x under picture of Mohammed] what this was. This is what you shall try in the next

```
task, number 28 has basically the same thing because
it is about the same but for the next task I would
like you to write using a variable when you describe
how you know
```

Here the teacher expresses that in the continuation of their work on a task that `has basically the same thing because it is about the same`, students should use `a variable` in order to describe how they `know`. Thus, here she points out that the concept/idea of a variable should be part of the solution and a part of `how you know`. However, she does not connect this to the idea of formulating a mathematical expression that is generative and that helps you to "know" in a mathematical sense, nor does she explain that addressing this issue is somewhat different from establishing the ages of the persons in the example. Also, it is interesting to note that she says `a variable` without connecting this to forming a mathematical expression, and it is far from clear if the students grasp this shift in discourse where a generalization involving mathematical expressions is being made, and when the specific ages of persons in the example in a sense no longer are the topic of discussion.

Analysis of Class B

The Nature of Mathematizing

This activity engages students in small group discussions concerning a task about age relations. The task is easy to solve arithmetically without introducing the concept of a variable. However, the task is framed by the book, and even more so by the teacher, to be about using a variable, and writing that variable as x. The task itself does not supply opportunities for learning about variables, since the questions can be answered through straightforward arithmetic procedures. And, from an everyday perspective, the students' approach to the task and their calculations make perfect sense; they find what is being asked for.

The teacher tries to make use of the affordances of the task by making it a question to be discussed in groups, and she tries to turn the focus away from the immediate numerical answers to other aspects of the problem. However, there is no indication that the students understand this intended shift in the level of discourse. Throughout this work, they do not appear to see any point in using the idea of a variable to deal with the task, nor do they seem to think in terms of the generalities and potentials of forming mathematical expressions when dealing with problems of this kind. When the teacher closes the activity, and points out that in the next round the students should use `a variable` to `describe how you know`, she is saying something that is very fundamental from the point of view of learning. Here you should "know" in a specific manner and by means of a particular conceptual resource. But this is mentioned implicitly, and so far in the discussion there is no indication that the students have realized the opportunities for mathematizing and for learning about variables and expressions in these tasks. The “x” is still treated as a number that stands for a real age that is fixed and that should be found. They address the problems through “everyday concepts” in the Vygotskian sense, and they calculate the answers that are asked for in an accurate manner and as you would if the main purpose of the task was to establish the ages of the persons.

The Nature of Participation

Again, we find that the students are active and contribute willingly to the discussions. They come up with suggestions and they co-construct the dialog. This testifies that the students know how to form a culture of participation when solving problems in school. The social norms of how to collaborate seem well established and are adhered to. In this flow of the conversation, the sociomathematical norms are not very explicit. Or to put it differently, the norms that are followed imply that you solve the problems that are presented one by one by providing the information asked for. The discourse does not imply that students are aware of the nature of the communication expected where the specific information provided is to be used as a means of mathematizing in a specific manner by using variables and formulating mathematical expressions. Thus, it is not that they would not understand this idea, the issue is more that this level of discourse—involving the potential use of “scientific concepts”—is not established in an explicit manner.

Discussion

The empirical illustrations we have offered here seem to indicate a fairly consistent pattern in terms of how participation frameworks are established, how the interaction is coordinated and the nature of mathematizing that goes on. In the first example, the activity is rich, with many opportunities for learning about variables in expressions and equations, but most of these opportunities are not exploited for this purpose. Little overt mathematization takes place. In the second classroom the mathematical activity is focused around manipulatives. A lot of time is spent working with these and talking in pairs, but the activity itself is limited to certain types of equations, and there is no transition from solving these particular equations to equation solving in general. In the third case, the problem also triggers student actions of counting and calculating, but the task in itself appears to be so simple that there is no functional need on the part of the students to engage with the concept of variable. The problem is easily solved through simple arithmetic.

The point of our analysis is not to make any evaluative claims about whether the teaching we have observed is particularly good or bad. We assume that the participants, teachers and students, are rational in the sense that they adapt to the expectations about how to proceed when learning mathematics in classroom contexts. Also, we do not know what happens after the lessons we have recorded, and to what extent what we have documented is taken further. As can be seen, a culture of participation that includes collaboration and sharing of ideas in the classrooms has been established in these classrooms. The students know how to formulate ideas, listen to and respond to suggestions from their fellow students and the teacher, in whole class situations as well as in group work or pair talk. They engage in the exercises presented and they exert agency. In other words, they are very competent at this level and they know how to "interthink" (Littleton & Mercer, 2013) and to produce coherent and accountable talk when facing problems.

If we look at the lessons as steps toward mathematization and algebra learning, it is obvious that the excerpts provide little evidence that the conversations adhere to sociomathematical norms, i.e. norms typical of mathematical communities (Yackel & Cobb, 1996). Overall, there is a low level of mathematization, and the students hardly ever engage in discussions where mathematical argumentation and justifications are expected. Each problem is treated as unique, and all solutions producing expected answers are accepted as equally valid. There is little continuity between tasks in terms of mathematical properties. In all the tasks we have used as illustrations, ideas of what constitutes a mathematical expression, what a variable is and how one uses the equal sign in equations are potentially there, but there is little evidence that students see these connections. For instance, when using the beans and the boxes, there are no obvious signs that the students realize that the beans in the boxes can be conceptualized in terms of the concept of variable. Nor is it clear if students understand the idea of a mathematical expression. When interviewed, very few students talk about their activities during these lessons using these terms as glosses to what took place.

In terms of instruction, the activities seem to represent an inductive approach to teaching, where it is expected that mathematization follows from engaging with the tasks. Students successfully manipulate concrete objects and find the information asked for, but they do this without attending to the mathematical concepts and procedures which the specific exercises are intended to exemplify. They do this by using arithmetic or solving a problem as a missing value task. Thus, following our findings, there is little evidence that students learn much about how to argue within the context of sociomathematical norms during these lessons. The tasks are easily solved without invoking algebraic concepts and ways of reasoning, and they do not force students to go outside what they already know.

To some extent our findings are similar to those reported by Emanuelsson and Sahlström (2008) on what they call the price of participation. They argue that participation comes at the expense of the acquisition of mathematical content knowledge in the Swedish context. Our findings seem to support this assertion, but they also complicate the argument by highlighting the importance of the particular frameworks of participation in mathematics classrooms. Without consistent and explicit attention to mathematization, the kinds of sociomathematical norms needed to support mathematically productive participation do not seem to be established. What Wood, Cobb, and Yackel (1991) describe as engaging students in "genuine conversations" about mathematics does not only imply that students talk about mathematics, but also that teachers take students' ideas seriously and support conceptual learning. Kazemi and Stipek (2001) described four sociomathematical norms that worked together to create a press for conceptual learning among fourth and fifth graders:

> (a) an explanation consists of a mathematical argument, not simply a procedural description; (b) mathematical thinking involves understanding relations among multiple strategies; (c) errors provide opportunities to reconceptualise a problem, explore contradictions, and pursue alternative strategies; and (d) collaborative work involves individual accountability and reaching consensus through mathematical argumentation. (pp. 77–78)

For Kazemi and Stipek, the presence of these sociomathematical norms indicate "an intellectual climate characterized by argument and justification" (p. 79). Without the clear establishment of such sociomathematical norms, our data suggest that it is the nature of participation in the classrooms rather than the focus on participation itself that is of most concern. The students in these classrooms do not get enough support to shift the level of discourse from engaging in the activity of finding correct answers to specific items, to talking about the mathematical ideas involved. They need to begin talking about how the problems they work with may be addressed through the "scientific concepts" of algebra.

The situation described in these three Swedish classrooms also resonates well with findings in a Swedish study about primary school mathematics teachers' professional identity development (Palmér, 2013). Based on in-depth interviews with novice teachers just before graduation, Palmér found that they shared a distinct and concurrent picture of how mathematics teaching should be managed. Their stories about good and less good mathematics teaching presented a dichotomy in which good mathematics teaching was described in terms of reformative, creative, student-focused, cooperative and active instruction; in contrast to less good mathematics teaching which was described as conservative, text-book-focused, repetitive and passive (*ibid*., p 105). Thus, there is a consistent picture of teachers valuing the participatory aspects of mathematical teaching. However, in Palmér's study, mathematics itself was in "good mathematics teaching" described as reality-based, concrete and hidden, i.e. children were expected to learn mathematics by engaging in mathematical activities in situations where mathematics was not made explicit and clearly visible, as if a clear mention of mathematics would make it more frightening, less compelling and, consequently, harder to learn. Less good mathematics teaching was characterized as reality-distanced, abstract and visible. Palmér writes:

> The focus of the respondents is on how and why, not on what and why. When they talk about examples of good and less good mathematics teaching they focus on how the lesson is taught and experienced by the students but not on the mathematics content in the lesson or how it was understood. (*ibid*., p. 106)

Thus, what we find in empirical research seems to be a fairly consistent picture that may provide a background for discussing how sociomathematical norms should be made explicit to students, and how they should learn to mathematize. Mathematization of the kind expected here, involving formulating mathematical expressions, identifying variables and working with structural aspects of equations and equation solving, does not seem to emerge spontaneously, not even when problems have this potential and students engage actively in the tasks at hand. They clearly need a push where a more “knowledgeable” partner points out how such concepts and operations are illustrated by, and relevant for, dealing with specific kinds of problems. There is no indication in our data that students would not be able to put such a conceptual frame on what they see in front of them, if they were more clearly encouraged to do so. Identifying and adhering to more sophisticated sociomathematical norms in this case requires more than being able to solve equations and calculate the number of beans; one must learn to reflect on and talk about what one is doing in mathematically relevant conceptual terms.

Chapter 5
Designed Examples as Mediating Tools: Introductory Algebra in Two Norwegian Grade 8 Classrooms

Unni Wathne, Jorunn Reinhardtsen, Hans Erik Borgersen, and Maria Luiza Cestari

Introduction

The teachers Kari and Ola are introducing algebra in two Grade 8 classrooms. Kari holds up a set of large playing cards and she writes on the blackboard what is written in the corner of the cards. Ola gets the attention of the students and carefully starts walking in one direction in the classroom, asking the students to describe what he is doing. These are the starting points of two examples that each teacher has designed as a tool for communicating and explaining new algebraic ideas in their respective classrooms.

The aim of this chapter is to investigate two introductory algebra lessons. The passage from arithmetic to algebra in school mathematics is known to be challenging for students as has been pointed out repeatedly in this volume. The learning of algebra includes new symbols, new concepts and also new ways of thinking (Berg, 2009). In the two lessons, the teachers are presenting algebraic concepts through examples, using them as mediating devices, bridging between new concepts, on the one hand, and familiar situations and prior knowledge of the students, on the other hand. The purpose of our analysis is to capture how the teachers approach the complexity students meet in such learning situations and how they support learning. Both teachers introduce and illustrate the concepts of variable and algebraic expression and demonstrate processes of simplification and substitution in the introductory lesson. The topics coincide with the textbook; however, the teachers have chosen to design their own examples in order to engage the students in algebra.

U. Wathne (✉) · J. Reinhardtsen · H. E. Borgersen · M. L. Cestari
Department of Mathematical Sciences, University of Agder, Kristiansand, Norway
e-mail: unni.wathne@uia.no

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Examples play a central role in the teaching and learning of mathematics as described by, among others, Bills et al. (2006) and Rowland (2008). Paying attention to how examples are used offers both a practical and a theoretical perspective on the design of teaching activities and on the professional development of mathematics teachers. The whole point of giving worked out examples is that students appreciate them as generic, and even internalize them as templates so that they have general tools for solving classes of problems (Bills et al., 2006).

Bills et al. (2006) give a historical account and a categorization of the use of examples from the perspective of the mathematics teacher. They refer to Rissland-Michner's (1978) four epistemological classes of examples (not necessarily entirely separate): (a) start-up examples, which help motivate basic definitions and results; (b) reference examples, which are mentioned repeatedly in different situations; (c) model examples, which are generic examples, indicative of the general case; and, finally, (d) counterexamples. These classes represent what Rowland (2008) refers to as inductive examples for the purpose of abstraction. "Exercises", on the other hand, are examples used for practicing and rehearsal.

The worked out examples in this study have features that the literature points out as important. They provide an opportunity for the students to experience the mathematization of familiar situations (Bills et al., 2006; see also Chap. 4), including transactions with semiotic means such as spoken language, inscriptions (e.g. numbers, words, and items belonging to the algebraic symbol system), and gestures, to interpret and express mathematical meaning (Fried, 2009; Radford, 2003). They have epistemological qualities as they motivate basic definitions and concepts, they model central ideas in algebra, and they are referred to more than once and in different situations. Both are inductive examples in Rowland's (2008) sense, and mainly start-up and reference examples, and to a certain extent model examples applying Rissland-Michener's (1978) epistemological classes.

The concepts of mediation and mediating tool (Carlsen, 2010; Säljö, 2006; Wertsch, 1991; cf. Chap. 3, this volume) are central for the analysis of our empirical material. Leont'ev (1981) considers the use of artifacts and tools as mediational means and emphasizes that tools connect "humans not only with the world of objects but also with other people" (p. 56). The theoretical term of mediating tool facilitates our analysis in making a distinction between, on the one hand, the tools (designed examples, concretes and semiotic means) that the teachers employ in their interaction with the students and, on the other hand, the educational goals of the lessons (including the mathematical objects of variable and algebraic expressions).

In this chapter, we will use the term semiotic mediation, introduced by Vygotsky (1978), when we discuss the teachers' use of semiotic means in the designed examples. John-Steiner and Mahn (1996) refer to semiotic mediation as one of three major themes[1] in Vygotsky's theory regarding the interrelationship between the social and individual processes of knowledge co-construction. The semiotic means play a central role in the designed example as they are physical links (can be seen or

[1] Based on Wertsch (1991) who highlighted these three themes: *social sources of development*, *semiotic mediation* and *genetic analysis* (the method for investigating the former).

heard) between the students and the mathematical objects that the teachers are trying to explain. The semiotic means are not neutral but add meaning to the activity; however, they are also given meaning through the activity.

In the following, the reader is invited into two Norwegian Grade 8 classrooms to observe the practices of two colleagues; how they start their lesson, how they approach introductory algebra, and how they interact with their students. The purpose is to make visible the complexity of the learning situation and teachers' ingenuity in trying to make algebra accessible to their students. Our attempt to show this important interactive and instructional work has guided the organization and presentation of the empirical material. The object of inquiry is the teachers' introduction of algebra, and we focus our analysis on how the teachers mediate the new concepts through their designed examples. More specifically, we ask: *Which approaches do the teachers use to introduce the concept of algebraic expressions?*

National Curriculum, Classroom Environment and Textbook

Compulsory schooling in Norway starts at the age of six. Algebra enters the curriculum as part of the main subject area, *Numbers and algebra*, in grades 5–7. This continues to be a subject area throughout primary education, where algebra in school is described as generalized arithmetic. The curriculum lists specific educational goals in each subject area after 2nd, 4th, 7th and 10th grade. The goals referring to algebra state that the students shall be able to:

- *explore and describe structures and changes in simple geometric patterns and number patterns* (Grade 7)
- *process and factor simple algebraic expressions, and carry out calculations with formulas, parentheses and fraction expressions with a single term in the denominator* (Grade 10)
- *solve equations and inequalities of the first order and simple equation systems with two unknowns* (Grade 10)
- *use, with and without digital aids, numbers and variables in exploration, experimentation, practical and theoretical problem solving and technology and design projects* (Grade 10)

(The Ministry of Education and Research, 2013)

In Norway, most students go to comprehensive school until the age of 16 (Grade 10) and are taught in mixed-ability groups. According to Pepin (2011) there appears to be particular "customary ways" of conducting the teaching of mathematics in Norwegian classrooms. For example, most teachers ask their students to work on exercises from the textbook for a considerable amount of time during a lesson, so that the students can practice what has been explained, and the teacher can monitor the students' understanding. The textbook used in the classroom and at home is chosen by the school, which furthermore provides a copy for each student.

There are many textbooks available in the market, and they differ with respect to the grade in which algebra is introduced and also on the number of pages dedicated to this topic. Many textbook series only produce textbooks for grades 1–7 or 8–10, which mirrors a shift in the Norwegian educational system, in teachers' education and in the classroom culture. Although letters appear as variables, mainly in geometry chapters, in 6th grade textbooks, they are often not introduced as such. Some of the 7th grade textbooks have algebra as a specific topic but vary in the extent to which it is covered, from only a few pages to a whole chapter. It is more common to find an algebra chapter in 8th grade textbooks, though there are exceptions.

The two Grade 8 classrooms presented in this chapter are from the same junior high school ("ungdomsskole", grades 8–10). The textbook used by all mathematics teachers in the school is *Faktor* (Hjardar & Pedersen, 2006), which is widely used in Norway. *Faktor 1* (the book for Grade 8) has a separate chapter called *Numbers and Algebra.* The title reflects the subject area introduced in the National Curriculum. In the textbook analysis done by Reinhardtsen (2012), *Faktor* is interpreted as reflecting the traditional view of learning by instruction. Each subchapter first presents a kernel (definitions, procedures, etc.), an example and then tasks that are similar to the one presented in the example. This is a teaching and learning cycle that is common in many educational systems.

The goals for the teachers' presentations in this study concern processing simple algebraic expressions and carrying out calculations with formulas, as formulated in the second point in the National Curriculum, and treated in the first section on algebra in the textbook *Faktor 1*. The teachers follow the textbook in this respect.

Methods

In order to accomplish the aims of this study, we use a qualitative approach to collect and analyze the empirical data grounded in a sociocultural perspective on learning. The data have been collected according to the VIDEOMAT design (see Chap. 3, this volume): as in the other countries, we observed the first five algebra lessons in each classroom (videotaping), interviewed the teachers after the fifth lesson (audiotaping) and collected written material used in the classrooms (teacher and student materials). As a first analytical approach to the collected data, lesson graphs for each lesson were produced, and the first lesson in all classrooms was transcribed.

In this chapter, we have used an inductive approach to the video analysis as defined by Derry et al. (2010, p. 9), which is suitable when: "a minimally edited video corpus is collected and/or investigated with broad questions in mind but without a strong orienting theory." The two episodes presented and analyzed in this chapter were chosen from the lesson graphs and after several viewings of the video material. The designed examples stood out as unique in the international video material. In addition, the examples used are referred to in later lessons by the teachers, and they therefore play an important role in the introduction of algebra in these two classrooms.

The episodes, as part of the first lesson in each classroom, have been transcribed in their entirety. The national curriculum and the textbook are viewed as integrated parts of the classroom practice, and the designed examples are analyzed and related to these important didactical documents.

Participants and Context of Research

The two Norwegian Grade 8 classrooms (A and B), as mentioned earlier, are from the same junior high school. It is the main school at this level in the local community and has about 500 students. The school is located at the center of this community, which is situated outside a larger city. The main income in the area is from industries, trade and services.

In classroom A there are 21 students (age 13–14), 11 girls and 10 boys (the class holds 29 students but eight were not present at the time of the observation). The teacher is an experienced female teacher (with 9 years of experience). She has a master's degree in pedagogy and a specialization in mathematics. In classroom B there are 25 students, 14 girls and 11 boys (27 in the class, with two students not present at the time of the observation). The male teacher has 4 years of experience and was at the time when the recordings were made taking additional courses in mathematics. He has been educated as a general teacher (4 years, including half a year of mathematics).

Analytical Framework

According to the theoretical and methodological constructs used for analyzing talk-in-interaction, Linell (1998) identifies two building blocks of a dialog, namely turns and idea units. A turn is basically a period of time when one speaker holds the floor, while an idea unit refers to, as the name indicates, a specific idea within a turn. A turn can include several idea units. In line with the sequential organization of a dialog, each turn should be interpreted and understood in relation to the prior discourse, as well as being seen as creating conditions for the ongoing dialog.

A number of turns form larger units, which Linell (1998) refers to as topical episodes. We choose to call these units "episodes", and we divide an episode into fragments

.

Each episode in this chapter contains all the turns in a period of time (which are numbered chronologically), and each fragment constitutes a continuous flow of turns and idea units as methodological constructs used for structuring the data. The analysis performed does not focus on the constructs of the dialog. However, the ideas emerging in the classroom discussion are understood and presented within the analytical framework of Linell. The unit of analysis is the introductory algebra example as realized in the interaction between teacher and students.

Findings: The First Minutes of Algebra

The data is organized in two episodes taken from Classroom A and Classroom B, respectively. Each episode starts with the lesson graph from the first lesson, and is divided into fragments with headings that characterize the content. We have chosen to present the complete dialog of the episodes intertwined with short descriptions and limited analyses in order to preserve for the reader a more genuine experience of the teachers' examples. Further analyses will follow at the end of each episode. The chapter ends with a comparison between the two episodes.

Episode 1: Classroom A

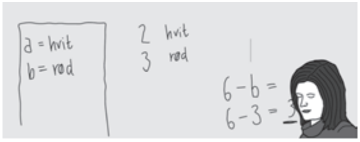
This episode is chosen from the very beginning of the first lesson on variable expression in classroom A. The teacher, Kari, uses playing cards to introduce the idea of using letters for numbers. The rest of the lesson is dedicated to an algebra game (see Table 5.1). The teacher returns to the playing cards in lessons 2 and 4. In the lesson graph (Table 5.1) the flow of the entire lesson 1 is presented.

In Fragments 1 to 7 we will present the full teacher-student conversation that takes place during the first 13 min of the lesson.

Fragment 1: Introduction of the Lesson

1.	T:	I know that you have been a little excited, because I have said that *no*, we will not begin with algebra before this week. Now we will start. But I have been asking whether anyone has had any experience with algebra before. There have been no hands raised, but perhaps you have some experience, only you don't know that it is algebra. And, among other things, we have indeed started ahead a little, for you have been doing some algebra, probably a lot, but at least one lesson, that we had two weeks ago. Do you remember? I think it was two weeks ago, that you brought some playing cards, and then we worked with those cards. We worked with positive and negative numbers first, then you got some cards and counted how many points you had. And the ones who got the highest number won. Do you remember that? (.) And then, afterwards, we played with the black cards as positive numbers and the red cards as negative numbers. And then you were to find out who came closest to zero when you added them together. Do you remember that? Yes, but then there was something else we also had to do when we were calculating with the cards…

Table 5.1 Lesson graph showing classroom A (lesson 1, 40 min)

[40 min]	
00:00 13 min	**Whole class: ITS** Introducing new topic, algebra. The teacher shows some large playing cards and asks the students if they remember how they used them when working with negative numbers. The teacher holds a hand of five cards and writes down what it says in the corner of each card. With the help of the students the letters are given a value. Then she makes an addition task replacing letters for numbers. She then picks a new hand of cards. Simplifies the expression with letters. The order of operations is mentioned. The teacher shows two sheets of paper with the letter ***a*** on one and ***b*** on the other. She makes the expression *a*+*b*, and then evaluates the expression for different values of the letters. Talks briefly about negative numbers 
13:13 6 min	**Whole class: IT** Introducing an algebra game. Handing out the game which is a printed sheet of paper to each pair of students. Explains the game (a is a white [hvit] dice, b is a red [rød]). Instructs the students to think out loud so the other student can hear their reasoning. A student asks what the expression *2a* means. The teacher explains with the help of another student 
19:40 16 min	**Student work: SGN** The students play the game 
35:30 2 min	**Whole class: FTS.** The teacher tells the students that she is happy with their work and that she has heard them explain their thinking while playing the game. She then asks for comments from the students about the game
37:01 3 min	**No mathematics: NM** Cleaning up. Preparing for the next lesson; finding the books and putting them on their desks

The teacher starts the lesson by relating algebra to prior activities in the classroom. She reminds the students that they used playing cards in a game where they performed calculations with negative and positive numbers. The teacher emphasizes the new topic, saying the word algebra four times. At the same time, she initiates the use of playing cards as a tool to appropriate algebra. She connects the word algebra to their work with whole numbers and operations with such numbers. In this way, she connects the word algebra with arithmetic, using playing cards as a mediating tool. The teacher does not explicitly mention her goals for the lesson, but implicitly she follows those of the textbook.

Fragment 2: Numbers and Letters in Playing Cards

1.	T:	… When we added the cards, I don't know if you at the back [of the classroom] can see, but here I have [showing the cards] an eight, or we can begin with a two, and then I have a four, and then I have an *A*, and then I have an eight, and then I have a *K* [writes on the blackboard: 2 4 A 8 K]. Do you see that? Here we operated [referring to a prior lesson] with both numbers and letters. We have an *A*, and we have a *K*. How did you do it, when you calculated how much you had altogether? Does anyone remember? Ola?
2.	S:	The *A* was one.
3.	T:	We said that the *A* was equal to one, yes. Great! So *A* is equal to one, [writes: A = 1] we said. Really, the *A* was one or fourteen. We could choose, but then everyone wanted to use *A* equal to one. I don't know if it is because it is easier to calculate with one or if that is the most common, that *A* equals one. Anyway, we used *A* equal to one. What about this *K* then? Alf?
4.	S:	Thirteen.
5.	T:	We said it was equal to thirteen, yes. The king was equal to thirteen [writes: K = 13]…

The teacher shows a hand of five large playing cards and asks the students what is written in the corner of each card. Some of the cards have numbers and some have letters. With the help of the students, the teacher reveals the hidden numbers behind the letters. The teacher carefully writes it all on the blackboard.

Fragment 3: Numerical Expression and Calculation

5.	T:	… And then, when you calculated [referring to the prior lesson], what did you do then to find out how many you had altogether? Do you remember that? What did you do to count them together? Ina?
6.	S:	We added all the numbers.
7.	T:	We added them all together. So we took two plus four plus *A*?
8.	S	We had said that it was one, so therefore we added one.

9.	T:	Great! So instead of *A* we put one, plus eight and then plus *K*? And that was thirteen. So instead of *K* we put in the number thirteen. Good! And then we calculated how much it was. Two plus four plus one plus eight plus thirteen [writes: 2 + 4 + 1 + 8 + 13]. It is? Kim?
10.	S:	It is (…) twenty-five.
11.	T:	Yes, seven, fifteen, it will be more.
12.	S:	I mean twenty-six.
13.	T:	Even more.
14.	S:	Twenty-seven.
15.	T:	He, he, he, even more.
16.	S:	Twenty-eight.
17.	T:	Yes, work it through one more time and see if you don't get twenty-eight (…) it could be that I'm also doing some wrong calculations, you know.
18.	S:	Twenty-six then.
19.	T:	Did you say twenty-six? Two plus four is six, plus one is seven.
20.	S	Twenty-eight.
21.	T:	Yes, plus eight is fifteen, plus thirteen is twenty-eight [writes: = 28]. Good! So then you see, when you added together, you put in, you replaced *A* with one. You replaced *K* with thirteen. Look here. You replaced a letter with a number. Good! We will erase this and then we will do another one…

In the dialogues in Fragments 2 and 3, the teacher identifies letters and numbers on the cards. Then she assigns fixed values (hidden numbers) to the letters and thus makes a correspondence between letters and numbers (see Fig. 5.1). The numbers and hidden numbers are arranged in a numerical expression, for which the sum is calculated.

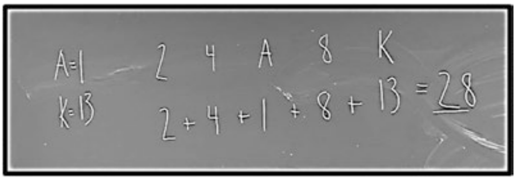


Fig. 5.1 Overview of what the teacher has written on the blackboard in Fragments 2 and 3 (letter K referring to King)

Fragment 4: Algebraic Expression

21.	T:	… Let's see if we can find some new cards here. Let's see, I think I will choose these cards [showing the cards]. Here I have, for those of you who cannot see, I have a six, then I have two queens and then I have two kings. Mmm. A six, two queens and two kings. So I have six plus queen plus queen plus king plus king. [Writes: 6 + D + D + K + K] If you are to calculate how much this is altogether, how should we calculate this? (.) Six plus queen plus queen plus king plus king. Ina?
22.	S:	The queen is twelve.
23.	T:	Great! The queen is twelve. So we write [writes: D = 12], we replace the queen with twelve. Good!
24.	S:	Then you add six plus twelve plus twelve, and then comes thirteen plus thirteen.
25.	T:	[writes: 6 + 12 + 12 + 13 + 13] Great! Good! …

The teacher picks a new set of cards and creates an algebraic expression. With the help of the students the letters are given values and the algebraic expression is made into a numerical expression without solving the addition task (see Fig. 5.2).

Fragment 5: Simplification of an Algebraic Expression

25.	T:	… If we now were to simplify these, now we are looking at the top one again, okay? Six plus *D* plus *D* plus *K* plus *K*. If we only were to add those, what would we get then? Tor?
26.	S:	Fifty-six.
27.	T:	Yes, and if we added, or if we add these together, a little more, but if we look at, if we have the number six [writes: = 6], and then we are to add queen and queen. Can we write it differently than *D* plus *D*? Ina?
28.	S:	D2

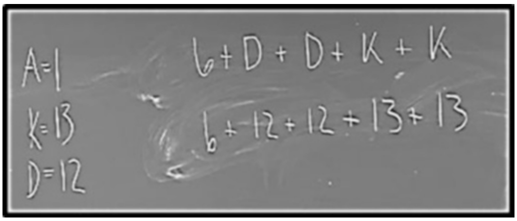


Fig. 5.2 Overview of what the teacher has written on the blackboard in Fragment 4 (letters D referring to Queen [Dame in Norwegian] and K to King)

29.	T:	Right, two times queen, for there are two queens [writes: + 2·D]. Good. Plus…
30.	S	Two times king.
31.	T:	Two times king, oi. Good! Two times king [writes: + 2·K]. It is, do you agree that twelve plus twelve is the same as if we now put it in here then, equals six plus two times twelve plus two times thirteen [writes: = 6 + 2·12 + 2·13]. Do you see that I replaced the queen with twelve and the king with thirteen? Do you see here that twelve plus twelve, do you agree that it is two times twelve? It is twelve two times. And thirteen plus thirteen, another way of writing that is two times thirteen. Right? It is the same that we are writing. But if we are to calculate this now, six plus two times twelve plus two times thirteen. Do you remember the order of operations when we were to calculate with addition, subtraction, multiplication and division in one expression? Arne?
32.	S:	(…) the multiplications first.
33.	T:	You have to do the multiplications first. And here we see clearly that the two queens belong together, because there are two of them. And the two kings belong together, because there are two of them. So we do the multiplications first. So it equals six plus, two times twelve, that is? Twenty-four, good, plus two times thirteen, yes. Six plus twenty-four plus twenty-six [writes: = 6 + 24 + 26], it equals?
34.	S:	Fifty-six.
35.	T:	Fifty-six. [writes: = 56] That is good. Great! Mm. So this was a little repetition, right. That we do the multiplication and the division first, and then the addition afterwards. And here we see clearly that the two queens belong together. The two kings belong together. Okay…

The teacher returns to the algebraic expression created in Fragment 4 and prepares a simplification. One student, Tor, responds to the teacher's question regarding how to simplify (25) by calculating the sum of the numerical expression, which is `fifty six` (26). The teacher does not follow up this response and instead starts to add the letters.

The teacher and the students replace the letters in the simplified algebraic expression with numbers, and simultaneously the teacher makes the link between addition and multiplication explicit (31). So, the teacher makes clear that the numerical

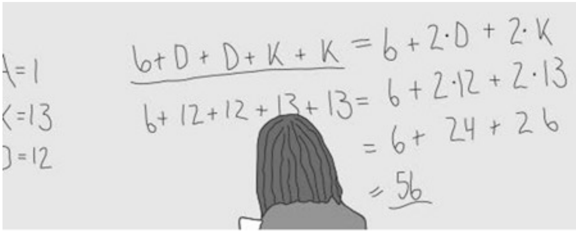


Fig. 5.3 Overview of what the teacher has written on the blackboard in Fragments 4 and 5

expression (31) is the same as the one developed in Fragment 4 (25). They calculate the sum (33, 34, 35) and get the same answer as Tor gave in (26) (see Fig. 5.3).

The choice of cards in Fragment 4, which includes two queens and two kings, gives an expression with two *D*s and two *K*s. The teacher follows the same approach of substituting the letters with fixed numbers as in fragments 2 and 3, but, in addition, she introduces an intermediate step of simplifying the algebraic expression. In doing both a vertical (on the blackboard, see Fig. 5.3) translation from an algebraic to a numerical expression and a horizontal translation to a simplified algebraic expression, the teacher sets the stage for showing the relationship between addition and multiplication and between the different numerical and algebraic expressions.

Fragment 6: Variables (on Sheets of Paper)

35.	T:	… Now I'm wondering, if I now had, instead, these are not playing cards [Shows two sheets of paper]. But if I, instead of *D* and *D* and *K* and *K*, which represent king and queen, if I now had an *a* and a *b*, do you see it, no perhaps you don't see it. But we have an *a* here and a *b* here.
36.	S:	Yes, we know that *a* is one (…) and *b* could be any number.
37.	T:	Great! You do know, good. If we now are to add these two cards, an *a* and a *b* [writes: a + b], then yes, we have said here that *a* is one, but letters can be variables and we can replace them with any number. So *a* is not always one. We can choose the numbers we want to replace the letters. So if I now say that we have the expression *a* plus *b* and then I erase this [erase: A = 1, K = 13, D = 12]. And now I put in that, for example, now we want *a* equals three, and *b* equals two [writes: a = 3 b = 2]. Can I now calculate how, what *a* plus *b* equals? If we have *a* plus *b*, and then I say that *a* equals three and *b* equals two, can you do this calculation? Ann?

38.	S:	Five.
39.	T:	You replaced *a* with, instead of *a* you put in?
40.	S:	Three.
41.	T:	Great, and you replaced *b* with two, and then you found that it was equal to five [writes: 3 + 2 = 5]. Good. Great. But if I now choose that *a* will equal 4, and then, oh, I want *b* to equal five [writes: a = 4 b = 5]. What do we get now? If *a* equals four and *b* equals five? Ulf?
42.	S:	Nine.
43.	T:	Because you replaced *a* with?
44.	S:	Four.
45.	T:	And what did you replace *b* with? And then, four plus five equals nine [writes 4 + 5 = 9]. Good…

The teacher shows two sheets of paper with the letter *a* on one of them and the letter *b* on the other. The teacher writes the expression $a + b$ on the blackboard, and introduces the term variables (37). She explains variables in this manner: ...letters can be variables and we can replace them with any number (37). Rather than further elaborating the concept of variable, she continues by giving the letters different values ($a = 3$, $b = 2$ and $a = 4$, $b = 5$) and calculates the sum in both cases (see Figs. 5.4 and 5.5).

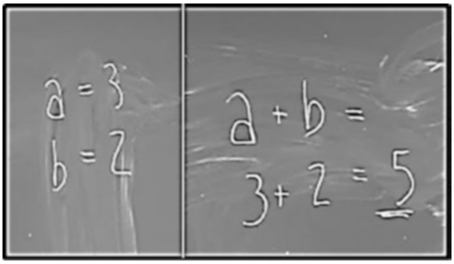


Fig. 5.4 Overview of what the teacher has written on the blackboard in Fragment 6

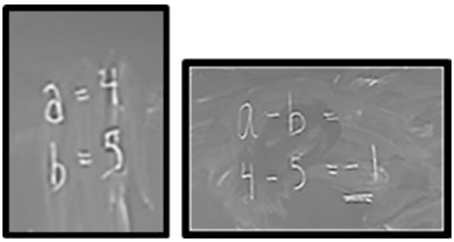


Fig. 5.5 Overview of what the teacher has written on the blackboard in the last part of Fragment 7

Fragment 7: Different Algebraic Expressions

45.	T:	… Great! What if I changed the expression to *b* minus *a*? [writes: b - a] *b* minus *a*. Ina?
46.	S:	Then *b* is five, so then (…) minus four, and that is one.
47.	T:	Great. *b* was equal to five, and *a* was equal to four. Five minus four equals one [writes: 5 – 4 = 1]. Good. If we did the opposite then, and used *a* minus *b*, how do we calculate that? Per?
48.	S:	Four minus five.
49.	T:	Yes. Then we get *a*, we replace *a* with four, and we replace *b* with five [writes: 4 – 5 =]. And four minus five, what is that? Ole?
50.	S:	Minus one.
51.	T:	It is minus one, yes [writes: -1]. Do you see that we have more negatives than we have positives? So then the answer has to be negative. It was the same with the cards, right? We added together, how many black we had, and how many red [the teacher uses her hands]... And the red was negative. So when we had most of those, we knew that the answer had to be negative. Good. Do you think this seems okay, or what? It wasn't so difficult to calculate with letters anyway. Was it difficult to calculate with letters, yes? Fortunately we will try this out now, because it is important to see how much of this you are able to do, and what I need to tell you more about.

At the end, the teacher varies the algebraic expressions and evaluates each one for the same set of values. It does not seem to confuse the students that the teacher again refers to the earlier use of the playing cards in relation to negative numbers. By introducing the two sheets of paper, the teacher extracts the letters *a* and *b* from the context of the cards. At this point she operates with independent letters and refers to them explicitly as variables.

The teacher continues the lesson by introducing the algebra game (see Table 4.1). The variables *a* and *b*, which have been replaced by different numbers, are now given additional meaning in the sense that they are explicitly connected to the number of eyes on the white and the red dice, respectively. The expressions on the game board determine how many steps a player can move in one turn. The evaluations of the expressions vary and depend on each throw of the dice.

Analysis: Classroom A

Our object of inquiry has been the teacher's introduction of algebra. More specifically, we asked: Which are the teacher's approaches when introducing the concept of algebraic expression? Although the teacher does not mention the textbook in the introduction, it is evident that she has considered it as she planned the first lesson in algebra. The algebra chapter opens with a repetition of numerical expressions with a focus on the order of operations, and it continues with an introduction to variables, algebraic expressions and then to equations. In line with the textbook's sequencing of algebraic topics, the teacher started by introducing the concepts of variable and algebraic expression, including demonstrations of simplification and substitution. But how she did this deviates from the activity in the textbook. Her mediating tool is a designed example, and we will focus our analysis on four concerns: (a) the manner in which the teacher introduces her lesson, (b) how she mediates her designed example, (c) what semiotic means she uses to introduce the concept of variable, and (d) how she interacts with the students.

Introduction of the Lesson

In the very beginning (Fragment 1), the teacher repeats the word algebra four times in order to emphasize the coming topic. Then she reminds the students that they have done algebra before (maybe without knowing it) in connection with an activity of adding positive and negative whole numbers by using playing cards. So, she brings the word algebra, calculation with whole numbers (arithmetic), and playing cards to the forefront of the students' attention. These are central elements in the coming activity with the designed example.

Mediating Function of Designed Example

The activity is based on a carefully designed example which is not taken from the textbook, but created by the teacher herself. Fragments 2 to 7 illustrate how the designed example is operationalized as a mediating tool for the teacher and the students in the learning situation. As well-known artifacts, the teacher uses playing cards and sheets of paper to bridge numerical and algebraic expressions. The approach used by the teacher to introduce algebraic expressions is intended to present the different components which constitute such expressions (numbers, letters as variables, operational signs) one by one. She is all the time linking the algebraic expressions with the numeric ones by substituting values for the variables and calculating the numerical expressions. So, together with the students she develops the concept of algebraic expression from numerical ones, as it is done in the textbook. The example plays several roles in the classroom, and we use Rissland-Michner's (1978) epistemological classes of examples to examine those. It is a start-up example as it motivates the basic algebraic ideas of performing operations on letters and that a letter can represent any number. The teacher's choice of designing an example to introduce the topic of algebra is also an effort to get the students' attention and a way of marking the shift from arithmetic to algebra.

In the first five fragments, the mediating tool of playing cards is used and the activity including them can work as a model example for letters representing numbers and the activity of developing expressions including letters. However, the letters on the playing cards are not variables but hidden numbers, and therefore the activity involving those cannot function as a model example for the notion of a variable. The teacher changes her mediating tool to sheets of paper featuring an *a* and a *b,* respectively, before discussing the concept of a variable which is first mentioned in Fragment 6. A student response shows that the transition from the playing cards and hidden numbers to letters as variables is not trivial (36): `Yes, we know that` *a* `is one (…) and` *b* `could be any number`. The teacher answers (37): `then yes, we have said here that` *a* `is one, but letters can be variables and we can replace them with any number`. The activity with the sheets of paper can play the role as a model example for the notion of variable as long as the difference between the two mediating tools is made clear. However, the shortcomings of playing cards as a model example may inhibit the conceptual development of students.

The designed example also plays the role of a reference example in the classroom as the teacher has used playing cards as the basis for an earlier activity. The cards are used again and in different contexts at the beginning of lessons 2 and 4. In lesson 2, the word variable is not mentioned in relation to the playing cards and the letters are assigned their numbers by the teacher: `We know now, I'm certain that you remember this now, but I write it anyway. J equals 11 and D equals 12`. The teacher then focuses on the rules of operations. In lesson 4, the cards are used to write an expression and again assigned their numbers however this time the teacher emphasizes that: `but such letters we call variables, right, we can put in almost what we want normally for letters, so that varies`. She lifts up the playing cards as a special case where the letters are assigned specific numbers. It is again clear that the activity with the playing cards does not function as a model example as it is not indicative of the general case.

Semiotic Mediation: Concrete Materials—Numbers—Variables

To interpret the steps taken by the teacher in a form of dialog when introducing variables, we use the concept of semiotic mediation to explain the passage between arithmetic and algebra, from numbers to variables. First of all she introduces playing cards, including letters and numbers, as a mediating tool. The material is suitable, as numbers and letters are part of the semiotic repertoire from algebra. Secondly, the teacher makes the values explicit for every letter (for example: $K=13$) as if the numbers are hidden. Thirdly, she introduces numerical expressions including numbers and operation signs and, then, fourthly, she calculates the sum. Only at the fifth step does she introduce algebraic expressions including both numbers and letters. Next, the teacher simplifies, adding similar terms and turning repeated addition into multiplication. Finally, she picks two sheets of paper with letters on (one *a,* and one *b*). She uses letters as independent entities. She substitutes *a* and *b* with different sets of values, objectifying in this way the idea of variables. And, at the very end, she varies the algebraic expressions as well. The teacher does not publicly elaborate on the concept of variable. In this example, the teacher is using numbers in a purely mathematical context. The variables *a* and *b* are substituted with numbers when first introduced in the classroom, but later, in relation to the algebra game, the variables are connected to quantities, i.e. the number of dots on the red and the white dice. Variables are, in this sense, therefore introduced in the classroom in an abstract, mathematical context and only later given a more concrete meaning for the algebra game.

Student-Teacher Interaction

Focusing on the teacher's role in the classroom interaction, we have identified the following steps:

- The teacher presents (verbally and visually) playing cards and two blank sheets of paper with only *a* and *b* written on them, respectively.
- She writes numbers, letters, operations and equal signs on the blackboard.
- The teacher poses mostly checking and controlling questions to the students.
- The students give short answers.
- The teacher writes the students' answers on the blackboard only if they are correct and proposed in a timely manner in order not to interrupt the flow of her presentation.
- The teacher openly asks (at the end): was it or wasn't it difficult to calculate with letters? And she announces that there are more exercises to come, so that she can see how much they can do themselves and what she needs to say more about.

We did not observe students taking notes or the teacher encouraging them to do so. The teacher seems to follow her plan for the presentation, and she keeps the students' attention by asking checking and controlling questions that mostly are returned with yes/no answers or facts. Even when a student gives an unexpected (but relevant) answer (as in turn 26), she neither comments on it nor follows up the possibilities it offers. Questions that require answers that the teacher has thought out in advance are, by Myhill and Dunkin (2005), referred to as closed questions, facilitating a procedurally oriented approach to teaching.

Episode 2: Classroom B

This episode is chosen from the very start of the first lesson on algebra in classroom B. The teacher, Ola, introduces the students to algebraic expressions and variables using body movements. The teacher continues the lesson with other examples involving expressions, and then the students work with tasks from the textbook. The textbook exercises will not be presented here. The flow of the lesson is described in the lesson graph (Table 5.2).

In fragments 1 to 8 we will present the full teacher-student discourse that takes place during the first 17 min of the lesson.

Table 5.2 Lesson graph showing classroom B (lesson 1, 42 min)

00:00 17 min	**Whole class** General information. New chapter: Algebra. Reading out aloud the goals for the chapter that are written in the textbook. The teacher does a demonstration of movement in the classroom and asks the students to describe what he did. The teacher writes it on the blackboard and comments that now they have done algebra. Then he adds to the procedure and writes it in the expression he is developing on the blackboard. The units he is using are "steps" and "foot". The teacher asks the students to write the expression in their notebooks, and elaborates on the notion of an expression. The teacher then asks the students if the expression can be written differently, shorter. He then demonstrates the movement again and takes note of where he ends up in the classroom. He simplifies the expression and then demonstrates that he ends up at the same place. The next step is that "skritt" and "fot" are shortened to *s* and *f*. The teacher asks the students that if one of them had done the movements would they end up in the same position, called *s* and *f* variables. The teacher then writes up the approximate sizes of his "skritt" and his "fot" in cm. The last step is to substitute those measurements for the *s* and *f* in the expression. The teacher then presents another example involving age differences. The students are given a task to discuss in pairs: a neighbor is 5 years older than me, write an expression that describes that the neighbor is 5 years older than me
16:42	**Student work**
17:14 9 min	**Whole class** The teacher asks the students for the solution. A student answers $x + 5$. The teacher reminds the students of a project they did some time ago involving wages. He gives another example, making an expression for how much someone would earn, working different numbers of hours, with an hourly wage. He also gives another example involving boxes of strawberries and making expressions. He refers to the *x* as an unknown. The teacher talks about the commutative property of multiplication and refers to the multiplication tables. Explains the invisible multiplication sign between the number and the letter
26:18 10 min	**Student work** In response to students' question, the teacher explains to the whole class the notion of a sum and a difference. The students work with tasks from the textbook *Faktor 1* (6.9–6.14). **6.10** Lotte is *x* years. Write an expression that shows how old (a) she was 5 years ago (b) she will be in 5 years
36:20 5 min	**Whole class:** Writes out the solution to some of the tasks from the textbook with the help of students

Fragment 1: Introduction of the Lesson

1.	T:	First of all, before I forget, you see that you have got a red folder on your desk, in front of you, everyone. You are to use that instead of your workbook this week, and you can write everything you do in school in it, and you can also take it home as a workbook and do homework in it. And then there are some cameras in here, but we are having a regular mathematics lesson, so nothing special besides that. And today we will start with a new topic, and that is Chap. 6, and it is called algebra. It is a chapter that we skipped that we are now going back to. If you turn to page one hundred and eighty-one, the one that looks like this, numbers and algebra it says. Then we will have a brief look at the goals for the chapter before we get started. In algebra we use letters as symbols for numbers. The value for the symbols can vary. Therefore we name the symbols variables. This may sound unfamiliar now, but we will talk a lot about variables in this chapter and what that means. Numbers that vary. And the goal for this chapter is that you will learn about simple, algebraic expressions, calculations with expressions or formulas and solutions of equations. This is the goal of the chapter. And here there were probably many unfamiliar words, but we will work with them in the following lessons this week, and I think we will use three or four weeks on this. Now everyone must pay attention to me and see what I'm doing now. It is a little, I will do it with my legs, but you have to see what I'm doing anyway. And then you will need to describe it afterwards. If I do this. Now you can see well enough. Are you ready?

The teacher starts the lesson with practical information regarding artifacts to be used (notebook, textbook), and he also informs the students to expect a regular lesson in spite of the cameras present. He introduces the new topic, algebra, by referring the students to Chap. 6, *Numbers and algebra*, in the textbook. He proceeds to talk about variables and introduces the term as follows: `In algebra we use letters as symbols for numbers. The value for the symbols can vary. This is why we call symbols variables.` Then the teacher again turns to the textbook and reads out aloud the goals for the chapter (see Fig. 5.6).

The teacher attempts to defuse the unfamiliar words by saying that they will be working with them over the next three or four weeks. The teacher continues by

Mål	Goals
I dette kapitelet vil du få lære om • enkle algebraiske uttrykk • regning med uttrykk eller former • løsning av likninger	In this chapter you will learn about • simple algebraic expressions • calculation with expressions or formulas • solving equations

Fig. 5.6 The goals for Chap. 6 in textbook Factor 8

introducing an activity, which we will follow in the next seven fragments. He informs the students that he is going to use his legs and that they need to follow closely what he does.

Fragment 2: Bodily Number Line; Unit (Step) and Direction

2.	S:	Yes!
3.	T:	What did I do now? Per?
4.	S:	You walked.
5.	T:	Yes, I walked. How did I walk?
6.	S:	Forward.
7.	T:	Forward. Yes?
8.	S	And then you first took three steps, and then two.
9.	T:	First I took three steps. Now I write exactly what you said. First three steps, and then two. Can I write plus then? [writes: 3 skritt + 2 skritt]. Will it be the same?
10.	S:	Yes.
11.	T:	I can write plus two steps…

The teacher performs a demonstration of body movements in the classroom. He walks three steps along a line in parallel with the blackboard, stops, walks two steps forward and stops again. Then he asks the students to describe what he did. The teacher formulates it as an expression on the blackboard. “Step” (“skritt”) is the quantitative unit he is using, and he writes out the word in his expression. Thus, the teacher is constructing an imaginary number line indicating unit and direction with his body.

Fragment 3: Doing and Speaking Algebra; Expression in Words (One Unit)

11.	T:	… Now you have to pay attention. Now I will do the same again. Now I will do exactly what it says there [walks]. What did I do in addition now? (…) what happened? Pia?
12.	S:	You took one step back.
13.	T:	Yes. First I did that, and then I took one step back. Can I write it as minus? [writes: 3 skritt + 2 skritt – 1 skritt]
14.	S:	Yes.
15.	T:	Now you have done algebra. Now I speak algebraically. Three steps plus two steps minus one step…

The teacher performs the same body movements again and adds one more step. He walks three steps forward, stops, walks two steps forward, stops, and walks one step back. Then he asks the students to describe what he did. The teacher adds another element to the former walking procedure and also includes it in the expression he is developing on the blackboard. He comments that now they have *done* algebra, and that he is *talking* algebra.

Fragment 4: Expression in Words (Two Units, Step and Foot)

15.	T:	… Did everyone follow what I have done now? Now I will do one more thing. (…) Now I first walked like this, and then I made it to here. And then I will do something quite smart here, but I will count aloud. One, two, three, and then I'm moving all the way up to the camera, four. What did I do now then? Was there any difference now?
16.	S:	(…) four.
17.	T:	I took four, what was it?
18.	S:	Steps!
19.	T:	Yes, but was it steps like before?
20.	S:	Half.
21.	T:	Half steps?
22.	S:	Feet? Foot lengths?
23.	T:	Feet? Mouse steps? Can I call it a foot? That was a nice example. Plus four feet. [writes: 3 skritt + 2 skritt – 1 skritt + 4 fot] Write it in your notebook. When you have written it, that expression, I call it an expression now. We can call every calculation task an expression…

The teacher makes exactly the same body movements as earlier and then takes some additional steps, and finally he takes some small steps in which he places one foot in front of the other (feet). He walks three steps, stops, walks two steps forward, stops, walks one step back, and walks four feet forward. Then he asks the students to describe what he did. The teacher again adds to the walking procedure and models it in the expression he is developing on the blackboard. The units he is using are "steps" and "feet"[2] ("skritt" and "fot"). The teacher then asks the students to write the expression in their notebooks. He elaborates on the notion of an expression by explaining that `We can call every calculation task an expression.`

Fragment 5: Simplification of Expressions; Letters for Words

23.	T:	… When you have written that expression, then you look at it and see if you can do something with it, so that it becomes a little shorter. Is it possible to shrink it so that it doesn't take up so much space? Write it differently? It is really another description so that I arrive at exactly the same position. Very open question. (…) Has everyone written it down? Good. Now I will walk down here, because there was not enough room here. Now I'm walking. First the three steps, two more, three plus two steps, one step back, and then four feet. One, two, three, four. Now I ended up about here. Ida, are you watching to make sure that it was right next to that stool?
24.	S:	Yes!
25.	T:	Three steps, plus two steps, minus one step. Is that the same distance as something else? Can I say that it is four steps?
26.	S:	Yes!
27.	T:	Three plus two steps, that is five steps total, minus the one [He gestures an equal sign after the expression 3 skritt + 2 skritt – 1 skritt + 4 fot]. Then I say that I have walked four steps and four feet. [writes: 4 skritt + 4 fot]. I start at the same place. One two three four one two three four. So, roughly, Ida. What is it?
28.	S:	You moved a little further.

[2] Not to be confused with the unit foot in the U.S. customary system of measurement.

29.	T:	I think I started a little further out. I should have marked where I started. But you agree that if I take four steps at once, or if I take five steps forward and then go back one, then I should end up at the same position. It is a little difficult to make exactly the same steps every time. Four feet, is it possible to shorten that expression? (…) Yes.
30.	S:	You can make feet into something else.
31.	T:	Is it possible to make feet into something else? Perhaps four feet are one step. It could be, but I don't know that. I have not measured it. So I don't know it. So I cannot do that. Perhaps I could have done it this way [writes: 4s + 4f]. But in addition I don't know how long a step is, or I don't know how long a foot is. Now I write down the equal signs [writes: = (after the expression 3 skritt + 2 skritt - 1 skritt + 4 fot); writes: = (between the expressions 4 skritt + 4 fot and 4s + 4f)], because now I see that this expression is equal to that expression. So, if you haven't written down the two expressions, then write them down in your notebook (…) It becomes a treasure hunt. One could have used a map, right? Then I could have said: four steps and then four feet and then you would end up where the treasure is, and then dig it out. I could have made infinite variations with this as long as the total was four steps in the end…

The teacher asks if the expression can be written differently, shorter. He then demonstrates the body movement again, now along a new "imaginary number line" between two rows of student desks, and takes note of where he ends up in the classroom. He simplifies the expression and then demonstrates that he ends up at the same place. He writes it on the blackboard. Then "steps" and "feet" are shortened to *s* and *f* (see Fig. 5.7).

