On Lindelöf Σ-Spaces

In Honour of Manuel López-Pellicer

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Abstract We revisit the notion of Lindelöf Σ-space giving a general overview about this question. For that, we deal with the Lindelöf property to introduce Lindelöf Σ-spaces in order to make a description of the "goodness" of such a type of spaces, making special emphasis in the duality between X and $C_p(X)$ respect to some topological properties, more specifically, topological properties in which different cardinal functions are involved. Classical results are linked with more recent results.

Keywords Lindelöf number · Cardinal inequalities · Topological properties

1 Notation and Terminology

The set-theoretic notation which will be used follows [19, 20]. Cardinal numbers κ and $\mathfrak{m}$ are the initial ordinals that will denote always *infinite* cardinals, ω is the smallest infinite cardinal number. The cardinal number assigned to the set of all real numbers is denoted by $\mathfrak{c}$. κ^+ is the smallest cardinal number after κ. The cardinality of a set E is denote by $|E|$, $P(E)$ is the power set of E and $[E]^n = \{A : A \subset E, |A| = n\}$. Respect to the notation refereed to topology the basic references used are [14, 22].

Let $(X, \mathscr{T})$ be a topological space, where X is a set and $\mathscr{T}$ is a topology. A family of sets in $\mathscr{N}$ it is called a *network* for X if for every point $x \in X$ and any neighborhood U of x there exists $N \in \mathscr{N}$ such that $x \in N \subset U$. The *network weight* of a space X, $nw(X)$, is defined as the smallest cardinal number of a network in X. A family of open sets in $\mathscr{B}$ it is called a *basis* if for every non-empty open subset $U \in \mathscr{T}$ of X can be represented as the union of a subfamily of $\mathscr{B}$. This definition is equivalent to the property that for each open set $U \in \mathscr{T}$ such that $x \in U$ there exists $B \in \mathscr{B}$ such that $x \in B \subset U$. It is clear that a basis is a network such that the

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J. C. Ferrando (ed.), *Descriptive Topology and Functional Analysis II*, Springer Proceedings in Mathematics & Statistics 286,
https://doi.org/10.1007/978-3-030-17376-0_14

elements of the family are open. The *weight of a topological space* $(X, \mathscr{T})$, denoted by $w(X, \mathscr{T})$, is the smallest cardinal number of the cardinality of a basis.

Let $x \in X$ be a fixed point of a topological space $(X, \mathscr{T})$ a family, $B(x) \subset \mathscr{T}$ of open subsets is called a *basis of neighborhoods at x* if for every open set $U \in \mathscr{T}$ such that $x \in U$, there exists $V \in B(x)$ such that $x \in V \subset U$. The *character of a point x*, denoted by $\chi(X, x)$ is the smallest cardinal number of the cardinality of a basis of neighborhoods at x. The *character of a topological space* $(X, \mathscr{T})$ is the supremum of all cardinal numbers $\chi(x, X)$ for $x \in X$, and it will be denoted by $\chi(X)$. We will write X a topological space instead of (X, τ) for short.

Definition 1.1 (*p. 12* [14]) A topological space X is said to be

1. *first-countable* or satisfies the *first axiom of countability* if $\chi(X) \leq \omega$, this means that each point has a countable basis of neighborhoods.
2. *second-countable* or satisfies the *second axiom of countability* if $w(X) \leq \omega$, that is, X has a countable basis.

The following definitions are standard and can be found in [14].

Definition 1.2 (*pp. 37–40* [14]) A topological space X is called a

1. T_1-*space* if for every pair of different points $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$.
2. T_2-*space*, or a *Hausdorff space*, if for every pair of different points $x, y \in X$ there exist open sets $U_1, U_2 \subset X$ such that $x \in U_1$, $y \in U_2$ and $U_1 \cap U_2 = \emptyset$.
3. T_3-*space*, or a *regular space*, if X is a T_1-space and for every $x \in X$ and every closed set $F \subset X$ such that $x \notin F$ there exist open sets U_1, U_2 such that $x \in U_1$, $F \subset U_2$ and $U_1 \cap U_2 = \emptyset$.
4. $T_{3\frac{1}{2}}$-*space*, or a *Tychonoff* space, or a completely regular space, if X is a T_1-space and for every $x \in X$ and every closed set $F \subset X$ such that $x \notin F$ there exists a continuous function $f : X \to I$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in F$. Every Tychonoff space is a regular space.
5. T_4-*space*, or a *normal space*, if X is a T_1-space an for every pair of disjoint closed subsets $A, B \subset X$ there exist open sets $U_1, U_2 \subset X$ such that $A \subset U_1$, $B \subset U_2$ and $U_1 \cap U_2 = \emptyset$.
6. T_5-*space*, or a *completely normal space*, if X is a T_1-space and for every pair of subsets A and B of X such that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ there exists open sets $U_1, U_2 \subset X$ such that $A \subset U_1$, $B \subset U_2$ and $U_1 \cap U_2 = \emptyset$.

All topological spaces in this chapter are supposed to be Hausdorff.

2 Lindelöf Spaces

It is well-known that a regular topological space X is a *Lindelöf space*, or has *the Lindelöf property*, if every open cover of X has a countable subcover. In particular, every compact space is a Lindelöf space. This is a descriptive well-known property that can be found along the bibliography thoroughly. According to R. Engelking [14], the notion of a Lindelöf space was introduced by Alexandroff and Urysohn in [1], although the property was named after Lindelöf [23] who proved in 1903 that any open covering of a subset F of $\mathbb{R}^n$ contains a countable subcovering.

