A Fixed Point Theory Linked to the Zeros of the Partial Sums of the Riemann Zeta Function

In Honour of Manuel López-Pellicer

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Abstract For each $n > 2$ we consider the corresponding nth-partial sum of the Riemann zeta function $\zeta_n(z) := \sum_{j=1}^{n} j^{-z}$ and we introduce two real functions $f_n(c)$, $g_n(c)$, $c \in \mathbb{R}$, associated with the end-points of the interval of variation of the variable x of the analytic variety $|\zeta_n^*(z)| = p_{k_n}^{-c}$, where $\zeta_n^*(z) := \zeta_n(z) - p_{k_n}^{-z}$ and p_{k_n} is the last prime not exceeding n. The analysis of fixed point properties of f_n, g_n and the behavior of such functions allow us to explain the distribution of the real parts of the zeros of $\zeta_n(z)$. Furthermore, the fixed points of f_n, g_n characterize the set $\mathscr{P}^*$ of prime numbers greater than 2 and the set $\mathscr{C}^*$ of composite numbers greater than 2, proving in this way how close those functions from Arithmetic are. Finally, from the study of the graphs of f_n, g_n we deduce important properties about the set $R_{\zeta_n(z)} := \overline{\{\Re z : \zeta_n(z) = 0\}}$ and the bounds $a_{\zeta_n(z)} := \inf\{\Re z : \zeta_n(z) = 0\}$, $b_{\zeta_n(z)} := \sup\{\Re z : \zeta_n(z) = 0\}$ that define the critical strip $[a_{\zeta_n(z)}, b_{\zeta_n(z)}] \times \mathbb{R}$ where are located all the zeros of $\zeta_n(z)$.

Keywords Exponential polynomials · Zeros of the partial sums of the Riemann zeta function · Diophantine approximation

1 Introduction

Since the non-trivial zeros of the **Riemann zeta function** $\zeta(z)$, until now found, lie on the line $\Re z = 1/2$ (the assertion that all them are situated on that line is the **Riemann Hypothesis**) and the trivial ones are on the real axis (they are the negative even numbers [9, p. 8]), it seems that the zeros of $\zeta(z)$ are situated on those two perpendicular lines. However that is not so for the zeros of the partial sums $\zeta_n(z) := \sum_{j=1}^{n} j^{-z}$ of the series $\sum_{j=1}^{\infty} j^{-z}$ that defines the Riemann zeta function $\zeta(z)$ on the half-plane $\Re z > 1$. Indeed, except for $\zeta_2(z)$ whose zeros all are imaginary

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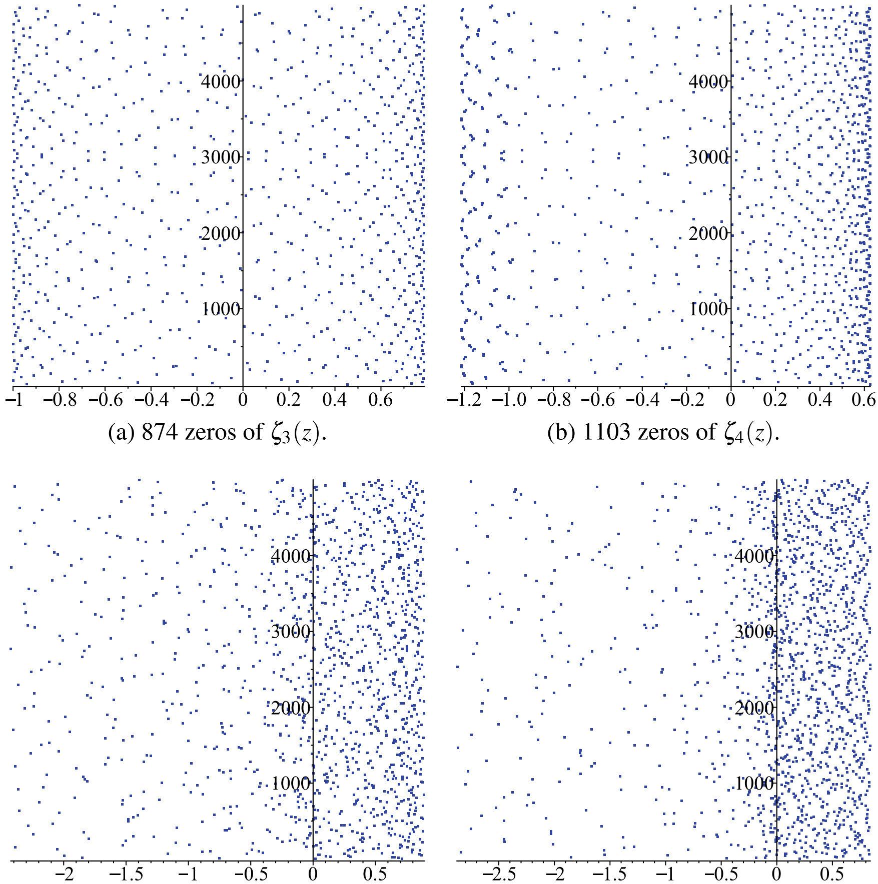


(a) 874 zeros of $\zeta_3(z)$.

(b) 1103 zeros of $\zeta_4(z)$.

(c) 1281 zeros of $\zeta_5(z)$.

(d) 1426 zeros of $\zeta_6(z)$.

Fig. 1 Graphs of the zeros of $\zeta_n(z)$ for some values of n, with $\Re z \in [-3, 1]$ and $\Im z \in [0, 5000]$

(it is immediate to check that the zeros of $\zeta_2(z)$ are $z_{2,j} = \frac{(2j+1)\pi i}{\log 2}$, $j \in \mathbb{Z}$), so aligned, the zeros of each $\zeta_n(z)$ for any $n > 2$ are dispersed in a vertical strip forming a sort of cloud, more or less uniform, that extends up, down and left as n increases, whereas at the right the cloud of zeros is upper bounded (essentially) by the line $\Re z = 1$ (see Fig. 1).

An explanation *grosso modo* why the zeros of the $\zeta_n(z)$'s are distributed of such a form is supported by the following facts:

(a) Any exponential polynomial (EP for short) of the form

$$P(z) := 1 + \sum_{j=1}^{N} a_j e^{-z\lambda_j}, \quad z \in \mathbb{C}, \quad a_j \in \mathbb{C} \setminus \{0\}, \quad 0 < \lambda_1 < \ldots < \lambda_N, \quad N \geq 1, \tag{1}$$

has zeros as a consequence of Hadamard's Factorization Theorem or from Pólya's Theorem [13, p. 71]. For $N = 1$, it is immediate that an EP of the form (1) has its zeros aligned. For $N > 1$, noticing that for any y,

$$\lim_{x \to +\infty} P(z) = \lim_{x \to -\infty} Q(z) = 1,$$

where $Q(z) := a_N^{-1} e^{z\lambda_N} P(z)$ (observe that $P(z)$ and $Q(z)$ have exactly the same zeros), it follows that the zeros of $P(z)$ are situated in a vertical strip. Therefore, for every EP $P(z)$ of the form (1), there exist two real numbers

$$a_{P(z)} := \inf\{\Re z : P(z) = 0\}, \quad b_{P(z)} := \sup\{\Re z : P(z) = 0\}, \tag{2}$$

that define an interval $[a_{P(z)}, b_{P(z)}]$, called *critical interval* associated with $P(z)$. Therefore the set $[a_{P(z)}, b_{P(z)}] \times \mathbb{R}$, called *critical strip* associated with $P(z)$, is the minimal vertical strip that contains all the zeros of $P(z)$.

It is immediate that any partial sum $\zeta_n(z) := \sum_{j=1}^{n} j^{-z}, n \geq 2$, is an EP of the form (1). Therefore the zeros of each $\zeta_n(z)$ are situated on its critical strip $[a_{\zeta_n(z)}, b_{\zeta_n(z)}] \times \mathbb{R}$ (a detailed proof on the existence of the zeros of $\zeta_n(z)$ and their distribution with respect to the imaginary axis can be found in [14, Prop. 1, 2, 3]). Regarding the bounds $a_{\zeta_n(z)}, b_{\zeta_n(z)}$, taking into account that all the zeros of $\zeta_2(z)$ lie on the imaginary axis, we get the property

$$a_{\zeta_2(z)} = b_{\zeta_2(z)} = 0; \quad a_{\zeta_n(z)} < 0 < b_{\zeta_n(z)}, \quad n > 2, \tag{3}$$

that will be proved below in Lemma 2.3, Part (ii). A much more precise estimation of such bounds is given by the formulas:

$$b_{\zeta_n(z)} = 1 + \left(\frac{4}{\pi} - 1 + o(1) \right) \frac{\log \log n}{\log n}, \quad n \to \infty, \tag{4}$$

obtained by Montgomery and Vaughan [12] in 2001, by completing a previous work of Montgomery [11] of 1983, and

$$a_{\zeta_n(z)} = -\frac{\log 2}{\log(\frac{n-1}{n-2})} + \Delta_n, \quad \limsup_{n \to \infty} |\Delta_n| \leq \log 2, \tag{5}$$

found by Mora [17] in 2015. Consequently, from (5) and (4), we have

$$\lim_{n \to \infty} a_{\zeta_n(z)} = -\infty, \quad \lim_{n \to \infty} b_{\zeta_n(z)} = 1,$$

what justifies the fact of the cloud of zeros of $\zeta_n(z)$ moves to the left as n increases but not to the right, where the cloud is upper bounded (essentially) by the line $\Re z = 1$ (it does not mean that some $\zeta_n(z)$ can have zeros with real part greater than 1; in fact, many works prove the existence of such zeros [10, 22, 23, 25], among others).

(b) Since the zeros of an analytic function are isolated, and all the $\zeta_n(z)$'s are entire functions, by taking into account the real parts of the zeros of each $\zeta_n(z)$ are bounded (the real parts are contained in the critical interval $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$ for every fixed n), their imaginary parts cannot be. Furthermore, as the coefficients of every $\zeta_n(z)$ are real, its zeros are conjugate. Consequently the zeros of the $\zeta_n(z)$'s are located up and down, symmetrically with respect to the real axis.

(c) From (3) we deduce that, for any $n > 2$, $\zeta_n(z)$ has zeros with positive and negative real parts.