The teacher simplifies the first expression (the first expression in the second line in Fig. 5.7). As a second step, he abbreviates the terms using only their first letters. The result is an expression that looks algebraic; however, the letters are still connected to the teacher's "steps" and "feet", and the expression is a mathematization of his movements in the classroom related to a specific distance. On the other hand, "steps" and "feet" are general terms which vary in lengths in relation to different people, and they can be talked about and operated on without first knowing their exact lengths. In this way, Ola develops the concept of variable from the students' everyday life. The teacher uses treasure hunt as a metaphor to explain the equivalent expressions. He does not comment on the contradiction the metaphor is to the development of the concept of variable; such a map is supposed to lead the readers to the same position i.e. all steps and feet are of equal length.

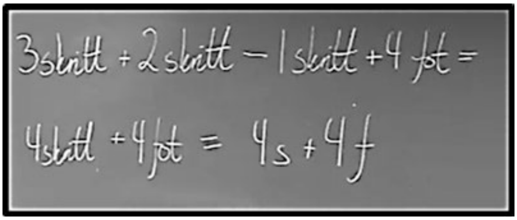


Fig. 5.7 Overview of what the teacher has written on the blackboard in Fragments 2 to 5

Fragment 6: Expression as Recipe; Variable and Constant

31.	T:	… Would it be the same if I do it as if… eh, Kai, if you do it? Would we end up at the same spot?
32.	S:	No.
33.	T:	Why not?
34.	S:	Your steps are longer than mine.
35.	T:	Are my steps longer than yours? Yes, Mia, would we end up in the same spot?
36.	S:	No.
37.	T:	Odd, me and you then? You are pretty tall.
38.	S:	Yes, that could be.
39.	T:	Could you say approximately?
40.	S:	Yes…
41.	S:	You and Pål.
42.	T:	Perhaps me and Pål, yes. And it is like this, someone might be walking a little like this, while another person takes shorter steps. And feet also, perhaps different shoe sizes. So that number, it can vary. Right? This number and this number, we call them a variable. [writes: variable] Because it doesn't need to be the same every time. It depends who is doing it, whether it is Kai, Mia, Oda or Pål, so it will never be exactly the same. A little different. But that, it is a constant. It is the same every time. The recipe is four steps plus four feet, no matter what…

The teacher asks the students whether, if one of them had done the walking, they would end up in the same position (31), and they answer No (32). The teacher asks for an explanation (33), and one student answers that the teacher's steps are longer than his (34). The teacher elaborates on the variations of the length of peoples' steps (35), and ends the discussion by labeling *s* and *f* as variables and 4 as a constant (see Fig. 5.8).

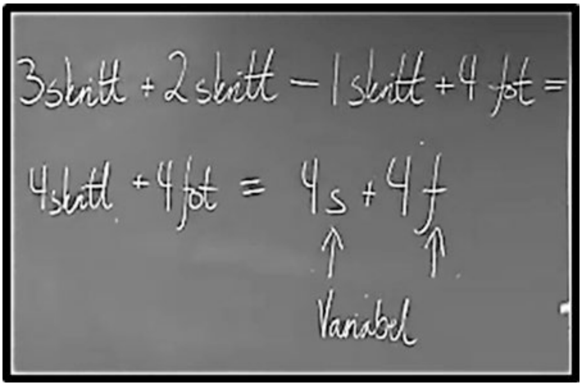


Fig. 5.8 Overview of what the teacher has written on the blackboard in Fragments 2 to 6

The teacher approaches the concept of variable by comparing steps and feet of students with his own. The students agree that the length of steps and feet varies among different people. To further exemplify the unique characteristic of variables, i.e. that they can take on different values, he points to the number 4 and names it a `constant`. He also makes an analogy between the expression and a recipe to further explain what remains the same and what varies; the constant and the algebraic expression in its entirety stay the same, while the variables *s* and *f* vary with the person who walks. Again, Ola relates expressions to a known concept from students' everyday life.

Fragment 7: From Algebraic to Numerical Expression; Letters to Values

42.	T:	… And then we will end up at different positions because the variables, which are the length of legs or lengths of the steps, are different for each of us. I could have made calculations if I say that one step for me was seventy centimeters, and one foot was twenty centimeters [writes: s = 70cm f = 20cm].
43.	S:	You need to know the shoe size?
44.	T:	The shoe size?
45.	S:	It is different.
46.	T:	Yes, but it is not marked in centimeters.
47.	S:	It is!
48.	T:	It is?
49.	S:	Yes!

50.	T:	Now I will not look so carefully at it. Twenty centimeters, I say that it is twenty centimeters, I have not measured, but yes, it is probably not enough.
51.	S:	It is forty-two.
52.	T:	If we say four steps plus four feet [writes: 4s + 4f], four, then I must multiply by seventy, right? Because I take four steps which are seventy centimeters plus four feet which are twenty centimeters [writes: 4·70cm + 4·20cm]. And then we have really made a calculation task of it. We have replaced the variables, replaced the *s*, because we know that one step was seventy centimeters for me. If it had been Mia, it would have been something different. Perhaps it would have been sixty, fifty. (…) Do you follow so far? Now I write an equal sign (.). Write it down in your book (…). [writes: 280cm + 80cm = 360cm] Two hundred and eighty centimeters. I walked with the four steps, plus the eighty centimeters I walked with the four feet. Did you follow that, the distance from here and down became about three hundred and sixty centimeters? Any questions? …

The teacher suggests approximate lengths of his "steps" and "feet" in centimeters (cm) and writes these on the blackboard. This provokes a student to request further accuracy and he suggests using the size of the teacher's shoe (43). Ola explains that the shoe size is not marked in centimeters but fails to convince the student. The teacher continues by substituting *s* and *f* for the given values in the expression. He bridges the algebraic and the numerical expressions by connecting the values to his prior walk. He also underlines that *s* is a variable by saying that the value would have been different if Mia had been walking (52). Ola calculates the numerical expression in two steps and continues by making explicit the connections between the numbers on the blackboard, the lengths of his steps and feet, and the previously marked distance in the classroom. The distance that was previously measured in steps and feet (Fragment 5, 25) is now described by the standard unit cm (see Fig. 5.9).

For the first time the teacher is not waiting for the students to respond to his question: `Any questions?` (52). He continues the lesson, introducing a new example presented in Fragment 8.

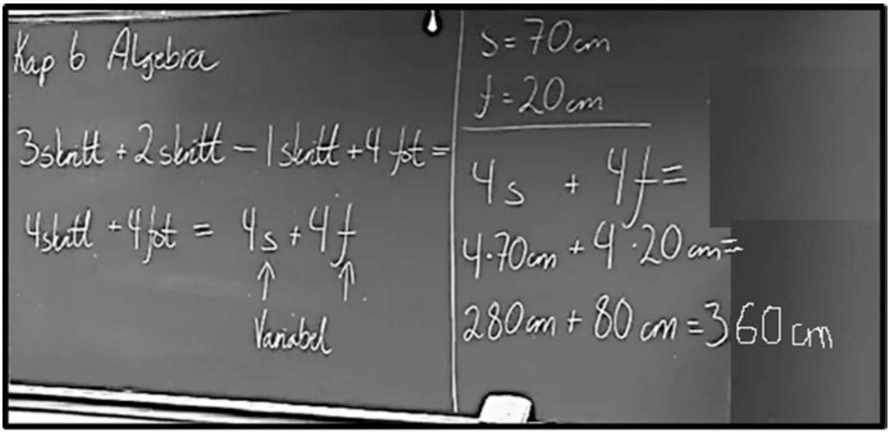


Fig. 5.9 Overview of what the teacher has written on the blackboard in Fragments 2 to 7

Fragment 8: From Expression and Recipe to Formula—Same Approach

52.	T:	… Now I'm starting to erase this, but you have written it down, so that will be fine. I have a little sister. She is seven years younger than me. Does anyone remember how old I am? Last time you guessed that I was about thirty-five, which was pretty rude. Do you remember?
53.	S:	Twenty-six.
54.	T:	Yes, that was rather young, which was nice of you. But I'm a little older. I'm twenty-seven.
55.	S:	Big difference!
56.	T:	Big difference. But I said that she is seven years younger than me. If I say that I'm twenty-seven, then you can easily calculate that she is…?
57.	S:	Twenty.
58.	T:	Twenty. Yes. Twenty-seven minus seven. Next year, how old am I then? You can also calculate that. Tonje?
59.	S:	Twenty-eight.
60.	T:	Yes, how old is she then?
61.	S:	Twenty-one.

62.	T:	Yes, twenty-eight minus seven. So, it is really my age minus seven [writes: min alder - 7]. Do you agree? This is the formula all the time. The expression is like this. And then I get my sister's age as an answer [writes: = søsters alder]. If you didn't know how old she was - now you knew approximately - then we could have said that my age was unknown, you don't know what it is, we could have called it *x* [writes: *x*]. Mister *x*. *x* is the, eh, letter we normally use in algebra, most often, when we have an unknown, something that we don't know the value of. So really it just means that we don't know that number. We only know that it is a number. But we know that my sister was seven years younger than me [writes: -7], and then you are able to calculate anyway. Should we put in twenty-eight? That becomes twenty-eight minus seven. Now I'm fifty, fifty minus seven. (…) Now you will get the next task. That one you will (…), now you are sitting two and two. You can turn around to the person sitting next to you also. I have a neighbor, a good neighbor; we live right next to each other. He is five years older than me. Can you try, two and two, to write down an expression like the one here which describes that my neighbor is five years older than me? You can talk across to each other here. You will not get much time.

This example is chosen from the textbook and mathematizes the relationship between the ages of two people. It is a regular paper and pencil problem that the teacher personalizes by using his own and his sister's ages as variables. He first models the relationship with an equation including words ("min alder", "søsters alder"), naming the expression on the left side of the equal sign a formula for his sister's age (see Fig. 5.10), which he then expresses in terms of the letter x. However, he first establishes a constant relationship between the ages by performing calculations with specific values for his own age. Then the lesson continues with the students working, in groups of two, on modeling a proposed problem of the same nature.

The teacher reconnects with the textbook by discussing one of its examples. There is a correspondence between the two introductory examples. Again, Ola writes the algebraic expression rhetorically before using letters as variables. And here, too, the variables represent quantities, which in this case have the inherent quality of changing with time. However, the two examples conceptualizes algebraic expressions differently. In the latter example, the two quantities are connected by a constant, and if one of them is given, the other can be found. This example features a special kind of algebraic expression in the context of an equation. The teacher also briefly talks

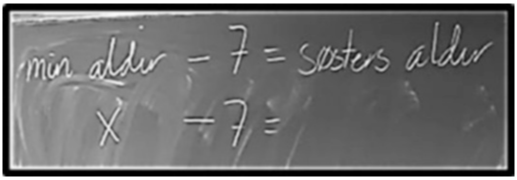


Fig. 5.10 Overview of what the teacher has written on the blackboard in Fragment 8 (*my age—7 = sister's age*)

about the *x* as an unknown, even though the task is focused on writing expressions. There is also a difference in how the two examples are mediated. In the latter example, Ola goes from the specific by comparing specific ages, to the general, writing an algebraic expression. While in the first example he did the opposite, moving from the general to the specific.

Analysis: Classroom B

Our object of inquiry has been the teacher's introduction of algebra. As was the case in the analysis of classroom A, we asked more specifically: Which are the teacher's approaches when introducing the concept of algebraic expression? Unlike Kari, Ola starts the lesson by having the students open the textbook to the new chapter called *Numbers and algebra*, and then he reads aloud the learning goals written there. He proceeds by introducing the same topics as Kari: the concepts *variable* and *algebraic expression*, and he also provides demonstrations of simplification and substitution. And, as Kari did, he designs his own example as a mediating tool. In order to compare the two designed examples, we will look at the same analytical concerns as previously presented: (a) the manner in which the teacher introduces his lesson, (b) how he mediates his designed example, (c) what semiotic means he uses to introduce the concept of variable, and (d) and how he interacts with the students.

Introduction of the Lesson

The teacher starts his lesson by giving practical information. He reminds the students about writing notes in the special folders handed out, and he urges them to write down everything done in school for later use, e.g. when doing homework. He also reassures the students regarding the cameras present, emphasizing that it is a regular lesson. He then continues by announcing that they will start with a new topic called algebra, and refers to the page and chapter in the textbook where the topic is presented. He points to the goals for the chapter, but before reading them aloud he explains the role of letters in algebra and why they are called variables. Reading the goals in the textbook aloud, he lists the contents to be dealt with in the coming three to four weeks, that is simple algebraic expressions, calculations with expressions or formulas, and equation solving. Finally, he asks for the students' attention, telling them that he will move his legs and that he expects them to observe carefully so that they can describe what he did subsequently. In short, the teacher prepares the students for his introduction of algebra in the coming weeks and finally for the lesson of the day. He shows consideration for the students' feelings as they are asked to participate in a new domain of mathematics, and he is explicit about what he expects from them, both in taking notes and in interacting with him in a specific way. The textbook has a clear role in the classroom and dictates the content of the lesson. However, the teacher adds what he thinks is important and his designed example shows an attempt to facilitate learning by relating to everyday experiences.

Mediating Function of Designed Example

The activity presented in fragments 2–7 is based on a designed example, which is not taken from the textbook, but created by the teacher himself. The teacher uses walking procedures to establish successively more extensive expressions, as he moves along an imaginary number line. These algebraic expressions are developed in interaction with the students, verbally and by writing the expressions in words (the units "step" and "foot") on the board and in their notebooks. In the same manner, a simplification of the final expression is also completed before the words are shortened to letters (*s, f*). Finally, in Fragment 7, the letters are given values, bridging algebraic and numerical expressions. The teacher uses the terms *expression*, *recipe*, and *formula* synonymously.

In moving along an imaginary number line, using his steps and feet to designate distance, the teacher operates with quantities and develops an understanding of variables and operations without using numbers and numerical expressions as motivators. Instead, he models the distance walked in the classroom with an algebraic expression that builds on relationships between quantities. He is therefore touching what Davydov, Gorbov, Mikulina, and Savaleva (1999) do in their approach to algebra in school, where algebra is introduced through working with quantities. The Russian curriculum developed by Davydov and his colleagues introduces algebra and its symbolism from first grade with numbers following as concrete applications of algebraic generalizations (Schmittau & Morris, 2004).

After having presented the concept of algebraic expression in terms of quantities, the teacher shows that numerical expressions are special cases of these. At the very end (Fragment 8), the teacher connects his way of introducing algebraic expressions with how it is done in a standard textbook problem, and thus bridges his own presentation and that of the textbook.

As Kari's designed example, Ola's example plays several roles in the classroom. It is a start-up example that motivates the use of the concepts *algebraic expression* and *variable*. The example signals a shift in the mathematics classroom in modes of working where numbers no longer have the central role and are replaced by letters as variables. The activity with the mediating tools of steps and feet can work as a model example for the concept of variable and also for building algebraic expressions and operating with letters. The image of how the length of steps and feet vary between different people is a very tangible reference for the meaning of variable.

The movements of the teacher in the classroom are described and written down as an expression of physical length in the classroom. The teacher walking a simplified version of the expression, and showing that it corresponds to the same length as the first one, is a physical demonstration that one can operate with letters (quantities). The example is also a reference example as the teacher uses this example again in Lesson 3, having a student perform the movements this time. The teacher marks the length in the classroom which corresponds to his own movements, and then he does the same for the student and shows that the lengths are different even though they made the same movements. The activity thus becomes a model example also in the sense that an algebraic expression can represent different lengths.

Semiotic Mediation: Body Movements—Quantities—Variables

To interpret the steps taken by the teacher in a form of dialog when introducing variables, we use the concept of semiotic mediation to explain the passage from body movement to variable. Firstly, the teacher demonstrates, by a walking procedure, an imaginary number line with direction (forward) and a unit (step). Movement (walking), is thus the first semiotic element that emerges. Secondly, the teacher repeats the students' description of his movement, and he formulates it as an expression on the blackboard (inscription). The semiotic means are spoken and written words, numbers, quantity (step), and plus and minus signs. Thirdly, the teacher adds a new unit (foot) first in his walking procedure and then on the blackboard. Fourthly, the teacher starts to operate with all the symbols and signs in order to simplify the algebraic expression. At this point the teacher abbreviates step and foot with the letters s and f. Fifthly, he points out s and f as variables. The teacher makes a connection between the concept of variable and letters in algebraic expressions through his inscriptions on the blackboard. So, altogether, he has made a passage from steps and feet as measuring units to seeing these units as variables, i.e. step, foot $\rightarrow s, f \rightarrow$ variables, mediated through body movements, spoken and written words, and inscriptions. At the end the teacher attributes values to letters/variables (measured in cm), showing that a numerical expression is a special case of an algebraic expression.

Student-Teacher Interaction

The teacher's way of interacting with the students goes through the following steps:

- The teacher walks back and forth along an imaginary number line indicating direction and units (step and foot).
- The teacher poses checking and controlling questions to the students.
- The students answer the specific questions from the teacher, and they describe the teacher's movements.
- The teacher writes the students' description of his walking procedure on the blackboard.

- The teacher and the students interact in dialogical modus at the end of the lesson (from Fragment 6).
- The students copy in their notebooks what the teacher has written on the board.
- The teacher summarizes and concludes what they have achieved together.

The teacher keeps the students' attention by asking checking and controlling questions and by challenging them to describe his movements. The last part of the episode is more like a dialog. They are asking and answering each other's questions. The teacher is specifically following up some of the students' responses (30, 43), but only for a short time before he brings the class back on track again. Only once does the teacher rush on to the next task without waiting for the students' response (52), which he had invited them to give. Following Myhill and Dunkin (2005), Ola used both closed questions, mostly returned with yes/no answers or facts, and open questions that invite the students to explore and investigate, the latter type facilitating a conceptual approach to teaching.

Comparison

From the interviews with the teachers, it is obvious that both teachers are interested in developing their mathematical competencies. They are also concerned with having a practical approach to the teaching and learning of mathematics, i.e. connecting mathematics to the everyday life of students. The female teacher, Kari, has 9 years of experience, and she is a coordinator between the leadership and the mathematics teachers in the school. The male teacher, Ola, has 4 years of experience, and he was at the time of data collection taking additional courses in mathematics. Ola was also involved in a national project which focuses on low-performing students in mathematics. Both teachers refer to professional development courses they have attended when explaining their viewpoints. In their mathematics teaching, they use the textbook, but often also other resources such as games, booklets, books, internet and digital tools (spreadsheets and GeoGebra).

Our focus has been on how the teachers introduce and mediate algebra through their designed examples. Their introductions of algebra have been analyzed with specific focus on four issues: introduction of the lesson, the mediating functions of designed examples, semiotic mediation in relation to the concept of variables and student-teacher interaction. In this section, the two approaches to introduce the concepts of variable and algebraic expression are compared.

Introduction of the Lesson

The two teachers introduce their lessons quite differently. Kari (the female teacher in classroom A) brings the word algebra, calculation with whole numbers, and playing cards to the forefront of the students' attention, providing only the information they need in order to follow the planned activity based on her designed example. Ola (the male teacher in classroom B) has a broader approach in his starting lesson. He introduces the new terms the students are expected to become acquainted with in the coming weeks by reading the goals of the algebra chapter, before narrowing down and focusing on the concepts for the actual lesson of the day.

Kari does not mention the textbook in the introduction of the lesson, and she does not bring it up after presenting the designed example. Instead, she provides copies of an algebra game as a group activity where the students work with the new concepts. Ola introduces his lesson by showing the students where they are in the textbook, and he points out what the students are expected to learn in the coming lessons by referring to the learning goals provided there. After the activity based on his designed example, he returns to the textbook by presenting one of its examples, and then the students work individually with tasks in the textbook. Although Kari does not use the textbook directly in her lesson, the content is the same as in Ola's lesson, and the same as the first algebraic topic in the textbook. In the interviews, the teachers do not refer to the national curricula when commenting on their teaching of algebra. The textbook is therefore interpreted as the enacted curriculum in the two lessons.

The role of affective factors in the learning of mathematics has been documented in research. It has been consistently shown that while confidence has a positive correlation with mathematical performance, mathematical anxiety has a negative effect (Schoenfeld, 1989). In the introductions of a new topic, both teachers address this aspect of learning. Kari only articulates a positive emotion when she says that she knows the students `have been excited` (Classroom A, Fragment 1) to start with algebra. However, as she continues it becomes clear that she has been preparing the students for this coming topic, i.e. she has previously asked them whether anyone has had any experiences with algebra. And now she reassures them that they probably have done a lot of algebra before without knowing it. She then proceeds to mention a specific example related to cards and negative numbers. In this manner, she attempts to address the feelings of anxiety students may have when a new branch of mathematics is introduced in the classroom. Ola also shows awareness about the issue of students' anxiety when encountering new concepts and terms. He says that now these words may appear unfamiliar but assures them that they will work a lot with them in the coming weeks.

Kari moves ahead with her designed example without making explicit her expectations about the students' role in the activity. Ola, however, explains what he is expecting from the students: they should take notes during the activity and observe, in order to describe afterwards, his exact movements as he walks in the classroom.

The Mediating Function of Designed Examples

Both teachers have chosen to follow the topical sequencing of the algebra chapter in the textbook and to introduce variables and algebraic expressions during the first lesson. However, they do not follow how this is done in the algebra chapter. Instead, they design their own examples as mediating tools in their presentations.

Kari has chosen playing cards, which the students are familiar with (she has used them earlier for adding whole numbers), in order to illustrate that letters can stand for numbers and be included in mathematical operations. The playing cards and two sheets of paper with the letters a and b respectively, are used to carefully develop algebraic expressions from numerical ones. Therefore, Kari's introduction is interpreted as an inductive approach to algebra that reflects the textbook's view of algebra as generalized arithmetic.

In the interview, Kari justifies her way of introducing algebra by underlining the importance of a `practical` approach. `So basically, I always think that I will follow the textbook, but this time I found the textbook's presentation of the algebraic concepts problematic`. She continues by explaining that she wanted to be `more practical`, as opposed to being theoretical, in her approach, especially since it is the students' first encounter with algebra: … `I said in the beginning also that I felt like, as a first approach [to algebra; as done in the textbook], that it was perhaps not the one I would have chosen, that it is very theoretical… in a way a little abstract in that they had not put it in an everyday context… but then again algebra [as a subject] is abstract. It is (.) but still I think the way it is done [in the textbook] is perhaps too theoretical and abstract in the way they have presented it. This is my initial impression.`

Kari explains that she therefore looked elsewhere for ideas: `and therefore I used something I have found in mathematics journals`. She mentions Tangenten, a Norwegian journal for mathematics teachers, as the inspirational source for using playing cards as concrete objects in her designed example: `When I read it I thought it looked like a nice approach and I decided to test it to see if it works`. Kari says that she does this a lot: `I have used ideas from different professional courses and from the specializing in mathematics (.) ... if I have learned something new I test it out and use it.`

Ola carefully establishes an imaginary number line with direction and two different units (step and foot), which allows him to introduce addition and subtraction of these quantities without using numbers. In their responses, the students appear to be able to follow the rather complex reasoning of the teacher, and we conjecture that this is facilitated by his pedagogical choices of mediating his ideas. That is, choosing the familiar activity of walking, involving the students in describing his movements, and writing their responses on the blackboard. In addition, he encourages the students to copy from the blackboard into their notebooks. Thus, Ola introduces algebraic expressions directly in terms of mathematizing a situation involving distance without first involving numerical expressions. At the end of his presentation he shows that numerical expressions are special cases of algebraic expressions. Ola's designed example is therefore interpreted as a deductive approach. In the last fragment from classroom B, Ola presents a new example inspired by the textbook as a part of his introduction. In this way he connects his designed example to the textbook's presentation of variables and algebraic expressions.

In the interview, Ola (like Kari) justifies his way of introducing algebra by underlining the importance of a practical approach. `I try to use examples that are realistic and that they can understand (.) Present examples that they can associate with daily life.`

Ola did not use any of the introductory examples in the textbook as he thinks they are too theoretical and artificial: `When I'm planning to teach a topic I tend to look in the textbook (.) I look at how it is organized, and I look through the examples to see if they are good (.) and decide whether I want to use any of them (.) This time I chose not to do it as in the textbook (.) I slightly rearranged the sequencing of concepts as I felt it would be a more appropriate way of doing it (.) Eh, additionally I have tried to use several other examples that are not in Faktor (.) I think the examples are too theoretical and artificially constituted, and they did not work for me as I wanted to find something that students can understand and that involves algebra, which is often a difficult and somewhat vague topic for them.`

Following the epistemological classification of Rissland-Michner (1978), both examples play several roles in the classroom; as start-up examples in which they introduce the definitions of variable and algebraic expression; and as reference examples since they are referred to in different contexts. However, the activity with the mediating tool of playing cards, in which the letters represent fixed numbers, cannot work as a model example for the concept of variable. The activity with the sheets of paper, which could play this role, seems to lack permanence (in the classroom) as they are only brought out briefly and not mentioned again. In contrast, the playing cards are familiar objects to the students; they are used repeatedly in the classroom, and have a central role in the designed example. Thus, the designed example dominated by the playing cards may not work effectively as a model example for the concept of variable. Ola's movements in the classroom, and the descriptive quantities of step and foot, which vary between different people, have qualities that correspond with variables and algebraic expressions. When used in the third lesson as a reference example, in which a student also walks in the classroom, it is used to deepen the understanding of algebraic expressions. Ola's designed example has the qualities of a model example in that it is indicative of the general nature of variables and algebraic expressions. This can be summarized by Mason's (1996) well-cited phrase of "seeing the general through the particular."

Semiotic Mediation of the Concept of Variable

The paths the teachers follow in order to develop the concept of variable are quite different. Kari starts by introducing playing cards, a concrete object, including letters and numbers, as a mediating tool. On the other hand, Ola walks along an imaginary number line, and it is the body movement that are intended to mediate the algebraic concept. The mediating tools the teachers use, concrete objects and the body, have different representational qualities involving affordances and constraints; in Kari's example the playing cards are only an entry point and the letters are not truly variables but hidden numbers (that do not vary); in Ola's example the movements in the classroom remain a visual demonstration of the work done on the blackboard, and the units step and foot used to describe the movements are variables from the very beginning. As we can see, the teachers use different semiotic means as starting points.

Secondly, Kari identifies hidden numbers represented by the letters in the cards. Immediately following this, she introduces numerical expressions from the hand of cards and then calculates the sum. Ola writes an expression on the blackboard related to his walking procedure. He introduces two units (step and foot). Ola takes account of the semiotic elements of an algebraic expression and operates with the letters (simplification). At this point Ola abbreviates step and foot as *s* and *f*, and then calls them variables (he also writes variable on the blackboard). He has made a passage of units: step, foot → s, f → variables. However, Kari follows the opposite approach, introducing the semiotic elements one by one, and develops algebraic expressions from the numerical ones.

Kari introduces algebra using playing cards, substituting letters for numeric values. The example works for evaluating expressions and for combining terms. However, the letters in this case are known values that do not vary. As an introductory example, they have the potential to create misconceptions among students if the teacher does not reflect on the concept of variable. There seems to be very little effort made by the teacher to address this possibility. In Fragment 6 (37), the teacher explains that `we have said here that` *a* `is one, but letters can be variables and we can replace them with any number`. This is a critical clarification in need of further elaborations.

As we have seen, these two teachers follow different semiotic pathways to introduce variables. Kari goes from the elements to the expression using an inductive approach, while Ola takes into account the whole expression, not pointing out the various elements until the end. We can characterize this as a deductive approach. Another difference is that while Kari's designed example only operates with numbers as mathematical objects in themselves, Ola operates with quantities and numbers that have a meaning.

Student-Teacher Interaction

When we compare these two lessons, some interactional features emerge clearly. For example, in the case of Kari, she made a lesson plan, and followed it carefully, using checking and controlling questions which require yes/no answers. Meanwhile, Ola also follows his plan and questioning mode. However, in addition, he also asks students to describe his movements, and they do that. In this way Ola goes further in the interaction by asking questions which require a more descriptive answer. He involves the students in the sense of stimulating them to find adequate words and to intuitively grasp the number line idea (including direction and units).

Both teachers use the blackboard for their inscriptions. Kari shows large playing cards and writes the numbers and letters included in the illustrations on the board. She asks the students what the hidden values of the letters are, and, after a brief discussion, she writes them in a separate place on the blackboard. She adds algebraic signs to the row of numbers and letters in order to create a numerical expression and then calculates the sum. Ola writes the students' answers systematically on the blackboard, thus acknowledging their contributions, and he also asks them to write what he has written on the blackboard in their notebooks. In Kari's class, the students are not required to make notes.

Both teachers initiate their lesson by speaking at length, leading and monitoring the discourse (an asymmetric interactional pattern). In Kari's case, this pattern continues throughout the whole episode. However, Ola changes his pattern by inviting more student input (transforming the relation into a more symmetric one) towards the end.

Conclusion

The aim of this analysis has not been to propose how algebra should be introduced in the classroom, but to carefully study how it is done in two specific cases. The analysis illuminates the complexity students meet when facing introductory algebra in school, and the challenge it is for teachers to make algebra accessible for all students. The main approach of the teachers has been to design and use examples, mediating the passage from the students' real world experiences, as well as the school mathematics they know, to algebra. There are similarities and differences in these teachers' ways of introducing algebra. We would like to close this chapter by answering the research question and pointing out implications for teaching: *Which approaches do the teachers use when introducing the concept of algebraic expression?*

The two teachers both design introductory examples that are used as their central means for explaining the same mathematical concepts (variable and algebraic expression). The examples are easily distinguishable in their use of concrete materials (playing cards versus the body); however, there are more fundamental differences in the example structures. Kari starts with numbers, number operations, and numerical expressions, and, based on their prior work in class, she makes generalizations introducing algebraic expressions. She continually connects the numerical and the algebraic elements, and explains variables as numbers. She goes from the specific to the general, and follows an inductive approach to introduce algebraic expressions (which adheres to the way it is done in the textbook). Ola, on the other hand, establishes an algebraic expression directly from the imaginary number line with given direction and units (first step, then foot) without using numbers. He builds the algebraic expression through a transformation chain following this path:

bodily movement—words—abbreviations—variables, and he sees variables as quantities. We are instantly immersed in algebra in general, with very little abstraction. Later he shows that numerical expressions are specific examples of more general algebraic expressions (Classroom B, Fragment 7). In this manner, he moves from the general to the specific. Therefore, Ola is following a deductive approach when introducing algebraic expressions.

In a summary of research addressing the teaching and learning of algebra in the elementary grades, Kieran (2007a) points out that the majority of this work is situated within the curricular approach of developing algebra from the experience of numbers and their operations. This body of work is mainly concerned with issues of how to engage students in the early grades in algebra thinking, with or without introducing formal algebraic notations. The Russian curriculum developed by Davydov and his colleagues represents an alternative position in which the study of algebra precedes the study of numbers and introduces algebraic notations in first grade. Schmittau and Morris, reporting on their study of Davydovs' curriculum and their adaption and implementation of it in a US school setting, explain that "algebra is developed from an exploration of quantitative relationships" (Schmittau & Morris, 2004, p. 61). As mentioned earlier, it is in this aspect that Ola in his designed example (unknowingly) touches the ideas of Davydov as he develops the algebraic concepts from quantities and not from numbers. In addition, introducing concepts by going from the general to the specific is also a trademark of the Russian curriculum, which is what Ola does in this specific example. The implications for student learning later in the curriculum are beyond the scope of this chapter but certainly in need of further exploration.

The teachers' strategy of designing their own examples as a first introduction to algebra has specific implications for their teaching. The teachers' use of concrete objects and body movements can make introductory algebra accessible to students by linking students' observations of real world activities to school mathematics. The examples themselves are well thought through. The teachers do not run into unforeseen limitations regarding their use of concrete objects and body movements, as is illustrated in the chapter from Sweden (see Chap. 4). Even if the presentations of the examples are well prepared, the teachers' flexibility is challenged when meeting the students' questions/answers. In her interaction with the students', Kari mainly uses questions that emphasize a procedurally oriented approach to teaching, and Ola mainly uses questions that emphasize a conceptually oriented approach.

Bills et al. (2006, p. 10) point out that "the art of constructing an explanation for teaching is a highly demanding task." In order to create the designed examples, it is evident that the teachers have reflected on the content they are about to teach and what they want the students to learn. The strength of Kari's example is the familiarity of the playing cards as she connects to the prior experiences of the students. While Ola's insistence on involving the students in developing his example is important as he directs the attention of the students to the creative process of building an algebraic expression and the meaning of its elements. Our data show that these examples become anchor points for the teaching of algebra in the sense that the teachers return to the examples in the following lessons. This in turn may be a product of the teachers' personal investment in the examples.

Our analysis is a response to what stood out as characteristic of the Norwegian classrooms in the international data of algebra teaching. We have discussed in detail the elements of the designed examples. The analysis has provided insights into the complexity of introducing algebra in the classroom. However, the phenomenon of teachers designing their own examples especially fitted for their classrooms deserves further attention from the research community: the role of this activity in the professional development of teachers; and which role these types of examples, with specific epistemological qualities as in start-up, model and reference examples, should play in student's concept formation. Discussing and comparing the epistemological underpinnings of examples as mediating tools in the learning process seem to be critical for advancing a shared understanding among teachers of how algebraic concepts can be made accessible to students.

Chapter 6
Learning to Solve Equations in Three Swedish-Speaking Classrooms in Finland

Ann-Sofi Röj-Lindberg and Anna-Maija Partanen

Introduction

The basic premise for this volume is that learning to solve equations is a critical moment in the school mathematical experiences of a student in many educational systems. A student may perhaps be able to figure out the missing value in an equation written as an open number sentence, but without understanding how and why the standard algorithm works for solving the very same equation. This particular situation has been shown to be associated with how student understand equality (Falkner, Levi, & Carpenter, 1999; Kieran, 1981; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2011; Knuth, Stephens, McNeil, & Alibali, 2006; Vieira, Gimenez, & Palhares, 2013). Successful equation solving is connected to a relational meaning of the equal sign and to understanding the notion of an equation as a statement about an equivalence structure (Stacey & MacGregor, 2000).

In this chapter, we draw on video recorded episodes occurring in three Swedish-speaking Grade 6 classrooms in Finland. We follow what happens when the three teachers Anna, Bror, and Cecilia in different ways try to account for the solving of arithmetic equations, where the unknown appears on one side of the equal sign.[1] We introduce the viewpoints of the teachers, and we will also reflect on how the teachers could have utilized the instances that appeared in the classrooms to bring their students' understanding of equality further. And finally, we comment on the approach to learning equation solving on which the textbook series used in the three classrooms seems to rely.

[1] The term arithmetic equation here used as proposed by Filloy and Rojano (1989).

A.-S. Röj-Lindberg (✉)
Faculty of Education and Welfare Studies, Åbo Akademi University, Vasa, Finland
e-mail: aroj@abo.fi

A.-M. Partanen
Faculty of Education, University of Lapland, Rovaniemi, Finland

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Three Perspectives on Teaching for Learning Equation Solving

In the literature, it is possible to discern three general perspectives on how to teach mathematics in general. These are (a) a procedures-first perspective, (b) a concepts-first perspective, and (c) a balanced approach. According to Rittle-Johnson, Siegler, and Alibali (2001) the procedures-first perspective and the concepts-first perspective are closely connected to the sharp distinction that traditionally has been made between two types of mathematical knowledge, procedural knowledge and conceptual knowledge (see, e.g., Hiebert, 1986), and to the competing theories that have been proposed regarding the developmental relations between these two types of knowledge.

From a "procedures-first perspective," a student who first develops procedural knowledge for equation solving, will over time develop a conceptual understanding from repeated experiences of solving different types of equations. Simply put, the assumption is that the student's understanding of the mathematical concepts will eventually emerge as the student grapples to make sense of why the procedures work, for instance by comparing solution methods to equations (Rittle-Johnson & Star, 2009).

From a "concepts-first perspective," the student's learning should start from developing conceptual knowledge, that is, knowledge of *why* procedures for solving equations work as intended. As this type of knowledge is flexible, and not tied to any specific type of equation, it is assumed that the student will be able to apply his or her conceptual knowledge to generate the proper procedures related to solving equations in general.

Advocates of a "balanced approach" argue that procedural flexibility and conceptual knowledge are aspects so closely related to each other that the procedural-conceptual dichotomy should be discarded. Kieran (2014) argues that there are conceptual components of procedures, and that even skilled procedural performance is constantly being updated by the conceptual. According to Kieran the very process of elaborating a procedure is a conceptually oriented activity. In relation to equation solving a balanced approach could support the students to see through the changes when the unknown is solved for, and contribute to increasing their capacity to reason about and explain the changes and to justify them mathematically (e.g., Anthony & Burgess, 2014).

Solving Linear Equations

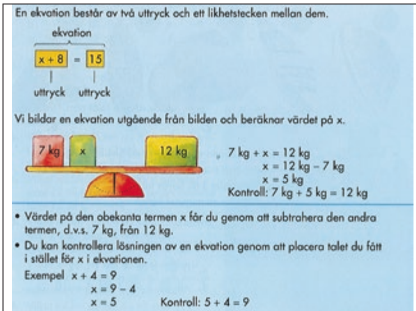
A distinction can be made between an arithmetic and an algebraic notion of equality and a corresponding difference in arithmetical and algebraic understanding. Following Filloy and Rojano (1989) and Vlassis (2002), if the unknown in a linear equation appears on one side of the equal sign only, e.g., $x + 5 = 8$, $13x = 39$, $6(x + 3) = 48$, the student has less need to operate on or with the unknown, or to deal with the equivalence structure of the expressions on both sides of the equal sign. For equations of this arithmetical type, the student manages to find the value of the unknown by applying known number facts or inverse operations, i.e., only working arithmetically (as is shown in the other empirical chapters in this volume). However, when the unknown appears on both sides of the equal sign, arithmetical understanding is no longer enough. Neither is arithmetical understanding enough in the abstract type of arithmetical equations where certain algebraic manipulations are needed, for example in the case of the presence of negative integers or subtracting the variable (e.g., $2 - x = 7$) or several occurrences of the unknown (e.g., $6x + 5 - 7x = 27$) (see Vlassis, 2002, p. 351). When solving such more abstract equations, a student with an algebraic understanding of equality first of all acknowledges that the expressions on both sides of the equal sign represent equal values, next that the solving process involves mathematical actions which preserve this balance and produce equivalent equations. Vlassis (2002) noted that concrete representations of equalities, like the two-pan balance model, may act as good tools for learning how to solve linear equations, but Vlassis also points at their limitations. For example, the balance model cannot represent the negatives in an equation. More generally, a true algebraic understanding of equation solving implies that the student can see through the equation as representing a concrete problem situation (for instance, that the expressions on both sides represent equal weights) and start to understand the equation as an equivalence structure maintained by the operations one has to apply on both sides to solve for the unknown. In arithmetic it is often enough to interpret the equal sign as an operator, as a *do something*-signal (Kieran, 1981). In algebra, however, the student should interpret equality between two expressions as an equivalence relation that does not change.

When problems situated in "day-to-day situations" (an expression used in the curriculum, National Board of Education, 2004) are used to introduce students to the standard algorithm of solving equations, students need to refrain from an arithmetical interpretation of the problem, a situation we will see appearing in the episodes discussed later. When solutions to these problems are found through the syntax of algebra, the meaning of the equal sign changes from announcing results to stating equivalences. Within an arithmetical interpretation, the meaning of the equal sign is dynamic, indicating that the problem has been solved. Within an algebraic interpretation the meaning of the equal sign is relational (Kieran, 1981), signaling an equivalence structure. Hence, the equal sign then carries a structural meaning as well. Furthermore, an algebraic interpretation implies that a student is able to refrain from immediately attributing a concrete meaning to a letter appearing in an equation. Instead, the student should interpret a letter as an unknown number, the value of which is not significant at the moment the equivalence structure is set up and manipulated (Vlassis, 2002).

The Notion of Equation in Curriculum and Textbooks

The mathematics syllabus in the current Finnish national curriculum for grades 1–9 is quite general (National Board of Education, 2004). It does not stipulate any particular teaching approach but, rather, gives overall comments related to the import of meaning to mathematics from applied contexts and external domains. For example, the syllabus highlights "day-to-day situations" as useful tools for developing mathematical knowledge, and it describes "concrete situations" as an aid for bringing together the students' experiences and systems of thought with the abstract system of mathematics (ibid., p. 158). The curriculum defines the core mathematical content separately for grades 1–2, grades 3–5, and grades 6–9 and thus overlaps the boundary between primary school with generalist teachers (grades 1–6, students from 7 to 13 years of age) and lower secondary school with subject teachers (grades 7–9, students from 13 to 16 years of age). The notion of equation appears in the curriculum for grades 3–5 as seeking solutions to equations by deduction, and for grades 6–9 as solving of linear equations, solving of incomplete quadratic equations and solving of pairs of equations algebraically and graphically. The teacher and the local school authorities have a common responsibility for deciding on the mathematical content for each grade. As a result, it is often the textbook and the teacher guide that actually define the mathematical content taught (Törnroos, 2001).

In the following, we briefly attend to how the notion of equation is approached in the textbook series used by the three teachers Anna, Bror, and Cecilia whom we meet later in this chapter. In this textbook series, each textbook section starts with a theoretical part, in the teacher guide described as a "box for teaching." Here we find examples and explanations (see Fig. 6.1).



En ekvation består av två uttryck och ett likhetstecken mellan dem.

ekvation

x + 8 = 15

uttryck uttryck

Vi bildar en ekvation utgående från bilden och beräknar värdet på x.

7 kg + x = 12 kg
x = 12 kg – 7 kg
x = 5 kg
Kontroll: 7 kg + 5 kg = 12 kg

- Värdet på den obekanta termen x får du genom att subtrahera den andra termen, d.v.s. 7 kg, från 12 kg.
- Du kan kontrollera lösningen av en ekvation genom att placera talet du fått i stället för x i ekvationen.

Exempel x + 4 = 9
x = 9 – 4
x = 5 Kontroll: 5 + 4 = 9

An equation consists of two expressions and an equal sign between them
Equation [$x + 8 = 15$]
Expression [$x + 8$]
Expression [15]

We form an equation from the picture and calculate the value of x.

$7 \text{ kg} + x = 12 \text{ kg}$
$x = 12 \text{ kg} - 7 \text{ kg}$
$x = 5 \text{ kg}$
Check: 7 kg + 5 kg = 12 kg

- The value of the unknown term x is found by subtracting the other term, i.e. 7 kg, from 12 kg.
- You can check the solution of an equation by placing the number you got for x in the equation.
  Example: $x + 4 = 9$
  $x = 9 - 4$
  $x = 5$. Check: $5 + 4 = 9$

Fig. 6.1 The box for teaching which begins the section "Addition with one term unknown" (Asikainen et al., 2008a, p. 160, our translation)

Each box for teaching is then followed by exercises to work on during the lesson and homework tasks, as well as supplementary tasks. To every textbook there is a teacher guide, which recommends the teacher to cover one textbook section each lesson. The teacher guide presents the main mathematical content, provides a suggested lesson plan as well as activities for each lesson.

In the Grade 5 material (Asikainen et al., 2007a, 2007b), the notion of equation is formally introduced only in the teacher guide and as consisting of "an expression, the equal sign = and the value of this expression" (Asikainen et al., 2007b, p. 98). With a few exceptions, all equations in the Grade 5 book are of a concrete arithmetical type, i.e., with a single occurrence of the unknown and only integers (Vlassis, 2002). On the right side of the equal sign there is always a single number. The numbers are small and the unknown can be found by applying known number facts or inverse operations. The Grade 5 book illustrates equality iconically as quantitative sameness with both two-pan balance scales and digital scales. The student is told to find the unknown intuitively by `figuring out` the weight of an unknown that keeps a two-pan scale in balance (see the example in Fig. 6.2), or that gives a certain numerical value for the digital scale (see the example in Fig. 6.3). The student is then asked to validate the solution of the equation by substitution as the following text exemplifies: "The solution to the equation $2 \cdot x = 12$ is $x = 6$ because $2 \cdot 6 = 12$" (Asikainen et al., 2007a, p. 187).

In the Grade 6 textbook, the content related to equation solving is divided into four sections (Asikainen et al., 2008a). The notion of equation is formally defined in terms of expressions and by reference to a relational meaning of the equal sign (Kieran, 1981) as consisting of "two expressions with an equal sign in between" (Asikainen et al., 2008a, p. 160). However, the example given to illustrate the definition is clearly an arithmetic equation. The separation of equation solving into four sections is based on the different roles of the specific unknown number in the equation: as an unknown term in expressions with addition, as an unknown term in expressions with subtraction, as an unknown factor in multiplication, and as an unknown dividend or divisor in division. Each type of equation is presented together with specific strategies for solving each type. Altogether six strategies are modeled in the "boxes for the teaching" of the four sections. For example, students are advised to use subtraction as a method when solving for the unknown term in expressions with addition (see Fig. 6.1). "You get the value for the unknown term x by subtracting the other term, that is 7 kg, from 12 kg" (ibid., p. 160). As an alternative method, the teacher guide says: "We can also subtract 7 kg from both sides of

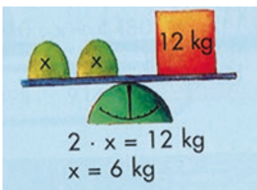


Fig. 6.2 Equation illustrated by means of a two-pan scale

Fig. 6.3 Equation illustrated by means of a digital scale

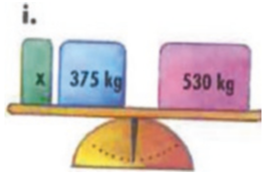


Fig. 6.4 Balance model illustrating addition

the equation. Then x is left on one side and 5kg on the other side" (Asikainen et al., 2008b, p. 83). The Grade 6 textbook also asks students to set up equations from word problems as well as from iconic representations. Most word problems invite the student to generate equations using a phrase-by-phrase translation from natural language to an algebraic syntax, for example "Write down the equation which corresponds to: the number x divided by the number 6 is equal to 9" (Asikainen et al., 2008a, p. 84).

In the Grade 6 textbook, the iconic representations appear as both digital scales and two-pan balance scales where equality is pictured as quantitative sameness. The balance model is used to represent equations where addition or multiplication is present, such as $x + 375 = 530$ (see Fig. 6.4) and $9 \cdot x = 63$ (see Fig. 6.5). A digital scale is used to represent equations where subtraction is present, for example $20 - x = 11$ (see Fig. 6.6) and $x - 5 = 13$ (see Fig. 6.7). Students are expected to infer the quantitative sameness from a problem situation represented with a digital scale together with a narrative that corresponds to the problem situation. But the quantitative sameness is expected to be transformed into *a particular* equation, one where subtraction is present. For example, for Fig. 6.6 the problem situation is described in the following way: "Sand has been taken away from the sack on the scale. When the sack was full its weight was 20 kg. Now the weight of the sack is 11 kg. How much sand has been taken from the sack?" and the expected equation is 20 kg – x = 11 kg. The teacher guide recommends subtraction as the solution method as follows: "From the original weight of the sack 20 kg you subtract 11 kg, which is 20 kg – 11 kg" (Asikainen et al., 2008b, p. 84).

The equations in the Grade 6 textbook continue to be predominantly concrete and arithmetical with a single occurrence of the unknown on the left side and one natural, fairly small, number on the right side of the equation. In general, the equations can be solved by figuring out the unknown value at a purely numerical level. Neither the textbook nor the teacher guide presents *doing the same to both sides* as a general strategy. The equations in the textbook do not put the teachers and students in situations where there is any obvious need to apply algebraic thought or to assume a more conceptual perspective in order to validate the solutions with reference to mathematical principles of solving equations.

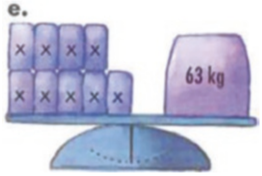


Fig. 6.5 Balance model illustrating multiplication

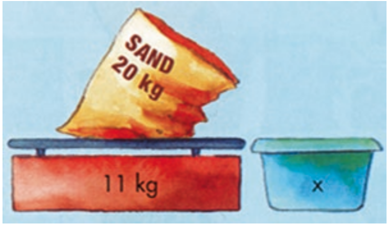


Fig. 6.6 Digital scale representing subtraction

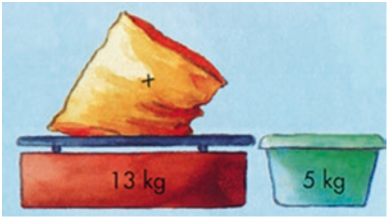


Fig. 6.7 Digital scale representing subtraction

Participating Schools, Teachers, and Students

In this chapter of the book, we will visit three schools and Grade 6 classrooms in the Swedish-speaking community of Finland. As in the other countries, consecutive lessons on equation solving were videotaped and transcribed. The teachers answered a few clarifying questions immediately after each lesson and participated in formal interviews after the last (fifth) videotaped lesson. The participating students speak Swedish and most of them have Swedish as their first language. Two schools, here called A and B, are situated in the countryside. School C is situated in a small town. The schools have between 100 and 250 students and are of an average size in relation to other primary schools in the Swedish speaking parts of Finland. The group size in the three Grade 6 classrooms varies from 27 students in school A to 17 and 16 students in schools B and C. However, in school A the group was split into halves for three out of four mathematics lessons each week, and in school C four students were regularly taught mathematics by a special education teacher. In Finland decisions about splitting classes as well as about allocation of special education resources are made by the municipalities as a result of the decentralized national system for funding schools.

The three teachers, Anna in School A, Bror in School B, and Cecilia in School C, voluntarily participated in the study. The teachers have similar educational backgrounds and have graduated from the same university as certified generalist teachers and Masters of Pedagogy. In addition, Cecilia is specialized in sloyd education (handicraft). At the time of our visits their teaching experiences varied from Bror's 5 years of experience, to Cecilia's seven and Anna's 15 years of experience. They have all stayed with their classes for at least one and a half year, teaching also other subjects besides mathematics. Hence, they know their students well. Since a teacher's relationship with his or her students is a critical factor in the students' learning, this is important to keep in mind (Black, Mendick, & Solomon, 2009).

Teaching Equation Solving from the Teachers' Perspectives

According to Anna, her students were a little bit withdrawn during the first recorded lesson. She thinks that, because of the video cameras, some `very competent students` did not dare to say anything in whole-class discussions even though she noticed that they sometimes looked perplexed. Anna describes her group as a mix of `strong` and `weak` students with no `really, really weak ones`. According to Anna the group is `on average really good` in mathematics.

Bror describes his students as `very silent` during mathematics lessons, some students are `strong` while others are `weak`. Although the students sit pairwise in the classroom, they often work individually and silently in mathematics. Two of the students used to get remedial teaching in mathematics but were at the time of our visits integrated into the regular teaching. Bror says that he attends more closely to the progress of these two students.

According to Cecilia, the 13 students in her mathematics group are `competent` with good work ethics as `they get a section done each lesson`. The students are `talkative`, they ask questions and give comments, and, contrary to the students in Anna's group, even more so because of the presence of video cameras.

In the following, we present the three teachers' accounts of how they teach equation solving in more detail. The presentations are based on the individual interviews the first author did with the teachers after the fifth video recorded lesson.

Anna's Account of Her Teaching

Anna describes flexibility and variation as aims for the affordances she makes available for her students. She wants to adapt to the reactions of her class. She cannot recall any school time memories related to equation solving, and normally she follows the textbook quite closely, however `not slavishly`. She seldom uses the teacher guide because, in her opinion, the guide `frees a teacher from thinking`. Every now and then she asks her students to practice mathematics with mathematical games or problem solving in small groups or individually on laptops.

> I try to teach without book sometimes; make it possible for the students to work with many different aspects. I want them to talk. There is not so much problem solving in the book, not so much text. I want them to work more with that, and also play games.

As Anna wants her students to talk during mathematics lessons, the exceptional silence she met during the first video recorded lesson on linear equations bothered her. But clearly this was not her only concern, and in hindsight she is critical toward the way she approached the lesson content. Equation solving was introduced by a substitute teacher the lesson before, and she was unsure about what the students actually knew about the topic. When asked if she would do anything differently if she taught the same algebra content again, she answers:

> Knowing what I know now I would have changed the first lesson completely. There were problems with the model of how to work with equations. They had not written any equations and the numbers were so easy. They immediately saw the value of x and just wrote 'x equals'. I would have made the first lesson much simpler. [During the interview it remained unclear what she meant by "simpler"]

Bror's Account of His Teaching

In his teaching Bror wants to support the mathematical thinking of all the students and he asks them to talk about their solutions. It is important for him to turn school mathematics into sense-making experiences and not into a subject where clarifying questions like: Why is it like this? are never asked. This goal is especially important for him in relation to the teaching of equations. Grappling with understanding the procedures for solving an equation was not an explicit issue during his own learning of mathematics at school, Bror argues. To solve an equation by transposition of terms from one side to the other and changing the corresponding signs was a technical procedure; the mathematical meaning of this activity, however, was not clear to him at that time.

> For me it has been like if you move x to this side it becomes plus. I was always, already in upper secondary school, often the one who asked why this works. But no teacher had the time to answer my question. Since then I have cared very much about, and as far as I have been able to, to try to create some sort of meaning in mathematics. Not turn it into something *because it has to be.* And of course this is easier on this level than on the upper secondary level; to always try to connect to some sort of reality.

When asked why he chose to work the way he did, he refers to having both the textbook and the teacher guide as his starting points for planning and executing his teaching. But as he finds the textbook `too limited with how it connects to reality` in terms of the tasks presented, he often starts a lesson with solving problems `belonging to the everyday of the kids`. This was also the case in the video-recorded lessons when he utilized such problems to introduce an algebraic interpretation of equation solving.

> I often try to start from something belonging to the everyday of the kids, like prices or fruit in a bowl, the prices of things and such. I try to at least. Sometimes my teaching gets mechanical and abstract. If you just start from equations it easily gets "numberish", but starting as I did you might get a broader understanding of the use of and where equations come from.

Cecilia's Account of Her Teaching

Cecilia is convinced that deviations from the usual school mathematical practice where a section each lesson sets the teaching pace would be confusing for her students. In her planning, she attends closely to the voice of the teaching guide as it tells her `what you are supposed cover with the kids`. However, it is also important for her to attend closely to the voices of her students. In her answer to the question How did the students engage in the tasks you gave them?, she refers to capitalizing on the range of contributions from students that occurred during the video recorded lessons.