The existence of a countable number of open sets is a level immediately close to the notion of compact space, in which a finite number of open sets are enough to cover it, and as we will see the number of open sets to cover a topological space is enough to establish bounds about other cardinal functions.

Basic properties related with axioms of separability follow. Thus,

Proposition 2.1 (Theorem 3.8.1 [14]) *Every regular second countable space is a Lindelöf space.*

The converse does not hold in general.

Example 2.1 Sorgenfrey line is a Lindelöf space which is not second countable.

Proof On the set of the real numbers X it is considered the right half-open interval topology, it means that τ is the family of all sets of the form $[a, b)$, where $a, b \in X$. The Sorgenfrey line $\mathscr{S} := (X, \tau)$ is a Lindelöf completely normal space which is not second countable since if $S = \{[x_i, y_i) : i \in \mathbb{Z}^+\}$ is a countable set of open sets, then there exists $a \in X$ such that $a \neq x_i$ for each $i \in \mathbb{Z}^+$, thus, for any $b > a$, we have that $[a, b)$ is an open set such that is not a union of elements of S [36, Counterexample 84, pp. 103–105]. Observe that Sorgenfrey line is not σ-compact since each compact set is countable and the real numbers is not countable. (X, τ) is Lindelöf. Let $\{U_\alpha\}$ be an open covering of X. Let $\{\text{int}(U_\alpha)\}$ be the family obtained considering the interior of U_α in the usual topology of the real numbers. Then $P = \cup_\alpha \text{int}(U_\alpha)$ is Lindelöf and there exists a countable subfamily such that $P = \cup_{n\in\mathbb{N}} \text{int}(U_{\alpha_n}) = \cup_\alpha \text{int}(U_\alpha)$. Let $A := X \setminus P$, then A is a countable set which can be covered by a countable subfamily of $\{U_\alpha\}$ and a countable subcovering can be obtained from the original one.

On the contrary, Lindelöf property implies T_4-space as it is stated in the following proposition. The proof can be found in [14].

Proposition 2.2 (Theorem 3.8.2 [14]) *Every Lindelöf space is normal.*

In the case of regular spaces to be Lindelöf is close to have the *countable intersection property*, namely,

Proposition 2.3 (Theorem 3.8.3 [14]) *A regular space X is Lindelöf if and only if every family of closed subsets of X which has the countable intersection, that is, each family $\mathscr{F}$ of closed sets such that for each countable subfamily $\mathscr{F}' \subset \mathscr{F}$ holds that $\cap_{F\in\mathscr{F}'} F \neq \emptyset$, has non-empty intersection.*

In the frame of locally compact space the Lindelöf property is characterized in the following proposition, see [14, Exercise 3.8.C, p. 195].

Proposition 2.4 *Let X be a locally compact space, that is, for every $x \in X$ there exists a neighbourhood U of the point x such that $\overline{U}$ is a compact of subspace of X. Then the following sentences are equivalent:*

1. *The space X has the Lindelöf property.*
2. *The space X is σ-compact.*
3. *There exists a sequence $A_1, A_2, \ldots,$ of compact subspaces of the space X such that $A_i \subset int(A_{i+1})$ and $X = \bigcup_{i=1}^{\infty} A_i$.*

Unless otherwise was indicated, we assume that all topological spaces are non-empty, completely regular and Hausdorff.

Respect to the stability properties, the Lindelöf property has a good behaviour respect to some operations but not all.

We have that every closed subset subspace of a Lindelöf space is a Lindelöf space. Every regular space which can be represented as a countable union of Lindelöf subspaces is Lindelöf. The continuous image of a Lindelöf space X onto a regular space Y is a Lindelöf space. Inverse images of a Lindelöf space under perfect mappings are also Lindelöf. In fact, inverse images of closed mappings with Lindelöf fibers are again Lindelöf. More about the "goodness" of the Lindelöf property comes from realcompactness. A topological space is *realcompact* if and only if it is homeomorphic to a closed subspace of a power $\mathbb{R}^{\mathfrak{m}}$ of the real line, for a cardinal number $\mathfrak{m}$, and it is known, that every Lindelöf space is realcompact [14, Theorem 3.11.12]. The *realcompactification* of a topological space X is denoted by νX, whereas the Stone-Čech compactification of X, is denoted by βX [14, Sect. 3.6]. As a good property we have that every open cover of a Lindelöf space has a *locally finite open refinement* [14, Theorem 3.8.11].

On the other hand, the Cartesian product of two Lindelöf spaces is not in general a Lindelöf space, considering again the Sorgenfrey line $\mathscr{S}$, then $\mathscr{S} \times \mathscr{S}$ is not Lindelöf although $\mathscr{S}$ is it. In [35, p. 632] it is proved that $\mathscr{S} \times \mathscr{S}$ is not normal and hence it is not Lindelöf.

2.1 The Lindelöf Number

Until now we have summarized some properties of Lindelöf spaces respect to separability axioms, countably axioms or stability properties, in a general frame. The following lines will be occupied on the relationship between Lindelöf property and other cardinal functions.