With the aim to understand what law controls the distribution of the real projections of the zeros of $\zeta_n(z)$, we introduce a Fixed Point Theory focused on two real functions, f_n and g_n, for every $n > 2$. Firstly, such functions, by virtue of a recent result [19, Theorem 3], allow us to have an easy characterization of the sets

$$R_{\zeta_n(z)} := \overline{\{\Re z : \zeta_n(z) = 0\}}. \tag{6}$$

Secondly, among others relevant results deduced from the fixed point properties of f_n and g_n, we stress those that characterize some notable *arithmetic sets* such as $\mathscr{P}^*$ and $\mathscr{C}^*$, the set of primes greater than 2 and the set of composite numbers greater than 2, respectively. In this way, we show how close the arithmetic sets $\mathscr{P}^*$ and $\mathscr{C}^*$ from the law of the distribution of the zeros of the partial sums of the Riemann zeta function are. Furthermore, our point fixed theory proves the existence of a *minimal density interval* for each $\zeta_n(z)$, that is, a closed interval $[A_n, b_{\zeta_n(z)}]$, with $a_{\zeta_n(z)} \leq A_n < b_{\zeta_n(z)}$ contained in the set $R_{\zeta_n(z)}$, for any integer $n > 2$, which means that there is no vertical sub-trip contained in $[A_n, b_{\zeta_n(z)}] \times \mathbb{R}$ zero-free for $\zeta_n(z)$. Then, since it is always true that $R_{\zeta_n(z)} \subset [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$, when the bound A_n coincides with $a_{\zeta_n(z)}$ it follows that $R_{\zeta_n(z)} = [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$. In this case we will say that $\zeta_n(z)$ has a *maximum density interval*, and it is exactly the critical interval $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$. Finally, we will give a sufficient condition in terms of the quantity of fixed points of f_n for $\zeta_n(z)$ have a maximum density interval.

2 The Functions f_n and g_n

The functions f_n and g_n that we are going to introduce below, are directly linked to the interval of variation of the variable x of the Cartesian equation of an analytic variety associated with the nth-partial sum $\zeta_n(z) := \sum_{j=1}^{n} j^{-z}$, $n > 2$. First we consider the EP

$$\zeta_n^*(z) := \zeta_n(z) - p_{k_n}^{-z}, \quad n > 2, \tag{7}$$

where p_{k_n} is the last prime not exceeding n. The bounds $a_{\zeta_n^*(z)}, b_{\zeta_n^*(z)}$ defined in (2) corresponding to $\zeta_n^*(z)$ satisfy the following crucial property (for details see [16, Theorem 15]) :

$$a_{\zeta_n^*(z)} = b_{\zeta_n^*(z)} = 0, \quad \text{for } n = 3, 4; \quad a_{\zeta_n^*(z)} < 0 < b_{\zeta_n^*(z)}, \quad \text{for all } n > 4. \tag{8}$$

Now our objective is to analyse the behavior of the end-points of the interval of variation of the variable x of the analytic variety, or *level curve* [24, p. 121], of equation

$$|\zeta_n^*(z)| = p_{k_n}^{-c}, \quad n > 2, \quad c \in \mathbb{R}. \tag{9}$$

To do it, we square (9) and by using elementary formulas of trigonometry we obtain the Cartesian equation of $|\zeta_n^*(z)| = p_{k_n}^{-c}$, namely

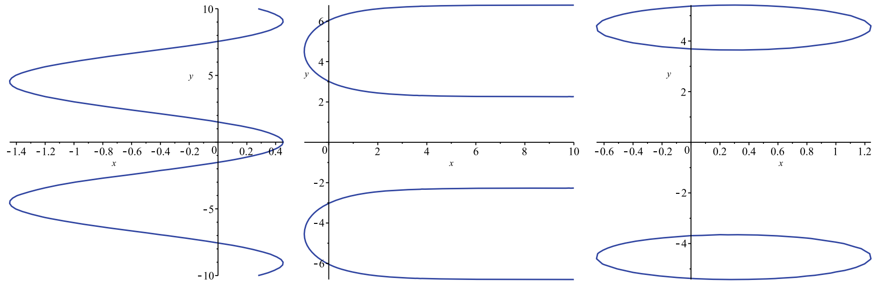
$$\begin{aligned} &\sum_{j=1, j\neq p_{k_n}}^{n} j^{-2x} + 2\cdot 1^{-x} \sum_{j=2, j\neq p_{k_n}}^{n} j^{-x}\cos(y\log(\frac{j}{1})) + \\ &2\cdot 2^{-x} \sum_{j=3, j\neq p_{k_n}}^{n} j^{-x}\cos(y\log(\frac{j}{2})) + \cdots + \\ &2(n-1)^{-x} \sum_{j=n, j\neq p_{k_n}}^{n} j^{-x}\cos(y\log(\frac{j}{n-1})) = p_{k_n}^{-2c}. \end{aligned} \tag{10}$$

It is immediate to see that for any value of y, the left-hand side of (10) tends to $+\infty$ as $x \to -\infty$. Then, as the right-hand side of (10) is a constant, the variation of x is always lower bounded by a number denoted by $a_{n,c}$. On the other hand, the limit of the left-hand side of (10) is 1 when $x \to +\infty$. Then, if $c \neq 0$, the variation of x is upper bounded by a number denoted by $b_{n,c}$. Therefore, fixed an integer $n > 2$, we have:

If $c \neq 0$, the variable x in the Eq. (10) varies on an open interval $(a_{n,c}, b_{n,c})$ satisfying the properties: (a) Given $x \in (a_{n,c}, b_{n,c})$, there is at least a point of the level curve $|\zeta_n^*(z)| = p_{k_n}^{-c}$ with abscissa x. Exceptionally $|\zeta_n^*(z)| = p_{k_n}^{-c}$ could have points of abscissas $a_{n,c}$, $b_{n,c}$. In this case we will say that $a_{n,c}$, $b_{n,c}$ are *accessible*. Otherwise the lines $x = a_{n,c}$, $x = b_{n,c}$ are asymptotes of the variety. (b) For $x < a_{n,c}$ or $x > b_{n,c}$ there is no point of the variety $|\zeta_n^*(z)| = p_{k_n}^{-c}$.

If $c = 0$, x varies on $(a_{n,0}, +\infty)$, so $b_{n,0}$ can be defined as $+\infty$, satisfying: (c) Given $x \in (a_{n,0}, +\infty)$, there is at least a point of the variety $|\zeta_n^*(z)| = 1$ with abscissa x. If there is a point of $|\zeta_n^*(z)| = 1$ with abscissa $a_{n,0}$, we will say that $a_{n,0}$ is accessible. Otherwise the line $x = a_{n,0}$ is an asymptote of the variety. (d) For $x < a_{n,0}$ there is no point of $|\zeta_n^*(z)| = 1$.

We show in Fig. 2 the varieties $|\zeta_n^*(z)| = p_{k_n}^{-c}$ for $n = 3$ and some values of c.



(a) Graph of $|\zeta_3^*(z)| = 3^{1/2}$. (b) Graph of $|\zeta_3^*(z)| = 1$. (c) Graph of $|\zeta_3^*(z)| = 3^{-1/2}$.

Fig. 2 Graphs of the varieties $|\zeta_3^*(z)| = 3^{-c}$ for some values of c

The end-points $a_{3,c}, b_{3,c}$ corresponding to the variety $|\zeta_3^*(z)| = p_{k_3}^{-c}$ can be easily determined by a completely similar way to those of the variety $|\zeta_3^*(-z)| = p_{k_3}^{c}$ (see [8, p. 49]). Each bound $a_{3,c}, b_{3,c}$ as a function of c is given by the formulas

$$a_{3,c} = -\frac{\log(1+3^{-c})}{\log 2}, \quad c \in \mathbb{R}; \quad b_{3,c} = \begin{cases} -\frac{\log(3^{-c}-1)}{\log 2}, & \text{if } c < 0 \\ -\frac{\log(1-3^{-c})}{\log 2}, & \text{if } c > 0 \end{cases}. \tag{11}$$

By virtue of above considerations (a), (b), (c), (d), and by using an elementary geometric reasoning, similar to that it was used to find the graphs of $|\zeta_n^*(-z)| = p_{k_n}^{c}$ (see [16, Proposition 8]), the graphs of the varieties $|\zeta_n^*(z)| = p_{k_n}^{-c}$ are described in the next result.

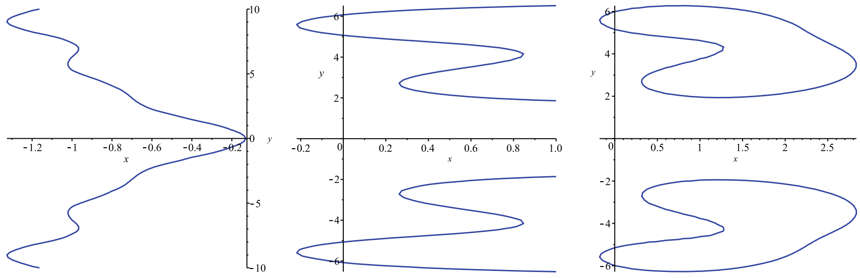
Proposition 2.1 *Fixed an integer $n > 2$, we have:*

(i) If $c > 0$, $|\zeta_n^*(z)| = p_{k_n}^{-c}$ has infinitely many arc-connected components which are closed curves and x varies on a finite interval $(a_{n,c}, b_{n,c})$, where $a_{n,c}, b_{n,c}$ could be accessible.

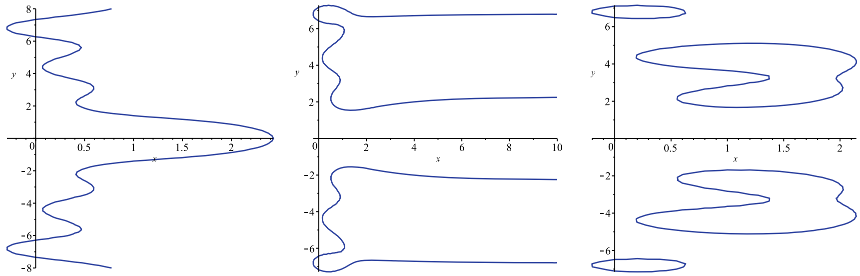
(ii) If $c = 0$, $|\zeta_n^*(z)| = 1$ has infinitely many arc-connected components which are open curves with horizontal asymptotes of equations $y = (2j+1)\frac{\pi}{2\log 2}$, $j \in \mathbb{Z}$, and x varies on the infinite interval $(a_{n,0}, +\infty)$, where $a_{n,0}$ could be accessible.

(iii) If $c < 0$, $|\zeta_n^*(z)| = p_{k_n}^{-c}$ has only one arc-connected component which is an open curve; x varies on a finite interval $(a_{n,c}, b_{n,c})$, where $a_{n,c}, b_{n,c}$ could be accessible. The variable y takes all real values. Furthermore, $|\zeta_n^*(z)| = p_{k_n}^{-c}$ intersects the real axis at a unique point of abscissa $b_{n,c}$, so $b_{n,c}$ is always accessible when $c < 0$.

In Fig. 3 we show the graph of $|\zeta_n^*(z)| = p_{k_n}^{-c}$ for some values of $n > 3$ and c. From Proposition 2.1, a simple consequence is deduced:



(a) Graph of $|\zeta_7^*(z)| = 7$. (b) Graph of $|\zeta_7^*(z)| = 1$. (c) Graph of $|\zeta_7^*(z)| = 7^{-1/15}$.



(d) Graph of $|\zeta_{12}^*(z)| = 11^{1/8}$. (e) Graph of $|\zeta_{12}^*(z)| = 1$. (f) Graph of $|\zeta_{12}^*(z)| = 11^{-1/12}$.

