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Descriptive Topology and Functional Analysis II

In Honour of Manuel López-Pellicer Mathematical Work, Elche, Spain, June 7–8, 2018

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To Manuel López-Pellicer

Preface

On 7 and 8 June 2018, the 2nd Meeting in Topology and Functional Analysis, dedicated to the mathematical research of Professor Manuel López-Pellicer, was held at the Operations Research Center (CIO) of the Miguel Hernández University (UMH) of Elche. This book is the result of this Meeting. Covering topics in descriptive topology and functional analysis, including topological groups and Banach space theory, fuzzy topology, differentiability and renorming, tensor products of Banach spaces and aspects of C_p-theory, this volume is particularly useful to young researchers wanting to learn about the latest developments in these areas.

I am grateful to Springer for the publication of the research results presented at the conference, as well as to the attendees, participants, anonymous referees and invited speakers, most of whose contributions have been collected in this book. I am indebted to the Directors of the CIO and the Department of Statistics, Mathematics and Informatics of the UMH, Professors Juan Aparicio and José Valero, for their help and financial support, as well as to Professors José Mas and Santiago Moll for their LaTeX assistance. I also want to express my gratitude to Professor López-Pellicer for his mathematical expertise, generosity and unwavering friendship over many years. Finally, I would like to acknowledge the tremendous work of Professor López-Pellicer as Editor-in-Chief of RACSAM, the mathematical publication of the Royal Academy of Exact, Physical and Natural Sciences of Madrid: he has managed to transform a national magazine of limited diffusion into an important reference of international mathematical research.

Elche, Spain
January 2019

Juan Carlos Ferrando

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Editor and Contributors

About the Editor

Juan Carlos Ferrando (JCF) earned his Ph.D. in Mathematics from the University of Valencia in 1987, under Prof. Manuel López-Pellicer. He is currently a Professor at the Department of Statistics, Mathematics and Informatics at the Miguel Hernández University (MHU) of Elche (Spain), and a member of the Operations Research Center Institute of the MHU. Professor Ferrando has published more than 100 papers in peer-reviewed national and international journals and proceedings, mainly on functional analysis and descriptive topology, is co-author of a research book on metrizable barrelled spaces and co-editor of the proceedings volume on descriptive topology and functional analysis I, published by Springer. He has delivered a number of invited talks at national and international conferences.

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On the Mathematical Work of Professor Manuel López-Pellicer

In Honour of Manuel López-Pellicer

Juan Carlos Ferrando

Abstract We examine selected topics of the research work of professor Manuel López-Pellicer. After an introductory section, the paper is divided in four main sections, which include his publications on Set Topology, Locally Convex Space Theory, C_p-theory and Descriptive Topology. We shall also glance at his work on Popular Mathematics.

Keywords Locally convex spaces · Strongly barrelled conditions · Closed graph theorem · Algebras with the Nikodým property · Normed spaces · Tychonoff spaces · K-analytic spaces · Spaces of real-valued continuous functions · Spaces of vector-valued functions

Classifications 46AXX · 54DXX · 54CXX · 46GXX · 46BXX

1 Ph.D. Dissertation and Early Work

Professor López-Pellicer is member of the Royal Spanish Academy of Sciences since 1998, where he served as Secretary of its Mathematical Section (2000–2007) and Editor in Chief of the Academy journal RACSAM since 2004 until today. He got two M.Sc. degrees, one in Physics and the other one in Mathematics and earned his Ph.D. in Mathematics in 1969 with a dissertation titled *Asymptotic expansions and compact families of vector-valued holomorphic functions* (Spanish), being his advisor Manuel Valdivia. Full Professor in the Department of Applied Mathematics (1978–2015), nowadays López-Pellicer is Emeritus Professor of Universitat Politècnica de València (UPV). Most of his 11 students are Full Professors, with 52 descendants so far. His mathematical genealogy from Gauss is depicted below, where each mathematician is

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the advisor of the one underneath, except Felix Kelin, also a descendant from Gauss through C.L. Gerling (Göttingen, 1812) and J. Plücker (Marburg, 1823).