We have considered in the beginning some cardinal functions as the weight and the network weight but a more formal definition is needed about what a cardinal function is. Recall that a cardinal function is a function that assigns to every topological space an infinite cardinal number which is invariant by homeomorphisms, it means that if X and Y are homeomorphic, the cardinal function of X is equal to the cardinal function of Y. In topology the descriptive properties of the spaces are mostly determined by different cardinal functions. The generalization of the notion of Lindelöf space

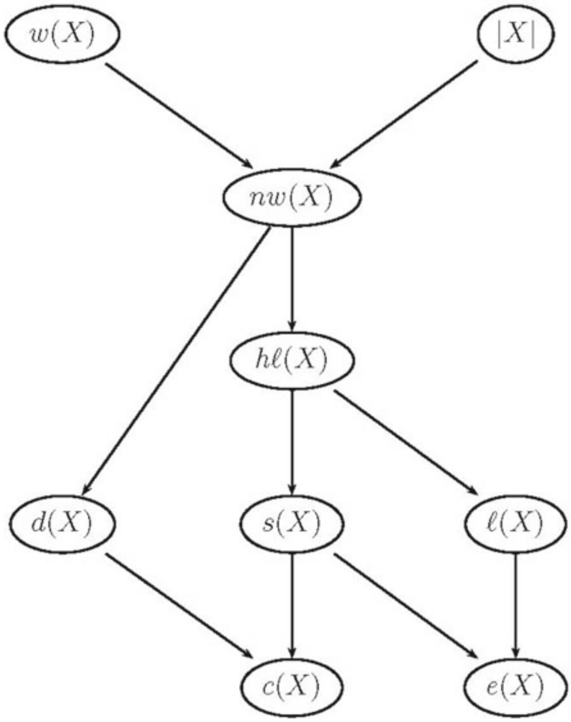

Fig. 1 General relationship between general cardinal functions. The arrow "→" means greater than or equal to. See [14] and [19, Sect. 3] for more details

gives us a new cardinal function defined for each topological space X. The *Lindelöf number*, denoted by $\ell(X)$, is the smallest cardinal number κ such that for every open cover there exists a subcovering of cardinality $\leq \kappa$.

Other cardinal functions are the following. The *density* of X, $d(X)$ is the smallest cardinal number of a set $S \subset X$, such that $\overline{S} = X$. The *Souslin number*, or *cellularity* of a topological space X, $c(X)$ is defined as the smallest cardinal number $\mathfrak{m}$ such that the cardinality of a family of pairwise disjoint non-empty open subsets of X is not greater than $\mathfrak{m}$. The *spread* of X, $s(X)$, is the smallest cardinal number $\mathfrak{m}$ such that the cardinality of every discrete subspace is not greater than $\mathfrak{m}$. While the *extent* of X, $e(X)$, is the smallest cardinal number $\mathfrak{m}$ such that the cardinality of a closed and discrete subset of X is not greater than $\mathfrak{m}$ [19, Sect. 3]. It is clear that $e(X) \leq \ell(X)$ and $e(X) \leq s(X)$. In other sense, as we have previously mentioned each closed subspace of a Lindelöf space is Lindelöf but the same does not occur for open subspaces, hence the *hereditarily Lindelöf number of* X, is defined as $h\ell(X) = \sup\{\ell(Y) : Y \subset X\}$.

Observe in Fig. 1 that the density character and the Lindelöf number are not closely related. The *Niemytzki plane* is an example of a separable space which is not a Lindelöf space [36, Example 82, p. 100]. Let $L = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. Let $L_1 = \{(x, y) : y = 0\}$ the real axis and $L_2 = L \setminus L_1$. In L_2, the topology τ is the Euclidean topology and τ^* is the topology in L generated by τ and all the sets of the form $\{(x, 0)\} \cup D$, where D is an open disc in L_2 which is tangent to L_1 at the point $(x, 0)$. The space (L, τ^2) is called the Niemytzki plane.

The rest of the results included in Fig. 1 are classical and can be found in [19, Sect. 3].

3 Lindelöf Σ-Spaces

A subclass of Lindelöf spaces is the class of Lindelöf Σ-spaces. Σ-spaces were introduced by K. Nagami in [25]. This class of spaces has been widely used not only in Topology, see [41] but also in Functional Analysis in which it is called the class of countably K-determined spaces [11, 12, 17, 37].

3.1 Definition and First Properties

The categorical definition of Lindelöf Σ-spaces can be found in [2, p. 6].

Definition 3.1 The class of Lindelöf Σ-spaces is the smallest class of spaces containing all compacta, all spaces with a countable basis and closed under the following operations: finite products, closed subspaces and continuous images.

This definition gives us the first difference respect to the Lindelöf property, namely, the finite product of Lindelöf Σ-spaces is again a Lindelöf Σ-space, although more can be done, since the countable product of Lindelöf Σ-spaces is a Lindelöf Σ-space, see [41, Proposition 3]. Nevertheless, the categorical definition is not operative to work with it.

Following M. Talagrand [37], we use the notion of upper semicontinuous map.

Definition 3.2 Let X and Y be topological spaces. A multivalued map $\phi : X \to 2^Y$ is said to be *upper semicontinuous* in $x_0 \in X$ if $\phi(x_0)$ is not empty and for each open set V in Y with $\phi(x_0) \subset V$ there exists an open set U of x_0 such that $\phi(U) \subset V$. A multivalued map ϕ is said to be *upper semicontinuous* if it is upper semicontinuous for each point in X. We will say that a multivalued map $\phi : X \to 2^Y$ is *usco* if ϕ is upper semicontinuous and the set $\phi(x)$ is compact for each $x \in X$.

The reader can find more information about usco maps in [9] and references therein.

The number of equivalent definitions for Lindelöf Σ-space has increased because the different situations in which it appears. In [41, Theorem 1], some equivalent definitions of Lindelöf Σ-space have been summarized.