Fig. 3 Graphs of the varieties $|\zeta_7^*(z)| = 7^{-c}$ and $|\zeta_{12}^*(z)| = 11^{-c}$ for some values of c

Corollary 2.1 *Fixed an integer $n > 2$, if $u \in \mathbb{C}$ satisfies $|\zeta_n^*(u)| < p_{k_n}^{-c}$ (in this case we will say that u is an interior point of the variety $|\zeta_n^*(z)| = p_{k_n}^{-c}$), then there exists a point w of $|\zeta_n^*(z)| = p_{k_n}^{-c}$, so $a_{n,c} \leq \Re w \leq b_{n,c}$, such that $\Re w < \Re u$.*

Definition 2.1 Given an integer $n > 2$, we define the real functions

$$f_n(c) := a_{n,c}, \quad c \in \mathbb{R}; \qquad g_n(c) := b_{n,c}, \quad c \in \mathbb{R} \setminus \{0\}, \tag{12}$$

where $a_{n,c}$, $b_{n,c}$ are the end-points of the interval of variation of the variable x in the Eq. (10).

We show in Fig. 4 the graph of the functions $f_3(c)$ and $g_3(c)$, defined by the Eq. (11), and the function $f_4(c)$.

Since $|\zeta_n^*(z)| = p_{k_n}^{-d}$ tends to $|\zeta_n^*(z)| = p_{k_n}^{-c}$ as d tends to c, it is immediate that f_n, g_n are both continuous on $\mathbb{R} \setminus \{0\}$, and f_n is continuous on whole of $\mathbb{R}$. For $c = 0$, by Part (ii) of Proposition 1 we can agree $b_{n,0} = +\infty$, and then we should define $g_n(0) := +\infty$.

Now we are ready to give a characterization of the set $R_{\zeta_n(z)}$, defined in (6), by using the functions f_n and g_n.

Theorem 2.1 *Let $n > 2$ be a fixed integer. A real number $c \in R_{\zeta_n(z)}$ if and only if*

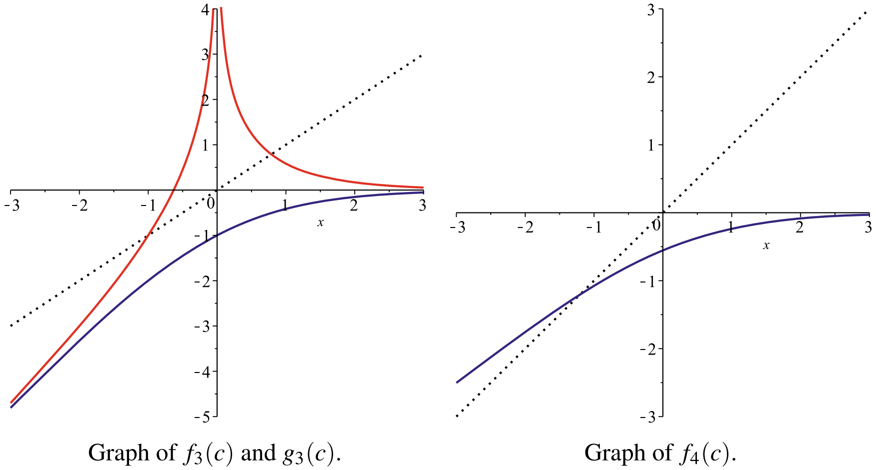


Fig. 4 Left: Graph of the functions $f_3(c)$ (blue), $g_3(c)$ (red) and $y = x$ (plotted). Right: Graph of the function $f_4(c)$ (blue) and $y = x$ (plotted)

$$f_n(c) \le c \le g_n(c). \tag{13}$$

Proof If $c \in R_{\zeta_n(z)}$, there exists a sequence $(z_m)_{m=1,2,\ldots}$ of zeros of $\zeta_n(z)$ such that $\lim_{m\to\infty} \Re z_m = c$. From (7), $\zeta_n^*(z_m) = -p_{k_n}^{-z_m}$ for each $m = 1, 2, \ldots$. By taking the modulus, we have $|\zeta_n^*(z_m)| = p_{k_n}^{-x_m}$, where $x_m := \Re z_m$. This means that each z_m is a point of the variety $|\zeta_n^*(z)| = p_{k_n}^{-x_m}$, so $x_m \in [a_{n,x_m}, b_{n,x_m}]$ and then we get

$$f_n(x_m) = a_{n,x_m} \le x_m \le b_{n,x_m} = g_n(x_m), \quad \text{for all } m.$$

Now by taking the limit when $m \to \infty$, noticing that $\lim_{m\to\infty} x_m = c$, because of the continuity of f_n and g_n, the inequalities (13) follow. Conversely, if $f_n(c) < c < g_n(c)$, by taking into account the definitions of f_n, g_n, the value c is in the interval of variation of x of the variety $|\zeta_n^*(z)| = p_{k_n}^{-c}$ and then the line $x = c$ intersects the variety. Therefore, by applying [16, Theorem 3], $c \in R_{\zeta_n(z)}$. If $f_n(c) = c$ or $g_n(c) = c$, the line $x = c$ intersects the variety $|\zeta_n^*(z)| = p_{k_n}^{-c}$ provided that $a_{n,c}$ or $b_{n,c}$ be accessible. Otherwise the line $x = c$ is an asymptote of $|\zeta_n^*(z)| = p_{k_n}^{-c}$. Therefore, in both cases, again by [19, Theorem 3], the point $c \in R_{\zeta_n(z)}$. □

As we can easily check, the function $f_3(c) := a_{3,c}$, with $a_{3,c}$ given in (11), is strictly increasing; this property is true for all the functions $f_n(c)$, $n > 2$, defined in (12), as we prove below.

Lemma 2.1 *For every integer $n > 2$, f_n is a strictly increasing function on $\mathbb{R}$.*

Proof Firstly, for each fixed $c \in \mathbb{R}$, we claim that f_n satisfies

$$\inf\{|\zeta_n^*(f_n(c) + iy)| : y \in \mathbb{R}\} = p_{k_n}^{-c}. \tag{14}$$

Indeed, we put $\lambda_{n,c} := \inf\{|\zeta_n^*(f_n(c)+iy)| : y \in \mathbb{R}\}$. By assuming $\lambda_{n,c} < p_{k_n}^{-c}$, there exists a point $z_c := f_n(c) + iy_c$ such that

$$\lambda_{n,c} \leq |\zeta_n^*(z_c)| < p_{k_n}^{-c},$$

and then it means that z_c is an interior point of $|\zeta_n^*(z)| = p_{k_n}^{-c}$. By Corollary 2.1 there exists w belonging to the variety $|\zeta_n^*(z)| = p_{k_n}^{-c}$, so $a_{n,c} \leq \Re w \leq b_{n,c}$, such that $\Re w < \Re z_c = f_n(c) = a_{n,c}$. But this is a contradiction, and then necessarily

$$\lambda_{n,c} \geq p_{k_n}^{-c}. \tag{15}$$

For $\varepsilon > 0$ sufficiently small, we consider the strip

$$S_\varepsilon := \{z \in \mathbb{C} : a_{n,c} \leq \Re z < a_{n,c} + \varepsilon\},$$

and put

$$\lambda_{n,c,\varepsilon} := \inf\{|\zeta_n^*(z)| : z \in S_\varepsilon\}.$$

From the definition of $a_{n,c}$, the set S_ε contains infinitely many points of the variety $|\zeta_n^*(z)| = p_{k_n}^{-c}$. Then $\lambda_{n,c,\varepsilon} \leq p_{k_n}^{-c}$ for all $\varepsilon > 0$, so $\lambda_{n,c} \leq p_{k_n}^{-c}$. Therefore, according to (15), $\lambda_{n,c} = p_{k_n}^{-c}$ and then (14) follows. Let d be a real number such that $d < c$, so $p_{k_n}^{-d} > p_{k_n}^{-c}$. Let η be such that $0 < \eta < p_{k_n}^{-d} - p_{k_n}^{-c}$. From (14), there exists some point $z_\eta := f_n(c) + iy_\eta$ such that

$$p_{k_n}^{-c} \leq |\zeta_n^*(z_\eta)| < p_{k_n}^{-c} + \eta < p_{k_n}^{-d},$$

so z_η is interior of $|\zeta_n^*(z)| = p_{k_n}^{-d}$. By Corollary 2.1, there exists a point w_η of $|\zeta_n^*(z)| = p_{k_n}^{-d}$, so $a_{n,d} \leq \Re w_\eta \leq b_{n,d}$, such that $\Re w_\eta < \Re z_\eta$. Then

$$f_n(d) = a_{n,d} \leq \Re w_\eta < \Re z_\eta = f_n(c),$$

which definitely proves the lemma. □

In the next result we prove that f_n is upper bounded by the number $a_{\zeta_n^*(z)}$ defined in (2) corresponding to the EP $\zeta_n^*(z)$, defined in (7).

Lemma 2.2 *For every $n > 2$, the function f_n satisfies*

$$f_n(c) < a_{\zeta_n^*(z)} \text{ for any } c \in \mathbb{R}.$$

Proof Let c be an arbitrary real number. By taking into account the definition of $a_{\zeta_n^*(z)}$, there exists a sequence $(z_m)_{m=1,2,\ldots}$ of zeros of $\zeta_n^*(z)$, with $\Re z_m \geq a_{\zeta_n^*(z)}$, such that

$$\lim_{m\to\infty} \Re z_m = a_{\zeta_n^*(z)}. \tag{16}$$

Since $\zeta_n^*(z_m) = 0$, we get $|\zeta_n^*(z_m)| < p_{k_n}^{-c}$, for all m. Then, from Corollary 2.1, there exists a sequence $(w_m)_{m=1,2,\dots}$ of points of $|\zeta_n^*(z)| = p_{k_n}^{-c}$, so $a_{n,c} \leq \Re w_m \leq b_{n,c}$, such that $\Re w_m < \Re z_m$, for all m. Therefore, since $f_n(c) = a_{n,c}$, we have

$$f_n(c) \leq \Re w_m < \Re z_m, \quad \text{for all } m.$$

Now, by taking the limit in the above inequality when $m \to \infty$, by (16), we get

$$f_n(c) \leq a_{\zeta_n^*(z)} \quad \text{for any } c \in \mathbb{R},$$

implying, noticing that by Lemma 2.1 f_n is strictly increasing, that $f_n(c) < a_{\zeta_n^*(z)}$ for any $c \in \mathbb{R}$. □

For every $n > 2$, let $a_{\zeta_n(z)}, b_{\zeta_n(z)}$ be the bounds, defined in (2), corresponding to the EP $\zeta_n(z)$. The function g_n, defined in (12), has the following properties.