> In some way they talked a little bit more (mathematics) than they usually do. They are of course quite talkative but there were students, who normally don't talk much and who don't put their hands up so often, who now perhaps were a little bit more active (mathematically). I think this is good because often the same ones talk, give answers, express opinions, reason and ask questions and they might perhaps express a reasoning which I would not have accounted for in my teaching. In some sense they go a bit further on. I think this is good.

Cecilia describes a mismatch between the learning goals she has set up for her students in relation to equation solving and her own school mathematical experiences. Now, from her perspective as a mathematics teacher, a first step in the students' learning of how to solve equations is to unlearn their use of equality to represent a string of calculations.

> `First they solve a task, and then they write the equal sign, then the answer, maybe add something on and a new equal sign, then they divide etc.`

The next step is to get the students to understand the equal sign as representing an equivalence structure. They should, she says, `understand that there has to be equal weights on both sides of the equal sign and that the letter is arbitrary. It doesn't have to be the letter x.` Cecilia remembers how she solved equations herself as a student by indicating inverse operations with a short vertical line at the right end of the equation. She describes this method as useful, but it did not carry any mathematical significance, and she does not connect it to a focus on the transformed equations as equivalent equalities.

> `This was the way in the book, and this is what the teacher did. I don't know why you write that line, no idea. If you multiply you write a line and times 2 or with addition plus 2 or subtraction minus 2 on both sides of that vertical line. I can't remember why the line appeared. But I have used it when I have had to solve equations myself. It is hard to know which way is the smarter. But for these students it was less abstract to do it the way we did.`

Introducing Methods for Solving Linear Equations

Since solving linear equations appears in the curriculum and in the textbooks for Grade 6, primary school teachers have to make decisions about how to introduce this topic to their students. In the following, we attend closely to what happens when the three teachers Anna, Bror, and Cecilia in different ways try to introduce how to solve linear equations to their students, emphasizing slightly different aspects of the equation solving procedures.

Why Can 7 + x *= 12 Be Solved by Rewriting It as* x *= 12 − 7?*

We start from an episode recorded in Bror's classroom in school B. As a part of the research project on which this book is based, each teacher was asked, on the basis of their recorded lessons, to formulate questions to be discussed with the other project teachers and to choose episodes to illustrate their particular concerns. Bror was especially puzzled by the method of solving equations when terms from one side of the equation are moved to the other while changing the corresponding signs. He wondered whether his students could grasp this procedure more deeply than just knowing how to execute the procedure. He asked himself if his students understood why x in the equation $7 + x = 12$ could be found using subtraction, that is, $x = 12 - 7$. However, he framed his concern in the episode we will discuss next, Episode 1, in more general terms. He asked: Is it necessary to explain the meaning of mathematical rules to students? And how can this be done?

In the beginning of the first video recorded lesson, Bror announced equations as the theme of the week and asked the students to recall and remember the definition of an equation as `We calculate with letters representing something unknown`. He then wrote on the whiteboard four uncomplicated and logically similar "day-to-day situations" as problems to be solved. The first problem situation was: `There are seven fruits in a basket altogether. And four of the fruits are apples, the rest are pears` followed by a question `How many pears are there in the basket?` Bror asked some students to give answers to the questions and then to explain how they got their answers. The first two explanations were given in the form of open number sentences, like Emmi's `Four plus something becomes seven`. The other two explanations were directly expressed as subtractions of two numbers, like `I took seventeen minus eight`. Bror then asked Emmi if the word `something` in her explanation could be replaced. Emmi suggested: `with x`. During the following minutes, Bror and his students, again, represented the four problems, now by equations written in a table on the whiteboard (see Fig. 6.8). Each equation was then solved by subtracting the number added to x from the number on the right side of the equal sign. Finally the answers were checked by substitution.

So far, the lesson had progressed smoothly. All four problem situations were modeled as additions with one unknown and a resulting value on the right side of the equal sign. The students answered Bror's questions, and they did not seem to be intrigued by the construction of the equations and the method of solving by subtraction. After getting the answers and explanations to the four questions, Bror focused his students' attention on the content of the box in the beginning of the textbook section (see Fig. 6.1). Bror read the text in the box, which defines an equation as equality of two expressions. He then described the problem situation represented as a scale in balance, as well as the corresponding equation 7 kg + x = 12 kg and its solution. He continued by drawing the students' attention to the method for solving this particular type of equation. The method was described in the box as `you get the value of the unknown term by subtracting the other term, which is 7 kg, from 12 kg`. After stating: `this might sound like`

	1	2	3	4
Equation	4 + x = 7	15 + x = 21	x + 8 = 17	x + 25 = 39
Solution	x = 7 − 4 x = 3	x = 21 − 15 x = 6	x = 17 − 8 x = 9	x = 39 − 25 x = 14
Check	4 + 3 = 7	15 + 6 = 21	9 + 8 = 17	14 + 25 = 39

Fig. 6.8 Equation table written on the whiteboard by the teacher. Exactly the same table appears in the teacher guide (Asikainen et al., 2008b, p. 83)

Hebrew, but we shall have a look at it, Bror started explaining why, in the equation $7 + x = 12$, the unknown x can be found through a subtraction, $x = 12 - 7$.

Episode 1

In the beginning of this episode, through a series of questions, Bror tries to make his students notice what happens to the number seven when the equation $7 + x = 12$ is solved. He wants the students to notice and formulate that this number has moved from the left side to the right side of the equal sign.

1. Bror: We have seven plus x and that equals twelve [Bror writes the equation on the whiteboard] And how do we get the answer? Once more, Lovisa?
2. Lovisa: Twelve minus seven.
3. Bror: Um. Does someone see something special when I do it this way (…) I mean, what is it that we actually do here? What happens to this seven? That is what I would like to discuss. What happens to this seven? Monika?
4. Monika: It was put last.
5. Bror: Yes. If we, if you say, with respect to the equal sign, what happens to it?
6. Monika: It goes beside twelve, no?
7. Bror: Yes, but with respect to the equal sign (…) Now you have seven there but then you have seven there. [Bror points to the sides of the equation.]
8. Monika: [inaudible]
9. Bror: What did you say?
10. Monika: Well, that you have taken it away.
11. Bror: Um. You, you think, or you, you mean the right thing, for sure, but you don't say it the way that I would like you to say it. [he laughs]. What I would like you to say is that we have seven (…) then we put it on the other side of the equal sign. First you have seven on the left side of the equal sign, but then you have it on the right side of the equal sign. [Pointing to the whiteboard] That is what happens.

After getting to a conclusion about the new position of the number seven, Bror continues in the same utterance by focusing on its sign.

But what else happens to this seven? It changes place.
Leif, stop calculating, you will have enough time afterwards. [Leif was working in his notebook.]
What else happens to this seven? (…) No hands.
This seven has changed place. [Bror points to the whiteboard.]
Don't you see anything else? [Two students raise their hands.]

Two see something else.
Maja, does the seven look like the same on the right side of the equal sign?

12. Maja: Yes, but it is minus
13. Bror: Yes! Good! It becomes minus. And this is the way we solve equations.

Maja noticed that the number 7 now was preceded by a minus sign, and Bror elevated her answer into a general description of a method for solving equations. He then attempts to explain the observation that the sign of the number seven changes from plus to minus when the number moves from one side to the other. Bror clearly has a particular explanation in his mind as he continues.

If you, you put it, maybe a little bit carelessly, yes, seven moves to the right side of the equal sign and then it becomes minus. [Bror points to the whiteboard.]
But why does it happen (…)
Why is it so?

14. Monika: Because it always, when it is twelve minus seven, then twelve is a bigger number than seven, for that reason it must be minus
15. Bror: Um. But it's not for that reason that it becomes, that I must change its sign.

Bror then rejects Monika's arithmetical interpretation without any further comment and changes the focus of his questioning to the meaning of the equal sign.

Well, what if we say that on the left side and the right side of the equal sign. This here and this here. [Pointing to the equation on the whiteboard.]
What can you say about it?
What is on the left side and what is on the right side of the equal sign (…)
What must they be like? It is self-evident, isn't it? Equal sign. What must they be like, those which are on the left side and on the right side of the equal sign? Yes, they must be (…) [Only a few students raise their hands.]
Now I want to have more hands raised. Wille, you haven't said anything today.

16. Willie: Oh
17. Bror: What must they be like (…) if I put an equal sign?
18. Willie: Oh … like
19. Bror: If I have an equal sign, what does it mean? [Pointing to the equal sign in the equation on the whiteboard.]
20. Willie: What the answer was.
21. Bror: What did you say?

22. Wille: `What will be the answer.`
23. Bror: `Um. Yes, we are used to put, we are used to put it there to write our answer. But what (…) What is on the left side and what is on the right side of the equal sign, they must be then? [Bror points with his pen, first to the equation on the whiteboard and then to the students.]`
24. Leif: `Equal`

Leif filled in with the word `equal` which Bror wanted to hear. Then Bror continues his questioning. He wants the students to notice that to preserve the equality, if you do something to the one side of the equation, you have to do the same thing to the other side as well.

25. Bror: `Equal. That's what I was searching for, Wille. They must be equal, mustn't they?`

    `And if we take, what we have done here with this expression, seven plus x. [Bror sweeps with the pen towards the left side of the equation on the whiteboard.]`
    `So what have we done here to get x alone. [Bror points to the x.]`
    `What have we done to get x alone, what have we done? Monika (…)`
    `How are these two different? This here and this here. [Bror draws a red circle around 7 + x and around x on the whiteboard.]`
    `What's the difference?`

26. Monika: `I don't know [Monika smiles.]`
27. Bror: `Yes you do know.`
28. Monika: `No.`
29. Bror: `So seven plus x, what's the difference between seven plus x and x? (.) I have taken away`
30. Monika: `Seven [Monika laughs a bit.]`
31. Bror: `I've taken away seven. From this expression, I have taken away seven and got x. Well, then, it means that if this still should be true, I have to do the same thing on the other side as well. Right?`
    `Are you following? Um (…) Somewhat uncertain. So if I take away seven from this expression, it means that I have to, so that it still would be true, so that I can still put the equal sign, then I must take away seven from the other side as well. [Bror sweeps with his hand towards the equation on the whiteboard.]`

Now it seems, contrary to what happened at the beginning of the lesson, that there is an emerging mismatch between the expectations of Bror and the participation of the students in answering his questions (utterances 3, 11, 13, 15, 19, and 25).

As it seems from the interaction, the students tried hard to answer the questions posed to them, but they did not succeed in producing the kind of answers that Bror was satisfied with. A few times, by giving hints, Bror narrows his demands in order to get the desired answer from the students (utterances 5, 11, 15, and 23). When he finally asks the students whether they had been following his teaching, he gets responses which he interprets as expressions of uncertainty. Immediately after the end of the episode, Bror continues by stating that he made the situation very difficult for the students because he `wanted to explain why we can do it this way` (i.e., use subtraction to find the value of x).

So what happened in this episode? Why did the students not participate in a manner Bror seemed to expect, and as they did at the beginning of the lesson? When solving the equations at the beginning of the lesson, the class used subtraction, which is the inverse operation of addition, to find the value of the unknown. In many cases the solution was also easy to see, to deduce from the situation. The students' interpretations of the equal sign can be seen as purely arithmetical; there was no real need to refrain from seeing the equal sign as a symbol for announcing a result. In their answers, the students referred several times to the equal sign as a do something-signal like in the answers `Four plus something` *`becomes`* `seven` and `What will be the` *`answer`*. The students in this class were familiar with arithmetical equations where the unknown appears on the left side only and with just one number on the right side. This is obvious also in the equations the students themselves designed during the second video recorded lesson (see extract from Alf's notebook in Fig. 6.9). The equations constructed by the students were to a large extent more diverse than the textbook equations with both small and large numbers and a variety of operations, but all were of the arithmetical type.

However, the explanation given by Bror referred to the algebraic way of interpreting equations which is known to be based on a very different conceptualization of equality than the arithmetic one (Herscovics & Linchevski, 1994). Even when algebraic methods for solving equations are the main topic of teaching, students often have difficulties in realizing the conceptions needed (Filloy & Rojano, 1989; Herscovics & Linchevski, 1994; Knuth et al., 2006), and intentional and extensive, even innovative, teaching is recommended to bridge the gap. The students in this class, during this particular lesson, met the algebraic way of interpreting equations suddenly and they were unprepared. They were not given a chance to be involved in reasoning about *different* ways to represent the same problem situation in the form of equalities, and Bror's attempt to support his students in making sense of the method of solving an equation with addition ended up in the mismatch observed. Rittle-Johnson and Star (2009) argue that comparing essential features in solution methods to the *same* problem may be a good way to support a student's mathematical competence related to equation solving.

One can also wonder whether the students were motivated at all to make sense of the new complicated way of thinking about equations, since the solutions to the problem situations in the beginning of the lesson were obtained more economically by arithmetical means. In fact, the equations were never used by the students to find solutions to the problems they presented. Thus, it was of no advantage to take theo-

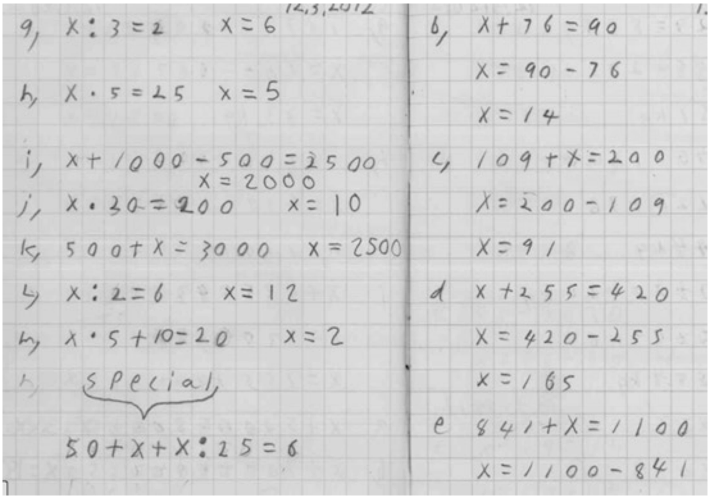


Fig. 6.9 A page in Alf's notebook. To the left are equations constructed by Alf; to the right are textbook equations

retical control by validating the solutions algebraically. Stacey and MacGregor (2000) emphasize that from the students' point of view, using algebra to solve easy arithmetical problems is an "extra difficulty imposed by the teachers for no obvious purpose" (p. 165). To appreciate the value of algebra as a problem-solving tool, they argue, students should work on problems which are not easily solved without algebra and ask several questions about one problem instead of changing the problem situation (ibid., p. 165).

"How Can I Get Plus Twelve to Zero?"

Our second example is from Anna's classroom in school A, where the first video recorded lesson started with Anna approaching equations arithmetically: she wrote an open number sentence on the whiteboard: 4 + _ = 9. As answers to Anna's questions about the missing value, the students answered with the missing number, and they named the object on the board an equation. Then Anna gave a rather abstract, lecture-type introduction to equations using a ready-made presentation on the whiteboard. She read aloud: `An equation is an equality between two mathematical expressions, which are called the left side and the right side. It includes one or more unknown numbers. If there is one unknown number, you normally use the letter x.` She then filled the place holder in the open number sentence already written on the board with the letter *x*.

Next, Anna continued to read aloud the text on the whiteboard: `An equation is an equality between two sides. The two sides are separated by an equal sign`. She illustrated this statement with the eq. $4 + x = 9$ and emphasized that both sides of the equation must be equal. In her presentation, she then pointed to a numerical equality, $4 + 2 = 7 - 1$, and stated that the value of both sides is six. Then Anna asked the students to solve two open number sentences and stated that instead of a place holder, you can use a question mark or the letter *x*. She continued to talk about the convention of using the letters *x*, *y*, or *z* for an unknown. All the equations she has shown to her students so far included only one number on the right side, except for the equality $4 + 2 = 7 – 1$, which she used to indicate a new meaning of the equal sign: the equal sign as a signal of an equivalence structure.

Before the start of the following episode, Anna has referred to the procedure of solving equations step by step as a `mathematical strategy`. She writes the equation $x + 12 = 18$ on the whiteboard and starts describing the procedure for solving it.

Episode 2

The first step is to get *x* alone on the left side.

1. Anna: `An example. x (…) plus (…) twelve is equal to eighteen [writes on the whiteboard], and we know that x should be six, this is what we know. But also this way of thinking about how to do it. What we have to aim at is, I want to have (…)`
   `If I have an equal sign in the middle, then I shall aim at having x alone in the left side (…) But now I have plus twelve there, what do you think, the way of thinking, how can I get this plus twelve away from there? I want to have x alone on the left side of the equal sign. How can I get it away? Janne`
2. Janne: `Eighteen minus twelve`
3. Anna: `Oh yeah, but (…) now I have it there. What should I do, just to fling it away? How can I get plus twelve to zero? How can I get plus twelve to zero? Well (…) Nelli.`
4. Nelli: `Maybe add [inaudible]`
5. Anna: `No, if I have, how can I make plus twelve into zero?`
6. Anna: `How shall I get plus twelve to zero, nothing? Olle`
7. Olle: `Maybe change it to x [inaudible]`
8. Anna: `No. Helena`
9. Helena: `[inaudible]`
10. Anna: `How shall I get plus twelve to zero? [Anna draws a minus sign after the number 12 on the left side of the equation.] I have helped you a little bit on the way. Nelli.`
11. Nelli: `Minus twelve.`

The message is: what you have done on the one side, you also must do on the other side.

12. Anna: `Minus twelve. But twelve minus twelve is zero, isn't it? But now the thing here, now I had, when I do something on the left side, so what do you think I should do on the right side? (…) Tor`
13. Tor: `Take away from there, that twelve`
14. Anna: `What did you say?`
15. Tor: `Take away the twelve from there`
16. Anna: `Exactly. I have to do the same thing here, now I have got eighteen, what should I also do then, here, on the right side? Well, now, Mimmi`
17. Mimmi: `Minus eighteen.`
18. Anna: `No, not minus eighteen, the same thing as on the left side. Mimmi.`
19. Mimmi: `Minus twelve.`
20. Anna: `Minus twelve, well let's check, x, twelve minus twelve is zero, so then, now I've got x on the left side, eighteen minus twelve is (…) quickly Mimmi`
21. Mimmi: `Six.`
22. Anna: `Six. Now I have, stepwise, through mathematical steps, done this equation. You could quickly see that it must be six. You could do it just like that. But now I have shown how it actually goes step by step. I want to have x alone on the left side, so that I get that x equals to. And then, I just have to look what I have on that side, what I need to do. In this case, I had plus twelve, then I have to take minus twelve so that it becomes zero. But when I do something on the left side, I also have to do the same thing on the right side. Do you understand? Did you follow?`
23. Students: `Yeah, um.`

In this episode we can see the same phenomenon occurring as in Bror's class, maybe it is now a little bit less dramatic. When Anna starts teaching the steps of solving the equation $x + 12 = 18$ (see Fig. 6.10), the students do not contribute with the answers she seems to expect (utterances 1, 4, and 9). The answers the students give also show some uncertainty: two students start their answers with `maybe`. After receiving a hint from Anna in the form a minus sign drawn after number twelve in the equation, Nelli gives the expected answer, `minus twelve` (utterances 11, 12). Like in Bror's class, solving an equation by algebraic means, and with an algebraic interpretation of equality, is unfamiliar to the students who most obviously are thinking about equations in an arithmetic way. They, however, try to fulfill Anna's expectations when answering her questions.

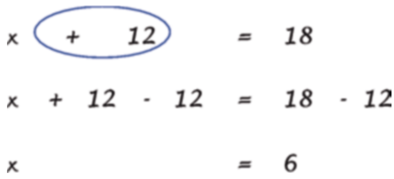


Fig. 6.10 Solving $x + 12 = 18$ on the board

Immediately after the previous episode, Anna and her students started solving the equation $y - 6 = 11$. In the following episode, the steps in the procedure for solving a linear equation are repeated and the importance of "doing the same on both sides" is stressed again.

Episode 3

1. Anna: y minus six equals eleven, an equation. Now, I know that you can, you're quick, you know the answer. But, now, we shall think about the mathematical steps. What do I want, I'll put the equal sign here, what do I want to have alone on this side of the equation? What am I aiming at?
2. Sofia: x
3. Anna: In this case?
4. Sofia: y
5. Anna: y, okay, I've got y there. But it is not yet ready, I've got the minus six, what shall I do then, what do I want to do then? Now I've got minus six. Cecilia.
6. Cecilia: You want to make it zero.
7. Anna: And how can I get it?
8. Cecilia: Plus six.
9. Anna: Plus six. Okay. And then on the right side I had eleven. Are we finished with it, or shall I still do something?
10. Anna: What does Janne say?
11. Janne: Plus six.
12. Anna: Plus six, too. Why Janne, plus six there too?
13. Janne: We have to do the same thing on both sides.
14. Anna: The same thing on the left and right sides. What I do on the left side, the same thing on the right side, or on the right and left sides. And it's plus six, now, because I had minus six. Okay. Then I have got y. Those two cancel each other out. Yes, then I've got y there. And what will there be on the right side (…) Vanja
15. Vanja: Seventeen
16. Anna: Seventeen. And I know that you could have been able, you could find it already in a few seconds, but now we did the mathematical steps, again. Are you following? [The

three equalities y – 6 = 11, y – 6 + 6 = 11 + 6, y = 17 are now written beneath each other on the blackboard.]

17. Students: Yeah, yeah.
18. Anna: Beginning to understand this, although these are easy numbers (…) This is what you will practice in the book. This is then, now you have solved equations, easy equations. Later, there will be a little bit harder ones, but now we'll begin with these.

In this second episode from Anna's classroom, we can notice how the students and Anna use the same wordings as when solving the equation $x + 12 = 18$ in Episode 2. In her first utterance in Episode 2 Anna states her expectation very clearly when she says: I want to have x alone on the left side of the equal sign. Now in this second episode she formulates the same expectation as a question to the students: What do I want to have alone on this side of the equation? The student Sofia remembers that the teacher wants to have x alone, but in this case the letter happens to be y (2). In the second episode Anna asked many times: How shall I get plus twelve to zero? In this episode she asks (5): What do I want to do then. Now I've got minus six. And Cecilia answers (6): You want to make it zero. In her summary at the end of Episode 2, Anna reminded the students that when I do something on the left side, I also have to do the same thing on the right side. In this episode Janne repeats that: We have to do the same thing on both sides (13), and Anna confirms that Janne remembers correctly when she says The same thing on the left and right sides. What I do on the left side, the same thing on the right side, or on the right and left sides (14).

Through the two episodes above, we see how teacher Anna and her students are using language, key terms, and phrases, to support the recall of the procedure for solving this type of equations. Language functions as a tool for memorizing the procedure. Although Anna repeatedly emphasized that you have to do the same thing to both sides of the equation, she did not focus on the mathematical reasons for doing this. There is a strong procedural emphasis (Hiebert, 1986; Kieran, 2014; Rittle-Johnson et al., 2001) in her teaching, and she does not, in this sequence, offer possibilities for her students to construct mathematical meaning for the steps. There is the possibility that the activity was reduced to performing operations on symbols the students did not understand (Herscovics & Linchevski, 1994). And when the only option is to memorize a series of rules, students are highly likely to forget them or remember them incorrectly at some later point in time (Falkner et al., 1999).

An analysis of the students' notebooks reveals that only a few students used adding or subtracting the same number on both sides as an equation solving method. The method was used only for equations in which a number was subtracted from the variable (e.g., $x - 10 = 15$), indicating that those who adopted the new method assimilated it into their existing idiosyncratic repertoire for finding an unknown number in an equation. Anna's communication with her students did lead the students into an algebraic way of solving equations.

To Know the Amount of Sand Used, We Have to Subtract

The third approach to equations comes from a lesson in school C taught by Cecilia. She started the first video recorded lesson by discussing the meaning of the equal sign with her students. She referred back to the mathematics test she had just marked, and told the students that they, again, had forgotten something. She wrote an equal sign on the blackboard and asked what it means (see Fig. 6.11). A student answered that `it means that there is the same number on both sides, so that it is the same thing`, and another student added `they must weigh the same.` As a response to a further question posed by Cecilia, a student read the sign as `is equal to` (in Swedish: `är lika med`). Cecilia then wrote this below the equal sign and drew an arrow from the text to the sign.

Cecilia continued with an explanation of what she meant by her comment that the students had forgotten something. On the blackboard she copied a solution to a task where the students were asked to calculate the area of a triangle (see Fig. 6.12): 3 m · 4.2 m = 12.6 m^2/2 = 6,3 m^2. She asked the students to comment on the solution. A student answered that the text on the board tells `that three meters times four point two meters equals six point three meters` and continued `it is not true`. Cecilia asked how they could correct the statement, and a student suggested that you should start writing a new row with 12.6 m^2 divided by two.

Fig. 6.11 Connecting "Is equal to" to the sign =

Fig. 6.12 Discussing expressions relevant for calculating the area of the triangle

Cecilia insisted that she wanted to do the correction in the expressions already written on the board, and a student suggested that 3 m · 4.2 m should be divided by two. Cecilia agreed and emphasized that because the area of a triangle is base times height divided by two, it should already be there in the first expression. And, she continued; now that they are beginning to study equations, the students should not by any means forget what the equal sign written on the board means. Otherwise there will be problems, she warned.

After the discussion about the equal sign, Cecilia wrote the equation $x + 8 = 15$ around the equal sign already existing on the blackboard. She named the parts in the equation by pointing to the different sides and the equal sign and saying `this expression is equal to this expression` as well as rounding the whole equation by her hand and saying `then this is an equation`. One student was ready to offer a solution to the equation and suggested that `to get to know the answer for x we have to take eight minus fifteen which is seven`. Cecilia wondered if it was possible to take eight minus fifteen. The student corrected her answer to be fifteen minus eight. Together with the student, Cecilia concluded that it must be fifteen minus eight, because otherwise the answer would be negative. After a short discussion about the right way of writing the solution, Cecilia asked the class to check the solution.

Cecilia also illustrated equality by a two-pan balance scale. She placed five blue and five pink blocks in one pan and ten green blocks in the other pan. A student suddenly added a rubber gum to the pan with the green blocks. Cecilia wrote on the board the contents of the pans of the balance: `10 green +1 rubber gum` and `5 blue +5 pink`. She asked the students: `What went wrong?` The students answered that the rubber gum is wrong. It made the other side `too much`. Cecilia asked the student to take the rubber gum away and illustrated equality as balance. She then said `ten here and ten here`, raised up the blocks, and kept them up in front of the pupils. `Five plus five equals ten, they must weigh the same`.

After demonstrating with the balance scale, Cecilia told the students to start working with an exercise including similar addition equations as in the previous example ($x + 8 = 15$), giving instructions to write down the solution. She stressed that the students should write the original equation in their notebooks, and that they also should write all the steps, just like in the example on the board. Only an answer was not sufficient. After some negotiation about the accepted way of writing down the solution the students started working.

In line with her goal to help the students unlearn their earlier use of the equal sign to represent a string of calculations, Cecilia focused on the relational meaning of the equal sign in several ways: by discussing it explicitly with her students, by referring to a solution a student had made in the test, and by representing both inequality and equality with a balance scale. These situations were familiar to the students, who also participated in the corresponding discussions in seemingly relevant ways. An explicit attention to the equal sign in teaching materials offers students possibilities to develop their understanding of the equal sign as expressing a relation (Knuth et al., 2006). Especially, the balance scale model as an analogy for an equation allows students to form strong and long-lasting mental images of equality and of the operations that can be applied to both sides of the equal sign to maintain the equality (Vlassis, 2002).

The introduction to equations in Cecilia's class might well have functioned as a starting point for introducing algebraic methods for solving equations. However, Cecilia did not utilize these ideas in the solution procedure of the addition equations in this lesson. As in the introductory example $x + 8 = 15$, all the exercises included only expressions with addition of one unknown, and they were solved through arithmetical means, by subtracting the known term in the expression on the left side from the number on the right side, without any further discussions.

Immediately after the first lesson about addition equations with one unknown, Cecilia taught another lesson focusing on subtraction equations. In the textbook, they are separated into two types and exemplified by the following equations (see also Figs. 6.6 and 6.7):

1. 20 kg − x = 11 kg, which is solved as x = 20 kg – 11 kg
2. x – 5 kg = 13, which is solved as x = 13 kg + 5 kg.

While teaching methods for solving the two different types of subtraction equations, Cecilia grounds her discussion on the context of taking sand or concrete powder out of a sack.

Episode 4
Cecilia starts by presenting the problem situation depicted in the box for teaching (see Fig. 6.6), posing the corresponding question.

1. Cecilia: `When you look at that blue box, you can see that there is a sack on a scale, and that the original weight of the sack was 20 kg. They had 20 kilos sand in it, and now it weighs 11 kilos, and we would like to know.`
2. Student 1: `No, this is difficult.`
3. Student 2: `No, it isn't`
4. Cecilia: `What, what`
5. Viktor: `[inaudible] 20 kilos and on that 11 [Viktor refers to the scale], it makes 31 kilos.`
6. Student: `No, but that,`

Viktor does not interpret the problem in an expected way, and Cecilia illustrates the situation in words and by drawing on the blackboard.

7. Cecilia: `No Viktor, think about it this way. [Cecilia draws on the board.] We buy a sack of sand from a shop; it weighs 20 kilos when we buy it, right?`
8. Viktor: `And then it is.`
9. Cecilia: `And you put sand in the ground, right? You put sand there (…)`
10. Viktor: `Well, this is for real.`

11. Cecilia: And then you put it on a scale. [She draws the scale.] We have a digital scale today, and you put it there.
12. Viktor: Yes, and there it says 20 kilos, so it has to be 20.
13. Cecilia: And it still says 20 kilos sand here, but you have just put sand on your pathway so that you wouldn't slip and get wet trousers.
14. Viktor: What is that 11 then?
15. Cecilia: And now when you have it on the scale, you want to know how much sand I have left in my sack.
16. Viktor: Yes, so it is 11 kilos.
17. Cecilia: Then it weighs 11 kilos, that sack, now, right? How much sand did you use to put on your pathway? Elli?
18. Elli: [inaudible]
19. Cecilia: What did you say?
20. Elli: Nine.

Elli answers the question correctly and Cecilia writes the equation 20 kg – x = 11 kg to describe the situation and starts discussing the solution process of the equation (see Fig. 6.13).

In her explanation, Cecilia uses the method for solving addition equations as a starting point and shows to her students that now they cannot solve the equation in the same way.

21. Cecilia: If I put (…), aa, I can't, now if we do it the same way like when we added, that we put, put this way. [Cecilia writes x = 11 kg - 20 kg.]
22. Student: But it says that we have to
23. Student: But shouldn't you now put?
24. Cecilia: What's wrong here? [Cecilia points to the equation x = 11 kg - 20 kg written on the board.]
25. Student: It becomes negative.
26. Cecilia: Yes.

Fig. 6.13 The situation on the blackboard after utterance 35

27. Student: `But then it must be 20 minus 11, otherwise.`
28. Cecilia: `Um, exactly. [Cecilia wipes out 11 kg - 20 kg.] So we have to switch`
29. Student: `But we did that also when we did plus.`
30. Cecilia: `No [Cecilia writes 20 kg - 11 kg on the board.]`
31. Student: `[inaudible]`
32. Cecilia: `And what does x become then?`
33. Student: `[inaudible]`
34. Student: `Nine kilos`
35. Cecilia: `Um, [Cecilia writes on the board.] nine kilos. Now we know that we have used nine kilos of sand on our pathway. Um.`

After this, Cecilia, together with her students, solves the other problem from the box for teaching (see Fig. 6.7). The problem situation is that 5 kg concrete powder has been taken away from a sack and there is 13 kg left. How much concrete powder was there originally in the sack? Cecilia presents the problem as an equation $x - 5$ kg = 13 kg and, again, she illustrates the situation by drawing and explaining it thoroughly. One student volunteers to answer and says that to know the original amount of concrete powder they have to add 13 kg and 5 kg. Cecilia agrees, but wonders why now, suddenly, they have to add although in the previous example they had to subtract (see Fig. 6.14).

36. Cecilia: `What makes the difference between this equation here and this equation here [Cecilia points to the two equations on the board] is that here we know how much we had in the sack from the beginning, we have used x kilos and we have 11 kilos left. For that reason we have to subtract what is left from the original to get to know how much sand we have`

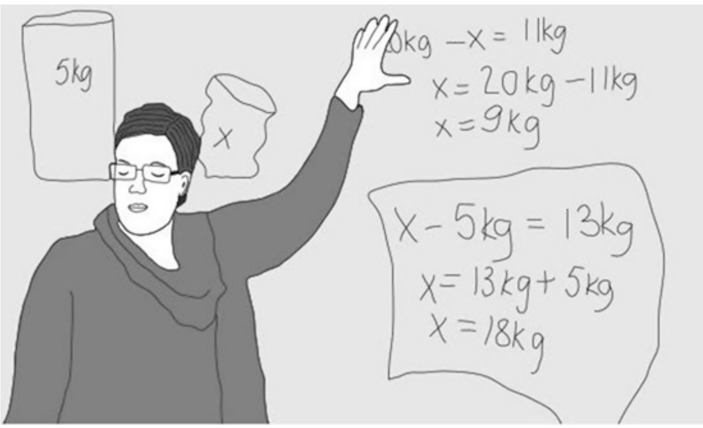


Fig. 6.14 Cecilia explains why the unknown is solved by subtraction

used. Here, we don't know how much we had in the beginning, but we know that we took 5 kilos from the sack and have 13 kilos left, and then we have to add (…) Do you follow?

37. Students: [both yes and no-answers]

When concluding her teaching, Cecilia justifies the use of different methods for solving the two subtraction equations by referring to the necessities of the real-world situations. The students did not, however, enter into any discussion about the underlying principles. Balacheff (2001) refers to this kind of approach as symbolic arithmetic. Typical of that genre is that arithmetical problems are symbolized in algebraic form, but a parallel relationship between the symbolic manipulation and the referent world is maintained as long as possible. Thus, the problem solution is validated through a pragmatic control; it is justified by the initial contexts of the problem. This approach is in line with the syllabus which describes "concrete situations" as an aid in bringing together the student's experiences and systems of thought with the abstract system of mathematics (National Board of Education, 2004). However, as Balacheff (2001) reminds us, symbolic arithmetic is not enough to help students enter the world of algebra.

In the beginning of her introduction to equations, Cecilia grounded the algebraic way of interpreting equations through activities that the students readily contributed to. But her teaching of methods for equation solving relied heavily on either the inverse operations or features of a particular problem situation.

Discussion

All the three teachers took initiatives to lead their students forward, from an arithmetic, and everyday, interpretation of equations to an algebraic way of solving them. Bror and Anna tried to teach according to the strategy of `doing the same thing to both sides of the equation`, and Cecilia emphasized the necessity of understanding the structural meaning of the equal sign. However, in general, the discussion of the topic in the lessons we have recorded, and the tasks given to the students by the teachers, did not consistently support moving to an algebraic way of understanding and solving equations. Despite the relational definition in the Grade 6 book of an equation as "two expressions with an equal sign in between," the analyses indicate that the textbook continues to expect equation solving to proceed within an arithmetic domain where the equal sign is interpreted operationally as a dynamic "*do something*-signal" (Kieran, 1981) and as indicating the answer to a problem. The book presents arithmetic equations which are logical equivalents of missing value problems. In these arithmetic equations, the unknown is always on one (mostly the left) side and there is a single number on the other side.

Neither the teachers, nor the authors of the Grade 6 textbook, seem to be aware of the underlying conceptual differences between solving equations within an arithmetical interpretation of equality as opposed to an algebraic interpretation. The difficulties of appropriating algebraic ideas from the teaching illustrate why a transition from arithmetic to algebra may not proceed as smoothly as intended. The students had encountered missing value problems in the textbooks every now and then from the first grade onward. They are familiar with the logic of that type of task. When introducing equations, the Grade 6 textbook writers seem to build on this basis. However, the students' earlier firm conceptions about equations were not challenged in the teaching observed. There was no real need for students to adopt algebraic ways of thinking about equations. At best, solving equations by adding or subtracting the same term from both sides of the equation seems to be used by students as one procedure among others and applied solely to one particular type of equation. Thus, the data indicated that the students did not experience a need for an algebraic interpretation of the equal sign to solve the tasks, and the book did not explicitly encourage students to expand their mathematical knowing into operating with or on the unknown. Instead of a smooth transition from arithmetic to algebra, Balacheff (2001) recommends that students should experience a clear rupture between arithmetic and algebra. The rupture might, for example, be introduced through a strong emphasis of the newness of the situation or by giving more complex equations to be solved.

Even first and second graders have been observed to construct the structural meaning of the equal sign when presented to suitable missing number problems and given the possibility to discuss their interpretations of the tasks (Falkner et al., 1999). Carraher, Schliemann, Brizuela, and Earnest (2006) found that students in grades 2 to 4 in their study could learn to construct and compare additive algebraic expressions and develop an algebraic interpretation of a variable as a general number. We hypothesize that in many cases minor changes and extensions in the types of tasks presented to children in the elementary school arithmetic, and the whole class discussions held, can help young children construct important ideas behind solving equations in algebra. Instead of "operation on two numbers equals an answer"-type of equalities, children should meet versatile missing value problems (such as $5 = 12 - \_$, $5 + 7 = \_ + 6$, and $2 + \_ = \_ - 4$) presented using multiple representations, e.g., manipulatives, drawings, and symbolic expressions. The structural meaning of the equal sign should be discussed in making sense of those situations. If children are asked to investigate and construct true and false equalities of different types with two-pan balance scales, they could investigate how a true equality remains true when the same number is added to both sides or subtracted from both sides, and they could investigate the influence on the equation of multiplying and dividing both sides by the same number. With game-like activities, the concept of variable could be developed. Affording the children investigations of series of arithmetic tasks with a pattern and discussing the pattern would raise the level of discussion in the classroom and help children to think generally about numbers. With Schliemann, Carraher, and Brizuela (2013), we call for teaching experiments and research on Early Algebra Education, aimed at laying a foundation for algebra already when very young children study arithmetic.

Chapter 7
How Teachers Introduce Algebra and How It Might Affect Students' Beliefs About What It Means to "Do" Mathematics

Karen B. Givvin, Emma H. Geller, and James W. Stigler

Introduction

When we think of the learning that takes place in classrooms, we normally think first of content knowledge—the knowledge of facts, concepts, theories, and principles. And, indeed, it is normally this kind of subject-matter knowledge that we assess on exams. However, what students learn in school includes also attitudes and values associated with the content. In mathematics, for instance, they learn what it means to *do* mathematics. We assume that students' beliefs about mathematics and how to learn it are, at least in part, a consequence of the socialization they receive through participation in classroom mathematics. Mathematics—as distinct from some other school subjects— is primarily learned in school. So it is in school, we believe, that students are most likely to develop their views of mathematics. These views might be inferred from the kinds of tasks they are assigned, the expectations set for how they should work on the tasks, or the kinds of feedback they receive for their performance; or they might be explicitly taught, by a teacher or by peers (Cobb, 1987; Franke & Carey, 1997; Schoenfeld, 1983; Stodolsky, 1985; Stodolsky, Salk, & Glaessner, 1991; Yackel & Cobb, 1996).

Beliefs about what mathematics is and how it should be done might have consequences for students' long-term learning and retention. In the U.S., students entering community college take placement tests in mathematics. Based on these tests, 59% of these students nationwide—most of whom are high school graduates—are deemed unprepared for college-level mathematics (Bailey, Jeong, & Chi, 2009). They are placed into remedial classes where they repeat the mathematics they

K. B. Givvin (✉) · J. W. Stigler
Department of Psychology, University of California, Los Angeles, Los Angeles, CA, USA
e-mail: karen.givvin@ucla.edu

E. H. Geller
Department of Psychology, University of California, San Diego, La Jolla, CA, USA

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presumably have already been taught: beginning algebra, pre-algebra, and even basic arithmetic. Most of these students never make it into a college-level mathematics course or graduate from college.

When we interviewed a group of these students in order to better understand their beliefs and what they do and do not know about mathematics (Givvin, Stigler, & Thompson, 2011), we found three things: First, even when these students can produce a correct answer to a mathematical question, further probing reveals a fundamental lack of understanding of even basic mathematical concepts. Most of what they know, it seems, was learned by rote, unsupported by conceptual understanding. Second, we found this lack of conceptual understanding to be associated with a fragile and bug-ridden knowledge of mathematical procedures, which the students often applied inaccurately and inappropriately. Third, we found that most students share a disturbing view of what it means to do and learn mathematics.

The majority of students we interviewed view mathematics as a collection of procedures to be applied and rules to be memorized, not as something that can be figured out by effort and thinking. They discount the value of conceptual understanding and reasoning—indeed, they do not appear to expect mathematics to make sense—and show a compulsion to calculate, even when calculation is unnecessary. When asked what their instructors could do to better help them learn, they never mentioned explanation or understanding. Instead, they suggested that procedures be broken down into smaller steps, demonstrated at a slower pace, and repeated more times. Beliefs and actions similar to these are not limited to college students or those struggling with mathematics. They have been found across grades and skill levels and exist not only among students, but among teachers as well (Dossey, Mullis, Lindquist, & Chambers, 1988; Frank, 1988; Garofalo, 1989; Schoenfeld, 1989; Stipek, Givvin, Salmon, & MacGyvers, 2001; Stodolsky et al., 1991).

These views of what it means to do and learn mathematics would seem to be highly detrimental to students' mathematical futures; one can only go so far in mathematics through memorization alone. Our interviews convince us that these students are capable of thinking and making sense of mathematics, yet they somehow do not see such behavior as necessary or appropriate in a mathematics classroom. Which raises the question: Where do these maladaptive beliefs come from?

We have argued elsewhere that the views we see expressed by struggling community college students are the result of a socialization process that begins in elementary and middle school mathematics classrooms. Students enter school with the expectation that the world—including mathematics—will make sense. However, when they encounter teaching that focuses on procedures disconnected from the concepts that underlie them, things begin to go awry (Givvin et al., 2011). Such experiences of mathematics—as a bunch of rules, procedures, and notations that need to be remembered but not necessarily understood—lead students to give up on the idea that mathematics is supposed to make sense. Conceptual explanations are seen increasingly as a waste of time, not nearly as important as getting the right answer. Eventually, even the basic understandings of mathematics students had developed prior to entering school, atrophy, leaving students with an ever-increasing (and ever-degrading) collection of disconnected facts to remember.

In this chapter, we take an up-close look at the kinds of socialization processes we presume are leading to the outcomes we have observed. As part of the VIDEOMAT project, we decided to focus on the initial introduction of algebra as a topic in school mathematics. Algebra is a key stumbling block for the students we have described. Yet the introduction of algebra offers great potential for students to have a fresh start in their study of mathematics. Taught well, algebra "opens the door to organized abstract thinking and supplies a tool for logical reasoning" (Stacey & MacGregor, 1997, p. 253). However, algebra, with its heavy emphasis on notation, also has the potential to heighten the focus of both teachers and students on the learning of rules for their own sake. Algebra could become a tool for reasoning, but it also could become one more domain for which a new collection of rules needs to be memorized (a tension that is discussed in the other chapters in this volume as well).

Using a qualitative case study approach, we studied two U.S. middle school classrooms at the very beginning of their study of algebra—the point in time when the door is first opened into the world of variables, expressions, and equations. We videotaped each teacher for five consecutive lessons, just as they formally introduced algebra for the first time. For the first four lessons, we asked teachers to teach as they normally would. At the conclusion of the fourth lesson, we provided teachers with a set of three mathematics problems to assign students during the fifth lesson. We also administered a questionnaire before we recorded the first lesson and interviewed the teachers after the last lesson. All of this was done in parallel across the four countries participating in the VIDEOMAT collaboration. Although we do not directly compare across countries in this chapter, our observations were no doubt colored and refined by the regular discussions we had with researchers in Finland, Norway, and Sweden.

In describing each case, we focus on three questions. First, what are teachers' conceptions of algebra as a subject matter? Second—and where we place the bulk of our attention—how do teachers introduce algebra to students? And third, how do students apply their early learning? Our small case-study design does not enable us to directly connect what we observe in the classrooms to student beliefs and learning outcomes. What we can do, however, is develop a rich understanding for these particular teachers of how their own conceptions of algebra and their instructional methods impact the learning opportunities they create for their students, and we can generate hypotheses to guide future, larger scale studies.

Participants

The two teachers who are the focus in this chapter were part of a larger study for which a total of four U.S. teachers were recruited, two at each of two middle schools. The sample was strictly one of convenience; we recruited teachers based on prior relationships with their school principals and mathematics departments. All four were volunteers, comfortable with us recording their instruction for a week. From the four, we selected the two teachers for the current chapter whose instructional approaches most differed from each other, thus giving us the widest possible range of approaches in our small sample. In spite of their differences, based on many years of work in U.S. classrooms, we would judge both of these teachers to be typical in terms of the instruction we see in American classrooms.

Ms. A taught seventh grade in a suburban middle school, composed of just over 1000 sixth- through eighth-grade students. The student body's ethnic make-up was mixed, but predominantly White. Seventeen percent of students at the school were classified as socioeconomically disadvantaged (i.e., eligible for the free or reduced-price lunch program). There were 35 students in her Mathematics 7 class, which was part of the college preparatory track. The class met daily for 50 min. Ms. A had a Bachelor's degree in Liberal Studies and a Master's in Education Technology. She had a multiple subject credential and 19 years of teaching experience, the last 12 of which were spent teaching mathematics.

Ms. B taught sixth grade in an urban setting. Like Ms. A, her middle school was composed of sixth- through eighth-grade. The student population was just over 1500 and was mostly Hispanic, but with large percentages of White and African-American students, as well. Forty-seven percent of students at the school were classified as socioeconomically disadvantaged. Her mathematics class (i.e., Mathematics 6) was mixed in terms of ability level and there were 30 students in it. The school worked on a "block schedule." On Mondays they met for 47 min and on Wednesdays and Fridays they met for 90 min. For the purpose of our work, we treated Wednesday and Friday as two lessons, each. Ms. B had a Bachelor's degree in mathematics and economics and a multiple subject teaching credential. She had been teaching for 16 years, all of which were spent teaching mathematics.

From our observation, both teachers appeared to have a good rapport with their students and to genuinely care about their students and the quality of instruction they provided. Engagement across all lessons was high; classroom management problems were virtually absent.

Teachers' Conceptions of Algebra

In a pre-survey given to each teacher, we asked, "What, in your opinion, does the introduction of algebra include? How much algebra is covered in your course?" Their answers were as follows:

Ms. A: Strong basic mathematics skills in the four operators (add, subtract, multiply, and divide). Knowledge and common usage of order of operations. Strong skills and facility with integer operations. Strong knowledge of vocabulary terms and their meaning.

Ms. B: An introduction to algebra includes:

- Developing students' understanding of patterns and functional relationships using words, tables, graphs, visual representations, and/or symbols.
- Developing an understanding of equality.
- Building an understanding of the concept of variable, including evaluating expressions and solving equations.

Algebra is embedded throughout our entire Mathematics 6 course. There is explicit instruction of the concept of variable, generating and evaluating expressions, pattern explorations leading to generalizations, and solving basic equations in the beginning of the school year. As students learn about different mathematical concepts (such as integers, fractions, proportional relationships, percents, etc.), connections are naturally made to how these concepts relate to variables, expressions, equality, and equations.

Ms. A's description included a strong emphasis on operations with mention also of basic skills and vocabulary. Furthermore, she discussed these things in terms of usage and facility with. The definition provided by Ms. B, in contrast, was much more conceptual. Her emphasis was on things like patterns and functional relationships leading to generalizations, the concepts of equality and variable, and of making connections. She discussed these things in terms of developing understanding and building understanding. Both teachers' beliefs were reflected in the way they introduced algebra, as described below.

Introduction to Algebra in Ms. A's Classroom

Rules About Mathematical Notation

Ms. A had barely begun her introduction of algebra before rules about notation became the object of discussion. Just 12 min into her first lesson and after having only discussed basic vocabulary (i.e., variables, expressions, and equations), Ms. A brought up possible errors in convention. For instance, she stated that it is incorrect to write x as $1x$ and $4x$ as $x4$. The rationale she provided students for the latter was that 4 is the coefficient and, by definition, a coefficient is the the number in front of the variable. She continued the rationale with:

00:15:49 Ms. A: The number in front of the variable automatically means to multiply. So remember, mathematics people, they like it neat, and clean, and tight with as little writing as possible. So $4 \cdot x$ would actually be incorrect right here. Now technically, does it mean the same thing? Take 4 and times it by *x*? Yeah, it means exactly the same thing, but it is not correct. Beginner move, right? So beginner. By the same token, 4(*x*) is not correct, even though it does mean 4 times *x*. It means exactly the same thing, but there is a certain way of doing algebra. Neat and clean and tight. Now *x*4 is just, "I missed the boat completely. I was there with you, but I didn't get your meaning." OK? Because this is just totally, just conceptually, incorrect. Number, letter. Number, letter. Always it's going to be number, then letter.

00:18:35 Ms. A: And then we went over 1. 1*x*. So, I'm going to give you a little bit of time to kind of get this down; but by half way through the school year if I see 1*x*, I'm going to have to mark it incorrect. But for our purposes right now, we're just kind of getting there. We're just learning it, so I'll kind of, I'll give it to you for a little while. But at a certain point, if you have the one, I'm going to have to mark it wrong. Right? Again, it means I kind of get it but not really.

The statements above illustrate that the teacher holds fast to rigid rules of mathematical notation, placing such a high value on them that they are almost the first things students hear when algebra is introduced. Ironically, the rules she conveys might lead a student to form a misconception of the meaning of the equal sign. If two values are equal (e.g., $4 \cdot x = 4x$), but only one of the two (i.e., $4x$) is correct, what does the equal sign mean?

Ms. A's emphasis on standard notation continued similarly in the second lesson. When she then discussed the product of *p* and 12, she stated that she would have to mark 12 · *p* wrong and characterized it as a beginner move.

00:38:20 Ms. A: When a number touches a letter it already means to multiply. So [12 · *p*] is redundant. It's saying "12 times times *p*." Think about math people. They like it neat and clean and tidy. Is [12(*p*)] neat and clean? No. Is [12 · *p*]? No. Okay now technically, mathematically, do they all mean the same thing? Take this and times it by that? Yes. But the mathematical conventions, right? They supersede that and we have to put things in a certain such kind of order that makes sense. Okay. Okay and this is common usage. You go to Asia, Europe, [Africa], this is how you do it.

Ms. A was unwavering about the importance of standard notation, herself stating that its use supersedes mathematical meaning.

Mathematical Procedures

Ms. A emphasized not only the importance and universality of notational rules (as illustrated above), but also the importance and universality of mathematical procedures. In Lesson 1, the class spent time evaluating algebraic expressions. Ms. A presented a ritualized approach for doing so and addressed the importance of the rituals she used, as for example in the following excerpt evaluating $x + 5$ (Fig. 7.1).

00:19:45 Ms. A: Okay now, right here, $x + 5$. Something plus five. Okay? And I know, you guys are like, "Oh my gosh [Ms. A]. I can do this in my head without even writing it down." This is true; but later your work is going to get very complex. And if you're doing the technique I'm showing you right now, your life later will be so much easier. Okay? So you should be using parentheses every single time you substitute, even on these simple problems. Do we even have to write it down? I know. It seems ridiculous. What I'm trying to do is teach you the process, not just, "Hey give me an answer." Because you could be done just like that [*snaps*]. The process: copy the problem, show your work, and then put your answer. Notice, everything goes under, under, under, under. Skip a line [before the next problem]. Do you notice? Substitute then solve. Okay? It looks a certain way. It is done a certain way. This is the way of algebra. And I know some of you are like, "Oh no, this is just [Ms. A's] way. She's like a control freak. She is making us do it." No, no. If you go to Asia, if you go to Europe, if you go to [Africa], they all do it like this. So this is *the* way. It's not mine. Ok? I'm just trying to show you *the* way. Substitute, then solve. So there's a procedure for doing this. That's what I'm trying to teach you. And most of you have found already if you show all your work you will not be making the errors. Okay? So the difference between an A student, in my mind, and everybody else, the A student is meticulous in their work. They write down all these details. That, by first glance, they look really unnecessary. You're looking at this

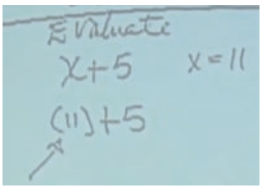


Fig. 7.1 On the board when evaluating the expression $x + 5$

going "Why? Why do you need parenthesis? I can do it in my head." Okay? But the A student, they will write down all these details.

In Lesson 2, Ms. A elaborated on the procedure for writing algebraic expressions, including not only the procedure to follow but also all of the exceptions to it. Ms. A began by soliciting from students the words that came to mind for each of the basic operations. For example, for subtraction, students supplied: `minus`, `decreased by`, `subtraction`, `less than`, `difference`, `subtracted from`, and `take away`. She continued:

00:27:55 Ms. A: So we're going to be turning the words into a mathematics problem. Now how do you do this? Well the standard operating procedure, everything goes in order from left to right. Just kind of like reading. Again you have to know a little bit of English to do a little bit of math. So everything goes in order from left to right, and in life aren't there always exceptions? So we do have a few exceptions. "Less than." "More than." "Added to." "Subtracted from." And then, of course, "increased by." "Increased by" and "decreased by." So everything will go in order from left to right. Except, when you see the words "less than," "more than." Except when you see the words "added to" or "subtracted from" or "increased by." And so these are the fine nuances of understanding English, and then translating it into a math problem. Oh, one other thing. This one: "sum." "Difference." And in my head I kind of see those two within parenthesis. Why? It's because in order to get the sum you have to add first, or to get the difference you have to subtract first.

Ms. A then demonstrated with the sentence `4 times the difference of n and 2`, and noted that it includes one of the exceptions (i.e., the `difference` indicates that parentheses are necessary). She continued with "`1 more than the product of` *p* `and 12`." She pointed out that it, too, has an exception (Fig. 7.2).