Proposition 3.1 (Theorem 1 [41]) *The following conditions are equivalent for a topological space X:*

1. *X is a Lindelöf Σ-space;*

2. *there exist spaces K compact and M second countable such that X is a continuous image of a closed subspace of $K \times M$;*
3. *there exists an usco map $\phi : M \to 2^X$, where M is a second countable space and $\cup\{\phi(x) : x \in M\} = X$;*
4. *there exits a compact cover $\mathscr{C}$ of the space X such that some countable family $\mathscr{N}$ of subsets of X is a network mod($\mathscr{C}$) in the sense that, for any $C \in \mathscr{C}$ and any $U \in \tau(X)$ with $C \subset U$ there is $N \in \mathscr{N}$ such that $C \subset N \subset U$;*
5. *there exists a compact cover $\mathscr{C}$ of the space X such that some countable family $\mathscr{Q}$ of closed subsets of X is a network mod($\mathscr{C}$);*
6. *there exists a countable family $\mathscr{F}$ of compact subsets of βX such that $\mathscr{F}$ separates X from $\beta X \setminus X$ in the sense that, for any $x \in X$ and $y \in \beta X \setminus X$ there exists $F \in \mathscr{F}$ for which $x \in F$ and $y \notin F$;*
7. *there exists a compactification bX of the space X and a countable family $\mathscr{K}$ of compact subsets of bX which separates X from $bX \setminus X$;*
8. *there exists a space Y such that $X \subset Y$ and, for some countable family $\mathscr{K}$ of compact subsets of Y, we have $X \subset \cup\mathscr{K}$ and $\mathscr{K}$ separates X from $Y \setminus X$.*

3.2 Generalizing Lindelöf Σ-Spaces

After characterization of Lindelöf Σ-space using usco maps, the cardinal functions *number of K-determination*, $\ell\Sigma(X)$ and *Nagami number*, $Nag(X)$, make sense.

Definition 3.3 (*Definition 2* [8]) Let X be a topological space.

(i) The number of *K-determination* of X, $\ell\Sigma(X)$, is defined as the smallest cardinal number $\mathfrak{m}$ for which there are a metric space (M, d) of weight $\mathfrak{m}$ and an usco map $\phi : M \to 2^X$ such that $X = \bigcup\{\phi(x) : x \in M\}$.

(ii) The number of *Nagami* of X, $Nag(X)$, is defined as the smallest cardinal number $\mathfrak{m}$ for which there are a topological space Y of weight $\mathfrak{m}$ and an usco map $\phi : Y \to 2^X$ such that $X = \bigcup\{\phi(y) : y \in Y\}$.

The following characterizations hold.

Proposition 3.2 (Proposition 6 [8]) *Let X be topological space and $\mathfrak{m}$ a cardinal number. The following statements are equivalent:*

1. *$Nag(X) \leq \mathfrak{m}$ (resp. $\ell\Sigma(X) \leq \mathfrak{m}$);*
2. *there is a family of closed sets $\{A_i : i \in \mathfrak{m}\}$ in βX, such that for every $x \in X$ there is a set $J \subset \mathfrak{m}$ (resp. with $|J| \leq \omega$) such that $x \in \bigcap_{i \in J} A_i \subset X$.*
3. *there exists a topological (metric) space Y such that $w(Y) \leq \mathfrak{m}$ and $\phi : Y \to 2^X$ an usco map such that $X = \cup\{\phi(y) : y \in Y\}$.*

Observe that $Nag(X) \leq \ell\Sigma(X)$ and $\ell\Sigma(X) \leq \omega$ implies that X is a Lindelöf Σ-space. Both notions are different, in [8, Example 9] is given an example of a space $\mathbb{Y}$ such that $Nag(\mathbb{Y}) \leq w(\mathbb{Y}) < \ell\Sigma(\mathbb{Y})$, [8, Proposition 10]. Figure 2 adds to Fig. 1

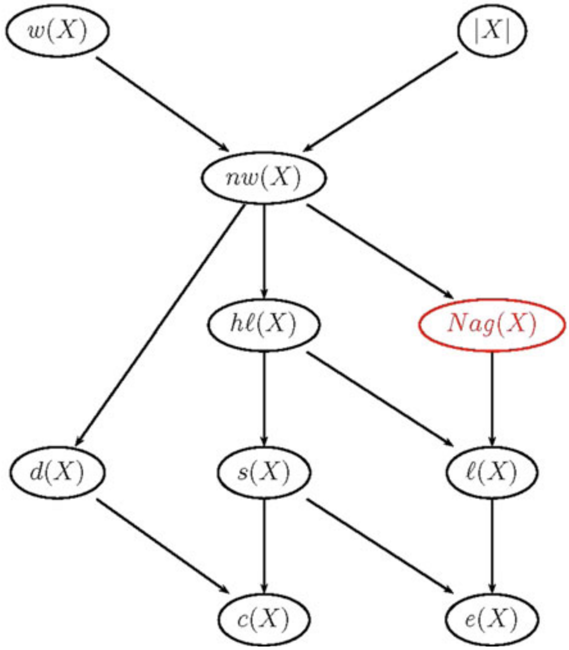

Fig. 2 Relationships between general cardinal functions for a completely regular topological space X including Nagami number. The arrow "→" means $\geq$

the cardinal function $Nag(X)$ and its relationships with other cardinal functions. Thus, it is known that $Nag(X) \leq nw(X)$ for X a completely regular space, see [8, Corollary 27]. In the class of infinite metric spaces we have that