Lemma 2.3 *For every $n > 2$, the function g_n satisfies:*

(i) *g_n is strictly increasing on $(-\infty, 0)$ and decreasing on $(0, +\infty)$.*
(ii) *If n is composite, then $c \leq g_n(c)$ for any $c \in (-\infty, b_{\zeta_n(z)}] \setminus \{0\}$ and the inequality is strict for all $c \in (-\infty, b_{\zeta_n(z)}) \setminus \{0\}$; if $c \in (b_{\zeta_n(z)}, +\infty)$, then $c > g_n(c)$.*
(iii) *If n is prime, then $c \leq g_n(c)$ for any $c \in [a_{\zeta_n(z)}, b_{\zeta_n(z)}] \setminus \{0\}$ and the inequality is strict for all $c \in (a_{\zeta_n(z)}, b_{\zeta_n(z)}) \setminus \{0\}$; if $c \in (-\infty, a_{\zeta_n(z)}) \cup (b_{\zeta_n(z)}, +\infty)$, then $c > g_n(c)$.*

Proof Part (i). Let c, d be real numbers such that $c < d < 0$. From Proposition 2.1, $b_{n,c}$ and $b_{n,d}$ are the unique points of $|\zeta_n^*(z)| = p_{k_n}^{-c}$ and $|\zeta_n^*(z)| = p_{k_n}^{-d}$ that intersect the real axis, respectively. Therefore $b_{n,c}$ and $b_{n,d}$ satisfy the equations

$$\sum_{\substack{m=1 \\ m \neq p_{k_n}}}^{n} m^{-x} = p_{k_n}^{-c}, \quad \sum_{\substack{m=1 \\ m \neq p_{k_n}}}^{n} m^{-x} = p_{k_n}^{-d}, \tag{17}$$

respectively. Each equation of (17) has only one real solution by virtue of [20, p. 46] and then, since $p_{k_n}^{-c} > p_{k_n}^{-d}$, the real solution of the first equation is obviously greater than the second one. Therefore $-b_{n,c} > -b_{n,d}$, equivalently, $b_{n,c} < b_{n,d}$. Consequently, $g_n(c) < g_n(d)$ and then g_n is strictly increasing in $(-\infty, 0)$. Let c, d be such that $c > d > 0$. From Proposition 2.1, $|\zeta_n^*(z)| = p_{k_n}^{-c}$ and $|\zeta_n^*(z)| = p_{k_n}^{-d}$ have infinitely many arc-connected components which are closed curves. Since $p_{k_n}^{-c} < p_{k_n}^{-d}$, any point of $|\zeta_n^*(z)| = p_{k_n}^{-c}$ is interior of $|\zeta_n^*(z)| = p_{k_n}^{-d}$, so $b_{n,c} \leq b_{n,d}$. That is, $g_n(c) \leq g_n(d)$, which means that g_n is decreasing on $(0, +\infty)$.

Part (ii). We firstly demonstrate that the bounds $a_{\zeta_n(z)}, b_{\zeta_n(z)}$ defined in (2) corresponding to $\zeta_n(z)$ satisfy the second inequality of (3), that is

$$a_{\zeta_n(z)} < 0 < b_{\zeta_n(z)} \quad \text{for all } n > 2. \tag{18}$$

Indeed, we introduce the EP

$$G_n(z) := \zeta_n(-z). \tag{19}$$

In [7, Chap. 3, Theorem 3.20] was shown that

$$b_{G_n(z)} := \sup\{\Re z : G_n(z) = 0\} > 0 \quad \text{for all } n > 2,$$

now we claim that

$$a_{G_n(z)} := \inf\{\Re z : G_n(z) = 0\} < 0 \quad \text{for all } n > 2.$$

Otherwise, if all the zeros of $G_n(z)$, say $(z_{n,k})_{k=1,2,\ldots}$, satisfy $\Re z_{n,k} \geq 0$, since $b_{G_n(z)} > 0$, there is at least a zero z_{n,k_0} with $\Re z_{n,k_0} > 0$. Then, as $G_n(z)$ is almost-periodic (see for instance [4, 5] and [10, Chap. VI]), $G_n(z)$ has infinitely many zeros in the strip

$$S_\varepsilon := \{z : \Re z_{n,k_0} - \varepsilon < \Re z < \Re z_{n,k_0} + \varepsilon\}, \quad 0 < \varepsilon < \Re z_{n,k_0},$$

and consequently

$$\sum_{k=1}^{\infty} \Re z_{n,k} = +\infty. \tag{20}$$

However, as all the coefficients of $G_n(z)$ are equal to 1, [21, formula (9)] applies and then we get $\sum_{k=1}^{\infty} \Re z_{n,k} = O(1)$, contradicting (20). Therefore the claim follows, that is, $a_{G_n(s)} < 0$ for all $n > 2$. By (19) we have $a_{\zeta_n(z)} = -b_{G_n(z)}$ and $b_{\zeta_n(z)} = -a_{G_n(z)}$, so (18) follows.

We now consider the point $b_{\zeta_n(z)}$. It is immediate that $b_{\zeta_n(z)}$ belongs to the set $R_{\zeta_n(z)}$ defined in (6). Then from Theorem 2.1 we have $b_{\zeta_n(z)} \leq g_n(b_{\zeta_n(z)})$, so the property $c \leq g_n(c)$ is true for $c = b_{\zeta_n(z)}$. From (18) and by using that g_n is decreasing on $(0, \infty)$ by virtue of Part (i), for any $c \in (0, b_{\zeta_n(z)})$ we have

$$0 < c < b_{\zeta_n(z)} \leq g_n(b_{\zeta_n(z)}) \leq g_n(c). \tag{21}$$

Consequently, Part (ii) follows for $c \in (0, b_{\zeta_n(z)}]$. We now assume $c < 0$ and n composite, so $p_{k_n} < n$. If $b_{n,c} \geq 0$, then $c < b_{n,c} = g_n(c)$ and again Part (ii) is true. Finally, we suppose $b_{n,c} < 0$. Since $c < 0$, $b_{n,c}$ satisfies the first equation of (17) and then $p_{k_n}^{-c} > n^{-b_{n,c}}$. Consequently $-c > -b_{n,c}$, so $c < b_{n,c}$ and then Part (ii) follows for $c \in (-\infty, b_{\zeta_n(z)}] \setminus \{0\}$. Finally, we claim that $c > g_n(c)$ when $c > b_{\zeta_n(z)}$. Indeed, because of Lemma 2.2 and (8), we have $f_n(c) < a_{\zeta_n^*(z)} \leq 0$ for any c. Therefore, since $c > b_{\zeta_n(z)}$, by (18) c is positive and then $f_n(c) < c$. Assume $c > g_n(c)$ is not true. Then we would have $f_n(c) < c \leq g_n(c)$ and by Theorem 2.1, $c \in R_{\zeta_n(z)} \subset [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$ which means that $c \leq b_{\zeta_n(z)}$. This is a contradiction because $c > b_{\zeta_n(z)}$, so the claim follows. This definitely proves Part (ii).

Part (iii). We first note that, since n is prime, $p_{k_n} = n$. Therefore the first equation in (17) becomes $\sum_{m=1}^{n-1} m^{-x} = n^{-c}$. By assuming $c < 0$, $b_{n,c}$ satisfies the above equation and then we have

$$\sum_{m=1}^{n-1} m^{-b_{n,c}} = n^{-c}. \tag{22}$$

For every $n \geq 2$, we consider the number $\beta_{G_n(z)}$, defined as the unique real solution of the equation $\sum_{m=1}^{n-1} m^x = n^x$ (see [20, p. 46]). By [6, Proposition 5], $\beta_{G_n(z)} \geq b_{G_n(z)}$ and the equality is attained for n prime. Therefore the set $\mathbb{R}$ of real numbers is partitioned in two sets:

$$(-\infty, \beta_{G_n(z)}] = \{x \in \mathbb{R} : \sum_{m=1}^{n-1} m^x \geq n^x\}, \tag{23}$$

and

$$(\beta_{G_n(z)}, \infty) = \{x \in \mathbb{R} : \sum_{m=1}^{n-1} m^x < n^x\}. \tag{24}$$

Now we claim that $c \leq g_n(c)$ when $a_{\zeta_n(z)} \leq c < 0$. Indeed, by (19), $b_{G_n(z)} = -a_{\zeta_n(z)}$, so c is such that $0 < -c \leq b_{G_n(z)} = \beta_{G_n(z)}$. Then, according to (23), we have

$$\sum_{m=1}^{n-1} m^{-c} \geq n^{-c}. \tag{25}$$

Therefore, if we assume $c > g_n(c) = b_{n,c}$, by applying (25) and taking into account (22), we get

$$n^{-c} \leq \sum_{m=1}^{n-1} m^{-c} < \sum_{m=1}^{n-1} m^{-b_{n,c}} = n^{-c},$$

which is a contradiction. Therefore $c \leq g_n(c)$ is true for c such that $a_{\zeta_n(z)} \leq c < 0$. Consequently, taking into account (21), it follows

$$c \leq g_n(c), \text{ for any } c \in [a_{\zeta_n(z)}, b_{\zeta_n(z)}] \setminus \{0\},$$

where the inequality is strict for all c of $(a_{\zeta_n(z)}, b_{\zeta_n(z)}) \setminus \{0\}$. Now suppose $c \in (-\infty, a_{\zeta_n(z)})$. Then, since $-c > -a_{\zeta_n(z)} = b_{G_n(z)} = \beta_{G_n(z)}$, by applying (24) we have

$$\sum_{m=1}^{n-1} m^{-c} < n^{-c}. \tag{26}$$

It implies that $c > g_n(c)$. Indeed, by supposing $c \leq g_n(c) = b_{n,c}$, from (22) and (26) we are led to the following contradiction:

$$n^{-c} = \sum_{m=1}^{n-1} m^{-b_{n,c}} \leq \sum_{m=1}^{n-1} m^{-c} < n^{-c}.$$

Therefore $c > g_n(c)$ if $c \in (-\infty, a_{\zeta_n(z)})$. Finally, if $c \in (b_{\zeta_n(z)}, +\infty)$, the reasoning used to demonstrate the end of Part (ii) of the lemma proves that $c > g_n(c)$. □

As a consequence of Lemma 2.3 we find the fixed points of the function g_n.

Corollary 2.2 *For every composite number $n > 2$, $b_{\zeta_n(z)}$ is the fixed point of the function g_n. If $n > 2$ is prime, $a_{\zeta_n(z)}$, $b_{\zeta_n(z)}$ are the fixed points of g_n.*

Proof Fixed an integer $n > 2$, by (18) $a_{\zeta_n(z)}, b_{\zeta_n(z)} \neq 0$, so g_n is well defined at $a_{\zeta_n(z)}$ and $b_{\zeta_n(z)}$. By applying Part (ii) of Lemma 2.3 for $n > 2$ composite, it is immediate, by the continuity of g_n, that the unique fixed point of g_n is $b_{\zeta_n(z)}$. If $n > 2$ is prime, by Part (iii) of Lemma 2.3, we get $g_n(a_{\zeta_n(z)}) = a_{\zeta_n(z)}$ and $g_n(b_{\zeta_n(z)}) = b_{\zeta_n(z)}$. Furthermore, Part (iii) of Lemma 2.3 also proves that $a_{\zeta_n(z)}, b_{\zeta_n(z)}$ are the unique fixed points of g_n. □

In the next result we obtain a characterization of $\mathscr{P}^*$, the set of prime numbers greater than 2.