00:37:00 Ms. A: So everything goes in order from left to right, except I see an exception. Do you notice? So do I start with [1 more than]? Or do I end with [1 more than]? Standard oper-

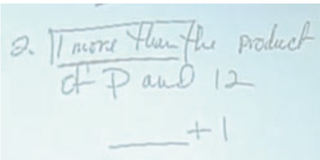


Fig. 7.2 On the board when demonstrating how to write an expressions

ating procedure says "no." You start with [1 more than] but then you go, "Wait a minute, this is an exception." Therefore, just like Zach said, you would have to end with [1 more than]. One more than something. In order to have more, don't you have to know what you started with first? Because otherwise you don't know if you have more or not. Fine nuances of the language.

The class continued writing algebraic expressions from verbal expressions the following day. When Ms. A asked students to write an expression for 1 more than the quotient of 21 and *b*, a student offered 1 + (21 x *b*), Ms. A responded

00:45:32 Ms. A: Okay standard operating procedure. Everything goes in order from left to right except, *except* "less than," "more than." And if it's an exception do we start with it? Or do we end with it? So I have 1 more than some thing.

Ms. A made no comment about the student's confusion of product and quotient, and instead focused on the placement of the "+1." Interestingly, the class had recently reviewed the commutative property of addition, but the teacher made no mention of the order of the addends having no effect on the value. Again, her attention to rules/procedures had the potential to lead to confusion about equality. And, like notation, the need to follow particular rules/procedures superseded mathematical reasoning.

In Lesson 4, Ms. A was specific about the steps involved in checking an answer with substitution and about how those steps were to be written out. There were, it seemed, three lines of work necessary to qualify as complete (Fig. 7.3).

00:35:25 Ms. A: Now some of you, your check yesterday was on one line and that [8(4) = 32] is what you did. In my mind, that is kind of ripping me off a little bit, okay? So I saw many people who had work of a very high quality. "Here is how [Ms. A] says to do it and I just do it. It is not [Ms. A's] way, it is *the* way." Some of you are a little resistant, but this is *the* way.

In sum, Ms. A presented the process of writing algebraic expressions in a very procedural way. According to Ms. A, there is a "standard operating procedure" for evaluating algebraic expressions: numeric values must be placed in parentheses. There is also a standard procedure for writing them: record from left to right, and be on the lookout for a series of exceptions (which, in fact, were evident in nearly every

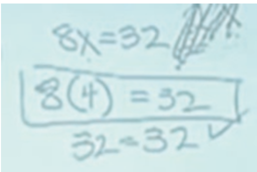


Fig. 7.3 Example of a student's correct solution of $8x = 32$, what Ms. A called "*the* way" to do it

example students saw). There is also a standard procedure for checking for accuracy: copy the original equation, substitute (complete with parentheses), and state the equality. Nowhere did she draw students' attention to the meaning conveyed by the words and how that meaning might be represented in mathematical notation. Nor did she discuss what the value of the variable means.

Finally, Ms. A promoted a clear procedure for solving equations. When she described it, she began by writing $x - 7 = 13$ on the board and next to it the steps necessary to solve it. The steps were as follows:

1. Isolate the variable
2. Do the inverse operation
3. Do it to both sides
4. Cancel cancel (zero-sum pair)
5. Solve

She used the steps to solve $x - 7 = 13$ and three other, similar problems. For none of the problems did she provide a conceptual explanation for either "do the inverse operation" or "do it to both sides." She explained "cancel cancel (zero-sum pair)" with "`What is 7 dollars minus 7 dollars?...` " `If it's really plus 0, it's like doing nothing.` During this demonstration, Ms. A conveyed the necessity of completing each of the five steps. She went through each step publicly, then asked if anyone got to the answer before she did. Several students raised their hands and she responded, "`Are you doing your steps? Because that's really important to me. Because that's how you're going to get credit.`"

Ms. A then assigned four problems for students to work on independently and stated:

`00:33:14 Ms. A: I know you can get the answer. What I want to see on your paper is the procedure. How many of you would agree these are not hard mathematics problems? Okay, because we use baby steps to teach you the procedure. Some of you, you're stuck up on the answer. 'Oh I already got the answer.' No. No, no, no. That's not all I'm trying to share with you, okay? I need you to get the procedure, not just the answer.`

Ms. A then worked through each of the four problems at the board and demonstrated how to substitute the solution into the initial equation to check. She informed students that they won't get credit if they don't show their work and show their check. Students were expected to use the solution steps on problems assigned for homework, and instruction on the following day began with Ms. A asking students to list the steps of the solution procedure, without referring to their notes.

It was possible for each of the four, simple problems to be solved by students through reasoning, thus enabling students to forego the steps Ms. A had listed. She made clear, though, that that approach was unacceptable. It was instead critical in her class to use these simple problems to practice procedures, the steps of which were disconnected from meaning.

Praise for Speed and Neatness

In classrooms (and elsewhere) values are often conveyed through praise. The two things that received praise most frequently in the lessons we observed in Ms. A's classroom were speed and neatness. Ms. A made several comments on the desirability of working quickly. For instance, in Lesson 1, when she assigned students a set of problems to complete on their own, she instructed them to `do them as fast as you can`. When she circulated the room as they worked, she said approvingly to one student (but at a volume audible to the class), "`speedy, speedy`." In Lesson 3, during private work, she commented to a student, "`Nice, nice. Wow you're speedy`." In Lesson 4, when she assigned a set of problems, she commented, "`And there is a certain amount of speed and accuracy that goes along with this. Because if you know what you're doing you can just crunch it out crunch, crunch, crunch. If you are stuck somewhere, then it goes a little bit slower, I think`."

As illustrated in the subsections above, Ms. A indicated that mathematicians like things "neat and clean" and she valued that in her students' work, as well. She spoke often of the need for students to be meticulous and fastidious. Ms. A also made repeated, vague comments about the desirability of "`work of a high quality`." In looking at the student work as she provided praise of this sort, we speculate that she was commenting on some combination of neatness, completeness, and generally following the steps provided. In Lesson 4, she commented to a student "`Wow, your work is so neat and extremely organized. Very nice Caitlin. That is going to take you far in and of itself`."

At no point did we hear Ms. A praise students for their attempts to find mathematical meaning in what they were doing (though admittedly, there were few opportunities to do so given the tasks assigned).

Mistakes

Rules, procedures, speed, and neatness all had a place in Ms. A's classroom. Mistakes did not. Student errors were rarely discussed, even when the opportunity to do so presented itself. For instance, in going over homework problems in Lesson 2, when students had errors they were directed to find the source of their error on their own. In Lesson 4, when the teacher called on a student to answer a homework problem and he answered it incorrectly, Ms. A simply moved to the next student. This was the case even when multiple wrong answers were provided to a single problem (which happened three times). Indeed, this was a consistent pattern throughout the lessons we observed. At best, mistakes received a "no," and at worst they received an "OK." In either case, Ms. A would then simply call on the next student. At no point did we see Ms. A take advantage of a student error to explore a mathematical misconception.

Through her comments as she distributed graded quizzes, students would have learned of her approval of high grades and correctness. As she returned the quizzes, for each that had earned an A (i.e., 90% correct), she announced to the class "`big, fat A`." Those students who scored 100% were greeted with a public "`flawless`" as they received their quizzes. When asked in the post interview what she looks for in high quality student work, she responded

`I think attention to detail and precision will get you accuracy. You know if you're kind of haphazard, you know and your work is not of a high quality. And I don't really mean neatness per se, although that is a huge factor. But I mean in the way of how you set it up, how you present it. The procedures and the quality of it I think will lead to greater accuracy. Or, I should say, less opportunity for error. So to me, it's a big deal. I mean that's one of the differences between an A student and a B or C student. You know we have kids of all similar test scores and similar grades, and yet, how come some of them get A's when their same, smart peer is getting a B? And if you actually look at the quality of their work, the students who are fastidious in their work just write everything down. Everything. They're going to be the ones who, consistently, get the A's.`

Ms. A: Post-interview

At the end of the week, in individual post-interviews, we asked teachers why they chose to introduce variables and expressions in the way that they did.

`Ms. A: My initial response is that's just how it's been done. We have these textbooks and they're organized in a certain order. We adopted the textbooks, meaning we've previewed this one and that one. We chose this one, and so for our district, do the book.`

The general content and order of Ms. A's introductory lessons indeed match closely the textbook adopted for the class (i.e., Bennett et al., 2008). The first chapter ("Principles of Algebra") has subsections for Evaluating Algebraic Expressions, Writing Algebraic Expressions, Solving Equations by Adding or Subtracting, and Solving Equations by Multiplying or Dividing, which mapped exactly onto the topics of the first 4 days of our video recording. The sample problems she used were similar to or the same as those in the book and the problems assigned to students were either from the text or its accompanying workbook. In that sense, as Ms. A indicated in her post-interview, she did "do the book." However, nowhere in those sections did the text describe nonstandard (and therefore "incorrect") notation, the use of parentheses when substituting, or a standard left-to-right procedure for writing algebraic expressions. In fact, it is noted in the text that because addition is commutative "1 more than the product of 12 and *p*" can be expressed either as $12p + 1$ or $1 + 12p$.

Introduction to Algebra in Ms. B's Classroom

Representing Patterns

The differences between Ms. A and Ms. B were evident from the start. Ms. A's definition of algebra made reference to fluency with operations. Ms. B's definition, on the other hand, was much more conceptual. As was the case with Ms. A, Ms. B's definition was very much in line with her instruction. Over the course of 4 days, the class completed several examples of patterns, both with and without visual representations, and generated algebraic expressions to describe them.

She started her first lesson by stating that

`00:33:14 Ms. B: Today we're going to look at number patterns. And, in fact, this whole week we're going to be exploring different types of patterns. But today we're going to focus specifically on ones that are represented as numbers to begin with. And then what I hope to do is to take these patterns, and be able to write them as variable expressions.`

After a brief discussion about the definitions of "variable expressions" and "numerical expressions," Ms. B assigned two problems on which students were to work independently.

Figure out the next three numbers in the sequence. Explain how you determined these numbers.

1. 4, 8, 12, 16, ___, ___, ___
2. 9, 14, 19, 24, ___, ___, ___

Ms. B began public work on the problems by calling on students to complete the pattern in problem 1. The student who named the numbers to fill in the blanks added that the numbers were multiples of 4. Ms. B then created a function chart in which students were to rewrite the pattern. She created also a function machine and described how it works with, "`So if I put the number 1 into my machine, out comes the first number in my sequence. So out comes 4.`" Students then helped to complete the function chart, from 2 to 6 (Fig. 7.4). She then asked students,

`00:24:16 Ms. B: When we put a number into our machine, what are we doing to it in this machine before it spits out? In other words, if I put 1 in, what's happening here so that the 4`

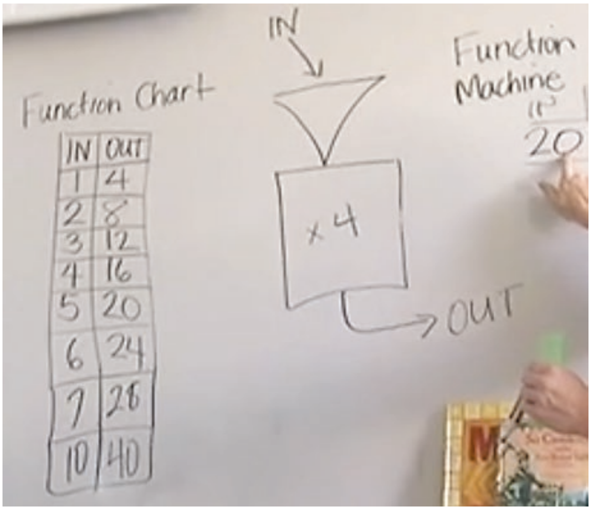


Fig. 7.4 Function chart and function machine representing the same number sequence

`comes out? And if I put a 2 in, what's going on in here, so that I get an 8 out? What's happening in this machine that represents this number sequence?`

Students were asked to briefly share with a neighbor their thoughts about what was happening in the function machine. When the class resumed public discussion, a student was called upon and stated that what goes in is multiplied by 4, and the class confirmed that that is the case for all numbers in their function chart. Ms. B then asked the class what comes out if 7 and 10 are put in, and the class responded in chorus. The class then considered 20 as an input and the student called upon replied that it would be 80. Finally, Ms. B asked, "`What if I put in any number?`" and added *n* to the "in" column of the function chart. She asked students to think back to what they did the prior week, to write 4 times any number, and asked students to talk it over with a neighbor. When they reconvened, a student offered $4n$ as the answer. Ms. B accepted it as correct and a brief discussion of multiplication notation ensued. Ms. B referred to a poster on the bulletin board indicating that $7 \times 5 = 7(5) = 7 \times 5$. She stated that any of those forms could be used, but that $4n$ is the standard algebraic form. Ms. B pointed out that they had written a variable expression and reminded them that that was the goal for the day.

Thirty minutes into the first lesson, discussion turned to the second problem (i.e., the number sequence 9, 14, 19, 24, ___, ___, ___). The class discussed patterns students had seen and then the teacher created a function chart and function machine. When they turned to writing the corresponding expression, the teacher began,

`00:35:52 Ms. B: Ok, this one's a little trickier. What's happening? Well, do you guys remember what we did last time where we plugged the number in and we said we thought last time we were multiplying by 4? Did you remember that? And,` did you remember this time we said there's something to do with 5? Five is being added over and over and over again? Did you notice that? So there's something going on with 5, isn't there? You agree with me? Ok. Well, let's see if it's just multiplying by 5. If I have a number, and I multiply it by 5, let's see if we get what we want to get. So let's try the first one. I plug a 1 in. What is 1 times 5?

Ss: 5.

Ms. B: Am I getting what I want to get?

Ss: No.

Ms. B: How much more do I need to get there?

Ss: 4.

Ms. B: I need 4 more, so I have to, if I multiply by 5, and then I add 4, am I going to get what I need?

S: Yes.

After testing with inputs of 2, 3, and 4, as a class, Ms. B and her students summarized that each time they multiplied by 5, then added 4. Ms. B had students talk with their neighbor briefly about how that would look as a variable expression, and then stated (and wrote) publicly that the variable expression is $5n + 4$. Finally, the class checked the expression with an input of 5.

The next class period was a double lesson and it continued in the same vein. It began with students working independently on the following questions:

1. Determine the next three numbers in the sequence. 3, 7, 11, 15, __, __, __
2. Describe the patterns you notice.
3. Make an "IN/OUT" function table for this data. Can you determine a variable expression for the sequence? How?

Ms. B began the discussion by asking students about the patterns they saw. The first student on whom she called said that she had seen the addition of 4 and as a class they confirmed its accuracy, using the rule also to identify the missing values. Ms. B indicated that, like yesterday, they had a pattern with 4. Ms. B then rewrote the pattern into a function table, and pointed out that the "IN" column indicated the values' place in the sequence. She asked the class for help completing the "OUT" column and they answered in chorus. With the values in a chart, she asked students if they saw new patterns. The first student called upon took a long time (27 s) to think before she answered that every time you add four to the number (the same pattern that had been identified earlier). The teacher praised her thinking and confirmed that the pattern can be seen in this representation as well. Ms. B followed this with

00:22:15 Ms. B: One of the things we were working on yesterday is trying to create a variable expression. And since we know that four is happening over and over and over again, we're adding four each time, we know our variable expression is going to have something to do with four. Right? Does that make sense? Okay. So it's going to have something to do with four. What do we do next? How do we start figuring out what's going on in this pattern? How do we figure it out? Danny what do you think?

S: Well, I was actually thinking of all of them, then I figured it out. You do: times 4, minus 1 and you get the other number.

Ms. B: So you're sort of thinking about our function machine from yesterday, maybe? And how we put a number in, which is the first column, and out comes something and in between that there's something going on. And you're feeling like it's what's happening there?

S: Times 4, minus 1.

Ms. B: You're noticing a times 4, minus 1 pattern. Can you talk us through a little bit about how you got that?

S: Well, um, I did, um, 4 times 1 equals—

Ms. B: Okay let me write that here. Okay so 4x1=4.

S: And then 4 minus 1 equals 3.

Ms. B: But—okay so if we just did times one we'd only get four and we really want to get an answer of three.

S: So you subtract one.

The class then tested Danny's rule with 2 and 3 (Fig. 7.5).

When Ms. B asked about writing an algebraic expression, which she explained as an expression that represents any number in the sequence, the student she called upon readily responded $4n - 1$. Ms. B asked students to use the expression to find the 20th term. They were to do so privately, in their notes, before checking with their neighbors. One student was then asked to describe what he did. The students responded quickly, in chorus, when the teacher asked what the 100th and 1000th terms would be. Ms. B then pointed out that the latter questions are made much easier when you know the rule to the pattern.

After completing three patterns problems that included only lists of numbers, Ms. B turned to sequences that were represented by visual representations. For one of these problems (Fig. 7.6), one student, Hannah, volunteered to have her work projected as the class discussed their work. Students were to have figured out the pattern of the number of small squares in each figure.

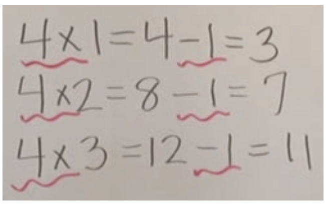


Fig. 7.5 Testing Danny's rule of multiplying by 4 and subtracting 1

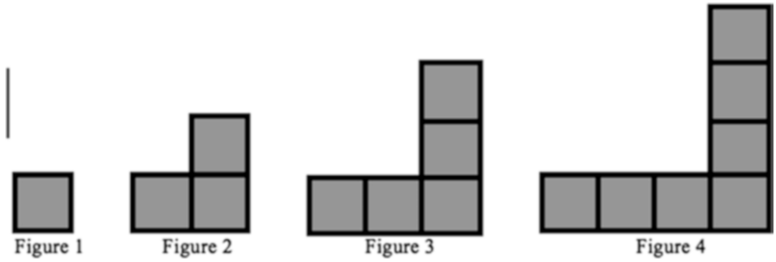


Fig. 7.6 Sequence of L-shapes

In completing the corresponding function table, one student pointed out that he added 2 each time. Referring to Hannah's work, Ms. B led a discussion of how to find the algebraic expression for the pattern:

00:13:50 Ms. B: You're noticing we're adding 2 each time? Great. All the way through, right? Can we expect that that's going to continue? Will that keep going on? If we made bigger and bigger figures, wouldn't that keep happening?

S: Yeah.

Ms. B: Absolutely, okay. So 2… There's a lot of 2's here. Yeah? So we come down to this part of the problem, where we're having to write an expression. Hannah's remembering [shown in her writing of *n* x 2 − 1] that it's got to have something to do with 2. Because that's the pattern we're noticing over and over again. And so what she said was it's 2 times the number that we're working with, the figure number, and then she also has this minus 1 here. So Hannah, I want to explore just a little bit as to how you got [*n* x 2 − 1]. Do you remember how?

S: Um, I tried it, I kind of did, um. I don't know how to explain it but I started with doing multiplying it by 1 and adding 2, but it didn't work. So I tried multiplying by 2 and subtracting 1.

Ms. B: Okay, so were you looking back at your numbers [in the function table]?

S: Yeah.

Ms. B: All right, so let's look at the numbers and maybe we can recreate some of the thought that Hannah went through. So, if she knows that it has something to do with 2, let's say she just tried 2 times the number and saw if that worked. So for every time I want put a number in for *n*, we're going to use the figure numbers over here. Let's start with the first one, It's the easiest one, right David? If I put a 1 in here David, what's 2 times 1?

S: 2.

Ms. B: Equals 2. Okay, so if we try, test out the number 1, 2 times 1 equals 2. Is that the answer we want to get for the first figure?

S: No.

Ms. B: No. What do we want to get?

S: 1.

Ms. B: Okay, how do we get to 1 from here?

S: Um, 2 minus 1.

Ms. B: We have to take away 1, okay. So let's take away 1. And that'll give us our 1. Okay, great, that one worked.

The class then checked the expression for the second and third figures and used it to find how many small squares would be needed to build Fig. 20. In explaining how to represent the repeated addition of a number in an expression, Ms. B stated that if we do something over and over and over and over again, a faster way than adding the same number over and over and over again is to just multiply by that number. At no point, though, did the class identify how the values in their algebraic expression related to the figures it represented.

Ms. B then presented the following diagram (Fig. 7.7):

Working in groups, students were to record the perimeter of the first three hexagon trains, use manipulatives to build the fourth train, and describe the fifth train. They were to then complete a function chart, search for patterns, and identify the variable expression. In giving instructions, Ms. B explained:

Ms. B: Well today I'm stepping back a bit. Today is your day to do similar things with a new assignment. Same type of stuff, but today I'm not going to be leading you through it. Today you're working with your group to work on this. And you're going to be guiding each other.

As students worked, Ms. B circulated to ask and answer questions. She reminded several groups of what they had done with the repeated addition of 2 in order to create an expression for the homework problem and asked them to use that knowledge to create an expression for the hexagon trains.

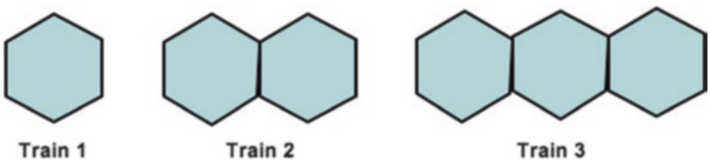


Fig. 7.7 Hexagon train

After 14 min of private work, public discussion resumed. With repeated student input, the teacher created a function table and called attention to the repeated addition of 4.

00:47:24 Ms. B: How does it help us create a variable expression, knowing that this pattern of 4 is going to keep going over and over and over again? Danny, what do we do?
S: Since you know you're using 4, you know that you have to use 4 in your equation.
Ms. B: Okay, good.
S: And since we're going, we're going up, so its multiplication.
Ms. B: Okay.
S: So we have 4 times.
Ms. B: So we're going to start with our 4 times a number and see if that works. Okay. So let's look at the first n. What if $n = 1$? David, if $n = 1$, what's 4 times 1?
S: 4.
Ms. B: And is that the answer we want? What do we want?
S: 6.
Ms. B: So what else do we have to do to get there?
S: plus 2.
Ms. B: Add 2. Let's try it. So we're saying, maybe we'll try this expression. 4 x 1 = 4, + 2 is 6. It worked. Um, can– Lily, can you tell me if it works for the second example?

The class confirmed that the same expression works for the second, third, and seventh trains and they were convinced. Finally, they used the expression to find the 10th and 100th trains.

Lost Opportunities for Conceptual Connections

A commonality across both teachers was that they prescribed a solution procedure and they presented well-organized steps for executing it. Ms. A went so far as to write the steps on the board. Ms. B never made the solution steps so explicit, but it was clear from multiple enactments what they were:

1. Complete the pattern
2. Create a function chart
3. Create a function machine (sometimes)
4. Find the pattern by looking at the numbers
5. Write general rule as variable expression

It was in Steps 4 and 5 that Ms. B lost opportunities for conceptual connections. When it came to finding the pattern and representing it as a variable expression, the process seems to have been (1) identify a value that you see repeatedly and multiply your variable by that value, and (2) if the result of your multiplication is not the desired value, add or subtract the appropriate amount. Ms. B conveyed these steps via multiple uses of statements like, "there's something to do with 5" or "there's something to do with 4." Only once, in the last lesson, was there a mention that we use multiplication to represent repeated addition. It is clear that students adopted Ms. B's practice without understanding it. Recall one student's explanation that "Since you know you're using 4, you know that you have to use 4 in your equation. And since we're going, we're going up, so its multiplication." The explanation for the use of multiplication to represent repeated addition is a particularly unfortunate omission, given the availability of the visual materials that accompanied the patterns. Ms. B might have used them to show why the value of an output increased by a particular amount each time the value of an input increased by one. Similarly, there was no discussion about why students should expect to add to/subtract from the product (e.g., 2 added to the product of 4 and *n*), if it did not yield the value they sought. Ms. B simply asked "how much more do I need to get there?" and the class followed along, not questioning why the addition/subtraction would be a sensible operation. Again, she might have used the visual representation of the pattern to explain the addition/subtraction of a constant. Students are likely to have come away from the series of lessons with a single image of what an expression looks like. When asked to create an expression, they could identify the coefficient as whatever number was recurring in the pattern and could use trial and error to determine the rest.

Ms. B: Post-interview

It came out in the individual post-interview that Ms. B had reviewed vocabulary with the students in lessons that preceded those that we captured. She had also spent time translating verbal phrases into algebraic notation, which is what we saw Ms. A do in her first videotaped lesson.

Ms. B: I find that when kids at this age first come in contact with variables, … they have a lot of questions about what is even happening. There is a lot of confusion, I think, around seeing letters now in their mathematics problems, and so I find that starting off early with this idea of patterns—with something that they can really grab onto and notice what's happening and then be able to explain sort of what's happening—in the future, tends to be a really nice way to bring in that idea of variable. What's happening with the *n*th number?

… They haven't thought about it that way, but I can lead them to that pretty easily. … [In the lessons before this study began], we spent some time talking about how to look at verbal phrases and translate them into symbols. We're talking about how mathematicians often times take ideas that are there but symbolize it and use the symbols for their work. So, breaking down common vocabulary words … but also and then looking at verbal sentences and bringing it into numerical expressions, but also what happens if there are some unknowns in there and so we kind of introduce the idea, the specific idea of variable, right prior to doing the pattern work.

A Brief Summary of the Two Classrooms

The teachers we observed took very different approaches to introducing algebra to their students. Ms. A's instruction heavily emphasized mathematical notation and procedures, with very prescriptive rules for "*the* way" to do mathematics. She praised her students for speed and neatness, but mostly ignored their mistakes, neither reprimanding nor exploring them. The focus of her lessons was on the correct way to write algebraic expressions and the solution procedure for solving algebraic equations.

Ms. A's lessons were entirely teacher-directed. Public student talk was minimal. The few questions students were asked to answer publicly were mostly ones that asked them to provide a numeric answer and nothing more. A few required a brief, procedural explanation. The work students completed privately began only after examples had been provided by the teacher.

In contrast, Ms. B focused on representing geometric patterns algebraically. She guided students through the process of identifying patterns in numeric and visual representations and then generating algebraic expressions for those patterns. She also structured her class differently, incorporating student work and public discussion.

Ms. B's recurring strategy for organizing students' work on problems involved posing a problem, allowing a brief period for students to chat with their neighbor at an adjoining desk (which, it should be noted, they all actively did), they then resuming public work. Ms. B frequently called on students randomly to share their answers. Those answers often required that students explain their thinking and sometimes involved projection of their work for the whole class to see. Ms. B provided correction, where necessary. But in spite of her attempts to make meaning of algebra, there were lost opportunities to make critical connections between ideas that may have limited the impact of Ms. B's efforts.

What We Might Speculate About Students' Early Learning

The activities described above were those that occurred in the first four lessons that comprised these students' introduction to algebra. In order to assess their students' learning at this point, we asked that both teachers give their students the same set of three problems to work on during Lesson 5. The way the students dealt with these problems reveals the early impact these two different classrooms had on students' learning. Here, we will limit our discussion to the first and third problems of that set, which represent problems both typical (problem 1) and atypical (problem 3) of U.S. classroom instruction. The problems were as described in Fig. 7.8.

In both the classrooms, students worked in groups of three or four. Problem 1 was an extension of the work Ms. A's students had done in Lesson 3 and Lesson 4. They had not yet discussed two-step equations, but students used what they knew about solving equations with addition and multiplication to figure out the problem. All of Ms. A's students had the correct answer in their written work (although only four students produced the check that had been required of the teacher on prior assignments).

Unlike Ms. A's class, the work Ms. B's students had completed on previous days led more toward an understanding of the third problem (on turning a pattern into an expression) than it did to the first problem (on solving a two-step equation). None of the groups in Ms. B's class were able to identify the correct solution to problem 1. Of the seven groups in the class, one failed to reach an answer. Students in another group answered 630, having multiplied 150 by 4 and adding 30 to it. Students in a third group answered 120, having identified the value of x in the equation, but misinterpreting its meaning. Students in the remaining four groups identified 30 g as the

1. In Zedland, the cost of shipping a package is calculated with the equation:

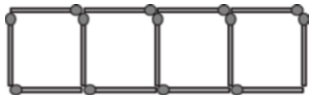
   $y = 4x + 30$, where x is the weight in grams and y is the cost in Zed dollars.

   A package that costs 150 Zed dollars to ship can be written with the equation:

   $150 = 4x + 30$

   How many grams does that package weigh?

3. In the figure below, 13 matches were used to make 4 squares in a row.

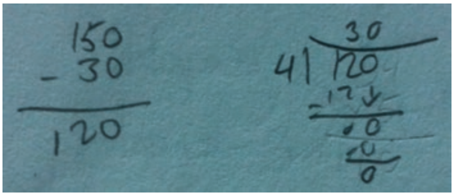


   How many squares in a row can be made in this way using 73 matches?

   How do you know?

Fig. 7.8 Two problems for group work during Lesson 5

Fig. 7.9 Student solution of the equation $150 = 4x + 30$



weight of the package. None of them reached an answer via formal steps. Rather, they informally "unwound" the problem, as can be seen in the student work in Fig. 7.9.

On problem 3, four groups in Ms. A's class reached the correct answer (i.e., 24) through arithmetic reasoning. That is, they either (a) subtracted the initial 13 matches from 73, divided the resulting 60 by 3, and added to it the initial 4 squares, or (b) subtracted the 4 matches that composed the initial square from 73, divided the resulting 69 by 3, and added the initial square. Two of these groups followed up their work with algebraic notation, albeit with the assistance of the teacher. Another group obtained 24 by dividing 73 by 3 and disregarding the remainder. The remaining four groups used arithmetic to arrive at an incorrect answer or no answer at all.

For the students in Ms. B's class, problem 3 should have looked familiar and they might have been predicted to perform well on it. Overall, though, they struggled. Students in only one of the seven groups made a function table. Those students not only identified that 24 squares could be made from 73 matches, but also arrived at $3n + 1$. One other group identified 24 as the answer, having found it by counting in threes. And members of two other groups found this answer by drawing out the matchsticks and counting them. The remaining three groups used an ineffective arithmetic strategy (e.g., 74/4).

In sum, given their work on the prior days, we expected students of Ms. A to have an advantage over students of Ms. B on the problem that involved solving an equation. In fact, all student groups in Ms. A's class solved the equation correctly. Just over half of the student groups in Ms. B's class did so. On the problem that involved the recognition of a pattern we expected the reverse to be true. In fact, four of the nine student groups in Ms. A's class reasoned their way to an answer. Just over half of the student groups in Ms. B's class arrived at the correct answer, but most found it through less mathematically sophisticated means (i.e., through counting).

What can we take away from these findings? Two patterns emerge: (1) at this early stage of learning about algebra, students are still willing to reason about mathematics, and (2) instruction that attempts to build reasoning skills has a tendency to nonetheless become quite procedural. It should be no surprise that Ms. A's students—who had spent four lessons practicing "the steps" of solving algebraic equations—were able to carry out those steps in a new problem. To enact this procedure does not require a deep understanding of what one is *doing* conceptually; strict adherence to a set of rules will lead to a correct answer, and this is exactly what they were being trained to do. However, when faced with a situation that did not obviously call for enacting those rules, Ms. A's beginning algebra students were still willing to reason about a pattern using appropriate logical inferences. This is further evidence that students are willing to reason about mathematics when problems do not obviously demand strict procedural approaches.

The more surprising findings come from Ms. B's classroom. If we suppose that building an understanding of mathematics by studying patterns should generalize to other patterns, their performance on the third problem presents somewhat of a conundrum. However, there is reason to think that what students had been doing was not so much reasoning as following a different kind of procedure. All of the problems Ms. B's students studied in their first four lessons fit the same general pattern: discover the output given the input, and then write an expression for the *n*th output. Problem 3 posed a different task: discover the input given the output. This is not a trivial difference. In all their earlier problems, students were provided with several examples of inputs and outputs and could use the same general solution strategy of finding the common difference between the outputs. When they were presented with a single data pair (4 squares and 13 matches), there was no obvious difference to identify. It also may not have been clear to those students what the inputs and outputs *were*. The inputs student encountered to that point could be reasonably interpreted to refer to the serial position of the number or drawing. It is telling that the only group to arrive at the solution was also the only group to generate a function table, which could provide them with the input/output system they had been using.

In spite of Ms. B's attempts to build a connection between patterns and expressions, she may have inadvertently communicated to her students that only certain kinds of problems can be solved using the "procedure" she taught them. When faced with a problem that is superficially different from those they had been studying, students may have abandoned previous reasoning strategies, believing that they no longer apply. In order to develop a deeper conceptual understanding of patterns and functions, students might need a better grasp on how one identifies inputs and outputs, as well as the "reversibility" of the function (i.e., that one could identify an input based on the output). Up to this point their instruction had not addressed those ideas, so it should not be surprising that they struggled to find an appropriate solution method.

Discussion

Our interest in the introduction of algebra stemmed from a desire to investigate how teachers inducted their students into a way of thinking about the content. Algebra, because it marks a shift in students' mathematics education, presents an opportunity for a new beginning. We sought to articulate what the teachers valued, as conveyed to us directly and to their students via assignments themselves, work on them, and commentary that accompanied their work. Our own and others' studies have found that U.S. students think of mathematics as a collection of rules and procedures and our assumption was, because teachers are a primary socializing agent, that we would see evidence of these ideas in their classrooms.

The two teachers whose classrooms we observed had different conceptions of algebra as a subject matter. In her description of it, Ms. A emphasized operations, basic skills, and vocabulary. In observing Ms. A's introduction of algebra, a few themes emerged. Most conspicuous was her emphasis on rules and procedures and "*the* way" of mathematics. Her pedagogical moves reflected an attempt to teach rules and procedures in a way that was clear and accurate and could be remembered by students. She gave the bulk of her attention to explaining the steps to solving various types of problems and checked that students could reproduce those steps. Also evident was repeated praise for speed and neatness. She never discussed student errors and it seemed that they were something to be avoided also by her students. She was the clear authority figure in the classroom and students were expected to passively follow along as she worked. We speculate that students in her class, completing their first week of algebra, would be developing a belief that algebra is a rigid set of steps and notations, and that their only access to them is through their teacher. Students' role, it would seem, is to memorize and reproduce them as quickly and thoroughly as possible.

In contrast, Ms. B's definition of algebra emphasized patterns, functional relationships, and making connections. Consistent with her definition, she used tables and visual representations in an effort to develop an understanding of functional relationships. She walked students through the process of identifying the pattern in the representation she provided and helped students arrive at a general rule for it. Students frequently discussed with each other their thinking about a step of the solution process. However, key connections were consistently missing. We speculate that students in her class had, in the end, limited opportunity to develop the understanding of functional relationships that was her goal. It is more likely that they took away from the many pattern problems a belief that the expressions used to represent patterns are discovered through a combination of a little magic mixed with trial and error. Though rules and procedures were not emphasized in Ms. B's class the way that they were in Ms. A's, neither were they wholly replaced by sense-making.

Students practice particular cultural routines in their classrooms and they get good at them. Assuming some of what we saw is consistent with the experiences of the community college students we interviewed, it is understandable that sixth- and seventh-grade students would come away thinking of mathematics as a collection of rules and procedures. If we are unhappy with U.S. students' performance in international comparisons, or the high rates of community college placement in developmental courses, or the low levels of quantitative literacy in members of the work force, perhaps U.S. teachers' conception of mathematics is a contributing cause. Less than desirable outcomes on achievement measures might be due not to poor teaching or poor student learning in the U.S., but rather to what we think it is that we ought to be teaching.

Chapter 8
The Fifth Lesson: Students' Responses to a Patterning Task Across the Four Countries

Jorunn Reinhardtsen and Karen B. Givvin

Introduction

In the prior chapters we have encountered different classrooms in Finland, Norway, Sweden, and the U.S. as teachers and students embark on the topic of algebra. We have seen how classroom cultures and the specific topics of the lessons both vary across classrooms of the same country and across countries, but that there are apparent similarities as well. In this chapter we wish to turn another stone and investigate the ways in which students are able to participate in the algebraic discourse.

In the VIDEOMAT project, five consecutive lessons in classrooms in the four countries were observed and video recorded. In the first four lessons the teachers proceeded with the teaching as they had planned, as has been shown in chapters four to seven. However, in the fifth lesson the teachers were asked to assign to their students three algebraic tasks adapted from the released set of TIMSS 2007 eighth-grade mathematics problems (TIMSS & PIRLS, 2009). Teachers were asked to have their students work on the tasks in groups. One of these tasks, a patterning task called *the matchstick task*, resulted in especially lengthy discussions among the students and will serve as the object of analysis in this chapter. In each of the four participating countries we examined four classrooms, and within each of those classrooms we focused on the work of a single group of students.

The interactions among the students in the target groups are interesting in three ways. First, a shared feature of the response to this task is that it, as we already mentioned, spurred an intense problem solving activity among the students—even among students in the U.S. classrooms for whom, based on their work in the

J. Reinhardtsen (✉)
Department of Mathematical Sciences, University of Agder, Kristiansand, Norway
e-mail: jorunn.reinhardtsen@uia.no

K. B. Givvin
Department of Psychology, University of California, Los Angeles, CA, USA

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preceding days, such a task might have been expected to have become routine (see Chap. 7). Second, the students' work with the matchstick task resulted in approaches ranging from basic ones (e.g., counting) to more sophisticated ones (i.e., algebraic). Third, because the students approached the task predominantly with the experiences and knowledge from arithmetic, the discussions offer an opportunity to analyze how the discourses sometimes became more formal—sometimes even algebraic. On all three counts we have looked for differences and similarities in the interaction of the different groups across countries.

In this chapter a four-part analysis will be presented. The first part will focus on the varied problem-solving approaches the students used as they tackled the matchstick task and on the progression of approaches within the student groups. In the second part we do an in-depth discourse analysis of four groups in which we separate the student's argumentations into different thematic discourses according to their focal objects. Part three juxtaposes the groups with respect to similarities and differences in the groups' thematic discourses. In part four we summarize and synthesize the prior analyses.

On an empirical level, it is interesting to observe that the approaches used in the different countries were similar, and, generally, one can find a group in one country that resembles groups in other countries in terms of the reasoning performed. As they solved the matchstick task, the students rarely applied, either spontaneously or at the suggestion of the teacher, the algebraic procedures and reasoning methods presented in the prior four lessons. Instead, they frequently used mathematics that they had previously internalized through their years of schooling (i.e., arithmetic). We therefore interpret this task not as an assessment of what the students had recently learned, but instead as offering a glimpse of the students' capabilities for mathematical reasoning and their level of mathematical discourse at a certain point in their learning trajectory/development: at the critical moment when they are expected to make the transition from arithmetic to algebra.

Methodological and Theoretical Considerations

Our main approach to the data is discourse analysis. The methodological and theoretical considerations in this chapter are based on Radford's theory of Knowledge Objectification (2002) and Sfard's theory of Commognition[1] (2008). Sfard's and Radford's theories are concerned with interactional forms of learning and build on a Vygotskian view of learning and development. We follow Radford in his attentiveness to the active relationship between words, gestures, and artifacts in the learning of mathematics, and we include this awareness into a Commognitive analysis of the students' discourse. Particularly, our method of transcribing the video recordings has its roots in Radford's conceptualization of learning processes.

[1] Sfard coined the term *commognition* to signal the close relationship between communication and cognition in her socio-cultural framework for the learning of mathematics.

Based on the theoretical underpinnings of Knowledge Objectification, we have identified the multimodal elements that play a role in the problem solving process of the groups (Reinhardtsen, Carlsen, & Säljö, 2015). The transcriptions include three categories of semiotic means: *inscriptions* such as drawings, texts, numbers, arithmetic, algebraic (including variable/s); *concrete material* (i.e., matchsticks); *gestures* such as pointing, tracing in air/figure/table, glance, raising hand. We view the activity of transcribing as a first step in the analytical process.

Sfard's theory of Commognition offers terminology regarding human thinking, which is operationalized in ways that are relevant for analyzing mathematical discourse. Sfard defines thinking "as an individualized version of (interpersonal) communication—as a communicative interaction in which one person plays the role of all interlocutors" (2008, p. 81). Thinking (i.e., cognitive processes) and communication are simply different manifestations of the same phenomenon—a phenomenon referred to as *commognition*. Commognition is defined as a patterned collective activity which involves reacting to certain actions in a distinct manner. These patterns are historically and culturally shaped. Discourses are defined as different types of commognition distinguishable by their objects, types of mediators used, and the rules followed by the participants (Sfard, 2008).

In Commognition, mathematics is defined as a discourse with unique qualities. Although mathematical discourses may differ strongly from one another, Sfard (2008) describes what she calls the "family resemblances" of these. The mathematical discourse is made distinct by the use of *words*, the *visual mediators* employed, the *narratives* endorsed, and the *routines* that are frequently practiced: The keywords in mathematical discourses often signify quantities and shapes and are used in a disciplined way. In the commognitive perspective of learning, *word use* is important as "it is responsible for what the user is able to say about (and thus to see in) the world" (op. cit, p. 133); *Visual mediators* are visible objects that are operated on in the process of communication. Colloquial discourses are often mediated through the image of material objects; however, mathematical discourses are mediated through signifiers of objects that exist only in that particular discourse. Mathematical objects or ideas are often mediated through algebraic notations or graphs; *Narratives* are series of utterances, written or spoken, about objects, relationship between objects or processes with or by objects; *Routines* are repetitive patterns in the discourse. There are explicit object-level routines, such as the distributive law in multiplication, and there are more implicit meta-level routines that make assertions about the discourse as a whole. The introduction of algebra in school represents a change in meta-level routines—for example in arithmetic there is a focus on processes of calculations and finding correct numerical answers, while in algebra one is required to reflect on numerical processes and describe these algebraically. These are very different ways of working with numbers.

In this chapter, we focus on the (mainly informal) discourse employed by students as they work with the matchstick task. Gestures and the figure provided in the task seem to play prominent roles in the problem-solving process. In order to scrutinize the dynamics of the semiotic resources in a local (both in time and content) learning process, we find it useful to incorporate Radford's (2002) multi-semiotic perspective.

On the other hand, Sfard's conceptualizations of the mathematical discourse and the process of discourse development provide a broader frame for the analysis of the students' discussions of the matchstick task. In particular, we interpret our data in light of a discursive model, developed by Caspi and Sfard (2012), which aims to depict the development of algebraic thinking in school. The concept of a multi-semiotic analysis, the model by Caspi and Sfard, and how we use these ideas in analyzing our data will be explained in Part Two.

In this chapter, we have a dual focus as we explore the groups' discussions of the matchstick task from two points of view. First, in a larger, developmental perspective, we see our data as a sample of classroom mathematical discourse at the time when students are about to make the transition from arithmetic to algebra. Our aim is to describe critical features of this transition. Second, in the short time span (i.e., 8–15 min) that the students discuss the task, there is a "local" discourse development and our interest is to explain the dynamics of this process.

In analyzing data from different countries there is a comparison aspect, and our initial response to managing and exploring the data was to identify the students' solution approaches to the task. Our findings are presented in Part One. Based on these, we chose four groups in Part Two, one from each country, for an in-depth discourse analysis. Furthermore, in Part Three we juxtapose the four groups. And, finally, in Part Four we summarize and synthesize our findings. Our first analytical steps are illustrated in Fig. 8.1. The terms are explained in their respective sections (Part One and Part Two):

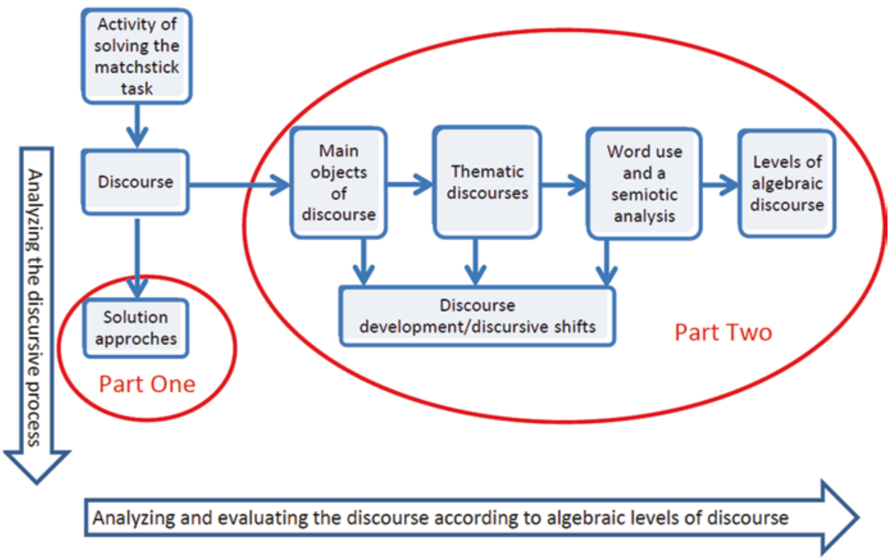


Fig. 8.1 Overview of analytical approach step by step

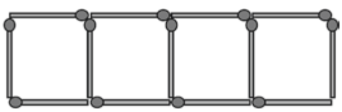
Part One: Student Approaches to the Patterning Task

Patterns and functions are recurring topics in students' elementary and secondary education across the participating countries. Young students may be asked to generate, describe, and extend patterns with numbers and shapes. With the onset of algebra, students are asked to translate patterns into expressions and equations with unknowns. The matchstick task that we asked teachers to present to their students offered an opportunity for students to make use of what they might have learned from the prior days' algebra instruction, yet not be stymied by the task if they failed to find an algebraic solution approach. The task itself is presented in Fig. 8.2.

Across the four participating countries, we identified a number of solution approaches from students' group discussions and written work. The matchstick task can be solved by brute force, as it were, extending the drawing until 73 matchsticks are used. We saw students in each country—indeed in all but 2 of our 16 target groups—use a drawing at some point in their discussion. Students frequently spoke of it as the easier way to solve the problem. However, there seemed to be a general impression among the students that the intent was that the problem be solved by something more sophisticated than a drawing. As an extreme example, one US student said to a group member concerning her drawing, `No offense, but your answer is pathetic`. The first student then sought an arithmetic approach. However rudimentary a drawing may have appeared, it frequently helped illuminate for students the "+3" pattern in the figure, a notion that then sometimes helped them arrive at successful arithmetic approaches. None of this is to say that students always used a drawing effectively. Some students' drawings formed a block of squares, rather than a single, long row, clearly illustrating their misunderstanding of the task. One group used an approach akin to drawing. They used toothpicks to replicate and expand the figure. The effectiveness of this approach was limited in that the group spent much of their time distracted from the task, making different patterns with the toothpicks.

Kaput (2000) describes algebra as the generalization and formalization of patterns, that is, explicitly identifying and exposing commonality across cases and then rendering them in some form. Although the matchstick task does not necessitate an algebraic solution, it does require that students recognize a repeating pattern, even when they solve it arithmetically. The approaches we saw across the 16 target

In the figure below, 13 matches were used to make 4 squares in a row.



How many squares in a row can be made in this way using 73 matches?
How do you know?

Fig. 8.2 The matchstick task (Adapted from the released set of TIMSS 2007 eighth-grade mathematics problems (TIMSS & PIRLS, 2009)) groups represented various attempts to generalize the pattern and, when a correct generalization was "seen" by students, different degrees of success in formalizing it mathematically.

The algebraic equation for the task (i.e., $3x + 1 = 73$) exposes the critical elements of the generalized pattern. Students must identify that for each square (x) there are three matchsticks, and one additional matchstick is required to close the final square. None of the target groups produced the full, algebraic equation to represent the pattern, but one came close. That group omitted mention of the one, additional matchstick and produced `3x = 72` (As will be seen below, the "1" in the pattern proved troublesome for most groups.). Three other groups made use of an unknown in some way, but without success. They produced the following equations:

```
3 · x = 73
4 · x = ?
13 · ? = 73
```

Each of these approaches suffered from either confusion about the one, additional matchstick or a misunderstanding of the repeating value in the pattern. When we coded these approaches, we included them among others with similar misunderstandings, rather than creating a coding category for erroneous algebraic expressions.

Many students saw the pattern in the matchstick figure and were able to formalize it arithmetically. Their renderings took a number of forms. Some students saw an initial, single matchstick:

```
73 – 1 = 72
72 / 3 = 24
```

Some students saw an initial, single square:

```
73 – 4 = 69
69 / 3 = 23
23 + 1 = 24
```

Still other students saw a complete, initial figure:

```
73 – 13 = 60
60 / 3 = 20
20 + 4 = 24
```

In each case, students took the number of matchsticks that remained after the initial unit was removed and divided it by 3, thereby determining how many squares could be created with the remainder. In Fig. 8.3, these three successful approaches are referred to as "single matchstick, +3," "single square, +3," and "complete figure, +3," respectively.

Although not by itself a complete solution, we also noted when groups were able to articulate the repeating (i.e., +3) pattern in words, whether or not they were able

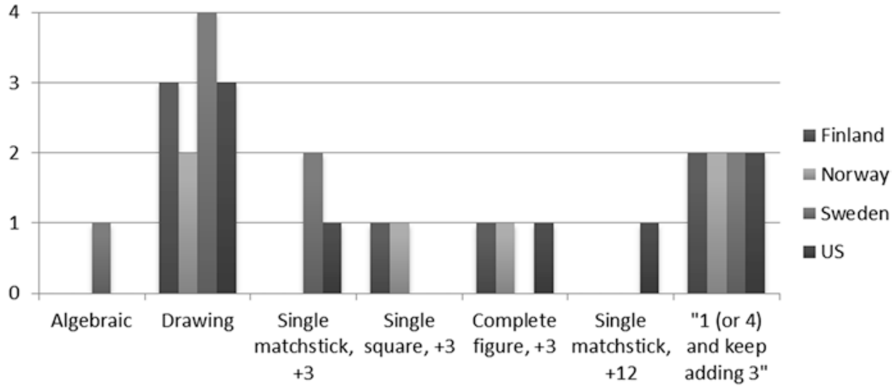


Fig. 8.3 Frequency of successful approaches to the matchstick task (i.e., approaches that lead to a correct answer) within each target group, by country

to communicate it with mathematical notation. For example, `You have 4 and keep adding 3.` In Fig. 8.3, this is referred to as "1 (or 4) and keep adding 3".

One group recognized that a larger quantity (i.e., groups of 12 matchsticks) could be repeated. In Fig. 8.3, this is referred to as "single matchstick, +12".

```
73 – 1 = 72
72 / 12 = 6
6 · 4 = 24
```

Of course, not all of the approaches students used led to a correct answer. Students sometimes produced a variety of inchoate approaches—approaches that were imperfectly formed but that might be developed into a successful approach. For instance, not all students who recognized multiples fully understood the pattern. As was suggested above, it was fairly common for students to see a repeating 3, but either not see or not know how to account for the additional 1 matchstick, as with the following five approaches. In Fig. 8.4, these are referred to as "See +3, but confused by or ignore 1."

```
73 / 3 = 24
73 / 3 = 24.333
73 / 3 = 24 R1
73 / 3 – 1 = 24
73 / 3 + 1 = 25
```

Other approaches reflected a recognition that *something* is repeated, but a misunderstanding about what that quantity was, whether it be 4, 13, or some combination of the two, as with the following seven approaches:

```
73 / 4 = 18
73 / 4 = 18.25
73 / 4 = 18 R1
```

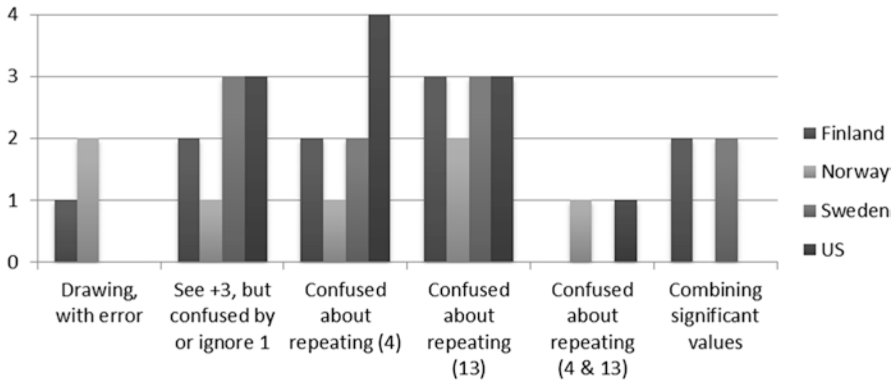


Fig. 8.4 Frequency of inchoate approaches to the matchstick task (i.e., approaches that are imperfectly formed but might be developed into a successful approach) within each target group, by country

```
4x = 73
73 / 13
13 · ? = 73
13 · 5 + 3 + 3 = 71
```

In Fig. 8.4, these approaches are referred to as "confused about repeating (4)," "confused about repeating (13)," and "confused about repeating (4 & 13)", respectively.

The student approaches furthest from being successful were those that reflected efforts to take significant values from the problem (i.e., 3, 4, 12, 13, 73) and combine them with various operations, as with the following two approaches. In Fig. 8.4, this is referred to as "combining significant values."

```
4 · 73 = 292
13/73
```

Use of the numbers 3 and 12 is more likely to represent some valid understanding of the task than use of the numbers 73, 13 and 4, which are all numbers given in the text of the problem. It is possible, of course, that in groups in which students' reasoning could not be verified through their discourse, the approaches identified further above (e.g., `73/3` and `73/4`) represent also random attempts to combine significant values, and that by indicating that the approaches reflect an understanding of a repeated pattern, we may overstate somewhat students' understanding.

From this assembly of student approaches, two questions arise:

1. Did the incidence of each type vary across countries?
2. Were there patterns in the orders in which the approaches appeared over time, within group discussions?

Comparing Student Approaches by Country

As we have seen across the prior chapters of this volume, algebra is introduced differently in the four participating countries. So, given brief exposure to those differing forms of instruction—just four lessons—did students from the different countries approach differently a problem for which they could draw on their burgeoning understanding of algebra? If we organize students' work on the problem in line with the approaches described above, the answer appears to be "no."