Carl Friedich Gauss (Helmstedt, 1799)
K.G.C. von Staudt (Erlangen-Nürnberg, 1822)
Eduardo Torroja Caballé (Universidad Central, 1873) and
Felix Klein (Universität Bonn, 1868)
Julio Rey Pastor (Madrid and Göttingen, 1909)
Ricardo San Juan (Universidad Central de Madrid, 1933)
Manuel Valdivia (Universidad Complutense de Madrid, 1963)
Manuel López-Pellicer (Universidad de Valencia, 1969)

Part of his doctoral dissertation was published in 1971 in RACSAM. In two 1972 papers a previous work of A. Plans on Hilbert space is generalized and some results of Valdivia on barrelled spaces are extended to the infrabarrelled case. The 1973 paper shows that a Tychonoff space X is realcompact if and only the space $C(X)$ with the compact-open topology τ_c is a Mazur space. This is the τ_c-version of a 1946 result on $C_p(X)$, originally due to S. Mazur [88], rediscovered by A. Wilansky in 1981, [112].

2 Research on General Topology

In 1929 Tychonoff gave an example of a regular space that is not completely regular. In [9] a new proof of this fact is provided and some problems are proposed. Among them, to characterize the compact spaces that have a stronger topology that is regular but not completely regular. Sharpening this question, professor López-Pellicer constructed in [72] the first example of a completely regular space (X, τ) whose associated k-space is not completely regular. Recall that the *k-extension* τ_k of a Hausdorff topology τ on X is the strongest topology on X that agrees with τ on compact sets.

Let ω_0 be the first infinite ordinal and ω_1 be the first uncountable ordinal. Denote by $[0, \omega_1]$ the set of all ordinals less than or equal to ω_1 equipped with the order topology and define $X = [0, \omega_1] \times [0, \omega_0]$ endowed with the product topology π, under which it is a Hausdorff compact space. For each (countable) limit ordinal $\alpha \in [0, \omega_1)$ a function $f_\alpha : X \to [0, 1]$ is defined in such a clever way that if

$$W_{\alpha,\varepsilon} := \{(\gamma, n) \in X : |f_\alpha(\gamma, n) - 1| < \varepsilon\} = f_\alpha^{-1}[(1 - \varepsilon, 1 + \varepsilon)]$$

then (i) $W_{\alpha,\varepsilon} \cap \{X \setminus ([0, \omega_1) \times \{\omega_0\})\}$ is π-open, (ii) $W_{\alpha,\varepsilon} \cap ([0, \omega_1) \times \{\omega_0\}) = \{(\alpha, \omega_0)\}$, and (iii) $\{\alpha\} \times [0, \omega_0] \subseteq W_{\alpha,\varepsilon}$ for each $\alpha \in \Omega$ and all $0 < \varepsilon < 1$.

Example 2.1 (López-Pellicer [72]) The family $\pi \cup \left\{W_{\alpha,\varepsilon} : \alpha \in \Omega, 0 < \varepsilon < 1\right\}$ is a subbase of a completely regular topology τ on X stronger than π such that (X, τ) is completely regular but, if τ_k denotes the k-extension of τ, the k-space (X, τ_k) is not completely regular.

Since (X, π) is completely regular, for each π-open set U there is a π-continuous map $g_U : X \to [0, 1]$ such that $U = g_U^{-1}([0, 1[)$. Hence, if Ω stands for the set of all countable limit ordinals of $[0, \omega_1]$, the initial topology on X determined by the family of real-valued functions $\{g_U, f_\alpha : U \in \pi, \alpha \in \Omega\}$ on X is a completely regular topology τ on X stronger than π (see [53, 3.7]). Property (i) shows that π and τ coincide on $X \setminus ([0, \omega_1) \times \{\omega_0\})$, property (ii) ensures that $[0, \omega_1) \times \{\omega_0\}$ is a discrete set in (X, τ) and property (iii) guarantees that the topologies π and τ coincide on each set $\{\alpha\} \times [0, \omega_0]$ *for all* $\alpha \in [0, \omega_1]$. If K is a τ-compact subset of X, if follows that $K \cap ([0, \omega_1) \times \{\omega_0\})$ is finite, which shows that $[0, \omega_1) \times \{\omega_0\}$ is a closed set in the k-extension τ_k of the topology τ. Now is not difficult to show that if h is any τ_k-continuous function from X into $[0, 1]$ there exists $\lambda \in [0, \omega_1)$ such that $h(\lambda, \omega_0) = h(\omega_1, \omega_0)$. Therefore, the point (ω_1, ω_0) is not separated in (X, τ_k) from $[0, \omega_1) \times \{\omega_0\}$, so (X, τ_k) is not completely regular.