Theorem 2.2 *An integer $n > 2$ belongs to $\mathscr{P}^*$ if and only if $a_{\zeta_n(z)}$ is a fixed point of the function g_n.*

Proof Assume $n > 2$ is prime, from Corollary 2.2, $a_{\zeta_n(z)}$ is a fixed point of g_n. Conversely, if

$$g_n(a_{\zeta_n(z)}) = a_{\zeta_n(z)}, \tag{27}$$

by supposing n composite, from Part (ii) of Lemma 2.3, we have $c < g_n(c)$ for all $c \in (-\infty, b_{\zeta_n(z)}) \setminus \{0\}$. From (18), $a_{\zeta_n(z)} \in (-\infty, b_{\zeta_n(z)}) \setminus \{0\}$. Then, $a_{\zeta_n(z)} < g_n(a_{\zeta_n(z)})$. This contradicts (27). Consequently n is a prime number and then the theorem follows. □

3 The Fixed Points of f_n and the Sets $R_{\zeta_n(z)}$

For every integer $n > 2$, the function f_n defined in (12) allows us to give a sufficient condition to have points of the set $R_{\zeta_n(z)}$, defined in (6).

Theorem 3.1 *For every integer $n > 2$, if a point $c \in [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$ satisfies $f_n(c) \leq c$, then $c \in R_{\zeta_n(z)}$.*

Proof We first claim that

$$a_{\zeta_n(z)}, 0, b_{\zeta_n(z)} \in R_{\zeta_n(z)} \quad \text{for every } n \geq 2. \tag{28}$$

Indeed, for $n = 2$, the claim trivially follows because as we have seen in Introduction all the zeros of $\zeta_2(z)$ are imaginary, so $a_{\zeta_2(z)} = b_{\zeta_2(z)} = 0$ and then $R_{\zeta_2(z)} = \{0\}$. Therefore we assume $n > 2$. By taking into account the definitions of $a_{\zeta_n(z)}, b_{\zeta_n(z)}$, both numbers obviously belong to $R_{\zeta_n(z)}$. Regarding the fact that $0 \in R_{\zeta_n(z)}$for all $n > 2$, it was proved in [18, (3.7)]. Then (28) is true. Hence it only remains to prove the theorem for $c \in (a_{G_n(z)}, b_{G_n(z)}) \setminus \{0\}$. But in this case,since by hypothesis $f_n(c) \leq c$, by using Parts (ii) and (iii) of Lemma 2.3 we are lead to $f_n(c) \leq c < g_n(c)$ and then, by Theorem 2.1, $c \in R_{\zeta_n(z)}$. □

An important conclusion is deduced from the above theorem.

Theorem 3.2 *For every integer $n > 2$, if c belongs to $R_{\zeta_n(z)}$ then*

$$[f_n(c), c] \cap [a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}. \tag{29}$$

If $n > 2$ is composite and c belongs to $R_{\zeta_n(z)}$, then

$$[f_n(c), c] \subset R_{\zeta_n(z)}. \tag{30}$$

Proof Assume $c \in R_{\zeta_n(z)}$. Then, by Theorem 2.1, $f_n(c) \leq c \leq g_n(c)$. Therefore the interval $[f_n(c), c]$ is well defined. If $f_n(c) = c$ the theorem trivially follows. Suppose $f_n(c) < c$. Let t be a point of $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$ such that $f_n(c) < t < c$. By Lemma 2.1, $f_n(t) < f_n(c)$. Therefore we have

$$f_n(t) < f_n(c) < t < c,$$

and then, by applying Theorem 3.1, $t \in R_{\zeta_n(z)}$. Consequently

$$(f_n(c), c) \cap [a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)},$$

and from the closedness of $R_{\zeta_n(z)}$, (29) follows.

Assume $n > 2$ is composite. Since $c \in R_{\zeta_n(z)}$ and

$$R_{\zeta_n(z)} \subset [a_{\zeta_n(z)}, b_{\zeta_n(z)}],$$

we have $c \leq b_{\zeta_n(z)}$. Furthermore, from Theorem 2.1, $f_n(c) \leq c \leq g_n(c)$. Then, if $f_n(c) = c$, (30) is obviously true. Suppose $f_n(c) < c$. Consider a number t such that $f_n(c) \leq t < c$. Then, we get

$$f_n(c) \leq t < c \leq b_{\zeta_n(z)}. \tag{31}$$

If $t = 0$, by virtue of (28), $t \in R_{\zeta_n(z)}$. If $t \neq 0$, from (31), $t \in (-\infty, b_{\zeta_n(z)}) \setminus \{0\}$. Then, as n is composite, by Part (ii) of Lemma 2.3, $t < g_n(t)$. On the other hand, since $t < c$, from Lemma 2.1, $f_n(t) < f_n(c)$ and then, again by (31), we have

$$f_n(t) < f_n(c) \leq t < g_n(t).$$

Now, by applying Theorem 2.1, $t \in R_{\zeta_n(z)}$. Consequently $[f_n(c), c) \subset R_{\zeta_n(z)}$ and then, since by hypothesis $c \in R_{\zeta_n(z)}$, we get $[f_n(c), c] \subset R_{\zeta_n(z)}$. The proof is now completed. □

As a consequence of the two preceding results we characterize the set $\mathscr{C}^*$ of composite numbers $n > 2$.

Corollary 3.1 *For every* $n \in \mathscr{C}^*$, $a_{\zeta_n(z)}$ *is a fixed point of the function* f_n.

Proof Assume $n \in \mathscr{C}^*$. From (28), $a_{\zeta_n(z)} \in R_{\zeta_n(z)}$. Since n is composite and greater than 2, by (30) we have $[f_n(a_{\zeta_n(z)}), a_{\zeta_n(z)}] \subset R_{\zeta_n(z)}$. Noticing $R_{\zeta_n(z)} \subset [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$, necessarily $f_n(a_{\zeta_n(z)}) = a_{\zeta_n(z)}$. □

In the next result we prove that $a_{\zeta_n(z)}$ is not a fixed point of f_n for any $n \in \mathscr{P}^*$.

Corollary 3.2 *For every* $n \in \mathscr{P}^*$, $f_n(a_{\zeta_n(z)}) < a_{\zeta_n(z)}$.

Proof For every $n > 2$, the variety $|\zeta_n^*(z)| = p_{k_n}^{-c}$, for arbitrary $c \in \mathbb{R}$, by virtue of equation (10) is not contained in a vertical line, so the interval of the variation of the variable x in the variety $|\zeta_n^*(z)| = p_{k_n}^{-c}$ is not degenerate. Therefore, taking into account (12), we have

$$f_n(c) < g_n(c) \text{ for every integer } n > 2, \quad \text{for all } c \in \mathbb{R}. \tag{32}$$

Assume $n > 2$ prime. By Corollary 2.2, $g_n(a_{\zeta_n(z)}) = a_{\zeta_n(z)}$. Then, by taking $c = a_{\zeta_n(z)}$ in (32), the corollary follows. □

As a simple consequence from Corollary 3.2 we obtain a characterization of $\mathscr{C}^*$.

Theorem 3.3 *An integer* $n > 2$ *belongs to* $\mathscr{C}^*$ *if and only if* $a_{\zeta_n(z)}$ *is a fixed point of the function* f_n.

Proof From Corollary 3.1, if $n > 2$ is composite, $a_{\zeta_n(z)}$ is a fixed point of f_n. Reciprocally, if $a_{\zeta_n(z)}$ is a fixed point of f_n, by assuming $n > 2$ is not composite, by applying Corollary 3.2 we are led to a contradiction. Therefore, the theorem follows. □

The bounds $a_{\zeta_n(z)}, a_{\zeta_n^*(z)}$ satisfy the following inequality.

Proposition 3.1 *For every integer* $n > 2$, $a_{\zeta_n(z)} < a_{\zeta_n^*(z)}$.

Proof By taking $c = a_{\zeta_n^*(z)}$ in Lemma 2.2 we have

$$f_n(a^*_{\zeta_n(z)}) < a^*_{\zeta_n(z)} \text{ for all } n > 2. \tag{33}$$

Again from Lemma 2.2, for $c = a_{\zeta_n(z)}$, we get $f_n(a_{\zeta_n(z)}) < a^*_{\zeta_n(z)}$. If n is composite, by Corollary 3.1 $f_n(a_{\zeta_n(z)}) = a_{\zeta_n(z)}$ and from (33) we then deduce that $a_{\zeta_n(z)} < a^*_{\zeta_n(z)}$. This proves the proposition for n composite.

Assume n is prime. Then $p_{k_n} = n$ and, from (7), $\zeta_n^*(z) = \zeta_{n-1}(z)$, so $a_{\zeta_n^*(z)} = a_{\zeta_{n-1}(z)}$. Now we consider the function $G_n(z)$ defined in (19). As we have seen in the proof of Lemma 2.3, because of [6, Proposition 5] we have $b_{G_n(z)} \leq \beta_{G_n(z)}$ for all $n \geq 2$ and the equality is attained for n prime. Noticing [17, Lemma 1], $\beta_{G_{n-1}(z)} < \beta_{G_n(z)}$ for all $n > 2$. Then we get

$$b_{G_{n-1}(z)} \leq \beta_{G_{n-1}(z)} < \beta_{G_n(z)} = b_{G_n(z)}, \quad \text{for all prime } n > 2, \tag{34}$$

or equivalently

$$-b_{G_{n-1}(z)} \geq -\beta_{G_{n-1}(z)} > -\beta_{G_n(z)} = -b_{G_n(z)}, \quad \text{for all prime } n > 2.$$

Now, since from (19) $a_{\zeta_n(z)} = -b_{G_n(z)}$ for all $n \geq 2$, from the above chain of inequalities we deduce

$$a_{\zeta_n^*(z)} = a_{\zeta_{n-1}(z)} = -b_{G_{n-1}(z)} > -b_{G_n(z)} = a_{\zeta_n(z)}, \quad \text{for all prime } n > 2.$$

The proof is now completed. □

Corollary 3.3 *For every integer $n > 2$, $a^*_{\zeta_n(z)} \in R_{\zeta_n(z)}$.*

Proof For $n = 3, 4$, because of (8) we have $a^*_{\zeta_n(z)} = 0$. Therefore, from (28), $a^*_{\zeta_n(z)} \in R_{\zeta_n(z)}$ for $n = 3, 4$. Assume $n > 4$. By Proposition 3.1, $a_{\zeta_n(z)} < a_{\zeta_n^*(z)}$ for all $n > 2$. Then, from (8) and (18), $a^*_{\zeta_n(z)} \in [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$ for all $n > 4$. Therefore, by using (33) and applying Theorem 3.1, $a^*_{\zeta_n(z)} \in R_{\zeta_n(z)}$ for all $n > 4$. This proves the corollary. □

In the next result we prove the existence of a minimal density interval for every $\zeta_n(z)$, $n > 2$.