We drew on both the videos of the target groups as they worked on the matchstick task and on copies of their written work to code each approach as it arose. We then tallied the number of target groups in each country that utilized each approach at some point during their work. In all countries, except Norway, drawing was the most frequently used among the approaches that resulted in a correct answer (see Fig. 8.3). Verbalizing the pattern of repeatedly adding three was another commonly used approach. Among the inchoate approaches used by students, confusion in the target groups primarily concerned what quantity was being repeated (see Fig. 8.4). Recognizing the correct repeating pattern, but being confused by the single matchstick or not included in it, was also common across the countries.

Order of Student Approaches

All of the student groups in the study tried more than one approach when attempting to solve the matchstick task. That is, the thinking exposed within each group changed over time. The classroom videos allowed us to investigate the order in which the various approaches appeared in students' work.

Of the 16 target groups, only one failed to identify a successful approach. One other group identified only successful approaches. The remaining 14 groups discussed a combination of successful and inchoate approaches. Interestingly, in all of those 14 groups, students continued to work with inchoate approaches even after a successful approach had been considered. Some groups identified what would be a successful approach but abandoned work on it in favor of using an inchoate one—sometimes revisiting the first approach and sometimes not. Other groups reached an answer with a successful approach and continued their work, exploring other approaches to the problem—some of which were themselves successful, others not. What the groups *did not* do was simply find a successful approach, carry it through to an answer, and halt their work on the task. Their work cannot be characterized as consistent progress toward a final, correct answer.

We repeatedly saw a pattern of problem solving that might explain this finding. Students with successful approaches frequently had difficulties convincing their peers of the value of their suggested approach, sometimes even after multiple attempts to explain their ideas. It may be that the task requires a degree of intellectual investment in order to make sense of it, and that this can be achieved only through wrestling with it personally. In some cases, students blindly followed the work of someone whom they saw as a more competent student, and in these cases it often appeared that they did not understand the solution they copied.

Nine of the focus groups identified more than one successful approach (with three groups identifying three successful approaches and one group identifying four). When groups identified more than one approach that led to a correct answer, making a drawing—the most frequent of the successful approaches—was very often one of them (i.e., seven of the nine student groups). The placement of making a drawing within the sequence of the groups' successful approaches was interesting. For three of the groups, a drawing was followed next by a verbal articulation of the pattern. For two other groups, the reverse was true: Their verbal articulation of the pattern was followed next by a drawing. Thus, for some groups, the drawing seems to have led to the discovery of the repeating pattern, and for others it served as a way to confirm the discovery. With respect to the latter, the incorporation of a drawing (or gesturing as an imitation of the drawing) appeared to provide an element of certainty among students and confidence in their solution. When the problem was solved correctly without at some point using a drawing, students were more prone to ask the teacher for confirmation. There was no detectable pattern in the sequences of less frequent, successful approaches.

Also of interest are the inchoate approaches students tried before a successful approach had been identified. Were there inchoate approaches that served as bridges to successful ones? Six of the groups tried inchoate approaches of some kind before trying a successful one. With so small a sample, it was difficult to detect a pattern. Among these six groups, students' last inchoate approach was, to the same extent, confusion over whether the repeating quantity was 4, whether the repeating quantity was 13, and how to manage the 1 matchstick that remains with a repeating quantity of 3. Half of these inchoate approaches led directly to a resolution by means of a drawing. Another two led to a successful verbalization of the pattern. In the end, this may indicate nothing more than that the highest frequency inchoate approaches were followed by the highest frequency successful approaches.

Part Two: An in-Depth Discourse Analysis of Four Groups

The approaches to the matchstick problem discussed above are part of and have been extracted from the discourse of the 16 groups from the four countries. There are two findings from the prior section that will form the background for the following analysis:

1. The students mainly use mathematics (arithmetic) they have previously internalized and not the algebra introduced in the four prior lessons
2. The groups, in spite of coming from different countries, apply similar approaches to the task.

Applying the theoretical framework of commognition (Sfard, 2008), the discourse of the groups is analyzed in a developmental perspective that explains learning as growth in discourse. Algebraic discourse can be either informal or formal, which is paralleled with the historical distinction between rhetoric and symbolic algebra (recognizing that the use of symbols is only one of a series of changes that are made in this transition). Caspi and Sfard (2012) argue that it is more likely for students who retain a connection between informal and formal algebraic discourse to learn algebra meaningfully than for students who do not (i.e., the latter might be able to manipulate symbols but not understanding the meaning of those actions). Students' learning of algebra is seen as a process of individualizing the formalized algebraic discourse they are exposed to in school.

A commognitive tenet is that mathematical discourses develop by formalizing and annexing their own meta-discourses. Following this, Caspi and Sfard (2012) define school algebra as a meta-arithmetical discourse described as "a sub-category of mathematical discourse that people employ while reflecting on arithmetic relations and processes" (p. 45). Our previous analysis, Part One, shows that the students mainly use arithmetic to solve the matchstick problem. However, in order to use multiplication/division correctly, they have to generalize the pattern and relate quantities of two different dimensions (one-dimensional matchsticks and two-dimensional squares). Thus, the students working with the matchstick task do at times engage in a meta-arithmetical discourse.

Given the similarity of student responses to the task, we have chosen four groups, one from each country, for an in-depth discourse analysis. These groups have been chosen for the purpose of illustrating and characterizing the meta-arithmetical discourse employed by the students of different classrooms as they discuss the matchstick task. We are also looking for patterns in a local (in time and content) discourse development in algebra.

There is a large body of research in school algebra (cf. Chap. 1). Here, we only touch the surface and bring to mind some well-documented issues. There are differing views regarding how to conceptualize the relationship between arithmetic and algebra in school and several different approaches to school algebra have been proposed and investigated, including functional, problem-solving, and generalized arithmetic (Carraher & Schliemann, 2007; Kieran, 2007b). The discussion also involves instructional timing. Stephens, Ellis, Blanton, and Brizuela (2017) argue that algebra should be considered a K-12 topic and proposes algebra in the early grades mainly in terms of generalized arithmetic. This contrasts with earlier proposals of pre-algebra approaches that are aimed at the middle grades in order to alleviate transitional issues as the students move from arithmetic to algebra (Kieran, 1992). The establishment of early algebra as its own field of research has further enriched the discussion on defining algebraic thinking as the use of alphanumerical signs is not necessarily involved (Kieran, 2018). This study follows the tradition of viewing arithmetic and algebra as two distinct and rich topics of school mathematics, in which there is an agenda to define and empirically explicate the nature of algebraic thinking as differentiable from arithmetic ways of working mathematically in school (Bednarz & Janvier, 1996; Filloy & Rojano, 1984; Radford, 2018; Vergnaud, 1982). We consider the use of algebraic symbols as the most visible change when algebra is introduced in the middle school classroom, but argue that there are many, more fundamental, changes that the students need to attune to in order to be successful in algebra.

Radford (2010) summarizes research on algebraic thinking in the 1980s and 1990s and proposes that although a minimal set of characteristics was not agreed upon, there was general consensus regarding two points: (1) algebraic thinking involves objects of an indeterminate nature such as unknown, variable, parameter, etc., and (2) these are dealt with in analytic ways. Radford (*op. cit.*) proposes that there are several semiotic ways of expressing indeterminacy, other than, and along with, algebraic symbols. From an ontological standpoint, Radford (*op. cit.*) advocates the usefulness of investigating what he calls the *zone of emergence of algebraic thinking*, instead of equating symbols with algebra.

Berg (2009), investigating the development of algebraic thinking of mathematics teachers at a lower secondary school, highlights the importance of the "discovery, exploration and investigation of patterns, aiming to grasp and express some algebraic structure" (p. 271), in this process. Further, she reflects on the role of algebraic symbols and characterizes the ability to use these to express observed structure as "a result of algebraic thinking and not as a condition *sine qua non* for it" (*op. cit.*, p. 271).

Mason (1996) has proposed the activity of generalizing as the essence of algebra and as a route to learning. He describes it as "detecting sameness and difference, making distinctions, repeating and ordering, classifying and labeling" (p. 83). These generalizing activities express an attempt to minimize demands of attention. Further, he proposes that students must develop interpretative flexibility regarding symbols which he calls necessary shifts of attention: (1) as expressions and as value; (2) as object and as process. The difficulty students have with seeing the duality of mathematical objects (as process and as object) has also been elaborated on by others (Gray & Tall, 1994; Sfard, 1991).

The central question is how to structure lessons in algebra so that the students can adopt its cultural ways of thinking and the algebraic syntax embedded in meaning. Some research has pointed out the important role of natural language in the learning of algebra (Freudenthal, 1983; Radford, 2000; Carraher, Martinez, & Schliemann, 2008). In the same vein, but put in a systematic structure, Caspi and Sfard (2012) propose that informal, meta-arithmetical discourse can provide the necessary background for a meaningful development of a formal algebraic discourse.

Caspi and Sfard (2012) present a hierarchical model for the development of algebraic discourse. The model is a discursive version of one elaborated earlier (Sfard & Linchevski, 1994) and is informed by historical, logical, and empirical considerations. Explaining the growth of algebraic discourse in school (informal and formal), using the metaphor of a tree trunk, it is separated into distinct layers according to what the specific discourse is about. Each layer is the meta-discourse of the preceding layer, and the layers represent rising levels of complexity; thus these discursive layers are described as levels of algebraic discourse. The main idea behind the model is the duality of mathematical objects—processes and objects. Sfard and

Linchevski (1994) argue that the processual use of mathematical objects must be developmentally prior to the objectified use of them.

Caspi and Sfard (2012) describe five levels of algebraic discourse as an attempt to depict school algebra (primary and secondary). The three lower levels are presented as *constant value algebra* in which the signifiers used for objects are interpreted as specific numbers, either known or unknown: (1) Processual level in which the focus is on numerical calculations that are described in the order of their execution. Examples are equations of the form $ax + b = c$, which are solved by the simple "undoing," and rules for patterns explained by listing calculations in their sequential order; (2) Granular level is also about numerical calculations but instead of focusing only on calculations one step at a time (73 – 4 = 69, 69/3 = 23, 23 + 1 = 24), it is a reflection on these calculations, in which some calculations are lumped together and bypassed (69 divided by 3 and add 1). Caspi and Sfard (op. cit.) explain that

> such expressions can be metaphorically called granular, because they can be seen as a result of shortcutting the chain of basic operations by tying parts of this chain into 'knots' or 'granules.' The granules are to be interpreted as results of auxiliary calculations rather than calculations themselves, that is, as objects rather than processes. (p. 50).

However, at this level, these objects only have a transient existence and even when expressed symbolically, they are not seen as legitimate answers to problems; (3) Objectified level is reached when complex algebraic expressions (verbal or symbolic) have the same status as a number and are used to describe relations between objects. The last two levels of algebraic discourses are described as *variable value algebra* and are developed as a response to the need for modeling processes of change; the objects of the discourse are variables and functions. Level 4 discourses are concerned with processes and level 5 discourses describe functions as objects.

The elements of each layer in the model increase with respect to their generalizing power (i.e., $3n + 1$ versus $f(n) = an + b$). It is a theoretical presupposition that in order for students to learn algebra meaningfully they have to pass through these levels, through a process of reification—replacement of talk about processes with talk about objects.

The model aims at including and depicting the spontaneously developed informal algebraic discourse at its lower levels. Two types of tasks are considered to give rise to this kind of meta-arithmetical discourse (Caspi & Sfard, 2012): (1) Questions about numerical patterns, and (2) Questions regarding unknown quantities involved in computations whose result is given.

Caspi and Sfard (2012) conducted task-based interviews with pairs of Grade 5 and Grade 7 students. In order to discern the developmental level of the students' informal algebraic discourse, the students' written and spoken formulations regarding a rule for a pattern were investigated in several aspects. Those included *means for saming* (expressing generality), *types of actions* described (algebraic operations versus actions such as finding, using, etc.), ways of dealing with *intermediary results* in complex calculations (listing calculations linearly or bypassing some calculations by using granules stated as for example: the product of a and b), *signifiers* used for constants and variables (evaluated regarding ambiguity), the extent of involvement of a *human actor* in the verbalizations, and the *lengths* (number of words used) of the replies. Task 1 and students' written responses to the last question posed are presented below (Caspi & Sfard, 2012, pp. 54 and 59):

Task 1, Type: (Informal, Generalizing, Abstract)

Given the sequence: 4, 7, 10, 13, 16 ….

1. Write the next three elements of the sequence
2. What number appears in the 20th place in the sequence?
3. What number appears in the 50th place in the sequence?
4. Write a rule for calculating any number [literally: a number that appears in any place] in the sequence

Two Written Rules by Seventh Grade Students:

1. To find a certain place in the sequence I need the place that I found (it better be round) and then the regularity (3 or any other number that is the regularity) times what must be added to the number you have now and then to add the number you have now and the product of the regularity and what you still need, and that's it.
2. Rule: place · regularity of the sequence +1

$\square \times 3 + 1$

Two Written Rules by Fifth Grade Students:

3. Rule—you need to start from the highest number you see (16) (the fifth in the sequence) and then you need to see which place you want to find (20th in the sequence). You do [the exercise of] subtraction between the highest place and the place you want to get to (20 − 5 = 15) and then you multiply the result you get by 3 (15 × 3 = 45). In the end you take the number you got and add it to the highest number you see (45 + 16 = 61).
4. $\underline{\ \ 3\ \ } \times \underline{\qquad} = \underline{\qquad} + \underline{\ 1\ } = \underline{\qquad}$

The two student groups in Caspi and Sfard's (2012) study had similar means for saming (mostly verbal and rather ambiguous); the solutions proposed were much the same, and the human actor often remained visible in the students' replays. However, several differences were documented: the rules of the seventh graders were shorter (thus more condensed), less processual (intermediary calculations momentarily bypassed by using granules), and the use of ideographs was more frequent; the fifth graders sometimes used a specific "generic" number instead of a variable and more frequently used verbs in their solutions. These differences Caspi and Sfard (op. cit.) interpret as marking the development of spontaneously developed meta-arithmetical discourse over the course of 2 years. They conclude that the seventh graders' rules are close to the second, granular level of algebraic discourse, while the fifth graders rules are characteristic of the first, processual level of algebraic discourse.

The matchstick task is similar, regarding mathematical content, to the task presented above. However, it also differs in several aspects: (1) it involves the same numerical structure, but it has to be applied to a geometrical figure; (2) the task is contextual in that it includes matchsticks, squares, and a geometrical figure; and (3) only one question is posed and therefore the task does not offer prompts that lead the students on a step-by-step journey from the specific to the general. The matchstick task includes a generalizing aspect (pattern) inserted into a question about an unknown quantity (number of squares) involved in calculations whose result is given ($3x + 1 = 73$). In comparison to the study by Caspi and Sfard (2012), our data are of a different nature as the students work in groups, organized by their teachers, in their ordinary classroom settings (as opposed to task-based interviews).

In the present study, we will focus on the three lower levels (constant value algebra) of Caspi and Sfard (2012) as our data may contribute to shed light on the early developments of algebraic thinking. A table exemplifying how we interpret the algebraic levels in relation to our empirical material is presented below (Table 8.1). Our table is based on the main ideas of Caspi and Sfard (2012) and Sfard and Linchevski (1994). However, focusing on one section of the discursive model, constant value algebra, we have attempted to add a dimension of algebraic thinking within a problem-solving context; when scrutinizing the meaning-making process, rather than its results only, we find it useful to look at how the students *model the problem* and how *relevant mathematical objects* (to the problem posed) are evoked.

Caspi and Sfard (2012) exemplify the three lower levels of algebraic discourse through *descriptions of a computational process* (column 2). We have sought to mirror this within *models of a text problem* (column 3). Caspi and Sfard (op. cit.) include the activity of generalizing in their description of *constant value algebra*. As an important aspect of algebraic thinking, we look at how students generalize relations between objects when only using signifiers that refer to specific numbers, in column 4. The ideas drawn upon here have been developed by Radford (2010) and will be explained later in this chapter. The last column (5) focuses on the relevant mathematical objects (algebraic) and their role within the problem-solving process. Rather than being a general table as the one presented in Caspi and Sfard (2012), our table emphasizes the particular problem posed and includes examples from the students' discussions. Even so, we see it as a part of the general mapping of elementary algebraic thinking.

We aim to further investigate informal algebraic reasoning using the work of Caspi and Sfard (2012) as a starting point and we ask:

1. What is the nature of the groups' argumentations regarding the matchstick task?

In addition, we also see the groups' work with the matchstick task as instances of (local) discourse development, of which we will explore the dynamics:

2. What characterizes the groups' processes of learning as they are introduced to algebraic ideas in a problem-solving setting?

In contrast to the study by Caspi and Sfard (2012), in which they describe changes in the discourse over a long period of time (2 years), our analysis maps changes in the discourse within a short period of time (i.e., 8–15 min.). A multisemiotic analysis (Radford, 2002) is incorporated in order to shed light on the role

Table 8.1 Levels of elementary algebra discourse (first three levels, only), based on Caspi and Sfard (2012, p. 48)

	Descriptions of a computational process	Mathematical models of a text problem	Generalize relations	Emergent mathematical objects of an algebraic nature. How are these evoked in a problem solving context, in students' discussions?
Levels	Theoretical examples from Caspi and Sfard (2012, p. 51): $3 + 2(n - 1)$	Examples from our data (the models are not necessarily correct)	How is indeterminacy addressed?	Relevant algebraic objects connected to the matchstick problem: numerical pattern,[a] rule (coefficient/rate of change, constant term), equation, unknown
"Constant value algebra"				
Level 1 Processual	Subtract 1 from *n*, multiply by 2, and add 3	73 − 4 = 69, 69: 3 = 23, 23 + 1 = 24	In-action-formulae[b] 73 − **52** = 21, 21/3 = 7; 73 − **55** = 18, 18/3 = 6; and finally 73 − **4** = 69, 69/3 = 23, 23 + 1 = 24 (Indeterminacy does not reach the level of discourse)	Intermediacy involved only in processes and not signified as mathematical objects. Examples from our data: intermediacy (variable) is present through some of its instances: 52, 55 and 4
Level 2 Granular	Multiply the difference between *n* and 1 by 2 and add 3	73 divided by 3 and then minus 4	Granular descriptions of calculations. Shortened recursion. Indeterminacy is named contextually (not identified in this study, but described in Caspi & Sfard, 2012)	Used as a means to solve a problem, i.e. finding a correct numerical answer; algebraic object not regarded as a fully-fledged object in itself. Examples of in situ objects: pattern as noticing structure; pattern verbalized as number sequences; the notion of rate of change expressed through gestures. The objects are momentary and contextual
Level 3 Objectified	The sum of 3 and the product 2 by the difference between *n* and 1	$3x = 73$	Talk about a rule or equation as an object in itself. Mainly recursive descriptions. Indeterminate quantities are identified and explicated as such	Recognizing the problem as a pattern problem. Consequently looking for a rule, making an equation and solving for *x*. Thus these objects are the primary focus and guide the solving process

[a]According to Carraher, Martinez, and Schliemann (2008) and Carraher, Schliemann, and Schwartz (2008) numerical patterns are not mathematical objects per se. However, with the inclusion of early algebra in primary school, there is an emphasis on students' learning to make generalizations about patterns; and we see numerical patterns, in the context of early algebra, as provisional mathematical objects, which later are absorbed into a discourse about functions
[b]The notion of an in-action-formula is explained on p. 218–219

of different semiotic means in a *process of knowledge objectification*. Radford's theory of Knowledge Objectification includes ideas which are linked to embodied cognition, which advocates that (1) sensory-motor experiences form the basis for abstract mathematical reasoning, and (2) gestures play an important role in the learning process (Hitt & González-Martín, 2016). Additionally, Radford (2002) emphasizes the role of history, and conceptualizes the learning of mathematics as becoming aware of the knowledge accumulated in the culture through social processes.

Radford (2002) proposes that knowledge objectification[2] happens through semiotic activity, that is, through "objects, artifacts, linguistic devices and signs that are intentionally used by individuals in social processes of meaning production, in order to achieve a stable form of awareness, to make apparent their intentions and to carry out their actions" (p. 14). The different semiotic means of objectification mediate activity, but are not equitable and play different roles in the learning process. A mathematical object may be referred to by a gesture or a drawing, phrased in natural language, and also implied by the use of mathematical terms and writing. However, that does not mean that the signs are equivalent. The semiotic systems include different *modes of signifying* and are thereby unique, i.e., the meaning encapsulated within one semiotic system is not directly translated into another one, but rather there is a transformation accomplished which includes an alteration (to varying extent) of meaning. Radford (2010) argues that the mode of signifying "characterizes the form and generality of the algebraic thinking that is thus elicited" (p. 2). The purpose of a semiotic analysis is to disentangle the dynamics of the semiotic means and shed light on the linking between them.

We see the theories of Sfard (2008) and Radford (2002) as both intertwined and complementary. Commognition conceptualizes mathematics as a discourse with distinct features, investigates student learning as an internalization of this discourse, and naturally emphasizes the particulars of discourse that are retained and refined through time. The theory recognizes that gestures play a role in the learning process, but these acts are not substantially theorized. The particulars of discourse are patterns of actions, and *word use* is one of the main categories. Radford (2009) points out that language has a "unique capacity for creating sustainable reference points with which to organize experience" (p. 124). Sfard (2008) has a well-defined and theoretically rooted analytical tool kit for investigating the use of words. The theory of knowledge objectification offers the possibility to investigate gestures as inventive and creative acts as they appear in interplay with different semiotic means in the students' discourse. De Freitas and Sinclair (2012, p. 138) highlight the creative role of gestures referring to the work of Châtelet: "The gesture is more than simply an intention translated into spatial displacement, for there is a sense that 'one is infused with the gesture before knowing it'." Radford (2010, p. 113) considers gestures as

[2] Radford (2002) use the wording objectification in a more general sense than Sfard (2008). The process of knowledge objectification that Radford delineates is an ongoing process of becoming aware of cultural ways of reasoning. However, Sfard uses objectification in order to describe the movement from talking about mathematical objects as processes to explaining them as objects.

"genuine constituents of thinking." However, gestures are often momentary. They may only occur once, or be repeated throughout certain learning sequences, but ultimately the experiences culminate into patterns of actions (word use, use of visual mediators, routines and narratives) as defined by Sfard (2008). Radford's (2014, 2018) multi-semiotic investigations of the immediate learning process show that these patterns of actions are deeply rooted in material and sensuous experiences.

As a first step, and in order to capture the significant changes in the groups' discourses as they evolve, the analysis will focus on the main objects of the discourse at different segments. The naturally occurring talk will be separated into different thematic discourses according to their focal objects. These will be further investigated regarding *word use* combined with a semiotic analysis focusing on the linking between the different semiotic means. The thematic discourses will also be evaluated according to the levels of algebraic discourse as proposed in the table presented above (Table 8.1). See Fig. 8.1 for an overview of the analytical process.

We will also be attentive to the initiation of discursive shifts. The term, *discursive shifts,* is used in Nardi, Ryve, Stadler and Viirman (2014) and is defined in a very broad sense as "the changes to the mathematical perspectives of those who act" (p. 182) and refers to changes in forms of discourse. We have defined discursive shifts in the context of our present study as *the replacement of talk about one type of objects with talk about another*. However, this replacement also includes changes in forms of discourse. For example, the talk about concrete objects is dominated by an extensive colloquial discourse, while the talk about numerical patterns is concise and centered on numbers. Therefore, our use of the term discursive shifts does not strongly deviate from its previous application and can be seen as a particular type of discursive shifts.

The excerpts will be introduced with a small summary of (1) what the students have been working with in the preceding four algebra lessons; and (2) a description of how the matchstick task was introduced to them. In the presentation of excerpts, we have attempted to preserve the naturally occurring discourse development within the groups.

Norwegian Group (N1): "Oh, You! 73 Divided by 3 and Then You Just Add 1!"

The group of students (Grade 8, 13 years), Ben (A), Ann (B), Trish (C), and Sam (D) presented here is part of one of the Norwegian classrooms. In the prior lessons in algebra, the teacher has been focusing on the concept of a variable, often as an unknown appearing in equations or expressions.[3] The students are arranged into groups and the teacher hands out one task at a time. The matchstick task is the last one and the group is given only two copies of the task. The teacher does not give any specific instructions regarding this particular task but hands out toothpicks as a con-

[3] See Chap. 5, this volume, classroom B for more information.

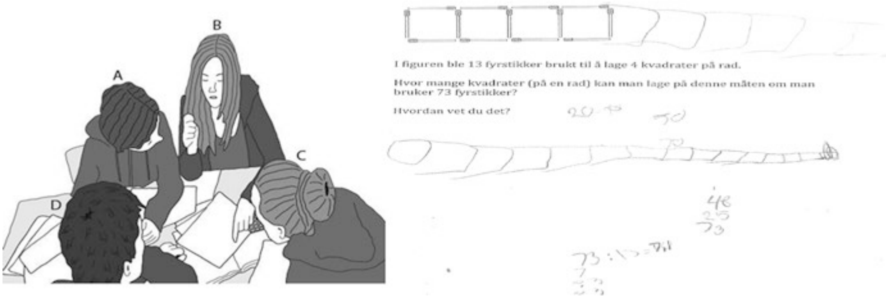


Fig. 8.5 Norwegian group working their solution to the matchstick problem

crete material the students may use in order to solve the problem. Only Ann writes on the task paper. The group spends in total about 8 min working with this task (Fig. 8.5).

Ann and Trish read the problem out aloud, and their first approach to the task is making use of the numbers provided within the task statement. They discuss dividing 73 by 4 or 13 and decide on dividing by 13. Ann explains …`if we divide by thirteen we find out if it becomes more, if it works`. Trish adds to the explanation: `how many more rows we can make`. The boys, at first, are distracted by a pen, and they do not yet take part in the discussion. Ann is struggling with performing the division, and when the teacher hands out the toothpicks, Trish suggests using them as another approach for solving the task.

23. Trish: `We can make them [`*`squares`*`] on the desk. But should we just use these or? [`*`Trish shakes the can of toothpicks she is holding in her hand`*`].`
24. Ann: `But see, we get 7.1 [`*`Ann points to the division, 73 divided by 13, she has been working on`*`], then if you have taken ( ) then you get 7.1 squares. 1, 2, 3, 4, 5, 6, 7 [`*`Ann points at the squares in the task paper as she counts them and continues by pointing at imaginary squares until she reaches 7`*`]. So then you get less than sev…then we get, if we make 7 squares. Ok, 4. [`*`The girls try to add a square to the figure using the toothpicks. They give it up quickly as they notice that the dimensions are different`*`].`
25. Trish*:* `Ha…ha`
26. Ann: `You, this didn't work`
27. Trish: `We'll draw it.`

Ann, in turn (24), is not willing to give up on their first idea and tries to make sense of her answer 7.1 by adding squares to the drawing. She relates the answer she found to single squares instead of rows of four squares. As she starts to draw, she seems to leave the first approach behind and does not bring up the number 7.1 again.

28. Ann: [*She adds a square to the figure by drawing three sides in one motion, she then points at each square as she counts them*] 1, 2, 3, 4…[*adds another square in the same manner*], 5. [*starts counting the matchsticks making up the squares*] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14… 17, 18. Ok, but see…ah…I got a good idea…look [*Now she only counts the horizontal matchsticks*] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12…[*adds more squares using the same motion*] 13, 14…15, 16…17, 18…19, 20. So if we take [*She now counts the squares*] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. When we have 20 rows we have [*writes 20 and then counts the vertical matchsticks silently*]…((then we have…then we have)) =
29. Sam: ((But what are we going to do with them…Ann?)).
30. Ann: = When we have 20 we have 50 pieces [*writes 50. It is unclear how she finds the number 50*]. Or, when we have 20, when we have 20 such things… [*she points or taps repeatedly at the figure*].
31. Sam: It is those [*Sam holds up a toothpick*].
32. Ann: Yes, matchsticks, then we have 50 altogether [*points to the number written*], used 50 such matchsticks [*points back at the figure*] and we are going to use 73, right? =
33. Sam: Just make…
34. Ann: = So then…
35. Trish: ((really one more will be 53 and then 56))
36. Ben: ((We are going to use…))
37. Ann: No, if we have one more with 10 in it, then it becomes… =
38. Sam: ((Yes because it is 4 in one)).
39. Ann: = So, then we get 20 more and it becomes 70 [*writes 70*]. ((It is 1, 2, 3…so then we get 70… No, now there is too much here)) =
40. Ben: [looks at Sam and responds to his comment] ((No, it is 3, it is 4 in one and 3…1, 2, 3, 4, 5, 6, 7, 8, 9))
41. Ann: = I think I sort of lost count of it.

Ann continues to take the lead in the problem-solving process as she adds squares to the figure, counts out aloud, and attempts to find a relationship between numbers of matchsticks and numbers of squares. She continues to confuse different units, now numbers of single matchsticks and numbers of squares. Sam is getting involved by asking questions (29). In turn 35, Trish makes a comment in which she converts the structure observed in the figure into a numerical sequence: one more will be 53 and then 56. This idea initiates a discursive shift as Sam follows her line of thought and says yes because there is four in one. Ben, in turn 40, corrects Sam's statement, combines Trish and Sam's observations, and makes explicit the numerical properties of the figure (40): it is 3; it is 4 in one and 3.

The objects of the discourse are no longer matchsticks and the processes of drawing and counting these, but the numerical structure of the figure, which takes the form of a numerical pattern. Ann appears unaware of the new discourse that has grown out of the one she initiated. Below, Trish attempts to include Ann in the discovery of the *4 3 3 3* structure by verbalizing the numerical sequence, which continually rises with 3 (42), then pointing out the numerical properties of the figure (44) and at last trying to get Ann to count the number of matchsticks in the second square in the figure (46).

42. Trish: No, 70, and then you should have 1 thing more and then it becomes exactly 73.
43. Ann: Ah, but see, oh yes because 20…
44. Trish: It is really only 3 in each, it is only the first there is 4 in, and then there is only 3 in each the whole time [*points at the figure while she explains*].
45. Ann: But see…
46. Trish: If you do like that then…4 [*she holds her finger over the first square*]
47. Ann: 1, 2, 3. [*counts three matchsticks in the first square, then pushes away Trish's finger and starts counting following the procedure she has developed, horizontal matchsticks first and then the vertical ones*] Ok, 1, 2, 3, 4, 5, 6, 7, 8, 9 ( ) 18, 19. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20. 20. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. 11. [While Ann is counting, Ben and Sam start paying attention to something that is going on in the classroom which is not relevant for the mathematical discussion. When the teacher approaches the group, the boys attend to solving the task again]
48. Trish: [*She traces the matchsticks in the squares using the same motion as Ann used earlier when drawing new squares, however she does not trace the first vertical matchstick*] 3, 6, 9, 12. Oh, you! 73 divided by 3 and then you just add one more! [*she picks up her calculator*]
49. Ann: There you said one. [While Trish is working on the calculator, Ann traces first the four matchsticks in the first square and then the 3 matchsticks in each of the following squares. She is using the same motion as earlier when drawing the squares].
50. Trish: No. [*The teacher comes over to the group, but Trish looks only at the calculator while she speaks*] 73 divided ((by 3, plus 1, 25)).
51. Ann: [*Ann looks at the teacher*] ((divided by…3. Is that right?))

When the teacher comes over to the group, Ann asks him if Trish's suggestion of how to solve the task is correct. The teacher fails to approve her approach and the group gives up on it. Ann resumes her approach of drawing and counting and finds that one can make 24 squares using 73 matches. Trish is unable to convince Ann to change her thinking. However, continuing her own line of thought in turn 48, after tracing squares and quietly listing the numerical sequence of continually adding 3, she excitingly exclaims: `Oh, you! 73 divided by 3 and then you just add 1!` This is a new change in the discourse. The pattern is no longer the object of discourse. The new object is a complex numerical expression which includes the coefficient (3) and the irregularity (`add one`) of a functional discourse.

The Meaning-Making Process

The discourse of the group is meta-arithmetical as the students attempt to relate numbers of matchsticks to numbers of squares (number of rows of 4 squares at the very beginning). This proportional reasoning is warranted in the text of the task, as 4 squares are related to 13 matchsticks. However, there are significant changes in the discussion as the students go from using the numbers provided in the text in a numerical expression, to drawing and counting, to the discovery of a pattern, and then using the coefficient and the irregularity from the pattern in a numerical expression.

Three thematic discourses have been discerned according to their focal objects: (1) *matchsticks* and *squares* in the procedures of drawing and counting; (2) the *numerical pattern* as a mathematical object; (3) a *complex verbal expression* which includes a *coefficient* and a *constant term*, as a mathematical object. Although the discourse is interpreted to be about the mathematical objects as listed above, no claim can be made that the students are aware of these in a general sense. On the contrary, the students are concerned with solving the problem and use the means readily available to them (arithmetic), and so they seem to be spontaneously touching mathematical objects of a discourse not yet individualized (functional).

The word use accompanying the process of drawing involves only basic arithmetic and is closer to a colloquial discourse (everyday talk) than it is to formal, mathematical discourse, as it includes an extended use of pronouns (I, we, you) and non-mathematical verbs (take, have, use). The conversation is extensive, with more than 200 words employed. It includes the physical movement of drawing first the top horizontal match, from left to right, then the vertical right match, and, finally, the bottom horizontal match in one motion as each new square is added to the figure. The students who are not drawing start to discuss the structure of the figure as Ann continues to draw and count. A discourse on a numerical pattern therefore seems to grow out of the discourse on matches and squares.

The mathematical object of a numerical pattern makes its first appearance in the discussion as a process of adding 3 to the prior number in a sequence, and is a response to Ann's drawing and counting process: `really one more will be 53 and then 56` (Trish, 35). This utterance initiates a discussion about the numerical structure that can be observed in the figure: `Yes because it is 4 in one` (Ben, 38); `No, it is 3, it is 4 in one and 3` (Sam, 40); `It is really only 3 in each, it is only the first there is 4 in, and then there is only 3 in each the whole time` (Trish, 44). The utterances are short and precise. All, but one (turn 48), are contextual; although the students do not directly refer to matches and squares they do so implicitly, using the words `in one` and `in each`. The students are looking at the figure as they discuss the pattern. Trish also traces the matches as she verbalizes the numerical sequence of continually adding 3 to the prior term in turn 48: `3, 6, 9, 12` (Trish). In this group, the pattern discourse involves both noticing numerical structure in the evolving figure (turn 38, 40, and 44), and expressing it as a numerical process of continually adding 3 (turn 35, 42, and 48). Noticing numerical structure in the figure is contextual, it retains in some sense the spatial properties of it. However the numerical sequence 3 6 9 12 is relational.

The numerical expression is verbalized as Trish reflects on the task involving elements from the prior thematic discourses: `[`*`She traces the matchsticks in the squares using the same motion as Ann used earlier when drawing new squares, however she does not trace the first vertical matchstick`*`] 3, 6, 9, 12. Oh, you! 73 divided by 3 and then you just add 1 more!` The gesture is the same motion as Ann used when drawing new squares and represents a more material and bodily insight into the problem. The numerical pattern verbalized as a process of continually adding 3 is its abstraction. The gesturing and the numerical pattern relay the significance of the number 3, which in a functional discourse is the rate of change. Trish continues by giving the coefficient a correct role in her expression.

The activity (coordination of gestures and verbal activity) taking place before Trish exclaims `Oh, you!` (turn 38), is what Radford (2009) has termed a *semiotic node*, i.e., "pieces of the students' semiotic activity where action, gesture, and word work together to achieve knowledge objectification" (p. 121). Trish saw something new regarding the problem, and she was able to formulate her idea in mathematical terms. Radford (op. cit) identifies this type of situation as an "Aha! Moment," that can be explained as a first rough idea of how to solve a problem; in this case, how to use the dynamics of the evolving figure and the numerical pattern to solve the task mathematically. Thus, in terms of early algebraic thinking Trish becomes aware of the role of the number 3 through the semiotic node. In a physical manner Trish grapples with the notion of rate of change. Expressed through gestures, the verbalization of a numerical sequence, and finally in an expression, the notion of rate of change is not general as in a functional discourse but instead sensual and contextual, i.e., rooted in the immediate experience with the problem at hand. Therefore, we evaluate the discourse regarding rate of change as being at a granular level of constant value algebra. The object is momentarily present in the discourse, however not explicitly verbalized as one and eventually only used in an expression—not as an algebraic model of the pattern but as a means to finding a numerical answer to the problem. According to Radford (*op. cit.*) the appropriation of cultural tools is an ongoing process and we interpret the objectification identified here as a step in the process of becoming aware of the cultural notion of rate of change.

Not concerned with calculations at this stage, Trish models the problem using a granule (`73 divided by 3`) in her expression, as explained by Caspi and Sfard (2012). The irregularity is included in the latter part of the expression: `add one more`. The expression is contextual and ambiguous as `one more` is a referent not only to a number but to a matchstick or a square. From her previous tracing of squares, in which she leaves the first match out, it is likely that it is the *one matchstick* she has in mind. However, when performing the calculations in turn 50, she adds *one whole square*. The verbal expression is interpreted as a granular informal algebraic discourse (level 2).

American Group (A2): "Each Box Also Is Interconnected with One"

There are three girls, Leah (A), Rachel (B), Christy (C) and one boy, Aron (D), in the group (Grade 7, 12 years) presented from the American classrooms. In the prior lessons the class has been working with algebraic expressions and equations.[4] The teacher hands out the three tasks at the same time and instructs the students that they need to work together and share their ideas with the group. She also says that they need to write down everything on their own paper. The group spends about 10 min solving the matchstick task. We have chosen to present Leah's solution-paper as she leads the discussion on the problem (Fig. 8.6).

1. Christy: Thirteen matches were used to make up four squares in a row [*She reads the task out aloud*].
2. Leah: How many squares in a row can be made in this way using 73 matches [*She continues the reading of the task*]. ( ) It's just am…each square is out of four matches [she points at the drawing with her pencil] …but they are all interconnected so how many matches would it make…((well… how))
3. Rachel: ((I was thinking we divide it)). [*Rachel has been writing on her paper while the others read the problem*].
4. Leah: how many boxes would 73 matches make if they are made out of four?
5. Christy: Oh.
6. Leah: So ((13 equals 4)).
7. Rachel: ((So that would mean that)) there be three boxes that were four [*She holds up four fingers*] and one box left that is just one. ( ) Because you will not get an even amount [*She stretches out her hand and draws it back supporting her meaning of even*].

[4] See Chap. 7 in this volume, Miss A's classroom for more information.

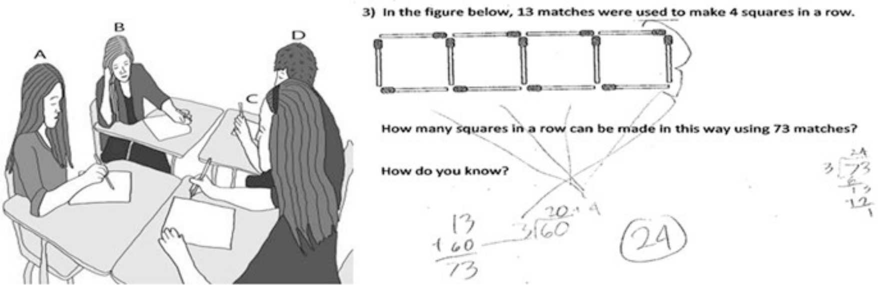


Fig. 8.6 American group working and Lea's solution to the matchstick problem

8. Leah: Well yea, but you can keep on adding boxes of matches [*She draws squares in the air with her pencil as if she was adding them to the figure: first the top horizontal, then the vertical one on the right side and then the bottom horizontal in one movement. She continues to use exactly this motion when she traces 3 matches in the air throughout the problem solving process*].
9. Aron: *Inaudible*
10. Leah: Yea you add on 3 until you get to 73…
11. Rachel: Oh [*She erases what she had written on the paper*].
12. Leah: … and then you count how many boxes you made out of [*She draws squares in the air with her pencil*] … but that would take too long I think, ha [*She smiles*].

Leah's first reaction to the task is to analyze the figure and incorporate what she sees into what she knows from the written text. She looks at the figure and notices two important features: 1 square is made of 4 matches; and the squares are interconnected (turn 2). Rachel, however, suggests an approach for solving right away, by using division (turn 3). In response to Leah's reframing of the question in the text (turn 4), she points out that if there are 4 matches in each square, then there must be one square that is made out of 1 match (turn 7); because 13 is not divisible by 4. Leah accepts this observation but is not willing to look at the case of only 4 squares and expands the figure verbally to include all the 73 matches (turns 8 and 10). In imagining the extended figure, by the use of words and gestures, Leah becomes aware that they add on 3 matches for each new square. She also realizes that a drawing of the completed figure would provide a solution but does not consider it appropriate as it would take too long (turn 12). Rachel initiates a shift in the discourse as she suggests an numerical expression as a solution to the task.

13. Rachel: So would you say 73 divided by 4?
14. Leah: I'd say 73 divided by 13… [*Rachel starts writing*] but I don't think that would be correct. ( )
15. Aron: *Inaudible*
16. Leah: Yea, 73 divided by 4.
17. Rachel: Yea [*She erases what she has just been writing*].
18. Leah: Wait.

The students now focus on how to find the number of squares that can be made using 73 matches. Two numerical expressions are suggested; 73/4 and 73/13. The discourse is short and decontextualized. It is interesting to note that the two first utterances are personalized: `So would you say 73 divided by 4` (Rachel, turn 13), and `I'd say 73 divided by 13 … but I don't think that would be correct` (Leah, turn 14). The students are uncertain and hesitant to make direct statements. Then Leah says `Yea, 73 divided by 4` (turn 16), but right after she says `wait` (turn 18). The uncertainty regarding the value by which to divide 73 makes the students return to the figure.

19. Rachel: Each box would have… [*As Leah interrupts her she listens to her for a while and then starts to work on her own paper. She underlines the numbers in the text and then taps her pencil inside the squares in the figure, one by one from left to right, she repeats this three times*].
20. Leah: yea, but each box also is interconnected with one, so that'd be 3, it'd be like 4 and then 3 and then 3 and then 3[*She traces in the air with her pencil the 4 matches in the first square and then the 3 and 3 matches making up the squares in the figure*], not 4 and 4 and 4 [*She traces the squares now with an almost circular motion signaling 4 matches in each square*].
21. Christy: That match has to be with this box [*She points to the figure on Leah's paper*].
22. Leah: So like if this was 4 [*She runs her finger (left hand) over the 4 matches that makes up the first square in the figure and holds one finger on the match that connects to the next square, then she traces the 3 matches that makes up the next square with her pencil (right hand)*], then this would already be used and this would be 3. *Inaudible* … (the turn continues below)

The students' discussion here is similar to the one in the beginning, but there is a slight change both in context and in topic: previously, the discourse was focused on making sense of the task, now it centers on how to solve it; before the number 3 was mentioned as the number of matches that was added for each new square until all 73 matches were used, now it is seen as the repeating number in a number sequence that reflects the geometrical pattern Leah sees in the figure: `but each box also is interconnected with one, so that'd be 3, it'd be like 4 and then 3 and then 3 and then 3` (turn 20). Again she uses the word `interconnected` and this time she specifies `with one` (turn 20). The objects of this discourse are the geometrical pattern that is transformed into a numerical sequence. After explaining to Christy the numerical properties of the figure as she sees it, Leah immediately suggests a solution to the task in the form of a verbal expression.

22. ...Let's do 73 divided by 3 and then minus 4. [*Then she says in a low voice, as if to herself*] Would that be right?

The expression in its written form is 73/3 − 4. It is incorrect, but it includes key elements of the algebraic rule for the pattern $3(n - 1) + 4$. In a discourse on functions, the 3 is a coefficient and the 4 is a constant term. The rule describes the relationship between the number of single matchsticks to the number of squares (n). Although Leah is not aware of a functional relationship, she assigns the number *3* and the number *4* the correct roles in the expression (coefficient and constant term, respectively) based on her understanding of the geometrical pattern.

Leah divides 73 by 3 on her paper. She gets the answer 24 with a reminder of 1. She pauses but seems unable to make sense of her answer and is quiet for a while. The two other girls also write on their paper. Christy stops as Leah stops. Rachel writes a little longer but stops and looks at Leah. Aron is not writing but seems to study the task. The group is quiet for about 30 s. Then Rachel turns to Aron who starts explaining what he is thinking. However Leah finds a way to avoid a complex expression in solving the problem.

23. Rachel: `Do you have an idea?`
24. Aron: `I want to keep adding 3 on this side (inaudible) so it'd be 12 divided by 3 because you only need 1 of these.`
25. Leah: `13 plus 60 equals 73 [`*`She says it in a low voice, as talking to herself`*`], oh! [`*`She looks up`*`]. I got it. It is 24 [`*`She points to the division she has performed on her paper`*`].`
26. Rachel: `It is?`
27. Leah: `Yeah because 13, 'cause we already have 13 [`*`she runs her pencil back and forth above the figure, then she writes numbers in the air with her pencil while she talks`*`], plus 60 equals 73 and so that'd be 60 divided by 4 for each box and that'd be 20- wait wait, how did I do that? No, that'd be 60 divided by … wait a second. Let me think about that one. (16 seconds)`
28. Rachel: `*inaudible* but you're using 13.`
29. Leah: `Well 13 plus 60 equals 73, so that'd be 13 plus 60 more [`*`She traces 3 and 3 matches in the air`*`]`
30. Rachel: `Wait what did you say? 13 times…`
31. Leah: `13 plus 60 is 73 [`*`She writes on her paper`*`]`
22. Rachel: `73`
23. Leah: `Yeah and so then that means, so this 13 we would have to add 60 more … [`*`She points to the figure and then traces 3 and 3 matches in the air as if she is adding squares to the figure`*`] 60 more 3 matches.`
24. Rachel: `3?`
25. Leah: `Yeah 60 more squares [`*`Most likely she means matches`*`] and each square is made up of 3 matches connected to the last one. So 60 divided by 3 is 20 and 20 plus the 4 that's already here [`*`She points her pencil one time at each square in the figure`*`] is 24.`

Leah calculates 73/3 and when the division leaves a remainder, she rethinks the problem. She realizes that by subtracting 13 from 73 she gets the number 60, which is divisible by 3. By doing this she avoids the troublesome irregularity and its placement in an expression. She finds, indeed, a geometrical solution by dealing with the figure as a special case in which the multiplicative approach of 3 does not work, however she can easily see that the 13 matches make 4 squares. She then divides 60 by 3 and finds that the rest of the matches will make 20 new squares. She adds the squares and finds that one can make 24 squares using 73 matches. The objects of this discourse are matches, squares, and the figure. She keeps the coefficient from the expression and includes it in her calculations. Although Leah has presented a successful approach to the problem, the group keeps discussing it; there are a total of 80 turns in the group's discussion of the problem. Leah explains her approach in its complete form twice again. Aron uses a different approach to solve the problem based on adding new units of 4 squares.

38. Aron: `What I'm thinking is if you take 13 and 13 you get 26 then you have a double line [`*He points to the end of the figure and traces two matches next to each other*`] since you have another one of those [`*lifts his hands as if he picks up the row of 4 squares and put it down again next to the original one*`]. But then you have a double line right there and it kind of makes the shape uneven, two lines there, shape uneven *inaudible* unless you draw it like this … [`*He draws on his paper*`]`

Aron solves the problem of getting 2 matches next to each other by using 12, and not 13, as a coefficient. He identifies the irregularity in turn 24, and although he does not use 3 as a coefficient, he acknowledges that there are 3 matches in each added square: `I want to keep adding 3 on this side (inaudible) so it'd be 12 divided by 3 because you only need 1 of these.` He subtracts one match from the total of 73, to account for the first 4 squares, which need 13 matches. On his paper, he writes 72/12 and gets 6, which he then multiplies by 4 and finds the solution of 24 squares. However, he is not able to fully explain his thought process to the other students and says: `I just found 24 boxes. Can we just put that as our answer?` (turn 63). Rachel listens to both Aron and Leah but is not able to make sense of their approaches. In the end, she instead checks the answer agreed upon by both Leah and Aron (i.e., 24 squares). First, she takes 24 times 4 and finds that it is 96, but with help from Leah, she then writes down 24 times 3 plus 1 and concedes, as she finds it is 73.

The Meaning-Making Process

In the first section of the discussion (turns 1–12), the task itself is the main object of discourse. After reading the problem, Leah reframes the question from the text and incorporates information from the figure (turn 4): `how many boxes would 73` matches make if they are made out of four? The students then contemplate the relationship between numbers of matchsticks and numbers of squares (turns 6 and 7). At the end of this section, Leah uses gestures and words to conjure up an image of a figure that is composed of all 73 matches. The act of conjuring up an image seems to play a role similar to the activity of drawing seen in the Norwegian group. Through it, Leah becomes aware that one adds 3 matches for each new square.

The discussion in the group is mainly meta-arithmetical throughout the problem-solving process, as the students focus on finding the relationship between single matchsticks and squares. However, the acts of imagining the completed figure (turns 8, 10, and12) are closer to a colloquial discourse than it is to formal, mathematical discourse, as it is about concrete objects. It is also personal, as it includes the pronouns "you" five times and "I" one time. Therefore, the acts of imagining the completed figure do not only play the same role in the problem-solving process as does the drawing, but also include the same type of word use as employed in the drawing approach.

Following the sense making section discussed above, four thematic discourses have been identified according to their focal objects: (1) The two *numerical expressions* (73/4, 73/13) as mathematical objects; (2) a *geometrical pattern* transformed into a *numerical sequence* as a mathematical object; (3) a more *complex numerical expression,* including the *coefficient* and the *irregularity* (73/3 − 4) as a mathematical object; and (4) the *figure* combined with an *numerical expression* that includes the *coefficient* from the prior expression as an object.

Rachel suggests a multiplicative approach very early in the problem-solving process: I was thinking we divide it (turn 3). However, Leah leads the group in a sense making discussion of the problem and Rachel's idea is put on hold until turn 13, when she suggests a numerical expression: So would you say 73 divided by 4? Leah responds: I'd say 73 divided by 13. The expressions are decontextualized and suggested without explanations. The statements are personal and the word use would you say and I'd say underlines the uncertainty of the students. Leah hesitates and returns to the figure. The question of what to divide 73 with initiates a pattern discourse in the group.

The students discuss the structure of the figure already in turn 2, when Leah says: each square is out of four matches [she points at the drawing with her pencil] … but they are all interconnected. The word use in this utterance is geometrical rather than numerical. Leah has also previously explained the nature of the problem as a physical process (turn 10): you add on 3 until you get to 73. She had therefore already identified the number 3 as having a role in the problem. Again, Leah uses the word interconnected before she describes the figure as a numerical sequence in turn 20: but each box also is interconnected with one, so that'd be 3, it'd be like 4 and then 3 and then 3 and then 3. In her next utterance, Leah proposes a complex numerical expression. Thus, the discourse development is a movement toward abstraction, which is necessary if the students are to solve the task mathematically. However, there seem to be remnants of the prior, more concrete, discourses which are carried along and play important roles in the continued meaning-making process.

In turns 20 and 22 we observe the same type of intense action as we identified as a semiotic node in the previous group (N1). Leah is tracing the squares and verbalizing the numerical structure of the figure before she immediately suggests a numerical expression that includes the coefficient and the irregularity: `Let's do 73 divided by 3 and then minus 4.` This is similar in its form to the one that appeared in the Norwegian group: `73 divided by 3 and then you just add 1 more.` Both include a granule. However, Leah's expression does not include a human actor or any referent to the context. Leah's numerical expression is also interpreted to be at a granular level of algebraic discourse (level 2), although it is slightly more reified. Leah writes and performs the calculation 73/3 and finds 24 with a rest of 1. She then abandons her complex expression and finds a geometrical and processual solution that includes the coefficient: 73 − 13 = 60, 60/3 = 20, 20 + 4 = 24. Leah explains why she subtracts 13 from 73 in turn 27: `'cause we already have 13 [`*she runs her pencil back and forth above the figure …*`].` It is clear to her that the 13 matches make four squares, and then she only needs to worry about the 60 that are to be added to it. Thus, including the figure as part of her solution, she avoids modeling the irregularity of the pattern. The calculations here are performed in their linear order and therefore this type of discourse is at a processual level of algebraic discourse (level 1).

When trying to explain her solution to the other students, Leah gets confused about the value by which to divide 60, and, following the group's discussion in turns 27, 29, 33, and 35, the gesture of repeatedly tracing 3 matches seems to have become an embodied part of her thinking, signifying the coefficient 3.

The same gesture is identified to play an important role in group N1. However, Leah's (A2) repeated use of the specific motion allows for a more vivid elaboration of its meaning in this particular group. Early in the meaning-making process it signifies the process of physically adding squares to the figure (turn 8). Later, the gesture is part of a semiotic node through which an (incorrect) complex expression is created (turns 20 and 22). Leah struggles to backtrack her steps after having confirmed 24 as the correct solution and uses the gesture actively in the continued meaning-making process: `Well 13 plus 60 equals 73, so that'd be 13 plus 60 more [`*She traces 3 and 3 matches in the air*`]` (turn 29); and, `Yeah and so then that means, so this 13 we would have to add 60 more…[`*She points to the figure and then traces 3 and 3 matches in the air as if she is adding squares to the figure*`] 60 more 3 matches` (turn 33).

The gesture is here part of a second semiotic node through which the role of the number 3 is brought to further light. What stands out as important in discerning the role of the gesture is how Leah struggles to express her insights about the problem verbally. The sentences immediately before the gesturing are complete in themselves and explain the roles of the numbers 73, 13, and 60 in the solution of the problem. Thus, the gesture does not appear as a replacement for a missing word but rather seems to refer to a bodily insight of the nature of the problem. Leah tries to decide what to divide 60 by, and lacking words to organize her thinking, she resorts to using the particular gesture. She finally uses the number 3 as part of a verbalization `60 more 3 matches` (turn 33). However, the wording is awkward and we interpret it, not as expressing the meaning of the gesture, but rather as a means by which to bring a deeper insight into the nature of the problem (the gesture as an embodiment of the rate of change) down to something concrete, which she is able to verbalize. The next utterance further contextualizes the number 3: ( ) `and each square is made up of 3 matches connected to the last one…` (turn 35). Finally, Leah uses the number 3 in her calculations: ( ) `So 60 divided by 3 is 20…` (turn 35).