$$w(X) = \ell(X) = d(X) = \ell\Sigma(X) = Nag(X).$$

When we consider $\aleph$-spaces the relationships between cardinal functions also allow us to have more information. The class of $\aleph$-spaces was introduced by P. O'Meara in [28]. A topological space X is called an $\aleph$-*space* if X is regular and has a σ-locally finite k-network. A family $\mathscr{F}$ of subsets of X is called a k-*network* in X, if whenever $K \subset U$ with K compact and U open in X, then $K \subset \cup\mathscr{F}' \subset U$ for some finite family $\mathscr{F}' \subset \mathscr{F}$. Because the regularity of the space, the collection of subsets which is a σ-locally finite k-network can be chosen to consist of closed sets. In the class of $\aleph$-spaces, $\ell(X) = Nag(X) = \ell\Sigma(X)$.

3.3 Lindelöf Σ-Spaces in C_p-Theory

The attempt to collect all the properties even in the particular case of Lindelöf Σ-spaces is not an easy task, because the large quantity of results related, see [41]. Thus, we will show up only some results that give us a general knowledge about the behavior of Lindelöf Σ-spaces. Let X be a topological space and $C_p(X)$ stands for the space of real-valued continuous functions endowed with the pointwise convergence topology. In this section we focus on how topological properties of both spaces are related in the framework of Lindelöf Σ-spaces, since there is a special relationship between properties of X and $C_p(X)$ when Lindelöf Σ-property is involved.

Additional notation and definitions are needed and they can be found in [2]. The *tightness* of a point x in a topological space X, $t(x, X)$, is the smallest infinite cardinal number $\mathfrak{m}$ such that for any $x \in \overline{A}$, there exists $B \subset A$ such that $|B| \leq \mathfrak{m}$ and $x \in \overline{B}$. The *tightness of a topological space* X, $t(X)$, is the supremum of all $t(x, X)$ for $x \in X$.

The following definition can be found in [2, Sect. 0.2, p. 5].

Definition 3.4 Let X be a topological space, we called $iw(X)$ the smallest cardinal $\mathfrak{m}$ for which there exist a topological space Y with $w(Y) \leq \mathfrak{m}$ and a one-to-one continuous map onto $f : X \to Y$.

A space X is said to be $\mathfrak{m}$-*stable* if for every continuous image Y of X if $iw(Y) \leq \mathfrak{m}$ then $nw(Y) \leq \mathfrak{m}$. The space X is *stable* if it is $\mathfrak{m}$-stable for any infinite cardinal number $\mathfrak{m}$. The following results get us a first example of what we mean about the relationship between X and $C_p(X)$ in the frame of Lindelöf Σ-spaces.

Theorem 3.1 (Theorem II.6.21 [2]) *Every Lindelöf Σ-space is stable.*

As a consequence of previous theorem the product of an arbitrary family of Lindelöf Σ-spaces is stable [2, Corollary II.6.27]. Moreover, a space is stable if and only if $C_p(C_p(X))$ is stable, [2, Corollary II.6.11]. Following [39], let $C_{p,0}(X) = X$ and $C_{p,n+1}(X) = C_p(C_{p,n}(X))$ for each natural number $n \in \mathbb{N}$. The following proposition holds.

Proposition 3.3 (Corollary II.6.32 [2]) *Let X be a Lindelöf Σ-space then $C_{p,2n}(X)$ is stable for any $n \in \mathbb{N}$.*

3.3.1 The Iterative Process

In [41] V. Tkachuck gave a description of all possible distribution of the Lindelöf Σ-property in the iterated spaces $C_{p,n}(X)$. Only the following cases can occur:

Proposition 3.4 (Corollary 2.10 [41]) *Only the following distributions of the Lindelöf Σ-property in iterated function spaces are possible:*

1. *$C_{p,n+1}(X)$ is a Lindelöf Σ-space for every $n \in \mathbb{N}$;*
2. *$C_{p,n+1}(X)$ is a Lindelöf Σ-space only for odd $n \in \mathbb{N}$;*
3. *$C_{p,n+1}(X)$ is a Lindelöf Σ-space only for even $n \in \mathbb{N}$;*
4. *for any $n \in \mathbb{N}$ the space $C_{p,n+1}(X)$ is not a Lindelöf Σ-space.*

An example of a non-Lindelöf space X such that $C_{p,2n+1}(X)$ is Lindelöf Σ-space for every $n \in \mathbb{N}$ but $C_{p,2n}(X)$ is not Lindelöf and a space Y such that $C_{p,2n}(Y)$ is a Lindelöf Σ-space for every $n \in \mathbb{N}$ and $C_{p,2n+1}(Y)$ is not Lindelöf are shown up in [41, Examples 2.9].

Previously, particular results in the framework of compact spaces had been obtained. The following definitions are well-known. A compact subset of a Banach space in the weak topology is called *Eberlein compact*. A compact space X is called *Gul'ko compact* if $C_p(X)$ is a Lindelöf Σ-space. Finally, a compact space is said to be a *Corson compact* if it can be embedded in the subspace of the product $\mathbb{R}^{\mathfrak{m}}$ of the real line consisting of functions vanishing at all but countably many points for an infinite cardinal number $\mathfrak{m}$ [2, p. 134].