Theorem 3.4 *For every integer $n > 2$ there exists a number $A_n \in [a_{\zeta_n(z)}, a_{\zeta_n^*(z)})$ such that $[A_n, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}$.*

Proof Firstly we note that, by Proposition 3.1, the interval $[a_{\zeta_n(z)}, a_{\zeta_n^*(z)})$ is well defined. On the other hand, by (18) $b_{\zeta_n(z)} > 0$ and, by (8) $a_{\zeta_n^*(z)} \leq 0$ for all $n > 2$, so by Proposition 3.1 we have

$$a_{\zeta_n(z)} < a_{\zeta_n^*(z)} \leq 0 < b_{\zeta_n(z)}, \quad \text{for all } n > 2. \tag{35}$$

This means that $[a_{\zeta_n^*(z)}, b_{\zeta_n(z)}]$ is a non-degenerate sub-interval of $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$ for any $n > 2$. By Lemma 2.2, we have $f_n(b_{\zeta_n(z)}) < a_{\zeta_n^*(z)}$. Then, according to (35), we get

$$f_n(b_{\zeta_n(z)}) \leq a_{\zeta_n^*(z)} < b_{\zeta_n(z)},$$

so

$$[a_{\zeta_n^*(z)}, b_{\zeta_n(z)}] \subset [f_n(b_{\zeta_n(z)}), b_{\zeta_n(z)}].$$

Now, since $b_{\zeta_n(z)} \in R_{\zeta_n(z)}$, because of Theorem 3.2 we obtain

$$[a_{\zeta_n^*(z)}, b_{\zeta_n(z)}] \subset [f_n(b_{\zeta_n(z)}), b_{\zeta_n(z)}] \cap [a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}. \tag{36}$$

This implies that $a_{\zeta_n^*(z)} \in R_{\zeta_n(z)}$ (observe that from Corollary 3.3 we already knew that $a_{\zeta_n^*(z)} \in R_{\zeta_n(z)}$)) so, again by Theorem 3.2, we have

$$[f_n(a_{\zeta_n^*(z)}), a_{\zeta_n^*(z)} \cap [a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}. \tag{37}$$

If $f_n(a_{\zeta_n^*(z)}) \leq a_{\zeta_n(z)}$, from (37) we deduce that $[a_{\zeta_n(z)}, a_{\zeta_n^*(z)}] \subset R_{\zeta_n(z)}$ and then, by (36) we get $[a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}$. In this case by taking $A_n = a_{\zeta_n(z)}$, the theorem follows. Moreover, $\zeta_n(z)$ has a maximum density interval and it coincides with its critical interval $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$.

If $f_n(a_{\zeta_n^*(z)}) > a_{\zeta_n(z)}$, from (37) we deduce

$$[f_n(a_{\zeta_n^*(z)}), a_{\zeta_n^*(z)}] \subset R_{\zeta_n(z)}. \tag{38}$$

Therefore $f_n(a_{\zeta_n^*(z)}) \in R_{\zeta_n(z)}$ and, again by Theorem 3.2, we have

$$[f_n^{(2)}(a_{\zeta_n^*(z)}), f_n(a_{\zeta_n^*(z)})] \cap [a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}, \tag{39}$$

where $f_n^{(2)}$ denotes f_n composed with itself. Then, if $f_n^{(2)}(a_{\zeta_n^*(z)}) \leq a_{\zeta_n(z)}$, from (39), we have $[a_{\zeta_n(z)}, f_n(a_{\zeta_n^*(z)})] \subset R_{\zeta_n(z)}$ and by (38), we get $[a_{\zeta_n(z)}, a_{\zeta_n^*(z)}] \subset R_{\zeta_n(z)}$. Therefore taking into account (36) we obtain $[a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}$. Consequently, by taking $A_n = a_{\zeta_n(z)}$, the theorem follows and $\zeta_n(z)$ has a maximum density interval that coincides with its critical interval $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$. If $f_n^{(2)}(a_{\zeta_n^*(z)}) > a_{\zeta_n(z)}$, from (39), we get

$$[f_n^{(2)}(a_{\zeta_n^*(z)}), f_n(a_{\zeta_n^*(z)})] \subset R_{\zeta_n(z)}.$$

Therefore $f_n^{(2)}(a_{\zeta_n^*(z)}) \in R_{\zeta_n(z)}$ and, again by Theorem 3.2, we have

$$[f_n^{(3)}(a_{\zeta_n^*(z)}), f_n^{(2)}(a_{\zeta_n^*(z)})] \cap [a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)},$$

and so on. Therefore, by denoting $f_n^{(k)} = f_n^{(k-1)} \circ f_n$ for $k \geq 2$ and repeating the process above, we are led to one of the two cases:

(i) There is some $k \geq 1$ such that $f_n^{(k)}(a_{\zeta_n^*(z)}) \leq a_{\zeta_n(z)}$. In this case, as we have seen $A_n = a_{\zeta_n(z)}$ and then $\zeta_n(z)$ has a maximum density interval that coincides with its critical interval $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$.

(ii) For all k, $f_n^{(k)}(a_{\zeta_n^*(z)}) > a_{\zeta_n(z)}$ and then, by virtue of Lemma 2.1 and (33), we have

$$a_{\zeta_n(z)} < \cdots < f_n^{(k)}(a_{\zeta_n^*(z)}) < \cdots < f_n^{(2)}(a_{\zeta_n^*(z)}) < f_n(a_{\zeta_n^*(z)}) < a_{\zeta_n^*(z)}.$$

Consequently there exists $\lim_{k\to\infty} f_n^{(k)}(a_{\zeta_n^*(z)})$ and then, by defining

$$A_n := \lim_{k\to\infty} f_n^{(k)}(a_{\zeta_n^*(z)}),$$

we have $a_{\zeta_n(z)} \leq A_n < a_{\zeta_n^*(z)}$. On the other hand, by reiterating Theorem 3.2, we get

$$[f_n^{(k)}(a_{\zeta_n^*(z)}), f_n^{(k-1)}(a_{\zeta_n^*(z)})] \subset R_{\zeta_n(z)}, \text{ for all } k \geq 2. \tag{40}$$

Then taking into account (36) and (38), by (40) we deduce that $[A_n, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}$. This definitely proves the theorem. □

Remark 3.5 Observe that if the case (ii) of above theorem holds, A_n will be a fixed point of f_n by virtue of the continuity of f_n. Then if $n \in \mathscr{C}^*$, by Theorem 14, the point A_n could be $a_{\zeta_n(z)}$. But if $n \in \mathscr{P}^*$, from Corollary 3.2, A_n can not be equal to $a_{\zeta_n(z)}$.

In the next result we prove that the number of fixed points of f_n influences on the existence of a maximum density interval of $\zeta_n(z)$.

Theorem 3.6 *For every integer $n > 2$, if f_n has at most a fixed point in the interval $(a_{\zeta_n(z)}, a_{\zeta_n^*(z)})$ then $\zeta_n(z)$ has a maximum density interval that coincides with the critical interval $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$ associated with $\zeta_n(z)$.*

Proof We first assume f_n has no fixed point in $(a_{\zeta_n(z)}, a_{\zeta_n^*(z)})$. Then we claim that $f_n(c) < c$ for all $c \in (a_{\zeta_n(z)}, a_{\zeta_n^*(z)}]$. Indeed, we define the function $h_n(c) := f_n(c) - c$. Then h_n is continuous on $\mathbb{R}$, and by virtue of Lemma 2.2 and (33), h_n is negative on $[a_{\zeta_n^*(z)}, \infty)$. Then, since f_n by hypothesis has no fixed point on $(a_{\zeta_n(z)}, a_{\zeta_n^*(z)})$, $h_n(c)$ has no zero on $(a_{\zeta_n(z)}, \infty)$. Consequently, $h_n(c) < 0$ for any $c \in (a_{\zeta_n(z)}, \infty)$ and in particular we have

$$f_n(c) < c \text{ for all } c \in (a_{\zeta_n(z)}, a_{\zeta_n^*(z)}]. \tag{41}$$

Hence the claim follows. On the other hand, by Corollary 3.3 $a_{\zeta_n^*(z)} \in R_{\zeta_n(z)} \subset [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$, so

$$(a_{\zeta_n(z)}, a_{\zeta_n^*(z)}] \subset [a_{\zeta_n(z)}, b_{\zeta_n(z)}].$$

Consequently, by taking into account (41) and by applying Theorem 3.1 we have

$$(a_{\zeta_n(z)}, a_{\zeta_n^*(z)}] \subset R_{\zeta_n(z)}.$$

Therefore, since from (28) $a_{\zeta_n(z)} \in R_{\zeta_n(z)}$, we get $[a_{\zeta_n(z)}, a_{\zeta_n^*(z)}] \subset R_{\zeta_n(z)}$ and then by (36) it follows that $[a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}$. As always is true that $R_{\zeta_n(z)} \subset [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$ we deduce that $R_{\zeta_n(z)} = [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$, i.e. $\zeta_n(z)$ has a maximum density interval. Then the theorem follows in this case.

We now suppose f_n has only one fixed point, say c_1, in $(a_{\zeta_n(z)}, a_{\zeta_n^*(z)})$. Then the function $h_n(c) := f_n(c) - c$, continuous on $\mathbb{R}$, is non-positive on $[c_1, +\infty)$ by virtue of Lemma 2.2. Therefore, in particular, $f_n(c) \leq c$ for all $c \in [c_1, a_{\zeta_n^*(z)}]$. Since $[c_1, a_{\zeta_n^*(z)}] \subset [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$, by applying the Theorem 3.1 at any $c \in [c_1, a_{\zeta_n^*(z)}]$ we have

$$[c_1, a_{\zeta_n^*(z)}] \subset R_{\zeta_n(z)}. \tag{42}$$

Now we claim that h_n is negative on $(a_{\zeta_n(z)}, c_1)$. Indeed, if we assume that h_n is non-negative on $(a_{\zeta_n(z)}, c_1)$, since c_1 is the unique fixed point of f_n in $(a_{\zeta_n(z)}, a_{\zeta_n^*(z)})$, then $f_n(c) > c$ for all $c \in (a_{\zeta_n(z)}, c_1)$. Then, by Theorem 2.1, $c \notin R_{\zeta_n(z)}$ for all $c \in (a_{\zeta_n(z)}, c_1)$. This means that $\zeta_n(z)$ has no zero on the strip $(a_{\zeta_n(z)}, c_1) \times \mathbb{R}$. But, taking into account that $a_{\zeta_n(z)} \in R_{\zeta_n(z)}$, $a_{\zeta_n(z)}$ would be an isolated point of $R_{\zeta_n(z)}$ and it contradicts [2, Corollary 3.2]. Therefore the claim follows. Consequently, $f_n(c) < c$ for all $c \in (a_{\zeta_n(z)}, c_1)$ and then, by Theorem 3.1, $(a_{\zeta_n(z)}, c_1) \subset R_{\zeta_n(z)}$. From the closedness of $R_{\zeta_n(z)}$, we have

$$[a_{\zeta_n(z)}, c_1] \subset R_{\zeta_n(z)}. \tag{43}$$

Then, from (43), (42) and (36) we deduce that $[a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}$. Consequently, $\zeta_n(z)$ has a maximum density interval and it coincides with its critical interval $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$. □

As a first application of the usefulness of Theorem 3.6 we prove a result on $\zeta_3(z)$ (the same result can be also deduced from others methods as we can see in [13, 15]).