It is interesting to note that the three students, Leah, Rachel, and Aron have different closures to the problem. And, although Leah explains her solution three times in full, there is no direct transfer to the other students. Even Rachel, who often takes part in dialog with Leah by asking questions or making suggestions, do not accept Leah's solution based on the reasoning provided by Leah. However, it is also evident that the students' utterances are contingent on each other, and that there is a shared pool of ideas among them. First, the students pursue a multiplicative approach (using division). Second, the squares are connected, and therefore one has to take into account that one matchstick is a part of two squares: the squares are `interconnected` (Leah) or, in Aron's words, if you put two rows next to each other you will get a `double line.` Therefore, one needs to subtract 1, and the resulting coefficients in this group are 3 or 12. Leah makes sense of Aron's approach in turn 62: `Yeah, like you added 4 more [squares] and then you took like 1 [matchstick] from each one.`

Another point that deserves attention is how the students deal with the irregularity. Leah finds a geometrical solution that helps her avoid the problem of modeling it mathematically. However, in helping Rachel check the answer, she is able to make an expression that includes both the coefficient and the irregularity (turn 70): `it's 24 times 3 plus 1.` Again, we observe Leah using a granule in her expression; `24 times 3.` Aron subtracts 1 from 73, but no explanation is given, either orally or in written form.

Finnish Group (F3): "So It Is Just 4 Plus 3 Plus 3 Plus 3…It Is 4!"

In the group chosen from the Finnish material there are two girls, Lana (A) and Sarah (B), and one boy, Bjorn (C) (Grade 7, age 13). The focus in the prior lessons has been on solving equations and the meaning of the equal sign. The teacher gives the instructions that the students are to work together and gives them the choice of submitting one common paper or one paper each. The groups of students are given one task at a time, which they complete and hand in to the teacher before starting on the next task. The matchstick task is the last one given and the group spends about 9 min solving it. In the transaction of papers, some confusion arises and the girls

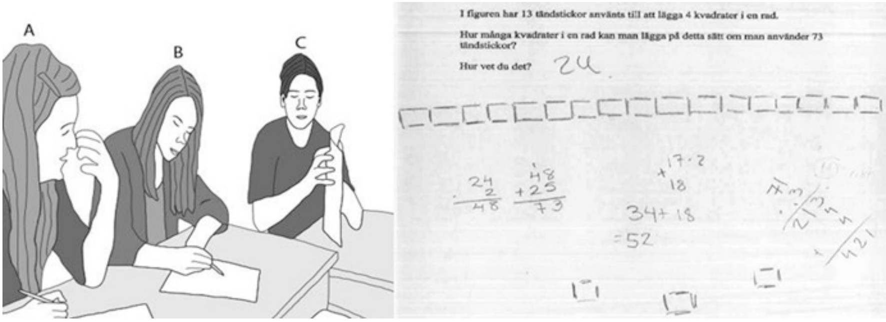


Fig. 8.7 Finnish group and one of their solutions to the matchstick problem

sort it out with the teacher. Meanwhile, Bjorn has been thinking about the task (Fig. 8.7).

22. Bjorn: `73! Fock me [in English] ( )It is very difficult.`
23. Sarah: `(inaudible)[`*`She starts to read out aloud but continues to read the task silently`*`]`
24. Lana: `[`*`she points her finger to the matches in the figure and counts them, first the ones making up the perimeter of the figure and then the 3 inside the figure`*`] Where are the squares? Aha.`
25. Sarah: `Mm It is just to start drawing [She starts to draw]. In case no one can do it mentally, which no one can.`
26. Lana: `Oh my God, 73 matchsticks.`
27. Bjorn: `73 divided by 13?`
28. Lana: `That is twenty…three or four. Ok, I'm going to draw too [she starts to draw, horizontal matches in a row first, then another row in a line underneath. She counts them and then draws the vertical ones so that it becomes squares in a row. She counts all the matches and then continues to draw]`

`[Bjorn writes 73/13 on his paper (in three different places) but is unable to perform the division.]`

The students read the task separately. Bjorn and Lana seem overwhelmed with the number of matches in consideration (turns 22 and 26). Lana pays attention to the figure and explores the connection between it and the text and it takes her a minute to see the squares in the figure (turn 24). Sarah has quickly decided that it is too difficult to solve the task `using only mathematics` and suggests they draw it all up (turn 25). However, Bjorn proposes an expression as an approach to solving the problem (turn 27): `73 divided by 13`. Lana considers it for a moment but decides to start drawing, just as Sarah is doing. The main focus of this section is the task and the two approaches suggested: drawing and a multiplicative approach.

The three students concentrate on their own papers for about a minute. Lana counts the matches in her drawing. First, she counts all of the upper horizontal ones and writes down the number 17. Then she counts the vertical ones (turn 18) and writes it down. She takes 17 times 2 and finds 34. Bjorn stops and looks up into the classroom. He talks to another student. Then he looks at what Lana is doing.

29. Bjorn: No, 34? Hah.
30. Lana: [*Sarah looks over at Lana's paper*] Don't look at my calculations because they are too bad for you, hah! Wait now [*Bjorn and Sarah turn to their papers*]
31. Bjorn: Hello, you divide 73 by 4. Then we include all the sides.
32. Lana: No, that does not work. You can't do such a thing.
33. Sarah: You can't do it, because there are 3 [*She holds 3 fingers up*] in some and 4 in some.
34. Lana: No there are 4 in two of them and that is at the ends.
35. Bjorn: Oh no, that is impossible to do. [*He looks up from his paper and points his pencil to Lana's drawing and seems to be counting the matches in it*]

Bjorn is still searching for an approach to solve the problem other than drawing. This time he proposes the expression: 73/4. The students then discuss not only the viability of this expression but also the applicability of a multiplicative approach: you can't do such a thing (Lana, turn 32), You can't do it (Sarah, turn 33) and that is impossible to do (Bjorn, turn 34). The girls point out that the expression and the approach cannot work as there are 3 in some and 4 in some (Sarah, turn 33). Lana makes it more precise and says No there are 4 in two of them and that is at the ends (turn 34). It is likely that the girls have gained these insights through the process of drawing. The students agree that the expression Bjorn suggested cannot work as the squares are made of different numbers of matches. The effort to disprove the expression leads to a discussion of the structures in the figure. This is a forerunner to the discussion of a pattern, which is initiated in turn 45. However, first they discuss how to interpret the task.

36. Sarah: It is easier if you just draw it, but it takes a fairly long time.
37. Bjorn: 10 [*He moves his pencil along Lana's drawing*]
38. Lana: Now there are 52 matchsticks here.
39. Sarah: Mm.
40. Lana: Sarah, they don't say… one could do it like this too [*She draws several squares that are not connected to each other on her paper. Bjorn and Sarah are looking at what she is doing*] And then you do one more… and here.
41. Sarah: But it is ((in a row))!
42. Bjorn: ((In a row))! ( ) Nice row.
43. Lana: But there is not enough space to continue making one row, but, Sarah can you continue?

Lana has run into a problem with her drawing, as she has reached the edge of her paper, and she has drawn only 52 matches. In contemplating how to proceed, she questions whether or not they have understood the task correctly (turn 40). However, Bjorn and Sarah use the wording in the task `In a row` (turn 41) to confirm that the squares are supposed to be connected to each other. Bjorn returns to his desk and looks at the task paper before he again turns to Lana to share a newfound insight regarding the structure of the figure.

44. Bjorn: So it is just 4 plus 3 plus 3 plus 3…it is 4! [*He points his pencil towards Lana and in the excitement he drops his pencil on the floor, which he ignores and keeps talking*] So see, 4 it is one square and then you only need to make 3 [*He points his finger to the figure and traces the 3 matches that make up a square in the row. With one movement he traces first the top horizontal one, then the right vertical one and finally the bottom horizontal one*],do you get it? [*He says it with a big smile*]
45. Lana: No [*She starts to draw a new row of squares*]
46. Sarah: Ok, Bjorn (inaudible)[*She is smiling and seems to enjoy Bjorn's enthusiasm*]
47. Bjorn: Look, look, "kato, kato kun minä" (in Finnish[5]: look, look when I) (inaudible)…look, you can first make a square with 4…
48. Sarah: Mm
49. Bjorn: Then you just need to make 3 when there is that one there "valmiiks" (in Finnish: ready) so you can take 4 plus…
50. Sarah: It is like 4, 4 [*holds up her left hand and then her right hand; implying one square with 4 matches at each end of the row*] and 3, 3, 3, 3, 3 [*moves her left hand in a rhythmic movement towards the right one which she holds still, signaling that the squares in between are made out of 3 matches*]

Bjorn has discovered that there are 4 in the first square and then one needs to only `make 3 matches to add another one`. He is beaming with excitement as he says: `So it is just 4 plus 3 plus 3 plus 3…it is 4!` (turn 44). It is worth noting that he uses the same gesture, tracing a square, as was part of the discourse in both the previous groups discussed. Lana is not interested and continues her approach of drawing all the 73 matches. Sarah is willing to engage in Bjorn's approach but retains the idea that there are 2 squares with 4 matches (see turns 33 and 34). Through her gestures and her words she is able to imagine the completed figure of 73 matches. Bjorn describes the numerical properties he sees in the figure

[5] These students normally speak Swedish, so the inclusion of Finnish words is a break in the ordinary discourse.

as a process of continually adding 3; 4 + 3 + 3 + 3, while Sarah describes it as a numerical sequence (an object); 4, 3, 3, 3, …, 4. Following the discussion of a pattern, Bjorn and Sarah both make calculations in order to find how many squares can be made using 73 matches.

51. Bjorn: 3, 60…3 divided by 69 [*He is probably thinking 69/3, but expresses it in reversed order*]
52. Sarah: It is [*She holds her hands in the same manner as in prior turn*] 73, 72, 71 of those with 3 [*She seems to mix up the units and instead of subtracting two squares from 24 she subtracts 2 matches from 73*]…71
53. Bjorn: No, but really one takes 4 minus…
54. Sarah: wait, wait, wait [*On Lana's paper she performs the calculation 71·3 = 213 and then she adds on the two squares with 4 matches (according to her prior explanation) and finds that 213 + 4 + 4 = 221*]
55. Bjorn: …73, because you already have made one there ( ) 69. But Lana, that takes such a long time [*He looks at Lana's paper as she is drawing squares*]
56. Lana: You think so.
57. Sarah: Hello. You can't make more squares than you have matchsticks [*She gives up on her calculations and returns to her drawing*]

Unlike the two prior groups, in which the students proposed expressions that included both the coefficient and the constant (irregularity), Bjorn and Sarah assign these objects their correct roles and use them directly in calculations. Bjorn subtracts the irregularity from the total (73 minus 4) and finds 69, which he then uses to suggest the expression 69 divided by 3. Bjorn does not pursue a solution using this approach, opting to follow what Sarah is doing on Lana's paper. Sarah writes down her own calculations, based on the structure she sees in the figure, but in addition to mixing up the units (squares and single matchsticks), she also uses multiplication instead of division and finds the number 221. She decides that this value does not make sense (turn 58) and abandons the multiplicative approach.

While Bjorn and Sarah discuss how to use a multiplicative approach to solve the problem, Lana has made a new row of squares and she pauses when she reaches 52 matches, which was how far she came in her first drawing.

58. Lana: look, Sarah, look, here we have 52… [*Bjorn listens to Lana and looks at her drawing but Sarah is busy with her own drawing*]
59. Sarah: mm
60. Lana: …matchsticks
61. Sarah: mm
62. Lana: …sooo [*She draws another square*]
63. Sarah: Wait, I shall draw this.

64. Bjorn: 21…7 times 3…7, 7 squares more [Bjorn is excited and points in Lana’s drawing], 7 squares more…7 squares more, yes, look
65. Lana: No
66. Bjorn: Yes, because when you have made 52, you just need to put 3 matchsticks for one square now. Do you get it? Look, now you have made 50 53
67. Lana: I have made 55
68. Bjorn: Have you?
69. Lana: Mmm. I have used 55 matchsticks.
70. Bjorn: Oh no ( ) and 73
71. Lana: But you just have to continue, just continue, now it works to go on [*This time she has drawn the squares smaller so that she has room for more of them in one row*]
72. Bjorn: 18
73. Lana: 56, 57, 58, 59, 60, 61 [She continues to draw squares]
74. Bjorn: No, but ((we get 18 divided by 3))
75. Lana: ((62, 63, 64)) [*Lana keeps drawing and ignores Bjorn who is getting frustrated*]
76. Bjorn: 6 squares more
77. Lana: 65, 66, 67
78. Bjorn: oh,((don’t you get it))!
79. Lana: ((68, 69, 70) now we have, [*She starts to count the squares that she has made using 73 matchsticks*]1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24. We get 24! Celia [the teacher], we are ready!
80. Sarah: Wait.[*She quietly counts*] 1, 2, 3, 4, 5, 6, 7, 8
81. Bjorn: Yes, do you get it? ((Because it was))
82. Lana: ((It is 24))It is 24 [*she rises her hand*] Celia [*teacher*], we are ready
83. Sarah: Wait, I just have to look, ((1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17))
84. Bjorn: ((I would have known a much simpler way to calculate it than what they did)) [*Bjorn talks to the teacher*]

Bjorn does not attempt to find the answer to the problem posed in the task immediately following the approach suggested in turn 52. Instead, he tries to convince the two girls that his approach is valid. When Lana announces that she has drawn 52 matches, Bjorn uses his method to find how many more she needs to draw. He takes 73 minus 52, which is 21, and then he divides 21 by 3 and says 7 squares more. As she draws another square, he does the same and finds that she needs 6 squares more. Lana gives him only a short moment to explain, and then quickly returns to her drawing. By drawing and counting she finds she can make 24 squares. Lana’s discourse consists mainly of counting the matchsticks she’s drawn, while Bjorn is verbalizing expressions that include the coefficient and performing mental calculations.

The teacher comes over to the group and she asks Bjorn about his approach to the problem. Sarah did not find the same answer as Lana in her first attempt of counting, and the girls are busy counting while Bjorn explains his thinking to the teacher. She encourages him to write his calculations on the paper and then compare his solution with the one the girls find.

123. Bjorn: `73 minus 4`[*He starts to write down his calculations*]
124. T: `Yes and that is?`
125. Bjorn: `69`
126. T: `mmm, and what did you do then after that`
127. Bjorn: `69 divided by`
128. Lana: [*The two girls agree on an answer after Sarah has counted first 73 matches and then 24 squares in Lana's drawing*] `Celia [teacher], but we have, now we have the right answer at least. It is the right answer?` [*She points to her paper. The teacher only glances over to her paper but mainly ignore her*]
129. Bjorn: `Twenty…twenty-three`
130. T: `mmm, and what is it you have calculated here?`
131. Lana: `It took ((quite a while to draw that))`
132. Bjorn: `How many…`
133. T: `Now I want, now I want, Bjorn`
134. Bjorn: `23 plus one … and then one` [*Excited he turns toward the teacher who is standing behind him*]
135. T: `Good!`
136. Bjorn: `23 plus one, 24.` [*He writes it on his paper*] `Oh! Look, look at what a short way I did it` [*He holds his paper in front of Lana*]

Bjorn writes and performs the calculations in the order of their execution (see Fig. 8.8): `73 minus 4` (turn 123), `69` (turn 125), `69 divided by` (turn 127), `Twenty…twenty-three` (turn 129), `23 plus one…and then one` (turn 134), `23 plus one, 24` (turn 136). When he finally writes up his solution, he has the units (squares and single matches) under control and he also accounts for the irregularity and uses the coefficient correctly and without hesitation.

The Meaning-Making Process

In this group, the two approaches (drawing and the multiplicative) run parallel throughout the problem-solving discussion, although not as separate processes. Bjorn refuses to draw 73 matches and continues to look for an arithmetic expression

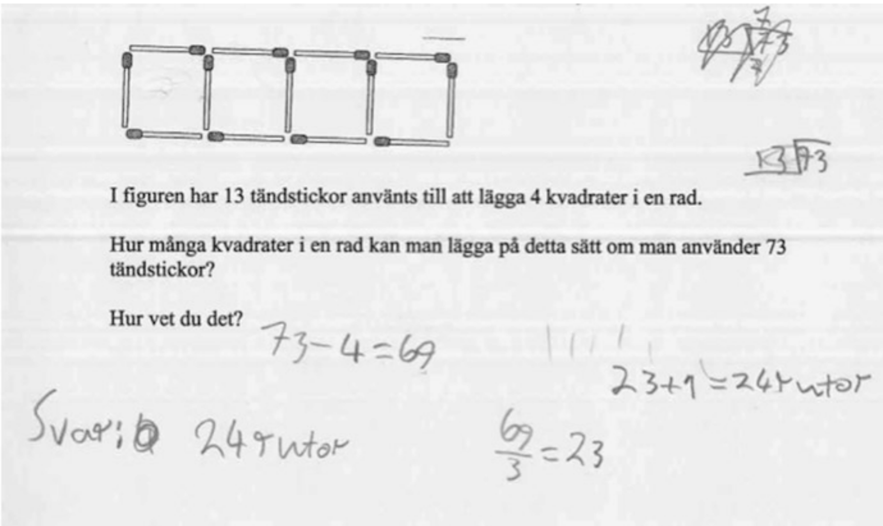


Fig. 8.8 Björn's solution to the matchstick problem

as a route to solving the problem. He tests his ideas on the girls, Lana and Sarah, who are in the middle of the drawing process. It is through the interaction between the two approaches (Bjorn discussing with the girls and tracing the drawings) that a discourse on a numerical pattern evolves in the group. The discourse of the girls as they draw involves only basic arithmetic and is mainly colloquial. However, when the students involve themselves in contemplating the structure of the figure and specifying a numerical pattern, they are modeling a relationship between quantities. Therefore, the students are engaging in a meta-arithmetic discourse as defined by Caspi and Sfard (2012).

The thematic discourses identified in the discussion of the group have been separated according to these focal objects: (1) *matchsticks* and *the figure* as concrete objects in the process of drawing and counting; (2) the mathematical objects of the *expressions* 73/13 and 73/4; (3) the mathematical object of the numerical structure of the *geometrical figure* that is transformed into a *numerical sequence*; and (4) the mathematical objects of *arithmetic expressions* that include a *constant* and a *coefficient*.

The girls decide to draw the 73 matches shortly after reading the task. They mostly work in silence but are interrupted by Bjorn and discuss the problem with him, sharing their insights about the numerical structure of their evolving figures. Lana uses addition to keep track of her counting; she writes down the number of horizontal matches and then adds the number of vertical ones. She runs into a problem as she reaches the edge of her paper and has drawn only 52 matches. She questions whether or not the squares have to be drawn in a row. Corrected by Bjorn and Sarah, she starts a new drawing in which the squares are made smaller. When she again reaches the number of 52 drawn matches, she announces it to the group (turn 58). Bjorn tries to engage her in his approach by calculating how many more squares she needs to draw, but she brushes him off as this time she has room enough to complete her figure. She now counts out aloud as she draws the matches and thereby verbalizes the approach. Sarah has not found the same answer and the girls engage in an intense counting process that goes on until the girls agree. Meanwhile, Bjorn has explained his solution to the teacher and written it up in his paper. The discourse of the students as they pursue the drawing approach consists mainly of counting words and non-mathematical verbs and is therefore close to a colloquial discourse. It also includes written numbers as aids in keeping count.

It is mainly Bjorn who pursues a multiplicative approach to the problem. In turn 27 he suggests the numerical expression 73/13. He writes it in three places on his paper but is unable to calculate it. Later in turn 31 he offers a different expression 73/4 and explains it saying `[t]hen we include all the sides`. He wants to divide 73 by 4 because 1 square is made up of 4 matches. He is thus modeling a relationship of the total number of matches to the total number of squares, using the number of matches in one square, and is therefore engaging in a meta-arithmetical discourse. The suggestion of this expression initiates a discourse on the structure of the figure.

As a response to Bjorn's expression, the girls insist that a simple expression cannot work as there are: `3 [`*`She holds 3 fingers up`*`] in some and 4 in some` (Sarah, turn 33). Lana makes a more precise statement about the structure of the figure: `No there are 4 in two of them and that is at the ends` (turn 34). Bjorn accepts their objections and returns to the figure. He points his pencil to Lana's drawing and counts the matches (turns 35 and 37) and then studies the figure on his own paper before he says: `So it is just 4 plus 3 plus 3 plus 3…it is 4` (turn 44). We interpret this as a discovery made possible through a semiotic node even though the process of coming to see the pattern is prolonged as he partakes in a discussion about Lana's drawing. In the semiotic node the process of counting successively the matches in the figure is aligned with the previously observed recurring numbers 4 and 3 and the observed structure of the figure. The connection made between these elements results in a new insight of the nature of the problem. In comparison with the semiotic nodes identified in the other groups, this one is different as the new insight is first verbalized followed by intense activity, which also include the familiar motion of tracing additional squares.

The number 4 and the following repetition of the number 3 signify and make existent the abstract object of a numerical pattern in the discourse. Sarah picks up on it and describes the numerical pattern she sees in turn 50: `It is like 4, 4 [`*`holds up her left hand and then her right hand; implying one square with 4 matches at each end of the row`*`] and 3, 3, 3, 3, 3.` These utterances are decontextualized and appear as an abstraction of the discussion about the structure of the figure. Bjorn describes the numeric pattern recursively as a process of continually adding 3, while Sarah describes the geometrical structure she sees as a numerical sequence in which 3 is the repeating number. The mathematical object of a numerical pattern appears in the discourse as the result of a process of noticing structure. In the following discussion, they use the coefficient and also the irregularity/constant in calculations.

In turn 51 Bjorn suggests the expression 69/3 which includes the coefficient and also, implicitly, the irregularity, as he has already subtracted 4 from 73. Instead of pursuing his approach in order to solve the problem, he uses it to predict how many more squares Lana has to make in her drawing. He performs these calculations correctly. When the teacher comes to the group, Bjorn finally writes down his solution as a linear process (listing the calculations in the order of their execution): $73 - 4 = 69$, $69/3 = 23$, $23 + 1 = 24$. Sarah also makes use of the coefficient and the irregularity in calculations but mixes up the units and uses multiplication instead of division: $73 - 2 = 71$, $71 \cdot 3 = 213$, $213 + 4 + 4 = 221$. In this group, the arithmetic expressions are used only in linear calculations and the discourse is therefore interpreted to be at a processual level of informal algebraic discourse. In this group, the discourse on arithmetic expressions is mainly short, concise, decontextualized, and impersonal.

*Swedish Group (S4): "First You Take 3…Times…*x *Equals…73"*

In the Swedish target group (Grade 6, age 12) there are two girls, Lori (A) and Tina (B) and two boys, Lars (C) and Ali (D). They have been working with variables in expressions and equations in the four prior lessons. The teacher hands out one task at a time. Before giving the students the matchstick task, the teacher says: `The last task can be conveniently solved by using an equation. You can solve it by drawing and trying different ideas but if you think about it for a while it is actually an equation that is the quicker solution even here.` The group spends about 12 min solving the task. Tina is on task continually, while the other students seem to partly rely on her to figure it out and then copy her. However, Lars and Lori do at times involve themselves in solving the problem. Ali is mainly off task, but asks other students (including students from other groups) for help. Lars' paper is presented in Fig. 8.9 and Tina's paper is presented at the end of the group analysis.

1. Tina: `In the figure, 13 matchsticks are used to make 4 squares in a row. How many squares… ((squares in a row can you make using 73 matchsticks)).`
2. Lars: `((how many squares in a row can you make using 73 matchsticks? How do you know?)) 73 … 13 times 4 what is that?`
3. Lori: `13 times 4`
4. Lars: `40 40 49`
5. Tina: `Wait, you ((have to))`
6. Lars: `((Wait wait 51 52))`
7. Tina: `Wait. 13 to 4 [`*she writes it on her paper*`]. First you take 3 … times … x equals … 73 [`*she writes it down as she speaks*`]`

Fig. 8.9 Swedish group and Lars's solution to the matchstick problem

The students do not discuss the problem but quickly propose ways of solving it. Lars appears to be using two numbers from the text and multiplies them. Tina takes more time before she expresses what she has understood of the problem. Using the same numbers from the text as Lars did, she focuses on the relationship between them: `13 to 4` (turn 7), implying that it takes 13 matchsticks to make 4 squares. Her equation $3 \cdot x = 73$ (turn 7) is likely to be based in the proportional reasoning displayed in her first statement, only here she relates 3 matchsticks to 1 square. A later statement indicates that she is using the x to represent the unknown (number of squares that can be made using 73 matchsticks) as she says: `That's what we have to calculate` (turn 22). She equates finding the value of x with solving the problem. However, the equation itself is incorrect, as it does not account for the additional match that is needed to form one square made out of four matches. The object of this discourse is the algebraic equation, which includes an unknown. In the next excerpt, Tina focuses on how to solve her equation.

19. Tina: `How many times can you do 3 in order to get 73?`
20. Ali: `((Do I have muscles here))?`
21. Lars: `((Inaudible))`
22. Tina: `That's what we ((have to calculate)).`
23. Lori: `((No, here)).`
24. Lars: `73 divided by something is 3.`
25. Tina: `60, how many times do you take … 20` [*writes in her paper*].
26. Lars: `20?`
27. Tina: `Then we get 20 and 10`[*A paper airplane comes flying and lands next to Ali. He gets up and walks over to another student group*]
28. Lars: `We get 23` [*He stretches out his hand and puts it down on the table*]
29. Tina: `It becomes a decimal number.`
30. Lars: `What?`
31. Tina: `It becomes a decimal number.`

32. Lars: `It does?`
33. Tina: `Yes, it does.`
34. Lori: [*She pulls on Tina's shirt to get her attention and she talks about what Ali is doing*] (inaudible) pull pants (inaudible) very funny. [*She starts to write on her paper. Ali comes back to his group*]
35. Tina: `Wait, 12, that is 3 times 4 then it becomes plus 4 [She starts writing on her paper] plus 4.`
36. Lars: `13 before 4` [*He leans over in order to see what Tina has written*]
37. Tina: `And then point 3 ((3 3 3 3 3))`

In this section, Tina is focused on determining the value of *x*, and Lars engages in the same pursuit. When finding it hard to determine a multiple of 3 that goes into 73, Tina chooses to find a multiple of 3 that is 60, which gives her the number 20 (turn 25). Lars has a different approach and says: `73 divided by something is 3` (turn 24). He finds the number 23 (turn 28), but Tina has become aware that 73 is not divisible by 3, and that the equation will give them a decimal number (turn 29). Again, she finds a number, 12, that is a multiple of 3, which gives her the number 4. She adds 20 and 4 (turn 35) and then she knows that 1 divided by 3 becomes 0.333333. She finds that $x = 24.333333$. The discussion in the group continues as the other students want to copy Tina's work. However, she tells them to think for themselves. When the conversation again returns to mathematics, Tina and Lars discuss the nature of the number she has found. Tina is not satisfied with simply writing *24.333333,* so instead writes `infinity` after it.

Tina does not seem concerned with how this number fits as an answer to the question: How many squares in a row can you make using 73 matches? In making the equation $3 \cdot x = 73$, Tina makes the problem mathematical and abstract. Trusting that her equation is correct, she does not return to the context from which it was extracted until she tries to explain her solution. She then turns her attention to the second question in the text and says: `Ok how do you know?`

Tina works on her paper in silence and writes an explanation for her algebraic equation: `one needs 3 because one uses the prior edge. Then one takes 3x = 73, then one starts with 60 = 20 squares.` She returns to the context only in order to explain the number 3. Meanwhile, Ali, Lars, and Lori are off task. The teacher comes over to the group. She relates the answer they have found to the context of the problem posed, and then she leads them in a discussion of the pattern in the figure:

70. Teacher: `How is it going?`
71. Tina: `I think it is going really well` [*She finishes writing and pushes her paper towards the teacher. The teacher only glances at it but does not offer any comments*].
72. Lars: `I think something is good. I calculated (inaudible) that is nothing` [*He crosses it out with his pencil*].
73. Teacher: `You have chosen to divide 73 by 3.`

74. Lars: No.
75. Teacher: [*She points to his paper*] It says there 3 times x will be 73.
76. Lars: Yes.
77. Teacher: So how are you thinking? So it won't be a whole number of, eh squares?
78. Tina: No.
79. Teacher: Does it become a strange square that does not tie together in the end?
80. Tina: Yes [*Smiles*].
81. Teacher: Aha, how many toothpicks [sic] do you use in each square?
82. Tina: ((3))
83. Lars: ((4))
84. Teacher: Is there any time when you use more than 3?
85. Lars: Yes, you have to make a square.
86. Teacher: No, then you just put 3. Think about what happens when you put down the first square?
87. Tina: Then you put down 4. [*Lars starts to look at and draw on the figure in the paper*]
88. Teacher: Aha. ((So we have done something wrong there))
89. Lori: ((And then you just put 3))
90. Teacher: The first time you need 4, how many ((times))…
91. Lars: ((3)) [*Lars says 3 every time he adds a new square to the figure*]
92. Teacher: how many toothpicks [sic] do you need after that?
93. Tina: ((3))
94. Lori: ((3)) 3 3
95. Teacher: How would you calculate an even number of squares?
96. Lars: 3
97. Teacher: One more time.

Tina's reply to the teacher; I think it is going really well (turn 71) implies that she has confidence in her approach and in her answer. Lars is more uncertain (turn 72), and as he cannot answer the teacher's questions it becomes clear that although he has participated in parts of Tina's reasoning, he has not been able to follow her train of thought. Tina shows that she is aware that there are 3 matches in each square (turn 82) (which is anticipated in her equation). Lars returns to the figure on the paper and starts to add new squares in order to explore the pattern. With the help of the teacher, Tina, Lars, and Lori seem to agree that the numerical properties of the figure can be described by the numerical sequence: 4 3 3 3. The teacher leaves the group with the challenge of calculating the whole number of squares.

The teacher initiates a new type of discourse in the group after pointing out that the approach proposed does not make sense when put in the context of the problem.

This discourse is about new objects that have not been visible in the prior discussion of the group: a geometrical pattern that is transformed into a numerical pattern in the form of a number sequence.

When the teacher leaves the group, Lori and Ali look to Tina, but she tells them to wait because she needs to think about it. Lars continues to draw and count matchsticks. Ali asks a student, Carl, who is walking past the group for help.

113. Ali: `What did you get? Come, come help me Carl.`
114. Carl: `72 divided by 4`
115. Ali: `Come here, come. What did you say? What did you say?`
116. Lori: `72 divided by`
117. Ali: [*He writes on his paper*] `Divided by?`
118. Tina: `3 toothpicks [sic]`

Carl offers an expression as a solution to the task without any explanation and thus initiates a shift in the discourse of the group. Lori and Ali are trying to register what he says, however Tina seems to contemplate the expression and incorporates it into her understanding of the problem. She gives a contextual answer to the other students' question regarding what number to divide 72 by: `3 toothpicks [sic]` (turn 12). In the continuing discussion the students keep debating using 73 or 72 divided by 4 or 3.

`Carl and another student, Tim, come over to Lars who is counting the matchsticks in his drawing and pointing his pencil to them as he counts.`

119. Tim: `What are you doing Lars?`
120. Lori: `Oi 73 I mean. Did you say 73?`
121. Tim: `There is a much, much easier way then yours, really.`
122. Lori: `But it is 73 matchsticks. That is what she says` [*She holds out her hand towards Tina*]
123. Lars: `But you have math-geniuses you two` [*He points out in the classroom*]
124. Tim: `What math-geniuses? ((Sarah and Lori? Number one; it was me who worked it out))`
125. Tina: `I understand how you think. Do it like this.` [*She puts her pencil to the paper*] `And then you do minus one. Because that one should be saved for the one with 4 isn't it?` [*She looks at Lori, then she writes 3x = 72 to the right in her paper*]
126. Carl: `It is 72 divided by 4`
127. Lars: `She is a math-genius.` [*He points to Tina*] `((She works out everything)).`
128. Lori: `((I'm done now))` [*She picks up her paper and puts it back, face down*]
129. Ali: `Me too` [*He picks up his paper and puts it face down*]

130. Tim: (Inaudible)[*Tim and Lars have turned to Tina and look at what she is doing on her paper*]
131. Lars: Hello. She calculates in a good way, so that you get something.
132. Carl: Lars says that you divide 73 by 3?
133. Tina: ((But it is really that and then it becomes plus one at the end in addition? [*She looks at the two boys, Lars and Carl*]
134. Tim: But ((73 divided by))… it is 72 divided by 3.
135. Tina: Thank you Tim.
136. Tim: 72 divided by 3 equals 24

The group now has three different accounts of how to solve the problem: 73 divided by 3, 72 divided by 4, and 72 divided by 3. Lori does not attempt to make sense of the expressions suggested by looking at the problem. Instead, she asks the other students to clarify and make a decision (turns 120 and 122). When this is not provided, she gives up on writing a solution for the problem (turn 128). Ali follows her example (turn 129). Tina is contemplating why one should divide 72 and not 73 by 3 and says: I understand how you think. Do it like this. [She puts her pencil to the paper] And then you do minus one. Because that one should be saved for the one with 4 isn't it? (turn 125). She makes sense of the number 72 by connecting it to the square made of 4 matches, however, later in the discussion it becomes clear that she has not solved the issue of modeling the irregularity.

The numerical expressions mentioned above are the objects of the discourse as long as the two boys are part of the group discussion. A discourse about the irregularity is developing intertwined with the one about the expressions. It first appears in the discussion of the numerical pattern, but now it plays a crucial role in determining which expression to use.

The two boys go away but return shortly. Carl is bringing a calculator. The students in the group want the two boys to leave them alone.

137. Tina: You know what? Find some other place to be. [*She clearly wants the boys to go away*]
138. Lars: 73 divided by something.
139. Tim: No, because in the first there are 4, aren't there?
140. Carl: 72 divided by 3 equals 24 [*He enters the numbers into the calculator and shows the answer to Lars*]
141. Tim: Idiot [*He gives Lars's hat a light knock*]
142. Lars: Yes, I know!
143. Tim: You said 73, so you did really know that
144. Lars: [*The two boys walk away*] Orrrh! We will do it in our own way.

145. Tina: Wait a bit. And then you have to take away one to save it for later isn't it?
146. Lars: Maybe.
147. Tina: Yes, you have to.

Lars does not continue to draw and count after the interruption by the two boys. He tries to engage in the discussion of an approach using division and says: 73 divided by something (turn 137). The two boys aggressively point out that one is not to divide 73 but 72 by a number and they explain this indirectly by referring to the first square, which is made up of 4 matchsticks (turns 139 and 141). Carl then uses the calculator to show that 72/3 = 24, implying that this proves they are right. Tina is still contemplating the connection between using 72 and the irregularity of the first square, and she repeats her prior conclusion: And then you have to take away one to save it for later isn't it? (turn 145); Yes, one must (turn 147). Tina tries out her ideas by sharing them with Lars but he does not offer any clarification.

149. Tina: 73 minus 1 becomes 72. [In her paper, she erases the 3 in the number 73 and replaces it with 2] How many times does 3 go into 72? That makes 60, 20. [*She writes in her paper*] 20. 4 and then the last one is left.

Tina now concentrates on working out her new ideas on the paper. When writing, Tina incorporates the discourse about the different numerical expressions into the prior, formal algebraic one, as she changes her previous algebraic equation into $3 \cdot x = 72$. Applying her previous method of division (60/3 = 20 and 12/3 = 4), she finds a whole number of squares, 24. On her paper she adds a sentence to her explanation: Then 4 because 4 · 3 = 12 then 20 + 4. However, she is still uncertain about the matchstick she has saved for later: and then the last one is left (see Fig. 8.10).

Tina is quiet for a while, and studies the problem. The other three are off task talking about other things. She then writes + *1* in her paper and places it in the middle of the table.

150. Tina: Ok I'm ready.

The other students are still talking about other things and do not pay attention. After a while, Lars takes an interest in her solution. He grabs the paper, but Tina takes it back and puts it under her elbow.

151. Lars: ((Let me see))
152. Tina: ((You can write)) your own calculations. You don't think the same way I do.
153. Lars: No, because I can't think math.
154. Tina: No, but think for yourself.
155. Lars: But why?

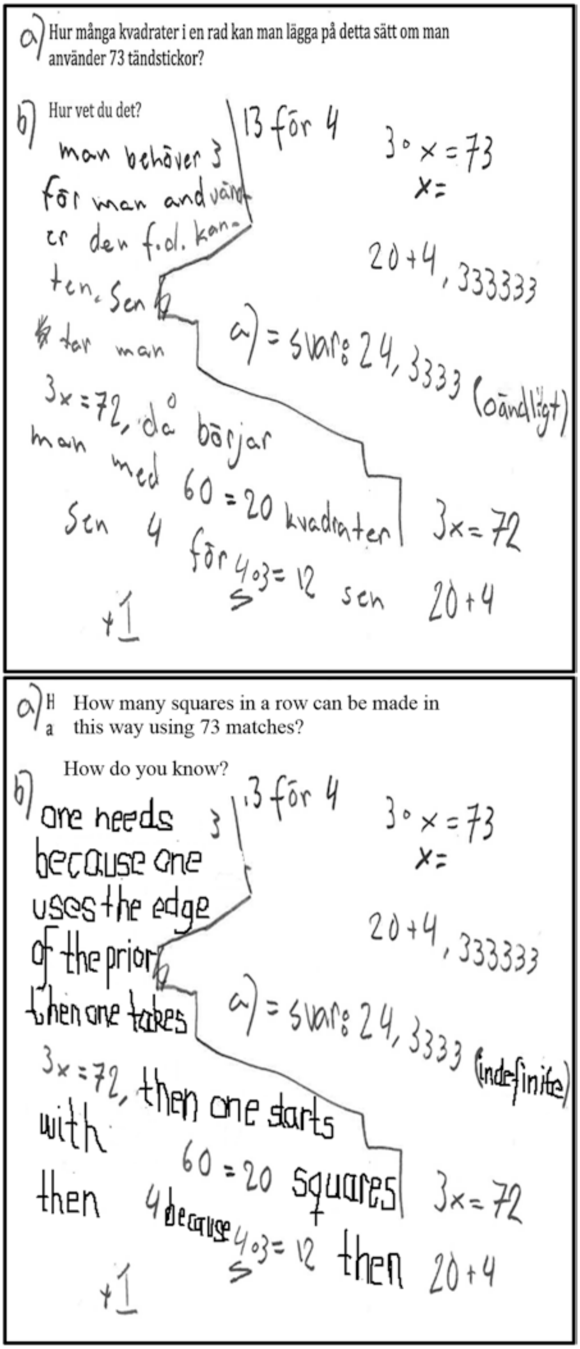


Fig. 8.10 (**a**) Tina's worksheet for the matchstick task, and (**b**) English translation

156. Tina: `Because we think differently. ( ) Ok, you solve it like this. You have to take 73 and minus 1 [`*She writes it on the desk as she explains*`] which you save for later, because there is going to be 4 in one. Then it becomes 72 and then you have to take 3 times something equals 72. What is the something?`
157. Lars: `24 ((24)).`
158. Tina: `((You take)) 60 divided by 3 which is 20. And then you get 12 and then it becomes plus 4 and then you get 24.`
159. Lars: `So that is like they told us?`
160. Tina: `Aha.`
161. Lars: `So then you can just write x, so then you can just write eh write that [`*He points to what she has written on the table*`]`
162. Tina. `No. You have to explain how you think when you calculate.`

Tina is satisfied with her approach. She has shown on paper how she finds 24 (even though she does not write the answer), and she seems to be confident that it is correct. She is still bothered by the one matchstick that she subtracted from the 73 but resolves it by simply writing +1 at the very end of her solution. Tina explains to Lars how she solved the task. He recognizes the solution as identical to the one that the two boys, Tim and Carl, proposed (turn 159). And when Tina asks him: `Then it becomes 72 and then you have to take 3 times something equals 72. What is the something?` (turn 156); he immediately replies 24. He finds Tina's method of division an unnecessary step, as he already knows that 72/3 = 24. She insists that it is important to show your calculations in your written work.

The Meaning-Making Process

This group work is different from those described from Norway, the U.S., and Finland, in that the teacher has suggested that the students should use an equation to solve the problem. The initial responses to the task differs: the students in N1, A2, and F3 struggle to make sense of the problem mathematically and search for ways to model it; however, Tina creates an equation immediately after having read the task. The group work is also influenced by other prompts. It is the teacher who points out that the first square is different from the following (when presented with the group's solution), and then she tells the students to find a whole number of squares. In addition, two boys from other groups suggest and argue for the use of the two expressions 72/4 and 72/3. The process of solving in this group therefore is in some sense "jolted" by specific clues. Characteristics of this group's work are the use of formal algebraic discourse and a lengthy discussion of the irregularity.

The discourse of the group is meta-arithmetical throughout the discussion as the students are concerned with relationships between quantities. Four different types of discourse according to their focal objects have been discerned: (1) the mathematical objects of an *unknown* in an *algebraic equation*; (2) the mathematical objects of a *geometrical pattern* transformed into a *numerical sequence*; (3) the mathematical objects of three *numerical expressions*: *72/4*, *73/3*, and *72/3*; and (4) the mathematical object of the *irregularity*.

Tina's sense making of the problem seems to be based in the *equation* as an algebraic object and three numerical relationships which model geometrical ones: *13 to 4*, *73 to x* and *3 to 1*. Tina's first sentence: `13 to 4` (turn 7) states a static relationship between two quantities (13 single matchsticks equal 4 squares). Her equation (which is her next utterance) is more complex as it combines 2 numerical relationships into a third one (*73 to x*, *3 to* 1 $\geq$ *3* $\cdot$ *x* = *73*).

Tina verbalizes and symbolizes (on paper) the inchoate algebraic equation $3 \cdot x = 73$. The x is a referent to an unknown, specific number and so the discourse belongs to the first three levels of algebraic discourse; constant value algebra. Tina is able to make sense of the problem in terms of an equation and thereby uses the cultural tool in an objectified way. Therefore, her discussion of the task is evaluated as an objectified algebraic discourse (level 3). Tina later changes her equation (turn 149) to $3x = 72$ by replacing 73 with 72 in her previously written one, and she finds the solution of 24 squares. However, it no longer fully models the problem and she struggles with how to account for the matchstick she has subtracted from 73.

The pattern discourse in this group is very limited and is initiated by the teacher. It appears in the students' discussion as noticing structure and is verbalized as a numerical sequence in which 3 is the repeating number (by the contribution of several students): 4 3 3 3. Tina, early in the discussion, formulated the algebraic expression $3 \cdot x = 73$ in which she already identified the number 3 as playing an important role in the problem. Therefore, the discussion of the pattern is mainly helpful in identifying the irregularity.

The numerical expressions in the discussion of this group play a different role in the problem-solving process than they did in the other groups. When Tina suggests her algebraic equation $3 \cdot x = 73$ in turn 7, she already presents a multiplicative approach to the problem. The numerical expressions (72/4 and 72/3) suggested by the two boys who are not part of the group have an effect mainly on how the irregularity is included in the calculations and modeled by the group.

After the two boys enter the group discussion (turn 119), the irregularity becomes a central topic of the group discussion. Tina struggles to make sense of why she needs to use 72 in her equation instead of 73: `And then you do minus 1. Because that one should be saved for the one with 4 isn't it?` (turn 125); `But it is really that and then it becomes plus 1 at the end in addition?` (turn 133); `Wait a bit. And then you have to take away one to save it for later isn't it?` (Turn 145); `73 minus 1 becomes 72 […] then the last one is left` (Turn 149). The irregularity is referred to as the square with 4 matches and as the *one* that is subtracted from 73, often in a combination where the square made out of 4 matches is the explanation for why one should subtract 1 from the total number of matches. Tina changes her equation to $3x = 72$ and finds the answer 24. Although Tina appears to be satisfied with her solution, the role of the irregularity in the problem and how to model it seem to remain a puzzle for her as she finalizes her written answer by simply writing +1 at the end of her solution. The irregularity is accounted for in the calculations; however, it is not modeled as a constant term in a rule and thus the discourse regarding it is evaluated to be at a processual level of algebraic discourse (level 1).

Part Three: Juxtaposing the Four Groups

The in-depth discourse analysis of the four groups from different school systems, Norwegian (N1), American (A2), Finnish (F3), and Swedish (S4), shows that the meaning-making processes of the different groups all have their own characteristics. Upon this acknowledgement, we find the similarities observed all the more intriguing. The work of the three groups presented first (N1, A2, and F3) is developing without prompts, and shares features that are worth further contemplation. In particular, in each, there is movement from a discourse about concrete objects such as matchsticks and figure, to a discourse about a geometrical/numerical pattern, and finally a discourse that includes complex expressions and calculations. Additionally, features of prior thematic discourses in the problem-solving process play prominent roles and mediate meaning in the critical moments of discursive shifts. The group S4 provides a contrast as we can observe the effects of specific prompts in a meaning-making process—and particularly how a student is able to use the concept of algebraic equation in solving the problem.

Part One did not show a consistent movement between approaches described, across the 16 target groups. This does not mean that our findings are contradictory, but rather it reveals important differences in the two ways of looking at the data. First, in Part One, we looked at the approaches to finding an answer; while in Part Two we focused on the way students find meaning. Second, there are multiple approaches that fall under "discourse about concrete objects" and multiple others that fall under "geometrical/numerical pattern," etc. This slightly larger grain size brings a trajectory to light. Third, in the analysis of approaches, we reported them completely linearly, from the introduction of one, discrete approach to the introduction of the next (without taking into account whether students went back and forth). However, in Part Two, we looked at how discourse builds upon itself in a more general sense applying analytical tools of Commogniton (Sfard, 2008) and the Theory of Knowledge Objectification (Radford, 2002).

In this section we will focus on the features of the meaning-making processes of the groups that show similarity in order to point out patterns of a learning process in algebra. Another objective is to describe and evaluate the discourse of the students according to the model of *constant value algebra* described previously (Table 8.1). In order to address these issues we will now present and discuss under one heading the thematic discourses identified separately in the four groups.

Common Characteristics of Thematic Discourses and Discursive Shifts

The discourse of the students has been categorized into five thematic discourses merging the ones occurring in the different groups that are similar:

1. *Concrete objects*: matchsticks, the figure; often involve drawing and counting but also include gesturing and word use in order to imagine a completed figure.
2. *Pattern*: numerical and geometrical patterns described in many different ways, for example: 4 + 3 + 3 + 3; 4 3 3 3; 3 6 9 12; 4 in one and then 3 3 3.
3. *Arithmetic (numerical) expressions*: expressions that include the coefficient and/or the constant from a pattern and those that do not.
4. *Algebraic equation*
5. *The irregularity*

Concrete Objects

This thematic discourse emerges for different reasons in the groups N1, A2, F3, and S4. However, it comes to play a similar and important role within their meaning-making processes. Recall from Part One that a drawing was used by ¾ of the groups in the full sample. N1 tries to find a numerical relationship between numbers of matches and numbers of squares while drawing and counting. Ultimately, this becomes the route to solving the problem (after giving up on their complex numerical expression), as is the case with F3 from the very beginning (the girls in F3 plan to draw all the matches in order to solve the problem). A2 imagines a completed figure using gestures and words when analyzing the task (Leah). Also, Aron (A2) uses this type of discourse later in the group work when he explains his thinking using a mix of gestures, drawings, and words. A discourse about concrete objects plays a limited role in S4, as only one student draws for a short period of time, initiated while the group discusses the pattern. The words used for the concrete objects are varied: matchsticks, squares, boxes, such things, rows, pieces, double line, one of those (referring to the figure), and uneven shape. It is clear that although the students are looking at the exact same figure what they see is not the same.

The discourse about the concrete objects often addresses actions and involves an extended use of verbs together with human actors: `we can make` (N1, turn 23), `we get` (N1, turn 24), `if you have taken` (N1, turn 24), `if we take` (N1, turn 28), `when we have` (N1, turns 28, 30, 32). Aron, in turn 38 (A2), uses these wordings: `if you take… you get… then you have… you draw`. The utterances that accompany the gestures as Leah is imagining a complete figure are similar: `you can keep on adding boxes of matches` (A2, turn 8), `you add on 3 until you get to 73` (A2, turn 10), `you count how many boxes you made` (A2, turn 12). Counting is part of the drawing process, and therefore the discourse becomes very extensive if that is the main approach applied and discussed in the group. Written calculations (i.e., addition) are also often part of this type of discourse, as the students count horizontal and vertical matchsticks separately, write down the numbers, and then add them (N1, F3).

The drawing on the paper, and the motions as students draw new squares, seem to be consequential for the development of a discourse about the numerical pattern in the groups. The observing students benefit from watching the evolution of the figure, in that they become aware of its numerical structure (N1, F3). The gesturing of tracing squares as if she adds them to the figure is part of Leah's (A2) awareness that one adds 3 matches for each new square. It is noteworthy that Trish (N1) and Bjorn (F3) use the exact same motion (as Leah) as part of their meaning-making process.

The type of discourse described here is mainly colloquial. It includes only basic arithmetic, such as counting and adding. In the groups N1, A2, and F3, the discourse on concrete objects forms the background for a discourse that indeed is an abstraction of the prior: the structure of the figure is discussed in numerical terms, forming the notion of a pattern. The transition between these different types of discourse is better described as fluid than distinct. The discursive shift is initiated as the students engage in and reflect on the process of adding new squares to the figure. That is, students look at the figure, often tracing the squares with their finger, or imitating the motion of adding new squares using gestures (N1, A2, F3). It sometimes also involves a different engagement with an ongoing counting process, i.e., counting by three, `53, 56` (N1), rather than counting by one. These reflections also seem to be spurred and influenced by the question of what to divide 73 by (A2, F3). Matches and boxes reside in the background, as something of an abstract nature comes to the forefront of the discussion and is described numerically. Although the students often look for a relationship between numbers of matchsticks and numbers of squares as they engage in a drawing and counting approach, there is no explicit evidence in the students' discourse that they are looking for a pattern. Indeed, not one of these groups uses the word "pattern" or a word that is synonymous. Yet, they describe the geometrical figures using numbers in several different ways and identify regularities and patterns observed.

Pattern

The geometrical pattern in the figure provided in the task is described numerically by the groups both as a process and as static properties of it. As the students engage in a discourse about a pattern, they focus on the paper. The semiotic means include: the figure provided in the problem; inscriptions, i.e., the extended figure in the forms of student-produced drawings; and gestures, i.e., tracing squares on paper or in the air, and also rhythmic hand movements representing squares in a row (which thereby also extends the original figure). In particular, the motion of first tracing an upper horizontal match, then a vertical right match and finally a bottom horizontal match, appears as an instance of embodied cognition, which relays the pattern as a process (N1, A2, F3).

In comparison with the discourse on concrete objects, the discourse on pattern is short and concise. It is descriptive rather than action-oriented, and the human actor is removed from the discourse: the pronouns I, you, we are not used. About half of the utterances regarding the pattern are contextual and describe objects in space. Even though the words "matches" or "boxes" are not used, they are indirectly referred to: `in some`, `in each`, `in them`. Often, these types of descriptions precede the utterances that are more strictly listing numbers such as 4 3 3 3, 4 + 3 + 3 + 3 and 3 6 9 12 (N1 and F3). The listing of the numbers 4 and 3 retains the spatial properties of the figure and therefore preserves a direct link to it. While the sequence 3, 6, 9, 12, (N1), leaving the first matchstick unaccounted for, translates the geometrical pattern into a numerical sequence which encompasses a functional relationship, i.e., the number of matchsticks vis-à-vis the number of squares. In the groups N1 and F3 the latter verbalizations of a numerical pattern are immediately followed by numerical expressions and calculations in the groups' discussions.

The mathematical object of a numerical pattern seems to appear in the discussions of the students (N1, A2, and F3) almost by coincidence. Its materialization (spoken words and gestures) in the group discussions seems more due to a need for making mathematical meaning through ordering, using numbers, than to students' awareness of the mathematical object of numerical patterns. This interpretation is supported by the fact that the students neither use the word "pattern" nor refer to similar mathematical problems in their discussions. If the students made this reference, tacitly, it is likely that it would initiate a certain course of action—formulating a rule for the pattern. No such attempts are found in the students' discussions. The mathematical object of a numerical pattern is evoked in the students' discussions through processes of noticing structure of the figure. The figure remains the object of the students' discussions, while the numerical patterns exist as momentary objects in the discourse and are not talked about as objects in themselves. We therefore evaluate the students' discourse about pattern to be at a granular level of algebraic discourse (level 2).

In N1 the discursive shift from a pattern discourse to numerical expressions and calculations is initiated by a semiotic node: tracing the figure while verbalizing a new numerical sequence in which the consecutive terms rises with 3. Bjorn, in F3, also looks at the figure, verbalizes the pattern as a process of continually adding 3, `4 plus 3 plus 3 plus 3` (turn 44), and traces the figure, immediately before suggesting arithmetic calculations. However, all groups, on several occasions, discuss the question of what to divide 73 by, and it seems to remain as an underlying force for discourse development in the groups, throughout their meaning-making process.

Arithmetic Expressions

The proposals of different arithmetic expressions represent, to a large extent, the students' attempt to use previously internalized mathematics to solve the problem. There are two main categories of arithmetic expressions, those that include the coefficient and the irregularity from a pattern discussion and those that do not. The first type of expressions is often suggested immediately following a semiotic node, which signifies 3 as a coefficient through which numbers of matches and numbers of squares can be related to each other. The latter expressions are arguably a product of limited meaning-making of the problem and putting the numbers in the text to use. In A2 and F3 the suggestions of this type of expressions initiate discussions about the structure of the figure.

If we look at Fig. 8.4, *Frequency of inchoate approaches to the matchstick task* (cf. p. 172), we find a high frequency of approaches that include the expressions 73/13, 73/4, and 73/3. The first expression differs in that it considers the complete (original) figure as a unit, while the other two use a square. However, all three expressions seem to be motivated by direct proportion and are simpler mathematical models than the correct one. Stacey (1989) problematizes what she sees as students' intuitive familiarity, from quite a young age, with the relations connected to direct proportion. Her mixed-method study of students' work with patterns includes students from ages 9 to 13, and she explains (op. cit., p. 160):

> The qualitative similarity in the responses of the primary and secondary students indicates that well before the formal study of relevant topics (ratio, linear functions and the distributive law), students have built up an understanding of certain nodes in the complex web of relationships that apply to direct proportion and linear and affine functions. In particular they overgeneralize these properties and employ them inappropriately.