The behaviour of iterated spaces for Lindelöf Σ-spaces have been widely studied. Gul'ko proved in [18] that for any Eberlein compact X the iterated function spaces $C_{p,n}(X)$, $n \in \mathbb{N}$, are Lindelöf. Sipachova [31] proved that $C_{p,n}(X)$ is Lindelöf Σ-space for any $n \in \mathbb{N}$ whenever X is an Eberlein compact space. On the other hand, Okunev [27] proved that if X and $C_p(X)$ are Lindelöf Σ-spaces then $C_{p,n}(X)$ is a Lindelöf Σ-space for each $n \in \mathbb{N}$. In general, when X is a Lindelöf Σ-space such that $X \subset C_p(Y)$, the following result is known.

Proposition 3.5 (Theorem 2.12 [27]) *Let X and Y Lindelöf Σ-spaces such that $X \subset C_p(Y)$, then $C_{p,n}(X)$ is a Lindelöf Σ-space for any $n \in \mathbb{N}$.*

More in this sense,

Proposition 3.6 (Theorem 4.3 [27]) *Let X be a Gul'ko compact space and K be a compact subspace of $C_{p,n}(X)$ for some $n \in \mathbb{N}$, then K is a Gul'ko compact space.*

A generalization of the previous result is the following one.

Theorem 3.2 (Theorem 4.4 [27]) *Let K be a compact subspace of $C_p(X)$ such that there exists a Lindelöf Σ-space Z such that $C_p(X) \subset Z$ then K is a Gul'ko compact space.*

The following result is a characterization of Gul'ko compact spaces.

Theorem 3.3 (Theorem 4.7 [27]) *Let X be a compact space. Then the following conditions are equivalent:*

1. *X is a Gul'ko compact space;*
2. *$C_{p,n}(X)$ is a Lindelöf Σ-space for some $n \in \mathbb{N}$;*
3. *$C_{p,n}(X)$ is a Lindelöf Σ-space for any $n \in \mathbb{N}$.*

This Theorem links to the following one proved by Sokolov [33, Corollary 2].

Proposition 3.7 *If X is a Corson compact space, then $C_{p,n}(X)$ is Lindelöf for each $n \in \mathbb{N}$.*

Similarly, Gul'ko [44, Problem 27, p. 610] conjectured that the Lindelöf property of all iterated continuous spaces characterizes Corson compact, nevertheless, Sokolov [34, Theorem 2.1] gave an example of a compact space X whose iterated continuous function spaces $C_{p,n}(X)$ for $n \in \mathbb{N}$ are Lindelöf but X is not a Corson compact space.

More similarities follow. If X is a Corson (Gul'ko) compact space and $C_{p,n}(X)$ is homeomorphic to $C_{p,n}(Y)$, for some $n \in \mathbb{N}$, then Y is Corson (Gul'ko). In 2018, see [6, Sect. 3] for details, it has been proved the same result for Eberlein compact space.

Recently (2017), Ferrando, Kąkol and López-Pellicer have characterized in [16] Gul'ko compact spaces considering the topology in $C(X)$ for σ_Y, where Y is a subset that separates the functions of $C(X)$ and σ_Y is the weak topology $\sigma(C(X), span(Y))$.

Theorem 3.4 (Theorem 4.1 [16]) *Let X be a compact space and Y be a G_δ-dense subspace. Then X is a Gul'ko compact space if and only if $(C(X), \sigma_Y)$ is a Lindelöf Σ-space.*

3.3.2 Σ_s-Products

The concept of Σ_s-product was used by Sokolov [32, Theorem 8] in order to give a different characterization of Gul'ko compact spaces. The following definitions are needed, see [30, Definition 3.1].

Definition 3.5 Let a be a point in the product space $X = \prod_{t \in T} X_t$.

1. The *support of* x, denoted by $supp(x)$, is the set $\{t \in T : x(t) \neq a(t)\}$.
2. The Σ-product of the family $\{X_t\}_{t \in T}$ centered at the point a, is the subspace of X given by
$$\Sigma(X, a) = \{x \in X : |supp(x)| \leq \omega\}.$$
3. The σ-product of the family $\{X_t\}_{t \in T}$ centered at the point a, is the subspace of X given by
$$\sigma(X, a) = \{x \in X : |supp(x)| < \omega\}.$$
4. Let s be a countable family of subsets of T and $s_x = \{E \in s : |supp(x) \cap E| < \omega\} \subseteq s$ for $x \in X$, then the Σ_s-product of the family $\{X_t\}_{t \in T}$ centered at the point a with respect to the set s is the subspace of X given by
$$\Sigma_s(X, a) = \{x \in X : T = \cup s_x\}.$$

If the point a in consideration is not relevant we will write $\Sigma(X)$, $\sigma(x)$ and $\Sigma_s(X)$.

Now the following characterization can be introduced.

Proposition 3.8 (Theorem 8 [32]) *A compact space X is Gul'ko if and only if X embeds into a Σ_s-product of real lines.*

An extension of the previous result is debt to Casarrubias-Segura et al. [6] in which it is proved that if X is a Lindelöf Σ-space contained in a Σ_s-product of real lines then $C_p(X)$ is a Lindelöf Σ-space.

More results related to Lindelöf Σ-spaces are known. Thus,

Proposition 3.9 (Theorem 3.2 [42]) *Every Σ_s-product of compact spaces is a Lindelöf Σ-space.*

Recently, in 2018,

Proposition 3.10 (Theorem 4.1 [6]) *If $X = \prod_{t \in T} X_t$ is a product, and every σ-product in X is a Lindelöf Σ-space, then each Σ_s-product in X is a Lindelöf Σ-space.*

Proposition 3.11 (Theorem 4.5 [6]) *Every Σ_s-product of K-analytic spaces is a Lindelöf Σ-space.*

In [6, Corollary 4.2] it has been proved that Proposition 3.9 holds for every Σ_s-product of σ-compact spaces.