Corollary 3.4 *$\zeta_3(z)$ has a maximum density interval and it coincides with its critical interval $[a_{\zeta_3(z)}, b_{\zeta_3(z)}]$.*

Proof The function $f_3(c) := a_{3,c}$ is explicitly given by the formula (11). Then it is immediate to check that $f_3(c) < c$ for all $c \in \mathbb{R}$. Therefore $f_3(c)$ has no fixed point and then, from Theorem 3.6, $\zeta_3(z)$ has a maximum density interval and it coincides with $[a_{\zeta_3(z)}, b_{\zeta_3(z)}]$. □

4 The Fixed Point Theory and the Maximum Density Interval for $\zeta_n(z)$

In this section our aim is to give a very useful result (see below Lemma 4.1) based on Kronecker Theorem [8, Theorem 444] that allows us to apply our fixed point theory to prove the existence of a maximum density interval.

Let $\mathscr{P} := \{p_j : j = 1, 2, 3, \ldots\}$ be the set of prime numbers and $U := \{1, -1\}$. For every map $\delta : \mathscr{P} \to U$, we define the function $\omega_\delta : \mathbb{N} \to U$ as

$$\omega_\delta(1) := 1, \quad \omega_\delta(m) := (\delta(p_{k_1}))^{\alpha_1} \ldots (\delta(p_{k_{l(m)}}))^{\alpha_{l(m)}}, \quad m > 1, \tag{44}$$

where $(p_{k_1})^{\alpha_1} \ldots (p_{k_{l(m)}})^{\alpha_{l(m)}}$, with $\alpha_1, \ldots, \alpha_{l(m)} \in \mathbb{N}$, is the decomposition of m in prime factors. Let Ω be the set of all the ω_δ's defined in (44). Observe that all functions of Ω are *completely multiplicative* (see for instance [1, p. 138]).

Lemma 4.1 *Let $n > 2$ a fixed integer, p_{k_n} the last prime not exceeding n and f_n defined in (12). Given an arbitrary $\omega_\delta \in \Omega$, the inequality*

$$p_{k_n}^{-c} \leq |\sum_{\substack{m=1 \\ m \neq p_{k_n}}}^{n} \omega_\delta(m) m^{-f_n(c)}|, \tag{45}$$

holds for all $c \in \mathbb{R}$.

Proof Because of (7), $\zeta_n^*(z) := \sum_{m=1, m \neq p_{k_n}}^{n} m^{-z}$. Therefore, given $c \in \mathbb{R}$ we have

$$\zeta_n^*(f_n(c) + iy) = \sum_{\substack{m=1 \\ m \neq p_{k_n}}}^{n} m^{-f_n(c)}(\cos(y \log m) - i \sin(y \log m)).$$

Then taking into account (14),

$$p_{k_n}^{-c} \leq |\sum_{\substack{m=1 \\ m \neq p_{k_n}}}^{n} m^{-f_n(c)}(\cos(y \log m) - i \sin(y \log m))|, \text{ for all } y \in \mathbb{R}. \tag{46}$$

Given $n > 2$, we define $J_n := \{1, 2, 3, \ldots \pi(n)\}$, where $\pi(n)$ denotes the number of prime numbers not exceeding n. As the set $\{\log p_j : j \in J_n\}$ is rationally independent, the set $\{\frac{\log p_j}{2\pi} : j \in J_n\}$ is also rationally independent. Then by Kronecker Theorem [8, Theorem 444] fixed an arbitrary set of real numbers $\{\gamma_j : j \in J_n\}$ and given an integer $N \geq 1$, there exists a real number $y_N > N$ and integers $m_{j,N}$, such that

$$|y_N \frac{\log p_j}{2\pi} - m_{j,N} - \gamma_j| < \frac{1}{N}, \quad \text{for all } j \in J_n. \tag{47}$$

For each $n > 2$, we define the set $\mathscr{P}_n := \{p_j \in \mathscr{P} : p_j \leq n\}$. Then, given a mapping $\delta := \mathscr{P}_n \to U$, we consider the set $\{\gamma_j : j \in J_n\}$ where $\gamma_j = 1$ for those j such that $\delta(p_j) = 1$ and $\gamma_j = 1/2$ for those j such that $\delta(p_j) = -1$. Then by applying the aforementioned Kronecker Theorem for $N = 1, 2 \ldots$, we can determine a sequence $(y_N)_N$ satisfying, by virtue of (47), that

$$\cos(y_N \log p_j) \to 1, \quad \sin(y_N \log p_j) \to 0 \text{ as } N \to \infty, \text{ for } p_j \text{ with } \delta(p_j) = 1,$$

and

$$\cos(y_N \log p_j) \to -1, \quad \sin(y_N \log p_j) \to 0 \text{ as } N \to \infty, \text{ for } p_j \text{ with } \delta(p_j) = -1.$$

Therefore for each m such that $1 \leq m \leq n$ we get

$$\cos(y_N \log m) \to \omega_\delta(m), \quad \sin(y_N \log m) \to 0 \quad \text{as } N \to \infty. \tag{48}$$

Now, we substitute y by y_N in (46) and we take the limit as $N \to \infty$. Then, according to (48), the inequality (45) follows. □

Theorem 4.1 *For all prime numbers* $n > 2$ *except at most for a finite quantity,* f_n *has no fixed point in the interval* $(a_{\zeta_n(z)}, a_{\zeta_n^*(z)})$.

Proof Corollary 3.4 proves the theorem for $n = 3$. Assume $n > 3$ prime. The numbers $n-2$ and $n-1$ are relatively primes and both cannot be perfect squares, so there exists $\omega_\delta \in \Omega$ such that $\omega_\delta(n-2)\omega_\delta(n-1) = -1$. Since n is prime, $a_{\zeta_n^*(z)} = a_{\zeta_{n-1}(z)}$ and $p_{k_n} = n$. By supposing the existence of a fixed point $c_n \in (a_{\zeta_n(z)}, a_{\zeta_{n-1}(z)})$ for the function f_n for infinitely many prime $n > 3$, we are led to the following contradiction:

By (45) we have

$$n^{-c_n} \leq |\pm((n-1)^{-c_n} - (n-2)^{-c_n}) + \sum_{m \in P_{n-3,\omega_\delta}} m^{-c_n} - \sum_{m \notin P_{n-3,\omega_\delta}} m^{-c_n}|, \tag{49}$$

where, for a fixed integer $n > 2$ and $\omega_\delta \in \Omega$, the set P_{n,ω_δ} is defined as

$$P_{n,\omega_\delta} := \{m : 1 \leq m \leq n \text{ such that } \omega_\delta(m) = 1\}.$$

On the other hand, $\lim_{n\to\infty} \frac{a_{\zeta_n(z)}}{n} = -\log 2$ (see [3, Theorem 1] and [17, Theorem 2]). Then noticing that $a_{\zeta_n(z)} < c_n < a_{\zeta_{n-1}(z)}$, we get

$$\lim_{\substack{n \text{ prime} \\ n\to\infty}} \frac{c_n}{n-1} = -\log 2.$$

Therefore, for each fixed $j \geq 0$, it follows

$$\lim_{\substack{n \text{ prime} \\ n\to\infty}} \left(\frac{n-j}{n-1}\right)^{-c_n} = 2^{-j+1}. \tag{50}$$

Now, dividing by $(n-1)^{-c_n}$ the inequality (49), we have

$$\begin{aligned} \left(\frac{n}{n-1}\right)^{-c_n} &\leq \left| \pm\left(1 - \left(\frac{n-2}{n-1}\right)^{-c_n}\right) \right. \\ &\quad \left. + \sum_{m \in P_{n-3,\omega_\delta}} \left(\frac{m}{n-1}\right)^{-c_n} - \sum_{m \notin P_{n-3,\omega_\delta}} \left(\frac{m}{n-1}\right)^{-c_n} \right| \\ &\leq \left| \pm\left(1 - \left(\frac{n-2}{n-1}\right)^{-c_n}\right) \right| \\ &\quad + \left| \sum_{m \in P_{n-3,\omega_\delta}} \left(\frac{m}{n-1}\right)^{-c_n} - \sum_{m \notin P_{n-3,\omega_\delta}} \left(\frac{m}{n-1}\right)^{-c_n} \right| \\ &\leq \left(1 - \left(\frac{n-2}{n-1}\right)^{-c_n}\right) + \sum_{j=3}^{n-1} \left(\frac{n-j}{n-1}\right)^{-c_n}. \end{aligned} \tag{51}$$

According to (50), by taking the limit in (51) for n prime, $n \to \infty$, it follows that the limit of the left-hand side of (51) is 2 whereas the limit of the right-hand side one is $1/2 + \sum_{j=3}^{\infty} 2^{-j+1} = 1$. This is the contradiction desired. Hence the theorem follows. □

As a consequence from Theorem 4.1, an important property of the partial sums of order n prime can be deduced.

Theorem 4.2 *For all prime numbers $n > 2$ except at most for a finite quantity, $\zeta_n(z)$ has a maximum density interval and it coincides with its critical interval $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$.*

Proof It is enough to apply Theorems 3.6 and 4.1. □

5 Numerical Experiences

Simple numerical experiences carried out for some values of n in inequality (45) joint with the application of Theorem 3.6 and Lemma 4.1, allows us to prove the existence of a maximum density interval of $\zeta_n(z)$ for all $2 \leq n \leq 8$. Indeed: For $n = 2$, we have already seen in the Introduction section that the zeros of $\zeta_2(z)$ are all imaginary, so the set $R_{\zeta_2(z)} = \{0\}$ and then $a_{\zeta_2(z)} = b_{G_2(z)} = 0$ which means that we trivially have

$$R_{\zeta_2(z)} = [a_{\zeta_2(z)}, b_{\zeta_2(z)}].$$

Therefore $\zeta_2(z)$ has a maximum density interval (in this case degenerate).

For $n = 3$, Corollary 3.4 proves that

$$R_{\zeta_3(z)} = [a_{\zeta_3(z)}, b_{\zeta_3(z)}]$$

and then $\zeta_3(z)$ has a maximum density interval. In this case the end-points $a_{\zeta_3(z)}$, $b_{\zeta_3(z)}$ can be easily computed, being $a_{\zeta_3(z)} = -1$ and $b_{\zeta_3(z)} \approx 0.79$. Thus, $R_{\zeta_3(z)} \approx [-1, 0.79]$.