Recall that we previously observed how Tina (S4) made her inchoate equation based on proportions: `13 to 4 [`*`she writes it on her paper`*`]. First you take 3…times…x equals…73 [`*`she writes it down as she speaks`*`]` (S4, turn 7). The implicit proportions involved are 3 (matches) to 1 (square) and 73 (matches) to *x* (squares). The students' intuitive tendency for using direct proportion that Stacy describes seems to permeate the students' initial work with the matchstick task, even when the students use formal algebraic syntax. However, the nature of the students' argumentations changes as they focus on the structure of the figure and grapple with the role of the number 3 in the problem.

In the complex expressions, the number 3, signified by the specific gesture, is reminiscent of a coefficient (rate of change) of a functional discourse as it signifies change—how the figure evolves. However, as the students in the groups N1, A2, and F3 do not attempt to formulate a rule for the pattern, there is no explicit evidence of an objectified use of the coefficient as rate of change. The coefficient is only evoked in processes, as part of numerical expressions and calculations, as the students discuss the problem.

Although students do not formulate a rule for the pattern, Bjorn (F3) develops what Radford (2010) has identified as an *in-action-formula*. In commenting on another student's evolving drawing and counting efforts, he calculates how many more squares that can be made using 73 matchsticks, if one subtracts the ones already used: 73 − **52** = 21, 21/3 = 7; and 73 −**55** = 18, 18/3 = 6 (turns 64–76). Finally, he uses the "formula" to calculate the answer to the problem, subtracting the one square made up of four matches and then adding it at the end: 73 − **4** = 69, 69/3 = 23, 23 + 1 = 24. Radford (op. cit.) explains this type of formula as an "embodied 'function' or 'predicate' with a tacit variable" (p. 7). A variable is implicitly present through some of its particular instances (52, 55, and 4). Bjorn's in-action-formula, $(73 - a)/3$, is limited in generality as it only applies to the particular case of 73 matchsticks, and the number that varies, a, is restricted to being a member of the numerical sequence described by the formula $a = 3n + 1$, $n < 24$. In the group discussion the formula is, in actuality, restricted to drawn and counted matchsticks, which further highlights the situated and concrete form of Bjorn's algebraic discourse. The in-action-formula exemplifies a type of generalization which belongs to constant value algebra as indeterminacy is not explicitly part of the discourse.

Across the groups, the students' discourse on arithmetic expressions is mainly at a processual level of algebraic discourse (level 1) as the students are focused on linear calculations. However, there are exceptions as Trish (N1) and Leah (A2) model the problem using granules in their verbal expressions before doing any actual calculations: `73 divided by 3 and then you just add 1 more` and `73 divided by 3 and then minus 4`. The calculations are still listed in the order of intended execution, however, in the sentence the result of 73 divided by 3 is bypassed and the next operation is listed as if the granule is a number, i.e., an object. The granule is created by the use of a preposition (divided by) which seems to signify both an action and its result. This seems to be an example of *presentation by prepositions*, one of two ways of dealing with intermediate results in complex expressions, identified in the study of Caspi and Sfard (2012). The other one is *presentation by nominalization*; the replacement of a verb, for example *multiply*, in a verbal/written expression with its nominalization, the noun *product*.

The result of using prepositions is that it gives the sentences (expressions) recursive depth. Caspi and Sfard *(op. cit.*, p. 58) define recursive depth of a sentence as "the length of the longest branch in the parsing tree of this sentence," as measured by the number of segments that constitutes this branch. The recursive depth of the students' granulated expressions in our study is 2 (see Fig. 8.11). Caspi and Sfard (op. cit) found that students employing a spontaneously developed meta-arithmetical

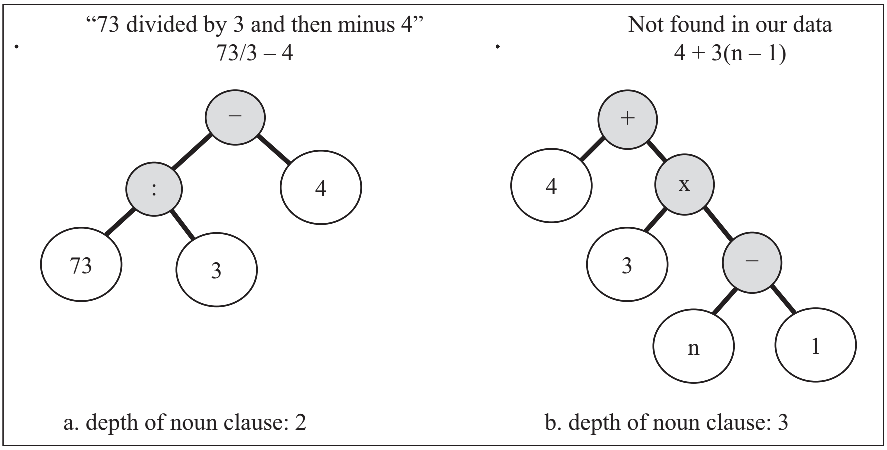


Fig. 8.11 The recursive depth of the expressions $73/3 + 4$ and $4 + 3(n - 1)$, based on Caspi and Sfard (2012, p. 58)

discourse created rules in which the recursive depth never exceeded 2. This may indicate that there are limitations regarding the complexity that students are able to handle analytically when employing this type of discourse.

Also, as the matchstick problem includes geometrical units; single matches (one-dimensional) and squares (two-dimensional), our data offer a contextual perspective on the students' expressions. In the granules above (73 divided by 3), the actions are performed with matches; however, the results are squares. In examining the students' discourse prior to the verbalizations we find that the second parts of these expressions seemingly also signify matches and not squares. If we now think of using an equation ($3x + 1 = 73$) to model the problem, the granule created here, $3x$, would refer to matches; as in Tina's equation (3 times x). Similarly, we find the same in the expression Leah creates after having found the unknown; `it's 24 times 3 plus 1`: the granule (24 times 3) and the second part of the expression both refer to matches. Using the cultural tool of an equation in solving this problem thus relieves the demands of attention in the particular manner described above. The second part of Trish's expression, `then you just add 1 more`, includes a human actor, and is ambiguous as it is unclear from the expression whether `add 1 more` refers to one match or one square. Leah's expression is slightly more reified; however, we evaluate both these expressions to be at a granular level of algebraic discourse (level 2, see Table 8.1).

Algebraic Equation

Only one of the four groups uses an algebraic equation[6] to solve the problem. The group (S4) belongs to the Swedish classroom where the teacher explicitly told the students, before handing out the task, that the problem could be solved most efficiently by making an equation: `an equation (…)is the fastest solution even here`. Tina makes an (incorrect) equation shortly after reading the task; $3 \cdot x = 73$. Using the algebraic symbol for an unknown in an equation, Tina employs a formal algebraic discourse as she solves the task.

The letter x is used to represent the unknown in calculations for which the result is given and the *unknown* is thereby used in an objectified way. As explained previously, the number 3 in Tina's equation seems to stem from an assumption of direct proportion (Stacey, 1989). In a sense, the discourse regarding the role of 3 lacks the dynamic quality that students in the other groups experience as they add 3 and 3 matches to the figure for each new square, which is an empirical example of the notion of rate of change. Tina's 3 appears rather as a static ratio between the two quantities single matches and squares. The prompt given (of making an equation) seems to guide the focus of this group and form the meaning-making process. A result is that the "equation part" of the problem is prominent while the functional

[6] Here the term "algebraic equation" is defined as an equation involving at least one alpha-numerical letter (variable).

aspect (pattern) resides in the background. In the other groups we see the opposite tendency.

The irregularity/constant term is not accounted for in Tina's first equation as she does not seem to be aware that one needs four matches to make the first square. After the teacher has pointed this out to the group, and the two students from other groups have given the prompt of using the number 72 instead of 73, Tina changes her equation to $3x = 72$. Thus, the algebraic equation provides a permanent platform for processing new information. The equation as an algebraic object guides the solving process. However, the students are following the teacher's instructions and do not apply the cultural tool on their own accord. Additionally, the students do not create an equation which fully models the problem. Their discussion is therefore evaluated to be only partly at the level of objectified algebraic discourse (level 3).

The Irregularity

As noted also in Part One, the irregularity of the numerical pattern is a point of struggle for all the groups and this is a very decisive element of the difficulties that the groups run into. Initially, it is the matter of identifying a pattern. To make the first (or final) square, one needs four matches, and in making all the others, one needs only three matches. Most of the groups come to this conclusion. However, to include the irregularity in mathematical models of the problem appears to be challenging for the students. In comparison, the coefficient 3, once identified, is usually applied correctly in calculations (as a multiplicative factor, *times 3*, in Tina's equation, or it's inverse, *divided by 3*). The irregularity is usually identified in the pattern discourse of the groups as 4, but modeled and accounted for in the calculations in several different ways.

In the group N1, the irregularity is modeled in the numerical expression `73 divided by 3 and then you just add one more`. The expression `add one more` is ambiguous, and, as explained earlier, Trish's gestures immediately before making this point indicate that it may refer to one *match* while in her calculations she adds one *whole square*. Group A2 creates a complex, numerical expression similar to that of N1, but accounts for the irregularity by subtracting four: `73 divided by 3 and then minus 4`. Both these complex numerical expressions are incorrect. The students' expressions show that they struggle (1) to correctly include two operations in one expression; and (2) to keep track of the different units. However, the type of problem posed (including calculations with an unknown for which results are given) also seems to play a role, as Leah (A2) is able to create the correct expression $24 \cdot 3 + 1$ after having identified the unknown. Although it might be a matter of making the numbers fit (i.e., $24 \cdot 3 = 72$, and so one has to add 1 in order to get 73), this model seems to correspond with Leah's meaning-making of the problem (see A2, turn 62).

Leah (A2) avoids the difficulty of modeling the irregularity by dealing with the original figure as a special case, in which division by 3 does not give a whole number of squares, but in which she can easily tell that 13 matches are needed to make

4 squares. She then finds the answer by doing the calculations 73 − 13 = 60, 60/3 = 20, 20 + 4 = 24. Bjorn, in F3, accounts for the irregularity in his calculations in the same manner, except that he subtracts 4 (instead of 13) from the total number of matches and then adds 1 square (instead of 4) at the end of his calculations: 73 − 4 = 69, 69/3 = 23, 23 + 1 = 24. Although this strategy of modeling the irregularity (subtracting a number of matches and then adding the corresponding number of squares) works well in finding the unknown, it is cumbersome in a generalizing aspect, as it would produce rather complex rules for the pattern of which the recursive depth exceeds 2 (see Fig. 8.11): $13 + 3(n - 4)$, $4 + 3(n - 1)$. These may be unattainable for the students. Indeed, in the study of Caspi and Sfard (2012) students modeled the irregularity in the same manner when writing a rule for the pattern (first subtracting parts of the number sequence and then adding it at the end of the calculations; see p. 178 in the text). These students did not use algebraic syntax when expressing the rule but instead explained it as a sequence of actions using many words. However, the rule of the seventh grader, in which the irregularity is modeled as +1, is closer to an algebraic rule.

In group S4, Tina starts the solving process by making the equation $3 \cdot x = 73$, in which the irregularity of the pattern is not included. The group does not become aware of the irregularity until the teacher points it out to them. Before Tina has time to incorporate this new insight into her solution, two boys from other groups insist that one must divide 72 by 3 (or 4), and not 73. Although correctly connecting the subtraction of 1 from 73 to the square with 4 matches (the irregularity), and accepting 24 as a correct solution, she is still confused about the 1 match she subtracted and seems to leave it as a reminder, which she signifies by writing +1 at the end of her written solution.

The irregularity is an element of the problem that adds complexity and takes the students into unfamiliar territory. In group F3 the girls insist that the task cannot be solved using a simple expression `because there are 3 [`*`She holds 3 fingers up`*`] in some and 4 in some` (F3, turn 33). The students who pursue a mathematical solution have to identify a common numerical property of the squares in the geometrical pattern and thereby also in what way this commonality does not hold for all the squares. In calculations, the students are successful when they separate one square or the complete figure (i.e., four squares) from the pattern in order to find a solution. It appears to be challenging for the students to think about the irregularity in terms of *one* additional matchstick in the first square (which would be the most efficient way of describing the pattern in order to make a rule for it). Stacy's (1989) observation that students in Grades 7 and 8, working with patterns, more frequently make mistakes in b than in a when using linear methods ($an + b$, $b \neq 0$), supports our findings that the constant term is more difficult for the students to model correctly than the coefficient.

Our study shows that the meaning-making processes of the students—all the way from a discourse on concrete objects, to a pattern discourse and to a discourse including numerical expressions—provide support and meaning for the coefficient but do not include a similar process of abstraction and refinement for the constant term. The students find contextual solutions in order to account for the irregularity and it remains in their discussions as numerical properties of concrete objects rather than, through processes of abstraction, becoming associated with a constant term in an algebraic rule for the pattern.

Part Four: Syntheses of the Analyses

In this study we have investigated the approaches taken of 16 groups, and the meaning-making processes of four groups, from different countries (Finland, Norway, Sweden, and USA), working with one specific patterning task; the matchstick task. Part One, in which we looked at the group work from the analytical level of approaches, shows the iterative nature of problem-solving processes. However, the in-depth discourse analyses in Part Two, which discuss how discourse builds upon itself in a more general sense, reveal that the discourse developments occurring in the groups N1, A2, and F3, follow a specific order. In this section we attempt to gather the different features that will help us substantiate what we see as patterns of a learning process in algebra. We also summarize our findings regarding the nature of the students' argumentations and discuss how these relate to formal algebraic discourse.

Patterns of a Learning Process in Algebra: Creating Algebraic Objects from Concrete Ones

The three groups (N1, A2, F3) that did not receive any particular prompts regarding how to solve the task engage in meaning-making processes that share similar features. In this summarizing section we will systematize these similarities while focusing on the construction of mathematical objects through discursive processes as elaborated by Sfard (2008).

Sfard (2008) proposes three different constructs through which discursive objects arise in discourse: by *saming*, by *encapsulating*, and by *reifying*. The manner in which these constructs are defined is influenced by Sfard's emphasis on word use, i.e., "it is responsible for what the user is able to say about (and thus to see) in the world" (*op. cit.*, p. 133). As discussed in the theoretical introduction to Part Two, *word use* is crucial in the long-term development of discourse as one is able to communicate, relate, and generalize particular experiences accumulated over time.

However, investigating a local discourse development which is multimodal in nature (Radford, 2014), we will add a multimodal view to Sfard's (2008) constructs. The semiotic means identified to play central roles in the objectification process are the discursive actions of drawing (extending the figure), gesturing (tracing squares in the air or on paper), use of linguistic classification categories (vocally creating numerical patterns), and modeling the problem in arithmetic calculations and expressions (vocal and written). We have identified Aha-moments in the students' discussions where several of these semiotic means are linked together and form semiotic nodes, through which a new objectification is accomplished. Keeping in mind that the semiotic node is a more general construct through which higher levels of awareness are reached, we also propose to include the construct of a semiotic node as a fourth discursive process through which mathematical objects arise in discourse.

Saming—the process of giving a number of things that have hitherto been seen as different the same name. We propose that this can also be evidenced in discursive procedures which are repeated in similar but different situations, even though it is not given a specific name by the interlocutors, i.e., in-action-formula.

Encapsulation—is the act of giving one name to a number of members of one specific set so that the stories that have previously been told in plural can now be told in singular, i.e., the counting of 3 matches in different squares is replaced by the noun *3 in each*.

Reification—is achieved when talk about processes is replaced by talk about objects. We suggest that this achievement can also be evidenced in gestures, i.e., the drawing and tracing of squares as a process of (physically or imagining) adding squares to the figure becomes a gesture that signify 3 as the rate of change.

Semiotic node—represents "pieces of the students' semiotic activity where action, gesture and word work together to achieve knowledge objectification" (Radford, 2009, p. 121).

Sfard (2008) defines mathematical objects as abstract, discursive objects (D-objects). Primary objects (P-objects) can be seen, heard or touched, and exist outside of discourse. D-objects exist only in discourse but are also conceived of by the senses (heard or seen). P-objects are therefore not separated from D-objects ontologically, but they are linked through discursive processes. Concrete D-objects include all the P-objects and all the nouns which arise through saming and encapsulating familiar P-objects (naming, e.g., Tom, cat, animal). Concrete D-objects are those that only involve saming and encapsulating while abstract D-objects also involve reification.

Figure 8.12 identifies the main features of the groups' discourse development systematized in four categories: discursive processes, modalities, word use, and discursive objects. The figure is three-dimensional in order to demonstrate how discourse builds upon itself. The three layers in Fig. 8.12 parallel the three first thematic discourses listed in Part Three: discourse on concrete objects (layer 1), pattern discourse (layer 2), and discourse including complex numerical expressions (layer 3). In the figure, the discussions of the three groups are presented in a compressed format, available by the prior analyses done, in the two categories: *modalities* (column 2) and *word use* (column 3). The figure summarizes our findings and includes a theoretically sharpened focus on discernable patterns of discourse, i.e., the creation of algebraic objects from concrete ones. Column 1, *discursive processes*, explains through theoretical constructs how the *abstract objects* (column 4) arise in the groups' discourse. We have also included the elements of discourse which initiate discursive shifts in the groups' discussions.

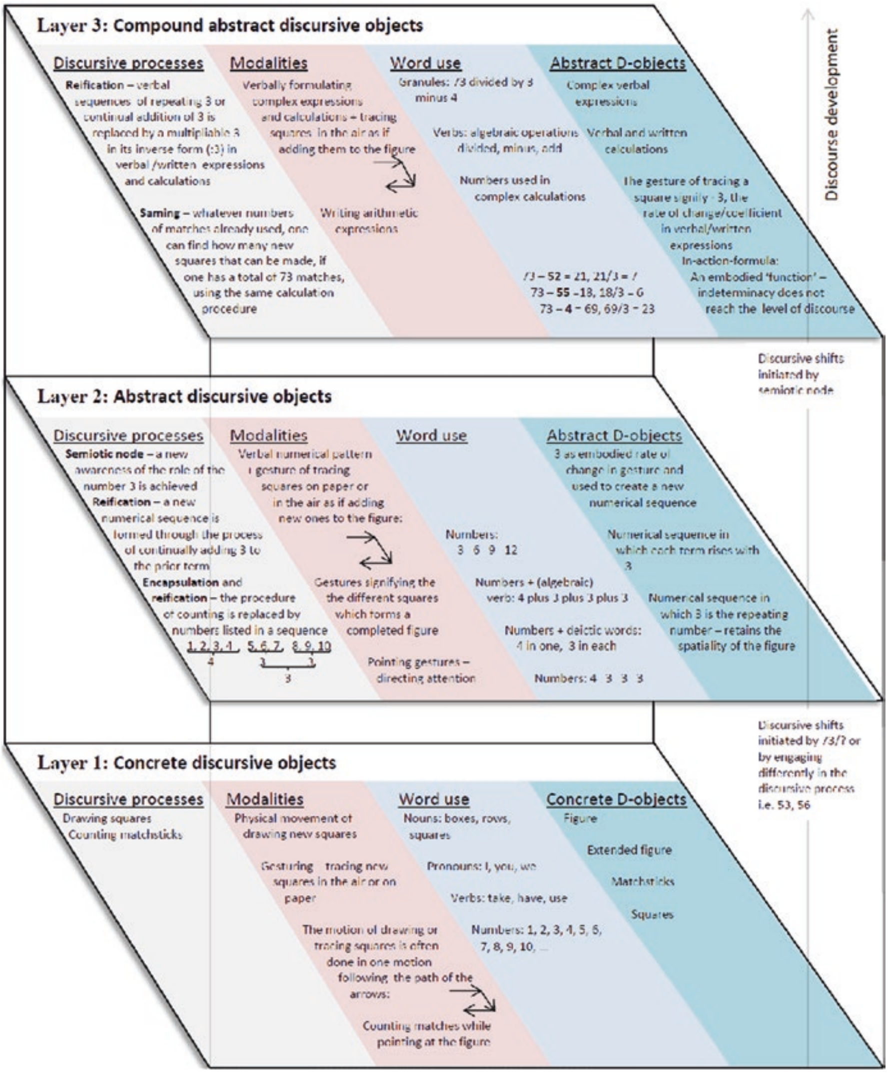


Fig. 8.12 Creating abstract objects from concrete ones

Figure 8.12 shows how students go from drawing and counting matches and squares to engaging in a more mathematical discourse in which numerical patterns emerge as objects. The discursive shift is initiated either by; students seeking a mathematical solution to the problem asking what to divide 73 by, or by; students engaging differently in the drawing and counting approach.

The discursive processes of drawing and counting which are creating new squares on paper are replaced by processes of reification and encapsulation, which result in the abstract objects of numerical patterns. Through reification the process of counting matches in each square is replaced by the numbers 4 3 3 3, which together form a numerical sequence in which 3 is the repeating number. The squares that were previously counted one at the time are now, through encapsulation, brought together and described by the same sentence: 3 in each. Students are then able to talk about the figure using only two numbers, the 4 and the 3. Another numerical sequence is created through a second reification in which the process of continually adding 3 produces the pattern 3 6 9 12.

The process of drawing new squares is replaced by tracing the squares in the figure, using the same motion. And, in one group, a student simultaneously verbalizes a numerical pattern which rises with 3. The motion is interpreted as a bodily knowledge of the problem, and signifies the additive 3 as an embodied rate of change. Through a semiotic node, i.e., the coordination of these semiotic means, a new objectification is achieved: students are able to create complex expressions and calculations, in which the numbers 4 and 3 are given new roles.

In one group, which creates an in-action-formula, the 4 becomes a particular instance of indeterminacy. And, in this formula the repeated addition of 3, `4 plus 3 plus 3 plus 3` (F3, turn 44), is reified to 3 as a coefficient in its inverse form: 73 – **4** = 69, 69/**3** = 23. The in-action-formula is created through saming: recognizing that similar questions, involving different numbers, can be solved by applying the same calculation procedure. The motion, of tracing squares in the air (or in the figure), is now used as a support in formulating expressions and calculations. The motion/gesture is reified as it no longer signifies the process of physically adding new squares to the figure, neither does it signify a repeated addition of 3, but instead the coefficient 3 as a contextual and embodied rate of change. We thus argue that reification is not limited to language as is claimed in the elaboration of Sfard's (2008) definition of the notion. The coefficient 3 is now the means by which the students relate the given number of matchsticks to the unknown number of squares.

We will suggest that the patterns of a learning process documented, rather than being the only way to learn, are describing learning in direct communication with peers in a problem-solving situation. It is inductive as the students start with the concrete and move toward abstraction. The constructs of saming, encapsulating, reification, and semiotic node seem as viable descriptions of discursive forces that achieve objectification and explain how compression and abstraction of discourse is achieved, i.e., how a discourse builds on itself.

The Nature of the Students' Argumentations

The analysis of a learning process in algebra described above gives credit to students as capable and inventive mathematical problem solvers. The students are able to transform a geometrical relationship (single matches versus squares) into a numerical relationship through the creation of numerical patterns. Furthermore, the students engage in an analytical practice with numbers, in which the same numbers are used in very different ways.

Recall from Part Two that we build our study on the definition of school algebra as a meta-arithmetical discourse that people employ while reflecting on arithmetic processes and relationships (Caspi & Sfard, 2012). Radford's (2010) summary, of what algebraic thinking entails, specifies the objects of this meta-arithmetical discourse: objects of an indeterminate nature (unknowns, parameters, variable) that are dealt with in analytic ways. From an ontological perspective, our findings show that students are able to participate in an algebraic discourse without using algebraic symbols. And, the students' algebraic thinking, as portrayed in this study, did not occur on the basis of goal-directed teaching, but rather as a result of students applying, in new ways, an already internalized arithmetical discourse. Our findings therefore supplement our knowledge of a *spontaneously developed meta-arithmetical discourse*, identified by Caspi and Sfard (2012).

In Table 8.1, *Levels of elementary algebra discourse* (p. 24), we attempt to sketch the development of constant value algebra through three different levels: processual (level 1), granular (level 2), and objectified (level 3). The significance of these levels of algebraic discourse, regarding the learning of algebra, is that the analytical possibilities, and, at the same time, the ability to handle complexity, rise with each level. In the students' discussions we see how they struggle to keep track of different units (squares and matchsticks) and the order in which to apply the different algebraic operations. Our analyses show that momentary mathematical objects arise in the students' discourse, through word use and other semiotic means. It is by using these objects that the students are able to solve the problem mathematically. Although these objects are of an algebraic nature, they are not recognized, described, or applied in the solving process as general algebraic objects. In Table 8.1, we identify three aspects of the students' discourse through which we can relate the students' argumentations to school algebra: (1) how a problem is modeled (how are expressions and calculations verbalized/written); (2) how relations are generalized; and (3) how algebraic objects are evoked in the solving process. In Part Two and Part Three we have referred to Table 8.1 as we analyze the students' discussions. We will now attempt to "flesh out" the levels of constant value algebra as we summarize these findings.

Processual Level

We find that several of the groups model the problem through linear calculations. Bjorn, in group F3, models the problem on paper writing the following sequence of calculations: $73 - 4 = 69$, $69/3 = 23$, $23 + 1 = 24$. The calculations are described and performed in a linear order and all intermediate results are listed. One could even discuss whether or not this model is meta-arithmetical. We will argue that it is because of how it arose in the discourse of the group; as shown previously, the model is part of an in-action-formula which is a product of reflection on numerical processes and relations. Leah's (A2) final solution is of the same nature: $73 - 13 = 60$, $60/3 = 20$, $20 + 4 = 24$. The models describe numerical relationships in the form of linear calculations (that are not previously described in any other form) and thereby exemplify the first level of constant value algebra.

The in-action-formula is not in any way discussed as an object, nonetheless it is embodied in the three calculation processes described previously. The in-action-formula is not the only type of generalization that we have identified in our material at this level. The students do generalize the one-to-one relationship between the two units, matches, and squares, through verbally listing numbers, 4 3 3 3, 4 + 3 + 3 + 3 and by contextual utterances such as `4 in one` and `3 in each`. The verbalization of the inaccurate sequence, 3 6 9 12, goes one step further and is similar to the in-action formula in that it does not only consider the relationship between the two numbers [unknown number of squares, 73], but several of the pairs to which the same relationship exists [(1, 3), (2, 6), (3, 9), (4, 12)]. However, instead of considering the relationship through a complex calculations process, it is here simplified (ignoring the irregularity) and considered recursively; applying the one operation, + 3, to the prior element of the sequence. Also, this relationship is only made explicit through gestures, i.e., tracing the matches in the figure synchronized with the listing of numbers. At this level, where processes are the main focus, mathematical objects of discourse are mainly numbers, often connected by actions, i.e., algebraic operations. However, intermediacy (variable) is present through some of its particular instances as they are listed in the calculation processes that embody the in-action-formula.

Granular Level

In groups N1 and A2 the students suggest complex verbal expressions before attempting to calculate a numerical answer: `73 divided by 3 and then you just add 1` (N1); `73 divided by 3 minus 4` (A2). These utterances are also about calculations; however, they bypass intermediate results using granules. In contrast to the models described above, no calculations are actually performed at this moment. We also found another expression that includes a granule in the work of the group A2. The verbal expression, `24 times 3 plus 1`, is created after the students have found that they can make 24 squares; and so the expression is a reflection on how the two numbers 24 and 73 relate to each other.

In the three expressions listed the granules are created by the use of the prepositions `divided by` and `times`. Although these expressions are only slightly granulated, we argue that there is a significant difference between the linear calculations and the complex verbal expressions listed here. There is a shift of focus that enables the students to model the problem, including all relevant numbers and operations at once, without getting involved with actual processes of calculations. The students only use these expressions as prescriptions for calculations and do not return to them when their calculations fail to bring acceptable numerical answers to the task. However, the ability to create such expressions does offer analytical possibilities that do not seem available when a problem is modeled through calculation processes only.

In the study of Caspi and Sfard (2012) the students created rules (complex expressions) as they reflected on processes of calculations they had already performed, however, in the present study the students created their complex expressions as they reflected on numerical processes and relationships prior to performing the calculations these describe. Looking at the two research studies combined we find that students in this age group (12–13 years) use granules in varying mathematical situations. Additionally, the study of Caspi and Sfard (2012) reports, as a main difference between the discourse of fifth graders and seventh graders, that the use of prepositions was frequent among the seventh graders; however, rare with the fifth graders. These findings seem to support the idea of a spontaneously developed meta-arithmetical discourse and document the granulation of complex expressions as a prominent aspect of it.

To solve the matchstick task, it is not necessary to generalize relations, i.e., to formulate a rule for the pattern (although it would be very helpful in order to create a correct equation), and we do not find any students' talk regarding rules. Generalizations at this level would be granular descriptions of for example the relationship displayed in the in-action-formula. The study of Caspi and Sfard (2012) provides empirical examples of these.

In this study we have discovered that the students create momentary objects of an algebraic nature that they use to solve the problem mathematically: numerical patterns and rate of change. The sequence 4 3 3 3 retains the spatiality of the figure while in the sequence 3 6 9 12 the functional relationship, $f(n) = 3n$, can be observed. These patterns only appear as objects through these verbal listing of numbers. The students do not talk about the patterns as objects nor do they describe them in the form of rules. Also, the gesture of tracing a square (on paper or in the air), is interpreted to embody the number 3 as rate of change. The coefficient 3 is later, in the context of discourse regarding arithmetic expressions, made permanent through writing; however, it only connects the two numbers 73 and 24. Rate of change is considered only in terms of processes and not as an object in itself. The objects described here are results of reflections on numerical relationships and processes, and they are contextual, situated, and momentary, i.e., not general algebraic objects. We evaluate the discourse in which these objects become visible and are applied to be at a granular level of constant value algebra. We suggest that objects of this nature are an important part of the spontaneously developed meta-arithmetical discourse.

Objectified Level

Tina (S4) models the problem using an algebraic equation; $3 \cdot x = 73$. In that classroom, the teacher told the students that using an equation is the most efficient way of solving the matchstick problem. In our data, we have found that none of the students do this on their own initiative, even though several of the classrooms were working with equations in the four prior lessons. However, this is not surprising at this early stage in the learning of algebra. The act of recognizing that the matchstick problem involves a question about an unknown, involved in calculations whose result is given, is an advanced form of algebraic thinking. That is, determining which cultural tool applies on the basis of discerning relational structures in a problem. The teacher does this for Tina. However, Tina is able to participate in a formal algebraic discourse regarding equation, i.e., her argumentation is not only rooted in a spontaneously developed meta-arithmetical discourse. Tina's first response to the task is to create the equation; $3x = 73$. As explained earlier the use of the number 3 arguably stems from direct proportion, which in this case is an overgeneralization and a simplification of the problem, similar in this way to the simple numerical expressions discussed in the other groups. The difference between the two forms of discourse is that in the case of the numerical expressions the intermediacy is implicitly present through the answer sought. However, in the equation it is made explicit by the use of x and it can therefore be part of an expression. Additionally, in creating an equation, Tina cannot focus only on calculation processes but has to structure her expressions in the form of an equivalence.

Tina clearly knows the basic structure of an equation and also correctly names the indeterminacy x. The equation is written down, revisited, and changed ($3 \cdot x = 73 \rightarrow 3 \cdot x = 72$) during the solving process and thereby is not a temporary object as is often the case at a granular level of algebraic discourse.

The use of algebraic syntax appears to have several effects, more alienated discourse (no human actor), disambiguation, and abstraction (no contextual references). These combined may be the explanation why Tina, in contrast to students in other groups when performing the same calculations, does not seem concerned by the appearance of a decimal number in the solution. Additionally, the equation built on the relational idea of direct proportion, both in its own right and as an approach confirmed by the teacher, is likely to present itself as a strong and convincing argument. Stacey (1989, p. 162) explains that:

> The models associated with direct proportion suggest themselves to students for strong cognitive reasons. When such an idea is found, students may be reluctant to question it, both because its effectiveness in supplying answers (and any answer is better than none!) and because of its simplicity.

We notice that Tina's arguments, after the teacher has pointed out her mistakes, are similar in their form to the other groups' discussions as they include contextual references and often a human actor. Furthermore, she is as much at loss for how to deal mathematically with a problem that demands a linear model including two operations, as the students of the other groups.

Although Tina is able to participate in a discourse that is partly at an objectified level of constant value algebra regarding an equation, she mainly follows the routines (meta-level rules) of arithmetic: (1) she focuses on a correct numerical answer rather than creating an equation that correctly models the numerical relationships in the problem; and (2) she insists that it is important to show the process of calculation, i.e., `60 divided by 3 which is 20. And then you get 12 and then it becomes plus 4 and then you get 24`. However, it would have been acceptable to write 72/3 = 24 as Lars in her group suggests. She thus appears to be more concerned with processes than with objects at this stage.

Pedagogical Implications and Looking Ahead

We will consider our last point of discussion, pedagogical implications, as we examine limitations and possibilities of the spontaneously developed meta-arithmetical discourse. We have only analyzed data from students' working with one particular task and do not expect to have exhausted the qualities of this type of discourse. On the other hand, our data includes students and classrooms from four different countries and our analyses therefore point to some common characteristics of students' discourse as they are entering the mathematical domain of algebra. Additionally, we have documented patterns of an inductive learning process in algebra which have implications for teaching. The issues raised in this section are brought to light through our analyses, however, in need of further investigations.

Radford (2010), investigating students' generalizations, argues that "the mathematical situation and the semiotic resources that are mobilized to tackle it in analytic ways characterize the *form* and *generality* of the algebraic thinking thus elicited" (p. 15). The matchstick task includes a geometrical figure and references to the extra mathematical objects of matchsticks. We have shown that in this particular situation students create contextual, situated and momentary algebraic objects that are signified through inscriptions, word use, and gestures. We have argued that the students' generalizations, regarding this task, are at a processual level of algebraic discourse. Also, the granulation of expressions appears to be a prominent feature of this type of discourse. These findings point to qualities of the spontaneously developed meta-arithmetical discourse: (1) it is mainly concerned with calculation processes; and (2) it employs a wide range of signs that are not part of the standardized mathematical discourse but which instead are closely connected to the particular problem at hand. Thus, it is questionable if a spontaneously developed meta-arithmetical discourse can reach the level of objectified constant value algebra, i.e., a discourse concerned with general algebraic objects. Rather, we suggest an initial approach to school algebra in terms of formalizing students' developing meta-arithmetical discourse, arguing that it offers possibilities for a meaningful learning of algebra.

Recall that we defined the learning of algebra in school as the individualization of the algebraic discourse employed in the classroom; in which formal algebraic syntax is inherent. We will now consider how the spontaneously developed meta-arithmetical discourse can serve as a platform for the growth of a formal algebraic discourse. However, we will not neglect the continued importance of natural language in the learning of formal algebra in school, as was pointed out in our literature review (Part Two). Caspi and Sfard (2012) suggest that the introduction to formal algebra may spur a growth in the informal discourse, and that maintaining connections between the two can ensure a meaningful learning of algebra. An important issue in this context seems to be in which ways the use of algebraic syntax changes the spontaneously developed meta-arithmetical discourse. In our data, we observed that Tina's use of an equation when solving the matchstick problem, in comparison to the other groups, altered the discourse employed: intermediacy made explicit, a focus on equivalence, and decontextualized arguments. Rather than exploring the issue posed above, Tina's equation provides one example that includes both changes and continuities (oversimplification using proportional reasoning) between the two types of discourse.

The majority of the students participating in our study expected direct proportion to be a relevant model for the problem and encountered difficulties in solving the problem mathematically when it did not work. The discussions show that the students cope with the unexpected complexity of the problem in different ways. Many students, like the girls in group F3 who said that such a model (of direct proportion) `won't work`, decide to draw and count. The creation of numerical patterns and the use of a gesture that signifies how the figure evolves have been interpreted as students' explorations of the mathematical structure of the task at hand (non-proportional linear function). The complex expressions suggested and the processual generalizations made, in immediate relation to these explorations, are interpreted as examples of students thrusting into a new domain of mathematics. However, as a linear model including two operations (a cultural notion that the students are likely to be unaware of) seems to be out of students' reach, they resort to contextual and processual solutions. The students are satisfied as they find numerical answers that are confirmed as correct. For many students, participating in the arithmetical discourse, this has been the main objective of their mathematization thus far.

Indeed, with the introduction of algebra the rules of school mathematics change. School algebra is less about specific numbers and processes of calculation, and more about making mathematical generalizations and chasing algebraic objects. From a historical perspective the development of algebraic syntax can be seen as a response to rising levels of complexity in problems i.e., problems in which algebraic objects support their solution or problems which solutions even necessitate their use. If students keep chasing numbers as their main mathematical objects they will struggle to participate in the algebraic discourse. It is of importance for the teaching and learning of algebra to explore how students may become aware of these changes in meta-rules. We will not attempt to tackle this issue here but rather use our analyses to highlight some aspects of it.

The characteristics of the students' responses to the matchstick problem are signs that the students engage in meta-level learning. Particularly as they intuitively apply an argument based on a strong association between multiplication and direct proportion that is not viable in the discourse in which they are to become participants. Our study shows that students' intuitive responses to the matchstick task often involve an application of multiplication that assumes a relationship of direct proportion. This expectation seems to drive the meaning-making process in the groups and at the same time leads to simplified conclusions regarding the problem. These conflicting arguments (direct proportion and a non-proportional linear model) can be interpreted as a rupture between arithmetic and algebraic thinking. It seems to be of importance to the introductory algebra classroom that the differences between the two types of arguments are explored. And further, to build an argument for the usefulness of applying a linear model in this type of problem.

Having considered some limitations of the spontaneously developed meta-arithmetical discourse, we will now return to discussing how it can play an important role in the introductory algebra classroom. The inductive learning process in the algebraic context allowed for students to explore new numerical relations with the means available, i.e., previously internalized mathematics. It thus appears as one way of breaking the inherent circularity of the learning of new mathematical objects (Nachlieli & Tabach, 2012), i.e., one can only come to know a new mathematical object by participating in its discourse, however, how can one participate in a discourse about objects one is not aware of? The analysis of patterns of a learning process in algebra identifies discursive forces (saming, encapsulation, reification, and semiotic node) through which in situ algebraic objects are created in the students' discussions.

Caspi and Sfard (2012) explain the formalization of the algebraic discourse historically as consisting of three processes: regulation, reification, and symbolization. The goal of formalization is to maximize the effectiveness of mathematical communication and the three processes respond to different needs. Regulation, the introduction of explicit discursive rules, prevents ambiguity and creates a standardized algebra. Reification, replacing talk about processes with talk about objects, achieves compression. Symbolization is an effective tool for both compression and standardization of a mathematical discourse.

In our analysis of patterns of a learning process we find that the students spontaneously create algebraic objects through processes of reification. However, if the students are to become full participants of the algebraic discourse in school, their talk about algebraic objects needs to change, from temporary, contextual, and situated descriptions, to become descriptions of general objects. By using granules in complex expressions the students achieved compression. We have shown that these expressions are often ambiguous. The historic analysis presented by Caspi and Sfard (2012) suggests that the response to the students' spontaneously developed meta-arithmetical discourse should be thought of in terms of regulating and symbolizing this discourse.

An investigation of the textbooks used in the classrooms of this present study (Reinhardtsen, 2012) shows that the algebraic syntax is often introduced in the form of simple problems. Problems that the students can easily solve without using algebra, and indeed the use of these cultural tools must appear as unnecessary, and as complicating, instead of aiding their solutions. Our study shows that the students are able to handle complex problems by the use of meta-arithmetical discourse. Further, complex task of an algebraic and problem-solving nature, can stimulate students' discussions which in turn provide ample *discursive entry points*. These are defined by Remillard (2014, p. 105) as following:

> That is, there are distinct characteristics of novice discourse that may lend themselves especially well to expert intervention. Utterances or patterns of utterances that indicate such intervention potential are herein named discursive entry points. Discursive entry points may represent valuable opportunities for helping novices to refine their discourse and advance their mathematical thinking.

The students' verbalizations of numerical sequences and the granulated expressions, the processual generalizations made and the gesture which embody the rate of change seem to be opportunities for an expert interlocutor to talk about general algebraic objects. Additionally, the inductive learning process allows the students to experience the shortcomings of their available mathematical arguments and may thereby motivate the learning of algebra.

However, as we recall the effects of teachers' comments in groups N1 and S4, these types of intervention should be approached with caution and respect for the processes of learning that students are engaged with. In N1 we observed the students' vulnerability as they explored unfamiliar numerical relationships and were dependent on an expert interlocutor to confirm their ideas. When the teacher failed to do so, the students abandoned their work. In S4 the teacher's prompt of using an equation seems to have entirely altered the students' experience with the task.

Our study suggests a democratized form of teaching in which student talk is allowed to shape and inform the introductory algebra classroom. Moreover, the study shows affordances of small-group work in a problem-solving setting. It makes evident how the students pool their ideas by verbally engaging with the problem, including raising and answering questions—a process that allows for the discourse in the different groups to build on itself. Particularly, we documented the students' observation of their peers' drawing process as playing an important role in the advancement of a mathematical solution of the problem. However, the analyses done also revealed that the students in one and the same group do not reach the same discursive level regarding the matchstick task. In adopting small-group work for teaching purposes one needs to consider how to alleviate this observed progressive unevenness. The students' spontaneously developed meta-arithmetical discourse appears to be an untapped source for the promotion of a meaningful learning of algebra. Our study has described features of this type of discourse and even pointed out what appears to be discursive entry points. However, how to intervene in a manner that initiates sustained changes in the students' discourse—toward further sophistication—remains an open question.

Chapter 9
Encountering Algebraic Reasoning in Contemporary Classrooms: Epilogue

Roger Säljö and Cecilia Kilhamn

In the preceding chapters, we have met students and teachers in four different countries as they begin engaging with algebra and algebraic thinking. Our ambition has been to focus on the participants' perspectives and activities as they—students and teachers—embark on this intellectual journey where both parties are challenged and have to learn; students about algebraic reasoning and teachers about how to instruct and guide students on a topic that for many present obstacles.

As we have pointed out, there are both similarities and differences between classrooms in different countries. Textbooks and curricula vary between the countries, but also within countries there are obvious variations. In the American data, for instance, we saw two teachers with markedly different conceptions of what it means to learn algebra, and mathematics more generally, and these differences correspond to rather different teaching strategies where the expectations about what students are supposed to do to learn algebra differ. Such variations in teachers' conceptions of what it means to learn mathematics and algebra, most likely, were present in the other countries as well. But, on the whole, the classrooms documented look familiar, and many people, also outside mathematics teacher circles, would agree that they correspond to what one would expect to find in contemporary Western societies.

As for the general instructional strategies, we see teachers engaging in a range of activities. They lecture in front of the whole class as they present concepts and procedures of early algebra. They assign tasks and exercises to groups that they interact with during the lesson. The exercises, taken from textbooks or, in some cases, designed by teachers, are from familiar everyday situations following the spirit of

R. Säljö (✉)
Department of Education, Communication and Learning, University of Gothenburg, Gothenburg, Sweden
e-mail: roger.saljo@gu.se

C. Kilhamn
Department of Pedagogical, Curricular and Professional Studies, University of Gothenburg, Gothenburg, Sweden

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the curricula, and, when mathematized, they are intended to serve as bridges to elementary forms of algebraic reasoning. In addition, teachers interact with individual students who are asking questions or struggling to come to grips with the concepts, notations, and formulas.

Another observation regarding the teaching is that there are similar ideas about instructional strategies suited to the teaching of introductory algebra and equations. For instance, the balance model appears in several classrooms communicating the idea that if something is added or removed on one side, the other side has to change as well. Although the balance model has some limitations, it serves well when reasoning about equivalence structures (Vlassis, 2002). However, though widely used in school algebra and in textbooks, it might not build on everyday experiences for students of today, since balance scales are rarely encountered in the digital era and may themselves appear quite abstract. Consequently, we may well ask if school algebra could benefit from incorporating other models or metaphors. The function machine, used in one of the US classrooms, is a model aligned with the industrial world of the twentieth century, and it is not obvious that this is a metaphor that enlightens students. Today, computer systems digital interfaces and algorithms dominate our everyday life. A challenge for future algebra teaching is to find ways of bridging this contemporary reality of young people with century-old algebraic thinking. Which are the models and metaphors from this era that could make their way into algebra teaching in the future? How will computational thinking relate to algebraic thinking? In what way will the introduction of variables in school algebra change when the most common use of the concept of variables is the one related to computer programming?

An interesting difference in the teaching strategies in terms of mathematical meaning-making between the classrooms is represented by the tension between an inductive and a deductive approach to algebra learning, respectively. This tension appears in several settings. In the Norwegian classrooms (Chap. 5), for instance, this is clearly illustrated in the nature of examples introduced to support learning. Both teachers introduce designed examples to mediate between the students' everyday experiences and algebraic expressions. These designed examples obviously are grounded in their previous experiences of teaching introductory algebra, and in this sense they represent a reflected alternative for achieving specific teaching goals. In addition, both teachers use physical resources (playing cards vs. the body) to bridge between an everyday interpretation of a problem situation and an algebraic one. But, in spite of this similarity in teaching, the approach to what it means to learn algebra is different. One of the teachers uses an inductive approach starting with numbers, numerical expressions and operations, and from this platform she generalizes to algebraic expressions and numbers. This inductive approach emphasizes the continuities between arithmetic and algebraic thinking, which is clearly in line with the predominant mode of reasoning observed among the students. The dilemma observed here, as well as in the other empirical settings (cf. the case of Sweden in Chap. 4), is that the concept of variable as a representation of a (given and hidden) number seems to be reinforced rather than challenged. The second Norwegian teacher, however, proceeds in a more deductive manner when attempting to communicate the concept of variable. In one example he moves from bodily movements (walking) to words ("steps" and "feet"), then to abbreviations (i.e., symbolic representations "s" and "f"), and further on to the concept of variable, which through the demonstration is a unit that may vary in terms of length, i.e., it has no fixed value. This fact that the value is not fixed is explicitly addressed as a topic of communication with the students, and in this sense a more generalized interpretation of what a variable may be is invoked. In the other example, he introduces the topic of how the comparison in age between himself and his sister can be represented by means of an algebraic expression. In both cases, the conceptual meaning of a variable and an algebraic expression is emphasized and provides the entry point into algebraic reasoning. It could also be seen that in these cases, the teacher attempted to maintain the conceptual focus of the argumentation and physical activity by trying to get the students to co-construct (a) how to model his walking in the classroom, and (b) the relationship between the ages of two persons.

To what extent this difference in approach has consequences for the appropriation of further elements of algebraic reasoning is an empirical question. As we have emphasized repeatedly, we have only been able to follow the first few hours of teaching and learning. Nevertheless, it is interesting to observe that such conceptual differences in mathematizing and instruction co-exist, and this also testifies to the considerable space which teachers have in organizing teaching and deciding on what challenges students should encounter in order to appropriate algebraic reasoning.

In the observed lessons, we have seen the students, mostly willingly, putting in considerable effort to understand and master concepts and procedures in what, for most of them, is unfamiliar territory. We also saw students occasionally distancing themselves from the ongoing problem-solving with their fellow students, talking off-task about things that temporarily occupied their minds or simply sitting passive. Engaging with abstractions and keeping focus over long stretches of time are challenging for many. We saw obvious differences between students in their preparation for algebraic concepts and procedures. Some, already at an early stage, come close to operating on the pattern task with matchsticks in Chap. 8 by focusing on transforming what they saw in front of them in the pictures to relevant objectified algebraic concepts. For other students this did not happen, and they attempted to follow their fellow students who were more active and conceptually prepared. Thus, inter-individual differences are a real part of classrooms and learning. All of these observations are expected features of classrooms.

But, in passing, there are also a few observations about what we did not see that are worth mentioning. One such observation is that the classroom work involved very little use of digital technologies. There were no traces of calculators, tablets, or computers in the activities and exercises. Even though such resources were not used, we know from our recordings, teacher interviews, and our observations that they were around in most cases. In most classrooms, there were interactive whiteboards, but they were hardly central for the instruction that took place. Rather, they were used as traditional blackboards for writing and drawing. The only relatively recent technology that played a visible role was the use of document cameras in some classrooms, which allowed student work to be easily presented to the whole class. However, even when this was done, we saw little use of such opportunities to extend and deepen the mathematical conversation about student solutions. Teachers across most of the classrooms and countries preferred the instruction for this particular curricular unit to be carried out in a traditional teaching format with lecturing, group work, individual instruction, and textbooks as core elements. In most of the classes, the textbook played a very central role. This is especially clear in the Finnish setting where the contingencies between the textbook and classroom practices are apparent. The Finnish curriculum gives ample space for local decisions about when and how to teach specific curricular units, and here it seems as if the textbook becomes the enacted curriculum. Obvious signs of this can be found in the other countries as well. But we also saw classrooms where the textbook was not at the center of attention during the lessons. In two of the Swedish classrooms, the textbook for this unit had been replaced by an "algebra activity box" providing resources for hands on and student active learning, designed by a major national (and international) project. Group work was a prominent feature also in one of the other Swedish classrooms where tasks from the textbook were used to initiate group discussions.

In the following discussion, we will comment on some of the major observations that follow from our study, while avoiding to repeat the analyses and syntheses that conclude the empirical chapters. We divide this discussion into three sections. The first section relates to the distinction made by Yackel and Cobb (1996) between various sociomathematical norms followed in the activities we have observed. This implies analyzing how students work, and the meaning-making they engage in. In this case, it is interesting to see what students bring with them into the classroom activities and the mathematical meaning-making that takes place in the classroom conversations. In the second section we discuss the meaning of the concept of variable that was made available for the students to learn in these classrooms. As a third topic, we want to raise the issue of the learning trajectories that can be observed. In other words, how far do the students get during the introductory lessons we have observed?

Student Engagement and Mathematical Meaning-Making

As we pointed out, in the classrooms we have documented students are active. They listen, ask questions, and discuss rather freely with the teacher and their fellow students. The social distance between the teachers and the students appears small in most cases. Thus, in terms of participation as a precondition for learning, all classrooms are characterized by a willingness to be involved and to share knowledge. The questions and comments students make also indicate that they understand that they are going to learn something new, a new kind of mathematical activity. Some of them report being familiar with elements of algebra symbolism, for instance that x can refer to an unknown number, but the general picture of the classrooms documented is that most of what is presented is new to the majority of the students.

It is clear that students in all classrooms rely on their experiences and knowledge of arithmetic and bring these with them into the lessons and group activities. They know how to add, subtract and so on, and they use these concepts and procedures as they take on the tasks presented to them. It is also obvious that many of the exercises and problems they meet during these lessons can be relatively easily solved through simple arithmetic procedures. In the extensive analysis of student work with the pattern task in Chap. 8, it is shown that some groups solve the problem of how many boxes can be made using 73 matches by extending the drawing and counting how many boxes you get. Given the focus on finding a result, this strategy is easy to understand and rational. This is also evident from the Finnish material when the students work with the balance model of an equation; they realize through arithmetic procedures what the expected answer is, and that in a practical sense concludes the task as a problem to be solved. In one of the Swedish classrooms, the boxes and beans supply a representation of equations limited to (small) whole numbers and the operations addition and multiplication. Although the equations the students create are algebraic in the sense that the variable appears on both sides (as described by Filloy & Rojano, 1989), they are solved by simple arithmetic procedures of using inverse operations. Thus, and as we have pointed out, the problems that are used to introduce algebraic reasoning do not really require algebraic solutions or thinking. If the focus is on finding a result, what the students already know is sufficient. This dilemma, that arithmetical procedures suffice, has been well documented in the literature for a long time, but these insights do not seem to have reached teachers. To function well, the problems, illustrations, and the instruction have to be understood as pointing more clearly to a different kind of activity where the goal is rather to discover patterns and express relationships.

The shift from arithmetic thinking to algebraic thinking, or, using the words of Keith Devlin,[1] from quantitative reasoning with numbers to qualitative reasoning about numbers, entails a shift in sociomathematical norms. In their seminal work from 1996, Yackel and Cobb describe the nature of sociomathematical norms and the difficult process of changing such norms. They write:

> What counts as mathematically different, mathematically sophisticated, mathematically efficient and mathematically elegant in classrooms are sociomathematical norms. Similarly, what counts as an acceptable mathematical explanation and justification is a sociomathematical norm. (Yackel & Cobb, 1996, p. 461)

What we see in all the chapters of this book are examples of classrooms where students are attuned to well-established sociomathematical norms related to arithmetic. In most of their previous experiences, they have been expected to complete calculations, focusing on numerical answers to clearly posed questions, and they operate within this established framework. Most likely, this attitude to how to approach mathematical tasks is grounded in an even broader everyday understanding of what mathematics is for; you calculate to arrive at numerical answers. Now the students are expected to engage in activities of representing and modeling, they are expected

[1] Keith Devlin, https://profkeithdevlin.org/2011/11/20/what-is-algebra/ (retrieved 15 Dec. 2017).

to reason *about* numbers rather than *with* numbers. In most of the classrooms, the teachers do not seem to lead the way here in a distinct way in order to make this shift in level of reasoning clear. Except for the introduction of variables through designed examples in one of the Norwegian classrooms (e.g., when referring to a variable [a step] that is agreed upon as not having a fixed value), the introductory algebra examples and tasks deal with specific numbers, albeit sometimes momentarily unknown and represented by a letter. This entry into algebra implies following the kind of arithmetical approach that students are already familiar with, and in this sense there is little incentive to go in a different direction. For example, in the Swedish material in Chap. 4 the two teachers in school A keep referring the students back to the specific numbers behind the symbols (class A1) and inside the boxes (class A2) rather than attempting to initiate a discussion about the structure of the equation and possible solutions or variations of it. Consequently, the goal of the mathematical activity is restricted to finding a numerical answer, and the reasoning is carried out in terms of calculations that provide answers, i.e., operating on numbers to generate numbers. The most important sociomathematical norm is perhaps the agreement of what counts as acceptable mathematics explanation and justification. Moving from arithmetic to algebraic thinking ought to entail attempting to move away from empirical justification, i.e., using specific numbers and particular cases to justify a generality inductively, toward justification based on logic and deduction. None of the classrooms in the VIDEOMAT material show clear signs of this change occurring or even explicitly attempted. Whenever a generality has been achieved, it is consistently justified through checking with specific numbers.