Different questions remain open in this framework, thus, in [6, Questions 5.6 and 5.7] the following questions are posed.

1. Let X be a Lindelöf Σ-space which admits a condensation in a Σ_s-product of real lines. Must $C_p(X)$ be a Lindelöf Σ-space?
2. Let X be a Lindelöf subspace of a Σ-product (or Σ_s-product) of real lines. Must $C_p(X)$ be Lindelöf?

Remind that a map $f : X \to Y$ is a *condensation* if it is a continuous bijection; in this case we say that X condenses *onto* Y. If X condenses onto a subspace of Y, we say that X condenses *into* Y (Sect. 2 in [6]).

3.3.3 Cardinal Inequalities

In this section we focus our interest on the relationship between X and $C_p(X)$ involving *different* cardinal functions. In [8] can be found some of them in which the number of K-determination and the Nagami number appear. Thus,

Proposition 3.12 (Proposition 16 [8]) *Let X be a topological space, then $t(C_p(X)) \leq \ell\Sigma(X)$. In particular, if X is a Lindelöf Σ-space, then $t(C_p(X))$ is countable.*

Involving the network of the space the following results give us information in the particular case of the Lindelöf Σ-spaces. Classical results of Arkhangel'skii follow.

Proposition 3.13 (Theorem 10 [3]) *Let X be a topological space such that $C_p(X)$ is a Lindelöf Σ-space and the spread of $C_p(X)$ is countable then $nw(X)$ is countable (X is cosmic).*

Proposition 3.14 (Proposition 12 [3]) *Let X be a topological space such that the spread of X is countable and $C_p(X)$ is a Lindelöf Σ-space then X is a Lindelöf Σ-space.*

Proposition 3.15 (Theorem 13 [3]) *Let X be a topological space such that the spread of $X \times X$ is countable and $C_p(X)$ is a Lindelöf Σ-space then X has a countable network (X is cosmic).*

More conditions to obtain X cosmic were obtained by Tkachuck.

Proposition 3.16 (Theorem 3.6 [40]) *Let X be a topological space such that $C_p(X)$ is a Lindelöf Σ-space and $s(X)$ is countable then $nw(X)$ is countable (X is cosmic).*

Proposition 3.17 (Theorem 3.30 [43]) *Let X be a topological space such that $C_p(C_p(X))$ is a Lindelöf Σ-space and $s(X)$ is countable then $nw(X)$ is countable (X is cosmic).*

When we consider subspaces as our goal, then the following definitions are needed. Let $\mathfrak{m}$ be an infinite cardinal number, a topological space X is said to be $\mathfrak{m}$-*monolithic* if for each $A \subset X$ such that $|A| \le \mathfrak{m}$ then $nw(\overline{A}) \le \mathfrak{m}$. A space X is called *strongly* $\mathfrak{m}$-*monolithic* if for every $Y \subset X$ with $|Y| \le \mathfrak{m}$, the weight of the space $\overline{Y}$ does not exceed $\mathfrak{m}$. A topological space X is said to be *monolithic* if X is $\mathfrak{m}$-monolithic for each infinite cardinal number $\mathfrak{m}$. Thus, if X is a monolithic space, then for each subspace $Y \subset X$, we have that $d(Y) = nw(Y)$, [2, p. 76].

The following result can be found in [8].

Proposition 3.18 (Proposition 17 [8]) *Let X be a topological space and $H \subset C(X)$ τ_p-compact, then H is strongly $\ell\Sigma(X)$-monolithic.*

The corollary which follows from the previous proposition is also an immediate consequence of [2, Theorem II.6.8].

Corollary 3.1 *Let X be a Lindelöf Σ-space and $H \subset C_p(X)$ then*

$$nw(H) = d(H).$$

In particular, if H is τ_p-compact subspace then H is metrizable, see [10, Corollary 1.2].

Recent work has established accurate boundedness of the weight in Lindelöf Σ-spaces. Tkachenko [38] has proved the following result.

Theorem 3.5 (Theorem 2.1 [38]) *Let X be a completely regular space then $w(X) \le |C(X)| \le nw(X)^{Nag(X)}$ holds.*

In [6, Theorem 8.2] this result has been proved with different arguments proving that the inequality $w(X) \le nw(X)^{Nag(X)}$ holds for regular spaces. In particular for a Lindelöf Σ-space X such that $nw(X) \le \mathfrak{c}$, then $|C(X)| \le \mathfrak{c}$ and $w(X) \le \mathfrak{c}$. Even more it is established,

Theorem 3.6 (Theorem 2.3 [38]) *Let Y be a dense subspace of a completely regular space X, then $|C(X)| \leq nw(Y)^{Nag(Y)}$ and $w(X) \leq nw(Y)^{Nag(Y)}$.*

Respect to the hereditarily numbers the following properties hold.

Theorem 3.7 (Theorem 5.2 [24]) *Let X be a topological space then:*

1. $h\ell(X) \leq \max\{Nag(C_p(X)), s(X)\}$.
2. $h\ell(C_p(X)) \leq \max\{Nag(X), s(C_p(X))\}$.

The corollary that follows can be found in [3].