For $n = 4$, we firstly claim that f_4 has no fixed point in the interval $(a_{\zeta_4(z)}, a_{\zeta_4^*(z)})$. Indeed, by (8), $a_{\zeta_4^*(z)} = 0$ and from (18), $a_{\zeta_4(z)} < 0$. Therefore we only study the behavior of $f_4(c)$ for $c < 0$. We recall that from (12) $f_4(c) = a_{4,c}$, where $a_{4,c}$ is the left end-point of the interval of variation of the variable x in the Cartesian equation of the variety $|\zeta_4^*(z)| = p_{k_4}^{-c}$. By taking into account formula (10) for $n = 4$, the equation of that variety is

$$1 + 2^{-2x} + 4^{-2x} + 2 \cdot 2^{-x}(1 + 4^{-x}) \cos(y \log 2) + 2 \cdot 4^{-x} \cos(y \log 4)) = 3^{-2c}. \quad (52)$$

By putting $\cos(y \log 4)) = 2\cos^2(y \log 2) - 1$ in (52) and solving it for $\cos(y \log 2)$ we have

$$\cos(y \log 2) = \frac{-(1 + 4^{-x}) \pm \sqrt{(2 \cdot 3^{-c})^2 - (\sqrt{3}(4^{-x} - 1))^2}}{4 \cdot 2^{-x}}.$$

Then the variable x must satisfy the inequality $(\sqrt{3}(4^{-x}-1))^2 \leq (2\cdot 3^{-c})^2$ which is equivalent to say that

$$4^{-x} \in [1-2\cdot 3^{-c-\frac{1}{2}}, 1+2\cdot 3^{-c-\frac{1}{2}}]. \tag{53}$$

Since $1-2\cdot 3^{-c-\frac{1}{2}} < 0$ for all $c < 0$, by noting that $4^{-x} > 0$ for any x, (53) is in turn equivalent to

$$-\frac{\log(1+2\cdot 3^{-c-\frac{1}{2}})}{\log 4} \leq x.$$

Hence the minimum value for x is $-\frac{\log(1+2\cdot 3^{-c-\frac{1}{2}})}{\log 4}$, so $a_{4,c} = -\frac{\log(1+2\cdot 3^{-c-\frac{1}{2}})}{\log 4}$ and consequently for $c < 0$ the function $f_4(c)$ is given by the formula

$$f_4(c) = -\frac{\log(1+2\cdot 3^{-c-\frac{1}{2}})}{\log 4}.$$

Then the fixed points of $f_4(c)$ are the solutions of the equation $f_4(c) = c$, that is

$$1+2\cdot 3^{-c-1/2} = 4^{-c}. \tag{54}$$

According to [20, p. 46] Eq. (54) has a unique real solution, say c_0, whose approached value is -1.21. On the other hand, since $n = 4$ belongs to $\mathscr{C}^*$, by Theorem 3.3 $a_{\zeta_4(z)}$ is a fixed point of the function f_4. Since c_0 is the unique solution of $f_4(c) = c$, necessarily $a_{\zeta_4(z)} = c_0 \approx -1.21$ and then f_4 has no fixed point in $(a_{\zeta_4(z)}, a_{\zeta_4^*(z)})$. Hence the claim follows. Then, by applying Theorem 3.6, $\zeta_4(z)$ has a maximum density interval and consequently

$$R_{\zeta_4(z)} = [a_{\zeta_4(z)}, b_{\zeta_4(z)}].$$

For $n = 5$ we take a mapping $\delta : \mathscr{P} \to U$ satisfying $\delta(2) = \delta(3) = -1$ and consider its corresponding $\omega_\delta : \mathbb{N} \to U$ defined in (44). Assume f_5 has some fixed point, say c_0, in the interval $(a_{\zeta_5(z)}, a_{\zeta_5^*(z)})$. By (8) $a_{\zeta_5^*(z)} < 0$ and then $(a_{\zeta_5(z)}, a_{\zeta_5^*(z)})$ contains only negative numbers, so $c_0 < 0$. By applying (45) for $n = 5$, f_5 and the above defined ω_δ, under the assumption $f_5(c_0) = c_0$, we have

$$5^{-c_0} \leq |1-2^{-c_0}-3^{-c_0}+4^{-c_0}|.$$

But this inequality is clearly impossible for any $c_0 < 0$. Hence f_5 has no fixed point in $(a_{\zeta_5(z)}, a_{\zeta_5^*(z)})$. Then, by applying Theorem 3.6, $\zeta_5(z)$ has a maximum density interval and consequently

$$R_{\zeta_5(z)} = [a_{\zeta_5(z)}, b_{\zeta_5(z)}].$$

For $n = 6$, we take a mapping $\delta : \mathscr{P} \to U$ satisfying $\delta(2) = -1$, $\delta(3) = 1$ and consider its corresponding $\omega_\delta : \mathbb{N} \to U$ defined in (44). Assume f_6 has some fixed point, say c_0, in the interval $(a_{\zeta_6(z)}, a_{\zeta_6^*(z)})$. By (8) $a_{\zeta_6^*(z)} < 0$ and then $(a_{\zeta_6(z)}, a_{\zeta_6^*(z)})$ contains only negative numbers, so $c_0 < 0$. By applying (45) for $n = 6$, f_6 and the above defined ω_δ, under the assumption $f_6(c_0) = c_0$, we have

$$5^{-c_0} \leq |1 - 2^{-c_0} + 3^{-c_0} + 4^{-c_0} - 6^{-c_0}|. \tag{55}$$

Regarding inequality (55) we consider the two possible cases: (a) $1 - 2^{-c_0} + 3^{-c_0} + 4^{-c_0} - 6^{-c_0} \geq 0$, (b) $1 - 2^{-c_0} + 3^{-c_0} + 4^{-c_0} - 6^{-c_0} < 0$. In (a), according to (55), we have the inequality

$$1 + 3^{-c_0} + 4^{-c_0} \geq 2^{-c_0} + 5^{-c_0} + 6^{-c_0},$$

that as we easily can check is not possible for any $c_0 < 0$. In (b), because of (55), we get

$$1 + 3^{-c_0} + 4^{-c_0} + 5^{-c_0} \leq 2^{-c_0} + 6^{-c_0}. \tag{56}$$

By a direct computation we see that (56) is only true for $c_0 \leq a_{\zeta_6(z)} \approx -2.8$ (observe that for $c_0 \approx -2.8$, inequality (56) becomes an equality and since $n = 6$ belongs to C^*, by Theorem 3.3, $a_{\zeta_6(z)}$ is a fixed point of the function f_6). Therefore for $c_0 > a_{\zeta_6(z)}$, (56) is not possible. Hence f_6 has no fixed point in $(a_{\zeta_6(z)}, a_{\zeta_6^*(z)})$. Then, by applying Theorem 3.6, $\zeta_6(z)$ has a maximum density interval and consequently

$$R_{\zeta_6(z)} = [a_{\zeta_6(z)}, b_{\zeta_6(z)}].$$

For $n = 7$, we take a mapping $\delta : \mathscr{P} \to U$ satisfying $\delta(2) = \delta(3) = \delta(5) = -1$ and consider its corresponding $\omega_\delta : \mathbb{N} \to U$ defined in (44). Assume f_7 has some fixed point, say c_0, in the interval $(a_{\zeta_7(z)}, a_{\zeta_7^*(z)})$. By (8) $a_{\zeta_7^*(z)} < 0$ and then $(a_{\zeta_7(z)}, a_{\zeta_7^*(z)})$ contains only negative numbers, so $c_0 < 0$. By applying (45) for $n = 7$, f_7 and the above defined ω_δ, under the assumption $f_7(c_0) = c_0$, we have

$$7^{-c_0} \leq |1 - 2^{-c_0} - 3^{-c_0} + 4^{-c_0} - 5^{-c_0} + 6^{-c_0}|. \tag{57}$$

We consider the two possible cases: (a) $1 - 2^{-c_0} - 3^{-c_0} + 4^{-c_0} - 5^{-c_0} + 6^{-c_0} \geq 0$, (b) $1 - 2^{-c_0} - 3^{-c_0} + 4^{-c_0} - 5^{-c_0} + 6^{-c_0} < 0$. In (a), according to (57), we have the inequality

$$1 + 4^{-c_0} + 6^{-c_0} \geq 2^{-c_0} + 3^{-c_0} + 5^{-c_0} + 7^{-c_0},$$

that is clearly impossible for any $c_0 < 0$. In (b), because of (57), we get

$$1 + 4^{-c_0} + 6^{-c_0} + 7^{-c_0} \leq 2^{-c_0} + 3^{-c_0} + 5^{-c_0}. \tag{58}$$

It is immediate to check that inequality (58) is false for any $c_0 < 0$. Hence f_7 has no fixed point in $(a_{\zeta_7(z)}, a_{\zeta_7^*(z)})$. Then, by applying Theorem 3.6, $\zeta_7(z)$ has a maximum density interval and consequently

$$R_{\zeta_7(z)} = [a_{\zeta_7(z)}, b_{\zeta_7(z)}].$$

For $n = 8$, we take a mapping $\delta : \mathscr{P} \to U$ satisfying $\delta(2) = 1$, $\delta(3) = \delta(5) = -1$ and consider its corresponding $\omega_\delta := \mathbb{N} \to U$ defined in (44). Assume f_8 has some fixed point, say c_0, in the interval $(a_{\zeta_8(z)}, a_{\zeta_8^*(z)})$. By (8) $a_{\zeta_8^*(z)} < 0$ and then $(a_{\zeta_8(z)}, a_{\zeta_8^*(z)})$ contains only negative numbers, so $c_0 < 0$. By applying (45) for $n = 8$, f_8 and the above defined ω_δ, under the assumption $f_8(c_0) = c_0$, we have

$$7^{-c_0} \leq |1 + 2^{-c_0} - 3^{-c_0} + 4^{-c_0} - 5^{-c_0} - 6^{-c_0} + 8^{-c_0}|. \tag{59}$$

Regarding inequality (59) we consider the two possible cases: (a) $1 + 2^{-c_0} - 3^{-c_0} + 4^{-c_0} - 5^{-c_0} - 6^{-c_0} + 8^{-c_0} < 0$, (b) $1 + 2^{-c_0} - 3^{-c_0} + 4^{-c_0} - 5^{-c_0} - 6^{-c_0} + 8^{-c_0} \geq 0$. In case (a), according to (59), we have the inequality

$$3^{-c_0} + 5^{-c_0} + 6^{-c_0} \geq 1 + 2^{-c_0} + 4^{-c_0} + 7^{-c_0} + 8^{-c_0},$$

which is clearly impossible for any $c_0 < 0$. In case (b), because of (59), we get

$$1 + 2^{-c_0} + 4^{-c_0} + 8^{-c_0} \geq 3^{-c_0} + 5^{-c_0} + 6^{-c_0} + 7^{-c_0}. \tag{60}$$

By an elementary analysis we can see that (60) is only true for $c_0 \leq a_{\zeta_8(z)} \approx -4.1$ (observe that for $c_0 \approx -4.1$ inequality (60) becomes an equality and since $n = 8$ belongs to C^*, by Theorem 3.3, $a_{\zeta_8(z)} \approx -4.1$ is a fixed point of the function f_8). Therefore for $c_0 \in (a_{\zeta_8(z)}, 0)$, (60) is not possible. Then, since by (8) $a_{\zeta_8^*(z)} < 0$, in particular (60) is not possible in $(a_{\zeta_8(z)}, a_{\zeta_8^*(z)})$. Hence f_8 has no fixed point in the interval $(a_{\zeta_8(z)}, a_{\zeta_8^*(z)})$. Then, by applying Theorem 3.6, $\zeta_8(z)$ has a maximum density interval and consequently

$$R_{\zeta_8(z)} = [a_{\zeta_8(z)}, b_{\zeta_8(z)}].$$

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