The question of what is *mathematically different* is an intriguing issue. In algebraic thinking *sameness* is a key aspect of generalization, sameness against a background of difference (Mason, 1996). Sameness appears in various ways. First, an equation is a statement that two different expressions are equal. What equality means here is that both expressions, on either side of the equal sign, can represent the same quantity, a statement we can check by transforming one into the other by following a set of pre-established arithmetical rules. Second, functions are defined for many, or at least several, values of an independent variable, and equations with two or more variables can have more than one solution, which effectively means that several different specific situations can have the same algebraic representation. Third, the same situation, for example the L-shapes and the hexagon train in Chap. 7 and the matchstick pattern in Chap. 8, can be represented in several different ways. Reasoning logically about these instances of sameness and difference is a truly algebraic activity, but does not occur unless the students are explicitly guided toward looking for sameness and difference. In their arithmetic experiences, they may have been used to seeing a different solution as a sign of error. Changing the norm to make differences and sameness in algebraic structure the focus of attention is therefore a challenging but important instructional enterprise, where the input of the teacher is necessary.

In the empirical chapters, we see several opportunities for teachers to initiate a renegotiation of sociomathematical norms in favor of more algebraic thinking about what is mathematically different. For example in class A1 in a Swedish classroom (Chap. 4), when one equation with three variables is produced by the student Anna: $a + a + b - c = 80$ (using geometric shapes instead of letters), the teacher chooses to explain that Anna has decided to use b and c to represent the same number. Consequently, she takes away the opportunity for the students to discuss the structure of the equation. Although the teacher does acknowledge several solutions to the problem, she is satisfied and moves on when Anna has revealed her solution. Thus, the conclusion is that there was a "correct" solution even when another solution was suggested. A different approach to Anna's equation could have been to discuss whether b and c would need to represent the same number, or, if not, what other possibilities there are. Posing such questions would have disrupted the dominant sociomathematical norm that every question has a correct answer consisting of specific numbers. The students initiate a shift in the norms through the conjecture that there could be many solutions, but without support from the teacher such ideas fail to flourish when arithmetical thinking is the norm. Perhaps the teacher herself was unprepared to change the sociomathematical norms, or, at least, she did not notice that there was an opening for addressing this critical issue.

Another example of a dominating arithmetical norm is visible in the two American classrooms in Chap. 7. Although quite different in many respects, both teachers seem to emphasize rules and procedures, directing the students to "*the* way of mathematics" (Ms. A), and teaching procedural steps to find "*the* general expression" of a pattern (Ms. B). The perimeter of the hexagon train that the students worked on in Ms. B's classroom (Fig. 9.1) could, depending on how the pattern is discovered and in which order the sides are counted, for example be represented by any one of the following expressions, where n is the number of hexagons in the train:

$4n + 2$ [4 on each of the hexagons and one on each end]
$4(n - 2) + 5 + 5$ [5 on the end hexagons and 4 on each of the others]
$2(2n) + 2$ [2 rows, top and bottom, with 2 on each hexagon and one on each end]
$6n - 2(n - 1)$ [6 on each hexagon minus all the middle edges]

But, although the students take the pictures of the hexagon trains as a point of departure for their work, they are taught to use the numbers in their function chart (t-chart) to find the correct expression through predetermined steps of reasoning.

A discussion about what could be seen as mathematically sophisticated, efficient, or elegant does not appear very often in the material described in the five

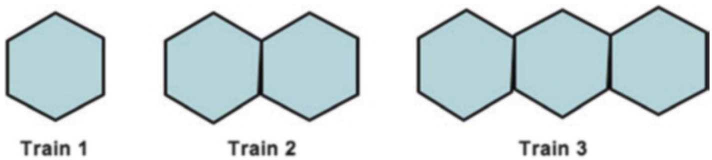

Fig. 9.1 Hexagon train (see also Fig. 7.7)

empirical chapters. When such evaluations do surface, the norm seems to be that using algebraic notation and learned procedures in a "correct" manner is more important than finding an efficient or elegant solution. One example of this is found in Chap. 4, where class A2 spends three lessons working with manipulatives in the form of boxes and beans to create and solve simple equations. This results in a situation where several students are unable to solve the equation $9 = 12 - x$, which they should have been able to solve at a glance if they asked themselves what the number x must represent. The question is not any more about solving this equation efficiently, or actually understanding the equivalence, but about working out how to apply the method of representing it as boxes and beans. The procedure taught is neither sophisticated, nor efficient, and what is meant as a concrete illustration becomes an obstacle. The students seem to get lost in their attempts to move between the equations and the boxes and beans.

Algebra as an efficient problem-solving tool does not come across in these introductory lessons. For example, the students in class B in Chap. 4 in Sweden are told to use a variable although they see no use for it. The teacher says she wants `a two-fold result`, asking the students to produce answers to the questions, but also to write down how they `reasoned using a variable`. The teacher conveys that the goal of the lesson is to learn to use algebra to solve a task for which they already have a solution. When she asks the students to `write using a variable when you describe how you know`, the arithmetic norm that mathematics is about quickly and efficiently finding a correct solution is violated. Instead, the teacher initiates a negotiation of a new norm, a norm that gives algebraic notation a value in itself, even when it does not make the solution of a problem more efficient. In Chap. 7, Ms. A makes this new norm explicit when she talks about the importance of learning the procedures even when the problems are easy. She says: `I know you can get the answer. What I want to see on your paper is the procedure. How many of you would agree these are not hard mathematics problems? Okay, because we use baby steps to teach you the procedure. Some of you, you're stuck up on the answer. […] I need you to get the procedure, not just the answer.` By using simple problems, the students do not get to experience that the goal of algebra could be to make mathematics more efficient. That remains a very distant goal. This is an example of when teaching violates what Harel (2000) calls the necessity principal: "*For students to learn, they must see a need for what they are intended to be taught*" (Harel, 2000, p. 185). According to Balacheff (2001), students are not likely to move beyond the arithmetic domain as long as an arithmetically validated solution is more economic from their point of view. The tension here lies between introducing small steps and simple examples that students can comprehend, and challenging students with problems where algebra shows its potential of being an efficient problem-solving tool.

The Meaning of the Concept of Variable

When viewing all introductory examples chosen by the teachers described in the empirical chapters, we notice that they rarely introduce students to the more advanced meanings of variables identified by Küchemann (1978, 1981) and described in Chap. 1. In Küchemann's studies, students achieved high scores on tasks which could be solved using less sophisticated interpretations of variable and low scores on tasks where more advanced meanings were necessary. Most of the examples and tasks described in the empirical chapters can be solved without the more sophisticated interpretation of letters as proper variables.

In the three Swedish examples (Chap. 4), all the tasks are solved by either evaluating the letter, giving the letter a specific value or ignoring it altogether. In the Norwegian case (Chap. 5), Kari's use of playing cards treats only variables as specific unknowns. In Ole's lesson, the letters *s* and *f*, used for steps and feet respectively, and their varying lengths are discussed, but since the questions only concern how to write and simplify an expression, it is possible to interpret the variables simply as objects or labels. Only when he moves on to the example of the relationship between two sisters described in a formula, (Chap. 5, Fig. 5.10) are students required to see the letter *x* as a variable since it clearly appears in a functional relationship. Both the Finnish teachers (Chap. 6) focus on equation solving where the letter is a specific unknown. In the American case (Chap. 7) the first teacher (Mrs. A) gives examples of creating expressions, evaluating expressions and solving equations. In all these examples, the letter can be understood as evaluated, a specific unknown or an object. The second teacher (Mrs. B) generates expressions in a functional setting where the variable is used as a generalized number. Although opportunities appeared in the Swedish classrooms, there were never any discussions about variables in terms of possibilities and constraints related to the domain and range of variation (cf. Mason, 1996). The variability and generality of a variable was clearly not paid much attention, and thus, the increase in the level of abstraction related to variables described by Treffers (1987) as vertical mathematization did not occur. Against this background, we wonder if a learning trajectory for conceptions of variable needs to go from less sophisticated to more sophisticated, or if more sophisticated uses of variable could serve as a productive entry point by emphasizing the conceptual dimension.

Using Usiskin's four conceptions of school algebra and use of variables to look at the introduction of variables in our data, we can identify some instances of *generalized arithmetic* in the construction of algebraic expressions and equations, i.e., translating from words to symbols. Usiskin writes: "The key instructions for the students in this conception of algebra are *translate* and *generalize*." (1988, p. 9). The translating part was much in focus, but the generalizing part was generally not attended to. Generating variable expressions could be used as an entry point to discuss arithmetic properties such as how addition and subtraction relate to each other, or why and how the distributive property works. The second conception, related to *algebra as a study of procedures for solving certain types of problems*, was visible in the equation solving procedures where variables were used as unknowns or constants. Also, the work with functional relationships in the American classroom was steered toward a procedure describing how to use the function chart to generate a variable expression. *The study of relationships among quantities* was touched upon in two different problem types: describing the relationship between the ages of two people, and describing the relationship between a term and the number of that term in a sequence. However, such problem types supply opportunities to study relationships between numerical quantities and to generalize them only if the variables are allowed to vary, and if that variation is also made a focus of attention in the instruction.

In addition, it is relevant to note in this context that, in recent decades, computer science has come to permeate all levels of society, with students growing up more accustomed to handling digital tools than writing with pens on paper. We have commented on this in the context of some of the illustrations introducing central concepts in early algebra, such as the balance scale as a representation of equality and equations, perhaps appearing somewhat unfamiliar to young generations. Also, ideas about computational thinking in the context of programming and design activities represent an important competence that is slowly being incorporated into school curricula (cf. Mannila et al., 2014; Wing, 2006). Even though the exact meaning of what constitutes computational thinking is disputed (Grover & Pea, 2013), the ideas, practices, and language of programming, which is an interface between human thinking and machines, are spreading. This implies that school algebra in the future will need to consider more closely the meanings of the concept of variable in computer science, and the ways in which variables are used in programming. Already in 1988, when describing conceptions of school algebra and the use of variables, Usiskin brought up the different syntax for variables used in computer science, where it is quite meaningful to write $x = x + 2$. Since this represents an equation without solution in traditional algebra, it would not be accepted as a correct way to write an equation in school algebra. Incorporating also the computer science use of variables in school algebra would imply rethinking the meaning of the equal sign as well as the meaning of variable. We see new challenges ahead as the teaching of algebra, what constitutes school algebra and what meanings students make of algebraic concepts and symbols, move further into the twenty-first century, where computational thinking and programming are incorporated into curricula and present competing uses of central terms and concepts.

Teaching and Learning Trajectories

One of the most interesting questions in the context of our documentation of the students and the classrooms in the different countries is what we can say about learning trajectories on the basis of the five lessons. What signs are there of students making progress in relation to understanding the central concepts and procedures they are to learn during these initial encounters with algebra? This is most clearly visible in the comparative analyses in Chap. 8. At a very basic level, and as is illustrated in Fig. 8.12 when the students are placed in a challenging situation, they realize that they can solve the problem by making a drawing which allows them to count the sticks. In this sense they show that they understand that the task can be solved in this concrete manner. This is a shared platform for how to operate in all groups. There are also indications in some of the groups that students realize that this is not the expected way in which a problem of this kind should be solved during a mathematics lesson. It is most clearly expressed in one of the American groups when one of the members tells a fellow student trying this concrete approach that this way of solving the problem is `pathetic`. This is an interesting expression, since it signals a value statement where the appropriateness and elegance of a suggested solution are questioned in a rather blunt way. Although there is a lot of drawing and even building with matchsticks going on, the assumption that this is not the expected manner of dealing with the problem is evident in the other groups as well since they all engage in attempts to find more sophisticated alternatives.

At the next level, the groups intuitively realize that there are numerical patterns and regularities involved, and that some semiotic work is expected and necessary. In this sense, they achieve some level of reification and encapsulation, where they transform the object of their discussion into numerical patterns and regularities, for instance by arguing that there are three in each square. In most of the groups, a substantial proportion of the discussion is at this level, and the discussion goes back and forth, sometimes in an objectified manner, sometimes by returning to the concrete level of matches and squares. The observation that the students struggle with the problem of how to mathematize the pattern in an appropriate manner indicates that they are working in some kind of zone of proximal development (Vygotsky, 1978), where they are operating at the boundaries of their knowledge. They have an intuitive understanding of where they should be heading, but they are uncertain about how to achieve their goal.

An obvious hurdle in the argumentation in relation to the particular matchstick problem is how to handle the observation the groups make that one side can be part of two squares, i.e., even though one square has four sides, one only has to add three matches to have a new square. Understanding how to model this irregularity is an obstacle in all groups, even in the Swedish group where the teacher at the start of the group work explicitly suggests that the problem should be solved through an equation, which also one of the students picks up by writing $3 \cdot x = 73$ (cf. Figs. 8.9 and 8.10). However, even when beginning the work at this level, it is obviously difficult at this stage of the students' learning trajectory to take that added step of introducing an expression that will take what is observed about the squares into consideration. It is interesting to note that this irregularity represents such a distinct challenge which is commented upon in many groups but not solved in a distinctive manner. We may speculate that if the teacher would have intervened at the particular point where the work with the equation goes on, the students would have been able to understand in what sense the equation would have to be modified in order to accommodate to what they have no difficulty observing.

To Learn, Unlearn and Relearn

To conclude, we would like to emphasize that the introduction of algebra seems to be a situation where students have to learn, unlearn, and relearn. They have to unlearn some of the deeply held habits in the contexts of solving mathematical problems that they have developed during schooling, where they assume that arriving at a correct numerical result is the expected endpoint of engaging in mathematics, and where the correct answer concludes the engagement with the tasks. As we have shown, a dilemma here is that the problems they encounter in the instruction we have documented can be solved by means of the arithmetical procedures they already know. The problems are not challenging enough in the sense of turning the attention to the modeling necessary for successfully solving the problems as instances of algebra. They have to relearn in the sense that modeling a problem situation per se is an important element in algebra and algebraic reasoning. This shift in focus of attention obviously takes time and exercise, and, most likely, for most students it will require extensive teacher support.

Also teachers in their work of introducing algebra are faced with situations where they have to learn. We have seen how they struggle with several dilemmas: the tension of choosing between clear examples that can be successfully handled by students, or introducing challenging problems that make algebra useful; the tension between letting students explore and notice relatively freely or telling them early on what they are expected to see and learn; the tension between introducing and illustrating variables as entities that may take on different values, or verifying by using fixed values, i.e., the tension between the general and the particular, which is at the very core of algebra. As part of the data collection in the project, teachers were interviewed several times, first individually after completing the recordings in each class, and later in focus group interviews where they were invited to discuss aspects of their algebra teaching based on the recordings of their lessons (see Nyman & Kilhamn, 2014, for more details). Given the opportunity to reflect on their teaching, the teachers gave voice to some of the learning they experienced. We will here let some teacher's voices round off the stories of teaching and learning introductory algebra told in this book.

Several teachers reflected on the instruction vs. exploration dilemma. For example, a Swedish grade 7 teacher highlighted the difficulty of focusing generality and structure in student-generated examples: `When students create math problems for their peers it turns into guess work. Like, 'no I didn't do` x`+13-2, I did` x`+11'. Well, the mathematics is correct, but all the same, for the students it's different. Then it is not the mathematics that's in focus, but more of a guessing task.` This quote relates to the necessary re-negotiation of sociomathematical norms, which at least some of the teachers pointed to. As one teacher said: `You want the students to understand that algebra can be used to interpret reality, but the problem is students only want to` calculate to get a numerical answer. The challenge for this teacher was how to go about this re-negotiation in order to get students into algebra.

Some of the teachers were quite explicit about what they learned, in particular after having watched the recorded lessons and, so to speak, become observers of their own lessons. For example, the Swedish teachers described in Chap. 4, who chose, in the focus group discussion, to highlight what they would want to change in their lessons. The teacher in class A2, who spent several lessons working with manipulatives creating equations using boxes and beans, exclaimed: I feel I simply get stuck somehow, in this manipulative swamp. I don't go on. That's what I feel when I see this—I'm stuck in those damn boxes the whole time. The teacher in class B realized that the tasks she used were far too trivial: Every task I chose missed the target I had for the lesson - to understand the usefulness of algebra. If I had been more alert I could have turned it around, but I didn't manage that here. This remark touches upon the very complexity of the teaching profession. In the very moment of teaching there are so many things to be aware of. When given the opportunity to step back, observe, and reflect, as these teachers were doing when watching their recorded lessons, new insights can be made. One of the Swedish teachers chose to discuss an episode when she was trying to help a student, saying that she only realized what the student's problem really was when she watched the video. She said: I think that I have some golden opportunities here to increase understanding of her ways of thinking, but I don't take them. What that teachers learned was that she had to try to listen more closely to the students, or, as another teacher put it: After watching all my videos I realize I really must give my students more time to work things out by themselves before I help them. [...] I need to slow down around each student. Teaching is more than instructing, it also involves bringing forth and relying on the students' own powers.

Content related dilemmas of teaching algebra conceptually or procedurally, issues about whether to be concrete or abstract, general or specific, were brought up in all focus group discussions. A Finnish teacher said about teaching equation solving that: You have to decide in the moment of teaching if you first should teach how to do it mechanically, the procedure, and then think about why. Or should you teach how and why in parallel. A Swedish teacher summarized what she had come to think about teaching algebra with the words: It's easier to understand when it becomes more abstract. [...] Algebra becomes more accessible when it becomes more abstract somehow. [...] It's easier to play around with numbers than with objects. Perhaps more of "playing around" with numbers and varying quantities could be a fruitful way of moving forward when attempting to bring across some of the basic ideas of how to engage in algebra.

References

Anthony, G., & Burgess, T. (2014). Solving linear equations: A balanced approach. In F. K. S. Leung, K. Park, D. Holton, & D. Clarke (Eds.), *Algebra teaching around the world* (pp. 17–37). Rotterdam: Sense.

Asikainen, K., Fälden, H., Nyrhinen, K., Rokka, P., Vehmans, P., Törnroos, S., et al. (2007a). *Min matematik 5 [My mathematics 5]*. Helsingfors: Schildts Förlag.

Asikainen, K., Fälden, H., Nyrhinen, K., Rokka, P., Vehmans, P., Törnroos, S., et al. (2007b). *Min matematik 5. Lärarhandledning [My mathematics 5. Teacher's study guide]*. Helsingfors: Schildts Förlag.

Asikainen, K., Fälden, H., Nyrhinen, K., Rokka, P., Vehmans, P., Törnroos, S., et al. (2008a). *Min matematik 6 [My mathematics 6]*. Helsingfors: Schildts Förlag.

Asikainen, K., Fälden, H., Nyrhinen, K., Rokka, P., Vehmans, P., Törnroos, S., et al. (2008b). *Min matematik 6. Lärarhandledning [My mathematics 6. Teacher's study guide]*. Helsingfors: Schildts Förlag.

Bailey, T., Jeong, D. W., & Chi, S.-W. (2009). *Referral, enrollment, and completion in developmental education sequences in community colleges* (CCRC Working Paper No. 15). New York, NY: Columbia University, Teachers College, Community College Research Center.

Bakhtin, M. M. (1981). *The dialogic imagination: Four essays* (C. Emerson & M. Holquist, Trans.). Austin, TX: University of Texas Press.

Bakhtin, M. M. (1986). *Speech genres and other late essays* (V. W. McGee, Trans.). Austin, TX: University of Texas Press.

Balacheff, N. (2001). Symbolic arithmetic vs algebra the core of a didactical dilemma. Postscript. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on school algebra* (pp. 249–260). Dordrecht: Kluwer.

Becker, H., Geer, B., & Hughes, E. (1968). *Making the grade: The academic side of college life*. New York, NY: Wiley.

Bednarz, N., & Janvier, B. (1996). Emergence and development of algebra as a problem-solving tool: Continuities and discontinuities with arithmetic. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 115–136). Dordrecht: Kluwer.

Bednarz, N., Kieran, C., & Lee, L. (Eds.). (1996). *Approaches to algebra: Perspectives for research and teaching* (Vol. 18). Dordrecht: Kluwer.

Begehr, A. (2006). Students' verbal actions in German mathematics classes. In D. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics classrooms in twelve countries. The insider's perspective* (pp. 167–182). Rotterdam: Sense.

Bellack, A. A. (1969). Methods for observing classroom behaviors of teachers. In K. Ingenkamp (Ed.), *Methods for the evaluation of comprehensive schools*. Berlin: Beltz.

C. Kilhamn, R. Säljö (eds.), *Encountering Algebra*,
https://doi.org/10.1007/978-3-030-17577-1

Bellack, A. A., Kliebard, H. M., Hyman, R. T., & Smith, F. L. (1966). *The language of the classroom*. New York, NY: Teachers College Press.

Bennett, J. M., Berger, E. B., Chard, D. J., Jackson, A. L., Kennedy, P. A., Renfro, F. L., et al. (2008). *California mathematics course 2: Pre-algebra*. Orlando, FL: Holt, Rinehart and Winston.

Berg, C. V. (2009). *Developing algebraic thinking in a community of inquiry: Collaboration between three teachers and a didactician.* Diss., The University of Agder, Kristiansand, Norway.

Bills, L., Dreyfus, T., Mason, J., Tsamir, P., Watson, A., & Zaslavsky, O. (2006). Exemplification in mathematics education. In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 126–154). Prague: Charles University.

Black, L., Mendick, H., & Solomon, Y. (Eds.). (2009). *Mathematical relationships in education. Identities and participation*. London: Routledge.

Blanton, M., Stephens, A., Knuth, E., Murphy Gardiner, A., Isler, I., & Kim, J.-S. (2015). The development of children's algebraic thinking: The impact of a comprehensive early algebra intervention in third grade. *Journal for Research in Mathematics Education, 46*(1), 39–87.

Britt, M. S., & Irwin, K. C. (2011). Algebraic thinking with and without algebraic representation. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 137–159). Berlin: Springer.

Brosseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.

Brown, A. L. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classroom settings. *The Journal of the Learning Sciences, 2*(2), 141–178.

Bush, S., & Karp, K. (2013). Prerequisite algebra skills and associated misconceptions of middle grade students: A review. *The Journal of Mathematical Behavior, 32*(3), 613–632.

Cai, J., & Knuth, E. (Eds.). (2011). *Early algebraization. A global dialogue from multiple perspectives*. Berlin: Springer.

Cai, J., & Moyer, J. (2008). Developing algebraic thinking in earlier grades: Some insights from international comparative studies. In *70th Yearbook Algebra and algebraic thinking in school mathematics* (pp. 69–182). Reston, VA: National Council of Teachers of Mathematics.

Carlgren, I., Klette, K., Mýrdal, S., Schnack, K., & Simola, H. (2006). Changes in Nordic teaching practices: From individualised teaching to the teaching of individuals. *Scandinavian Journal of Educational Research, 50*(3), 301–326.

Carlsen, M. (2010). Appropriating geometric series as a cultural tool: A study of student collaborative learning. *Educational Studies in Mathematics, 74*(2), 95–116.

Carlsson, S., Liljegren, G., & Picetti, M. (2004). *Matte Direkt Borgen 6B [Maths Directly The Castle]*. Stockholm: Bonniers.

Carraher, D. W., Martinez, M. V., & Schliemann, A. D. (2008). Early algebra and mathematical generalization. *ZDM: The International Journal on Mathematics Education, 40*(3), 349–361.

Carraher, D., & Schliemann, A. (2007). Early algebra and algebraic reasoning. In F. Lester (Ed.), *Handbook of research in mathematics education* (pp. 669–705). Greenwich, CT: Information Age Publishing.

Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education, 37*(2), 87–115.

Carraher, D. W., Schliemann, A. D., & Schwartz, J. (2008). Early algebra is not the same as algebra early. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 235–272). New York, NY: Routledge.

Caspi, S., & Sfard, A. (2012). Spontaneous meta-arithmetic as a first step toward school algebra. *International Journal of Educational Research, 51–52*, 45–65.

Cazden, C. (1988). *Classroom discourse. The language of teaching and learning*. Portsmouth, NH: Heinemann.

Cestari, M. L. (1998). *Communication in mathematics classrooms: A dialogical approach*. Diss., University of Oslo, Norway.
Clarke, S., Howley, I., Resnick, L., & Penstein Rosé, C. (2016). Student agency to participate in dialogic science discussions. *Learning, Culture and Social Interaction, 10*, 27–39.
Clarke, D., Keitel, C., & Shimizu, Y. (2006). *Mathematics classrooms in twelve countries. The insider's perspective*. Rotterdam: Sense.
Cobb, P. (1987). An investigation of young children's academic arithmetic contexts. *Educational Studies in Mathematics, 18*, 109–124.
Cole, M. (1996). *Cultural psychology: A once and future discipline*. Cambridge, MA: The Belknap Press.
Collis, K. F. (1975). *The development of formal reasoning*. New South Wales: University of Newcastle.
Da Rocha Falcão, J. (1995). A case study of algebraic scaffolding: From balance scale to algebraic notation. In L. Meira & D. Carraher (Eds.), *19th International Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 66–73). Recife: Universidade Federal de Pernambuco.
Daniels, H. (2008). *Vygotsky and research*. New York, NY: Routledge.
Davydov, V. V., Gorbov, S. F., Mikulina, G. G., & Savaleva, O. V. (1999). *Mathematics class 1* (J. Schmittau, Trans.). Binghamton, NY: State University of New York.
de Freitas, E., & Sinclair, N. (2012). Diagram, gesture, agency: Theorizing embodiment in the mathematics classroom. *Educational Studies in Mathematics, 80*, 133–152.
Derry, S. J., Pea, R. D., Barron, B., Engle, R. A., Ericson, F., Goldman, R., et al. (2010). Conducting video research in the learning sciences: Guidance on selection, analysis, technology, and ethics. *Journal of the Learning Sciences, 19*, 3–53.
Donald, M. (2018). The evolutionary origins of human cultural memory. In B. Wagoner (Ed.), *Handbook of culture and memory* (pp. 19–40). Oxford: Oxford University Press.
Dossey, J. A., Mullis, I. V. S., Lindquist, M. M., & Chambers, D. L. (1988). *The mathematics report card: Are we measuring up? Trends and achievement based on the 1986 National Assessment* (NAEP Report No. 17-M-01). Princeton, NJ: Educational Testing Service.
Edwards, D., & Mercer, N. (1989). *Common knowledge: The development of understanding in the classroom*. London: Routledge.
Emanuelsson, J., & Sahlström, F. (2008). The price of participation: Teacher control versus student participation in classroom interaction. *Scandinavian Journal of Educational Research, 52*(2), 205–223.
Even, R. (1998). Factors involved in linking representations of functions. *The Journal of Mathematical Behavior, 17*(1), 105–121.
Falkner, K. P., Levi, L., & Carpenter, T. P. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics, 6*(4), 232–236.
Filloy, E., & Rojano, T. (1984). From an arithmetical to an algebraic thought. In J. Moser (Ed.), *Proceedings of PME-NA VI* (pp. 51–56). Madison, WI: PME.
Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics, 9*(2), 19–25.
Flanders, N. A. (1964). *Interaction analysis in the classroom: A manual for observers*. School of Education, University of Michigan.
Flanders, N. A. (1970). *Analyzing teaching behavior*. Reading, MA: Addison-Wesley.
Frank, M. L. (1988). Problem solving and mathematical beliefs. *The Arithmetic Teacher, 35*(5), 32–34.
Franke, M. L., & Carey, D. A. (1997). Young children's perceptions of mathematics in problem-solving environments. *Journal for Research in Mathematics Education, 28*(1), 8–25.
Freudenthal, H. (1968). Why to teach mathematics so as to be useful. *Educational Studies in Mathematics, 1*(1), 3–8.
Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: Reidel.
Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: Reidel.
Freudenthal, H. (1991). *Revisiting mathematics education: China lectures*. Hingham, MA: Kluwer.

Fried, M. N. (2009). Similarity and equality in Greek mathematics: Semiotics, history of mathematics and mathematics education. *For the Learning of Mathematics, 29*(1), 2–7.
Gallardo, A. (2001). Historical-epistemological analysis in mathematics education: Two works in didactics of algebra. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on school algebra* (Vol. 22, pp. 121–139). Dordrecht: Kluwer.
Garfinkel, H. (1967). *Studies in ethnomethodology*. Englewood Cliffs, NJ: Prentice-Hall.
Garofalo, J. (1989). Beliefs and their influence on mathematical performance. *The Mathematics Teacher, 82*(7), 502–505.
Geertz, C. (1973). *The interpretation of cultures*. New York, NY: Basic Books.
Giroux, H. (2001). *Theory and resistance in education*. London: Bergin & Garvey.
Giroux, H., & Purpel, D. (Eds.). (1983). *The hidden curriculum and moral education*. Berkeley, CA: McCutchan Publishing.
Givvin, K. B., Stigler, J. W., & Thompson, B. J. (2011). What community college developmental mathematics students understand about mathematics, Part II: The interviews. *MathAMATYC Educator, 2*(3), 4–18.
Goffman, E. (1981). *Forms of talk*. Oxford: Blackwell.
Goldin, G., & Shteingold, N. (2001). Systems of representations and the development of mathematical concepts. In A. Cuoco & F. Curcio (Eds.), *The roles of representation in school mathematics* (pp. 1–23). Reston, VA: National Council of Teachers of Mathematics.
Goldman, R., Pea, R. D., Barron, B., & Derry, S. J. (Eds.). (2007). *Video research in the learning sciences*. Mahwah, NJ: Erlbaum.
Goodwin, C., & LeBaron, C. (2011). *Embodied interaction: Language and body in the material world*. New York, NY: Cambridge University Press.
Gray, E., & Tall, D. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education, 26*(2), 115–141.
Gresalfi, M., Martin, T., Hand, V., & Greeno, J. (2008). Constructing competence: An analysis of student participation in the activity systems of mathematics classrooms. *Educational Studies in Mathematics, 70*, 49–70.
Grover, S., & Pea, R. (2013). Computational thinking in K–12: A review of the state of the field. *Educational Researcher, 42*(1), 38–43. https://doi.org/10.3102/0013189x12463051
Gumperz, J. J., & Hymes, D. (1972). *Directions in sociolinguistics: The ethnography of communication*. New York, NY: Holt, Rinehart and Winston.
Hammersley, M. (1990). *Classroom ethnography: Empirical and methodological essays*. Milton Keynes: Open University Press.
Hansson, Å. (2010). Instructional responsibility in mathematics education: Modelling classroom teaching using Swedish data. *Educational Studies in Mathematics, 75*(2), 171–189.
Hansson, Å. (2011). *Ansvar för matematiklärande, effekter av undervisning i det flerspråkiga klassrummet [Responsibility for mathematics learning, effects of instructional responsibility in the multilingual calssroom]*. Gothenburg: ACTA Universitatis Gothenburgensis.
Harel, G. (2000). Three principles of learning and teaching mathematics. In J. L. Dorier (Ed.), *On the teaching of linear algebra. Mathematics education library* (Vol. 23, pp. 177–189). Dordrecht: Springer.
Heath, C., Hindmarsh, J., & Luff, P. (2010). *Video in qualitative research*. London: Sage.
Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics, 27*, 59–78.
Hewitt, D. (2014). A symbolic dance: The interplay between movement, notation, and mathematics on a journey toward solving equations. *Mathematical Thinking and Learning, 16*(1), 1–31.
Hiebert, J. (Ed.). (1986). *Conceptual and procedural knowledge: The case of mathematics*. Hillsdale, NJ: Lawrence Erlbaum.
Hitt, F., & González-Martín, A. S. (2016). Generalizations, covariation, functions, and calculus. In A. Gutiérrez, G. C. Leder, & P. Boero (Eds.), *The second handbook of research on the psychology of mathematics education: The journey continues* (pp. 3–38). Rotterdam: Sense.
Hjardar, E., & Pedersen, J.-E. (2006). *Faktor 1 [Factor 1]*. Oslo: Cappelen Damm.

Hohenwarter, M., & Jones, K. (2007). BSRLM Geometry Working Group: Ways of linking geometry and algebra, the case of Geogebra. *Proceedings of the British Society for Research into Learning Mathematics, 27*(3), 126–131.

Hoon, S. L., Kaur, B., & Kiam, L. H. (2006). Case studies of Singapore secondary mathematics classrooms: The instructional approaches of two teachers. In D. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics classrooms in twelve countries. The insider's perspective* (pp. 151–165). Rotterdam: Sense.

Hopkins, D. (2014). *A teacher's guide to classroom research* (5th ed.). New York, NY: Open University Press.

Jackson, P. W. (1968). *Life in classrooms*. New York, NY: Holt, Rhinehart & Winston.

Johansson, M. (2006). *Teaching mathematics with textbooks.* Diss., Luleå University of Technology, Luleå.

John-Steiner, V., & Mahn, H. (1996). Sociocultural approaches to learning and development: A Vygotskian framework. *Educational Psychologist, 31*(3/4), 191–206.

Jordan, B., & Henderson, A. (1995). Interaction analysis: Foundations and practice. *The Journal of the Learning Sciences, 4*(1), 39–103.

Kaiser, G., Luna, E., & Huntley, I. (Eds.). (1999). *International comparisons in mathematics education*. London: Falmer Press.

Kaput, J. (1987). Representation systems and mathematics. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 18–26). Hillsdale, NJ: Lawrence Erlbaum.

Kaput, J. J. (2000). *Teaching and learning a new algebra with understanding*. Dartmouth, MA: National Center for Improving Student Learning and Achievement in Mathematics and Science.

Kaput, J., Blanton, M., & Moreno, L. (2008). Algebra from a symbolization point of view. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 19–56). New York, NY: Lawrence Erlbaum.

Kaput, J., Carraher, D., & Blanton, M. (Eds.). (2008). *Algebra in the early grades*. New York, NY: Routledge.

Kaur, B., Anthony, G., Ohtani, M., & Clarke, D. (2013). *Student voice in mathematics classrooms around the world*. Rotterdam: Sense.

Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *The Elementary School Journal, 102*(1), 59–80.

Kendal, M., & Stacey, K. (2004). Algebra: A world of difference. In R. Stacey (Ed.), *The future of the teaching and learning of algebra* (pp. 329–346). Dordrecht: Kluwer.

Kentli, F. D. (2009). Comparison of hidden curriculum theories. *European Journal of Educational Studies, 1*(2), 83–88.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*(3), 317–326.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York, NY: Macmillan.

Kieran, C. (1996). The changing face of school algebra. In C. Alsina, J. Alvarez, B. Hodgson, C. Laborde, & A. Pérez (Eds.), *8th International Congress on Mathematical Education: Selected lectures* (pp. 271–290). Seville: S.A.E.M. Thales.

Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Teacher, 8*(1), 139–151.

Kieran, C. (2007a). *What do we know about the teaching and learning of algebra in the elementary grades?* (NCTM Brief). Retrieved from http://mason.gmu.edu/~jsuh4/impact/research.algebra.pdf

Kieran, C. (2007b). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester (Ed.), *Second Handbook of research on mathematics teaching* (pp. 707–762). Charlotte, NC: Information Age Publishing.

Kieran, C. (2014). The false dichotomy in mathematics education between conceptual understanding and procedural skills: An example from algebra. In K. Leatham (Ed.), *Vital directions in mathematics education research* (pp. 153–172). New York, NY: Springer.

Kieran, C. (Ed.). (2018). *Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice*. New York, NY: Springer.

Kieran, C., Pang, J. S., Schifter, D., & Ng, S. F. (2016). *Early algebra: Research into its nature, its learning, its teaching. ICME 13 Topical Surveys*. Hamburg: Springer Open.

Kilhamn, C., & Hillman, T. (2015). Structural and pedagogical diversity in Swedish Grade six algebra classrooms. In O. Helenius, et al. (Eds.), *Development of mathematics teaching; design, scale, effects. Proceedings of the 9th Research Seminar of the Swedish Society for Research in Mathematics Education*. Madif 9 i Umeå 2014. Linköping: SMDF.

Kilhamn, C., & Röj-Lindberg, A.-S. (2013). Seeking hidden dimensions of algebra teaching through video analysis. In B. Grevholm, P. S. Hundeland, K. Juter, K. Kislenko, & P.-E. Persson (Eds.), *Nordic research in mathematics education, past, present and future* (pp. 299–328). Oslo: Cappelen Damm.

Knuth, E. J., Alibali, M. W., McNeil, N., Weinberg, A., & Stephens, A. C. (2011). Middle school students' understanding of core algebraic concepts: Equivalence & variable. In J. Cai & E. Knuth (Eds.), *Early algebraization. A global dialogue from multiple perspectives* (pp. 259–276). Berlin: Springer.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*(4), 297–312.

Kramer, S. N. (1963). *The Sumerians. Their history, culture, and character*. Chicago, IL: The University of Chicago Press.

Kramer, S. N. (1981). *History begins at Sumer*. Philadelphia, PA: The University of Pennsylvania Press.

Küchemann, D. (1978). Children's understanding of numerical variables. *Mathematics in School, 9*, 23–26.

Küchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics: 11-16* (pp. 102–119). London: John Murray.

Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life*. Cambridge, MA: Cambridge University Press.

Leont'ev, A. N. (1981). *Problems of the development of the mind* (M. Kopylova, Trans.). Moscow: Progress Publishers.

Lesh, R., & Landau, M. (1983). *Acquisition of mathematics concepts and processes*. New York, NY: Academic.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 33–40). Hillsdale, NJ: Lawrence Erlbaum.

Leung, F. K. S., Park, K., Holton, D., & Clarke, D. (Eds.). (2014). *Algebra teaching around the world*. Rotterdam: Sense Publishers.

Linell, P. (1998). *Approaching dialogue. Talk, interaction and contexts in dialogical perspectives*. Amsterdam: John Benjamins Publishing Company.

Linell, P. (2011). *Samtalskulturer: Vol. 1 [Conversation cultures]. Studies in Language and Culture, 18*. Linköping: Linköping University.

Littleton, K., & Mercer, N. (2013). *Interthinking: Putting talk to work*. Abingdon: Routledge.

Lundgren, U. P. (1972). *Frame factors and the teaching process. A contribution to curriculum theory*. Stockholm: Almqvist & Wiksell.

Lundgren, U. P. (1977). *Model analysis of pedagogical processes*. Lund: Liber/Gleerup.

Lundgren, U. P., & Säljö, R. (2017). Skolans tidiga historia och utveckling – från skrivarskola till folkskola i Sverige (The early history and development of schooling – from scribal schools to folkschools in Sweden). In U. P. Lundgren, R. Säljö & C. Liberg (Eds.), Lärande, skola, bildning (Learning, schooling, Bildung) (pp. 29–64). Stockholm: Natur & Kultur.

MacGregor, M., & Stacey, K. (1997). Students' understanding of algebraic notation: 11-15. *Educational Studies in Mathematics, 33*(1), 1–19.

Mannila, L., Dagiene, V., Demo, B., Grgurina, N., Mirolo, C., Rolandsson, L., et al. (2014). Computational thinking in K-9 education. In A. Clear & R. Lister (Eds.), *Proceedings of the Working Group Reports of the 2014 on Innovation & Technology in Computer Science Education Conference* (pp. 1–29). New York, NY: ACM.

Mason, J. (1996). Expressing generality and the roots of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra. Perspectives for research and teaching* (pp. 65–86). Dordrecht: Kluwer.

Mason, J., Graham, A., & Johnston-Wilder, S. (2005). *Developing thinking in algebra*. London: Sage.

Mason, J., & Johnston-Wilder, S. (2004). *Fundamental constructs in mathematics education*. London: RoutledgeFalmer.

Mehan, H. (1979). *Learning lessons: Social organisation in the classroom*. Cambridge, MA: Harvard University Press.

Ministry of Education and Research. (2013). *Curriculum for the common core subject of mathematics* (MAT1-04). Retrieved from http://www.udir.no/kl06/mat1-04/Hele/?lplang=eng

Mook, I. A. C. (2006). Teacher-dominating lessons in Shanghai: The insiders' story. In D. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics classrooms in twelve countries. The insider's perspective* (pp. 87–98). Rotterdam: Sense.

Myhill, D., & Dunkin, F. (2005). Questioning learning. *Language and Education, 19*(5), 415–427.

Nachlieli, T., & Tabach, M. (2012). Growing mathematical objects in the classroom. The case of function. *International Journal of Educational Research, 51*, 10–27.

Nardi, E., Ryve, A., Stadler, E., & Viirman, O. (2014). Commognitive analysis of the learning and teaching of mathematics at university level: The case of discursive shifts in the study of calculus. *Research in Mathematics Education, 16*(2), 182–198.

National Board of Education. (2004). *The curriculum for basic education*. Helsinki: National Board of Education.

Ng, S. F. (2004). Developing algebraic thinking in the early grades: Case study of the Singapore primary mathematics curriculum. *The Mathematics Educator, 8*(1), 39–59.

Niss, M. (2003). Mathematical competencies and the learning of mathematics. The Danish KOM Project. In A. Gagatses et al. (Eds.), *The 3rd Mediterranean Conference on Mathematical Education* (pp. 115–124). Athens: Hellenic Mathematical Society.

Nyman, R., & Kilhamn, C. (2014). Enhancing engagement in algebra: Didactical strategies implemented and discussed by teachers. *Scandinavian Journal of Educational Research, 59*(6), 623–637. https://doi.org/10.1080/00313831.2014.965790

Ochs, E. (1979). Transcription as theory. In E. Ochs & B. Schiefflin (Eds.), *Developmental pragmatics* (pp. 43–72). New York, NY: Academic.

Palmér, H. (2013). *To become, or not to become, a primary school mathematics teacher: A study of novice teachers' professional identity development*. Diss., Linneaus University Press, Växjö.

Pepin, B. (2011). Pupils' attitudes towards mathematics: A comparative study of Norwegian and English secondary students. *ZDM: The International Journal on Mathematics Education, 43*(4), 535–546.

Philipp, R. A. (1992). The many uses of algebraic variables. *The Mathematics Teacher, 85*(7), 557–561.

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics, 42*, 237–268.

Radford, L. (2002). The seen, the spoken and the written. A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics, 22*(2), 14–23.

Radford, L. (2003). On the epistemological limits of language. Mathematical knowledge and social practice during the Renaissance. *Educational Studies in Mathematics, 52*(2), 123–150.

Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics, 70*, 111–126.

Radford, L. (2010). Algebraic thinking from a cultural semiotic perspective. *Research in Mathematics Education, 12*(1), 1–19.

Radford, L. (2011). Embodiment, perception and symbols in the development of early algebraic thinking. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 17–14). Ankara: PME.

Radford, L. (2014). Towards an embodied, cultural, and material conception of mathematics cognition. *ZDM: The International Journal of Mathematics Education, 46*(3), 349–361.

Radford, L. (2018). The emergence of symbolic algebraic thinking in primary school. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice* (pp. 1–23). New York: Springer.

Reinhardtsen, J. (2012). *The introduction of algebra: Comparative studies of textbooks in Finland, Norway, Sweden and USA* (Master's thesis at the University of Agder). Kristiansand: University of Agder.

Reinhardtsen, J., Carlsen, M., & Säljö, R. (2015). Capturing learning in classroom interaction in mathematics: Methodological considerations. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 1474–1481). Prague: Charles University in Prague.

Remillard, K. S. (2014). Identifying discursive entry points in paired-novice discourse as a first step in penetrating the paradox of learning mathematical proof. *The Journal of Mathematical Behavior, 34*, 99–113.

Rissland-Michner, E. (1978). Understanding mathematics. *Cognitive Science, 2*(4), 361–383.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93*(2), 346–362.

Rittle-Johnson, B., & Star, J. R. (2009). Compared with what? The effects of different comparisons on conceptual knowledge and procedural flexibility for equation solving. *Journal of Educational Psychology, 101*(3), 529–544.

Roser, M., & Ortiz-Ospina, E. (2017). *Global rise of education*. Retrieved March 13, 2017, from https://ourworldindata.org/global-rise-of-education

Rowland, T. (2008). The purpose, design and use of examples in the teaching of elementary mathematics. *Educational Studies in Mathematics, 69*(2), 149–163.

Rystedt, E., Kilhamn, C., & Helenius, O. (2016). What's there in an *n*? *Nordic Studies in Mathematics Education, 21*(1), 5–26.

Sacks, H., & Jefferson, G. (1995). Lectures on conversation. *Human Studies, 18*(2), 327–336.

Sahlström, F., & Lindblad, S. (1998). Subtexts in the science classroom—An exploration of the construction of science lessons and student careers. *Learning and Instruction, 8*(3), 194–214.

Säljö, R. (2006). *Læring og kulturelle redskaper; om læreprosesser og den kollektive hukommelsen [Learning and cultural tools: On processes of learning and the collective memory]*. Oslo: Cappelen Akademiske.

Säljö, R. (2009). Learning, theories of learning and units of analysis in research. *Educational Psychologist, 44*(3), 202–208.

Säljö, R. (2019). Materiality, learning and cognitive practices—Artefacts as instruments of thinking. In T. Cerrato Pargman & I. Jahnke (Eds.), *Emergent practices and material conditions in learning and teaching with technologies* (pp. 31–52). New York, NY: Springer.

Schliemann, A. D., Carraher, D. W., & Brizuela, B. M. (2013). *Bringing out the algebraic character of arithmetic*. London: Routledge.

Schmittau, J. (2011). The role of theoretical analysis in developing algebraic thinking: A Vygotskian perspective. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 71–81). Berlin: Springer.

Schmittau, J., & Morris, A. (2004). The development of algebra in the elementary mathematics curriculum of V. V. Davydov. *Mathematics Educator, 8*, 60–87.

Schoenfeld, A. (1983). Beyond the purely cognitive: Belief systems, social cognitions, and metacognitions as driving forces in intellectual performance. *Cognitive Science, 7*(4), 329–363.

Schoenfeld, A. (1989). Explorations of students' mathematical beliefs and behavior. *Journal for Research in Mathematics Education, 20*(4), 338–335.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1–36.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. New York, NY: Cambridge University Press.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification: The case of algebra. *Educational Studies in Mathematics, 26*, 191–228.

Shimizu, Y., Kaur, B., Huang, R., & Clarke, D. (Eds.). (2010). *Mathematical tasks in classrooms around the world*. Rotterdam: Sense.

Sinclair, J., & Coulthard, M. (1975). *Towards an analysis of discourse*. Oxford: Oxford University Press.

Skolverket. (2003). *Lusten att lära—med fokus på matematik [The desire to learn—Focussing mathematics]* (Report 221). Stockholm: Skolverket.

Skolverket. (2011a). *Curriculum for the compulsory school, preschool class and the leisure-time centre 2011*. Retrieved July 20, 2012, from http://www.skolverket.se/publications

Skolverket. (2011b). *Laborativ matematik, konkretiserande undervisning och matematikverkstäder [Experiment-based mathematics, embodied teaching and mathematics workshops]* (Report 366). Stockholm: Skolverket.

Skolverket. (2012). *Likvärdig utbildning i svensk skola: en kvantitativ analys av likvärdighet över tid [Equal education in Swedish school: A quantitative analysis of equality over time]*. Stockholm: Skolverket.

Snyder, B. R. (1971). *The hidden curriculum*. New York, NY: Alfred Knopf.

Stacey, K. (1989). Finding and using patterns in linear generalising problems. *Educational Studies in Mathematics, 20*(2), 147–164.

Stacey, K., & MacGregor, M. (1997). Building foundations for algebra. *Mathematics Teaching in the Middle School, 2*(4), 253–260.

Stacey, K., & MacGregor, M. (2000). Learning the algebraic method of solving problems. *Journal of Mathematical Behavior, 18*(2), 149–167.

Stephens, A. C., Ellis, A. B., Blanton, M., & Brizuela, B. M. (2017). Algebraic thinking in the elementary and middle grades. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 386–420). Reston, VA: National Council of Teachers of Mathematics.

Stevenson, H. W., & Stigler, J. W. (1992). *The learning gap: Why our schools are failing and what we can learn from Japanese and Chinese education*. New York, NY: Summit Books.

Stigler, J. W., Gallimore, R., & Hiebert, J. (2000). Using video surveys to compare classrooms and teaching across cultures: Examples and lessons from the TIMSS video studies. *Educational Psychologist, 35*(2), 87–100.

Stigler, J. W., Gonzales, P., Kawanaka, T., Knoll, S., & Serrano, A. (1999). *The TIMSS videotape classroom study: Methods and findings from an exploratory research project on eighth graders mathematics instruction in Germany, Japan and the United States*. Washington, DC: National Center for Education Statistics.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York, NY: Free Press.

Stipek, D. J., Givvin, K. B., Salmon, J. M., & MacGyvers, V. L. (2001). Teachers' beliefs and practices related to mathematics instruction. *Teaching and Teacher Education, 17*(2), 213–226.

Stodolsky, S. S. (1985). Telling mathematics: Origins of mathematics aversion and anxiety. *Educational Psychologist, 20*(3), 125.

Stodolsky, S. S., Salk, S., & Glaessner, B. (1991). Student views about learning math and social studies. *American Educational Research Journal, 28*(1), 89–116.

Sutherland, R. (1987). *A longitudinal study of the development of pupils' algebraic thinking in a logo environment*. Diss., University of London, Institute of Education.

Sutherland, R. (1993). Connecting theory and practice: Results from the teaching of Logo. *Educational Studies in Mathematics, 24*(1), 95–113.

Thomas, R. (1992). Literacy and orality in ancient Greece. Cambridge: Cambridge University Press.

Thompson, A. G. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics, 15*(2), 105–127.

TIMSS & PIRLS. (2009). *TIMSS 2007 user guide for the international database: Released items, mathematics eighth grade*. TIMSS & PIRLS International Study Center: Boston College. Retrieved from http://timss.bc.edu/timss2007/items.html

Törnroos, J. (2001, June). *Finnish mathematics textbooks in grades 5-7*. Presentation at the Second Scandinavian Symposium on Research Methods in Science and Mathematics Education, Helsinki, Finland.

Treffers, A. (1987). *Three dimensions: A model of goal and theory description in mathematics instruction—The Wiskobas Project*. Dordrecht: Reidel.

Treffers, A., & Vonk, H. (1987). *Three dimensions: A model of goal and theory description in mathematics instruction—The Wiskobas Project*. Dordrecht: Reidel.

Unesco. (2015). *Education for all 2000-2015. Achievements and challenges*. Retrieved January 12, 2017, from http://unesdoc.unesco.org/images/0023/002325/232565e.pdf

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. Coxford & A. Shulte (Eds.), *The ideas of algebra, K-12* (pp. 8–19). Reston, VA: National Council of Teachers of Mathematics.

Varadarajan, V. (1998). *Algebra in ancient and modern times*. Providence, RI: American Mathematical Society.

Vergnaud, G. (1982). Cognitive and developmental psychology and research in mathematics education: Some theoretical and methodological issues. *For the Learning of Mathematics, 3*(2), 31–41.

Vergnaud, G. (1998). A comprehensive theory of representation for mathematics education. *The Journal of Mathematical Behavior, 17*(2), 167–181.

Vieira, L., Gimenez, J., & Palhares, P. (2013). Relational understanding when introducing early algebra in Portuguese schools. In B. Ubuz, C. Haser, & M. A. Mariotti (Eds.), *Proceedings of the Eight Congress of the European Society for Research in Mathematics Education* (pp. 550–557). Manavgat, Side: Middle East Technical University, Turkey on behalf of the European Society for Research in Mathematics Education.

Vlassis, J. (2002). The balance model: Hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics, 49*, 341–359.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Vygotsky, L. S. (1981). The genesis of higher mental functions. In J. V. Wertsch (Ed.), *The concept of activity in Soviet psychology* (pp. 144–188). Armonk, NY: Sharpe.

Vygotsky, L. S. (1987). Thinking and speech (N. Minick, Trans.). In R. W. Rieber & A. S. Carton (Eds.), *The collected works of L.S. Vygotsky: Vol. 1. Problems of general psychology* (pp. 39–285). New York, NY: Plenum Press.

Waller, W. (1932). *The sociology of teaching*. New York, NY: Wiley.

Watanabe, T. (2011). Shiki: A critical foundation for school algebra in Japanese elementary school mathematics. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 109–124). Berlin: Springer.

Watson-Gegeo, K. (1999). Classroom ethnography. In N. H. Homberger & D. Corson (Eds.), *Encyclopedia of language and education: Research methods in language and education* (Vol. 8, pp. 135–144). Dordrecht: Kluwer.

Wells, G., & Mejia Arauz, R. (2006). Dialogue in the classroom. *The Journal of the Learning Sciences, 15*(3), 379–428.

Wertsch, J. V. (1991). *Voices of the mind: A sociocultural approach to mediated action*. Cambridge, MA: Harvard University Press.

Wertsch, J. V. (1998). *Mind as action*. New York, NY: Oxford University Press.

Wertsch, J. V. (2002). *Voices of collective remembering*. Cambridge, MA: Cambridge University Press.

Wertsch, J. V., & Addison Stone, C. (1985). The concept of internalization in Vygotsky's account of the genesis of higher mental functions. In J. V. Wertsch (Ed.), *Culture, communication, and cognition: Vygotskian perspectives* (pp. 162–179). Cambridge: Cambridge University Press.

Wertsch, J. V., & Kazak, S. (2011). Saying more than you know in instructional settings. In T. Koschmann (Ed.), *Theories of learning in studies of instructional practice* (pp. 153–166). New York, NY: Springer.

Westbury, I., & Bellack, A. A. (Eds.). (1971). *Research into classroom processes: Recent developments and next steps*. New York, NY: Teachers College Press.

Willis, P. (1977). *Learning to labour: How working class boys get working class jobs*. Farnborough: Saxon House.

Wing, J. M. (2006). Computational thinking. *Communications of the ACM, 49*(3), 33–35. https://doi.org/10.1145/1118178.1118215

Wood, T., Cobb, P., & Yackel, E. (1991). Change in teaching mathematics: A case study. *American Educational Research Journal, 28*(3), 587–616.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*, 458–477.

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https://doi.org/10.1007/978-3-030-17577-1

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