Corollary 3.2 (Proposition 9 [3]) *If X is a Lindelöf Σ-space and the spread of $C_p(X)$ is countable, then $C_p(X)$ is hereditarily Lindelöf, and $X \times X$ is hereditarily separable.*

Finally, the classical theorem of Baturov [4] states that

Theorem 3.8 (Theorem III.6.1 [2]) *Let X be a Lindelöf Σ-space and $Y \subset C_p(X)$ a subspace, then $\ell(Y) = e(Y)$.*

If X is a countably compact space Baturov's theorem fails. Buzyakova [5, Example 3.6] showed an example of a countably compact space X such that $e(C_p(X)) < \ell(C_p(X))$.

4 Characterizing When νX Is a Lindelöf Σ-Space

Realcompactification of a space is related with the space $C_p(X)$ and as an intermediate step in order to get any of the previous results. In fact when dual spaces are considered, *envelopes* play an important role.

Theorem 4.1 (Theorem 3.5 [27]) *Let X be a topological space. Then νX is a Lindelöf Σ-space if and only if there exists a Lindelöf Σ-space Z such that $C_p(X) \subset Z \subset \mathbb{R}^X$.*

As a consequence of the previous result we have that

Proposition 4.1 (Corollary 3.6 [27]) *Let $\nu C_p(X)$ be a Lindelöf Σ-space, then νX is a Lindelöf Σ-space.*

Theorem 4.2 (Theorem 3.5 [26]) *Let $C_p(X)$ be a Lindelöf Σ-space, then νX is a Lindelöf Σ-space.*

Theorem 4.3 (Theorem 2.3 [39]) *Let $C_p(X)$ be a Lindelöf Σ-space, then $C_p(\nu X)$ is a Lindelöf Σ-space.*

Now, it is clear that

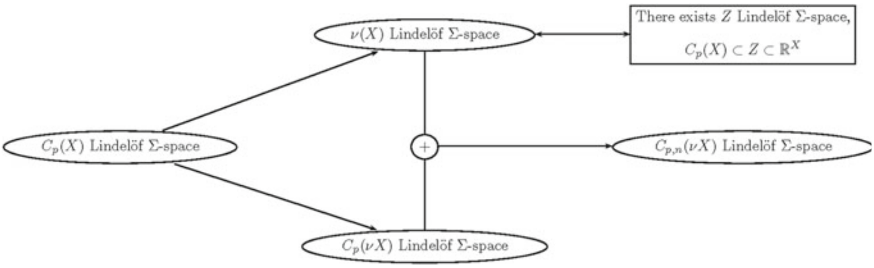

Fig. 3 Relationships between some notions that summarizes some properties involving realcompactification of a space. $A \to B$ means A implies B

Corollary 4.1 (Corollary 2.4 [39]) *If $C_p(X)$ is a Lindelöf Σ-space, then $C_{p,n}(\nu X)$ is a Lindelöf Σ-space for each $n \in \mathbb{N}$.*

Figure 3 summarizes the previous results.

Completely regular spaces X whose realcompactification νX is a Lindelöf Σ-space were studied by Ferrando in [15]. The characterization of topological spaces whose realcompactification is a Lindelöf Σ-space was also considered in [21] where the notions of *strongly web-bounded space* and *web-bounding space* are involved.

Definition 4.1 (*Definition 3* [7]) A locally convex space X is *web-bounded* if there is a family $\{A_\alpha : \alpha \in \Omega\}$ of sets covering X for some nonempty $\Omega \subset \mathbb{N}^\mathbb{N}$ such that if $\alpha = (n_k)_k \in \Omega$ and $x_k \in C_{n_1,n_2,\ldots,n_k} := \bigcup\{A_\beta : \beta = (m_k)_k \in \Omega,\ m_j = n_j,\ j = 1, \ldots, k\}$ then $(x_k)_k$ is bounded.

Definition 4.2 (*p. 150* [29]) A space X is *strongly web-bounding* if there is a family $\{A_\alpha : \alpha \in \Omega\}$ of sets covering X for some nonempty $\Omega \subset \mathbb{N}^\mathbb{N}$ such that if $\alpha = (n_k)_k \in \Omega$ and $x_k \in C_{n_1,n_2,\ldots,n_k} := \bigcup\{A_\beta : \beta = (m_k)_k \in \Omega,\ m_j = n_j,\ j = 1, \ldots, k\}$ then $(x_k)_k$ is functionally bounded, that is, $f((x_k)_k) \subset \mathbb{R}$ is bounded for each continuous function $f : X \to \mathbb{R}$.

Characterization of the realcompactification of a space X which is also a Lindelöf Σ-space was given by Kąkol and López-Pellicer in [21] giving a description of a web-bounded structure in the original space.

Theorem 4.4 (Theorem 1.2 and Corollary 2.6 [21]) *Let X be a completely regular space then the following sentences are equivalent.*

1. *νX is a Lindelöf Σ-space;*
2. *X is strongly web-bounding;*
3. *$C_p(X)$ is web-bounded;*
4. *there exits a Lindelöf Σ-space Z such that $C_p(X) \subset Z \subset \mathbb{R}^X$.*

Regarding the question if this property is in some sense "hereditary" when realcompactification is involved we have the following result.

Proposition 4.2 (Theorem 8 [13]) *Let X and Y be spaces and $h : C_p(X) \to C_p(Y)$ a surjective map that takes bounded sequences to bounded sequences. If νX is a Lindelöf Σ-space, then νY is a Lindelöf Σ-space.*

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