In [71] is proved that Kōmura's $\mathscr{T}^f$ topology (see [66, 21.8]) is regular, which reveals that the non compatibility with the linear structure of a topology is independent from regularity. In [87] an embedding theorem for regular not completely regular spaces in products of appropriate topological spaces is given. The definition of those spaces is motivated by Tychonoff's example of a regular not completely regular space. It is shown in [73] that under certain conditions a topological space X is Baire if and only if it has the *Blumberg property with respect to* Y, i. e., if for each function $f : X \to Y$ there is a dense subset D of X such that $f|_D$ is continuous.

3 Research on Topological Vector Spaces

3.1 Strong Barrelledness Conditions

Although we survey here the research of professor López-Pellicer on strong barrelledness, let us mention that he also wrote a paper on weakly barrelledness properties (namely, [37]). A locally convex (Hausdorff) space E is called *barrelled* if each *barrel* of E (i. e., each absolutely convex, closed and absorbing set) is a neighborhood of the origin. A locally convex space E is called *Baire-like* (BL for short) if given an increasing sequence of closed absolutely convex subsets of E covering E, one of them is a neighborhood of the origin, [100]. The classic Amemiya-Kōmura theorem [1] guarantees that (i) each metrizable locally convex E is barrelled if and only if it is BL, and (ii) if E is BL and F is a dense barrelled subspace of E then F is BL. A locally convex space E is called *suprabarrelled* in [107] (SB for short) or *db* in [102] if each increasing sequence of linear subspaces of E covering E has a dense barrelled member. This definition was generalized by transfinite induction by Rodríguez Salinas [99] as follows. If we call *barrelled of class* 0 to the barrelled spaces, for every successor ordinal $\alpha + 1$ a locally convex space E is *barrelled of* *class* $\alpha + 1$ if in each increasing sequence of linear subspaces of E covering E there is one of them which is dense and barrelled of class α, and for every limit ordinal α a locally convex space E is *barrelled of class* α if E is barrelled of class β for all $\beta < \alpha$. A locally convex space E is *totally barrelled* (TB) if given a sequence of linear subspaces of E covering E, one of them is BL, [110]. A locally convex space E is *unordered Baire-like* (UBL) if each sequence of closed absolutely convex sets which covers E contains a neighborhood of the origin [104]. Full account of strong barrelledness conditions is given in [96, Chap. 9] and [68].

Suprabarrelled spaces compose the class of barrelled spaces of class 1. Barrelled spaces of classes n and ω_0, the latter spaces called barrelled of class $\aleph_0$ in [44], fit in the scheme of strong barrelledness properties as depicted in the following diagram

$$\text{Baire locally convex space} \Rightarrow \text{UBL} \Rightarrow \text{TB} \Rightarrow \text{barrelled of class } \aleph_0 \Rightarrow$$
$$\text{barrelled of class } n+1 \Rightarrow \text{barrelled of class } n \Rightarrow \text{BL} \Rightarrow \text{barrelled.}$$

Metrizable (LF) spaces are BL but not SB and each non normable Fréchet space contains a dense BL subspace that is not SB. Every infinite-dimensional Fréchet space contains a linear dense subspace which is TB but not UBL, [102]. Examples of TB spaces that are not Baire can be found in [108]. In [102] Saxon and Narayanaswami proved that a metrizable barrelled space E is not SB if and only if there exists a linear subspace F of the completion $\widehat{E}$ of E such that $E \subseteq F$ and F is dominated by an (LF)-space, i. e., there is a stronger locally convex topology τ on F so that (F, τ) is an (LF)-space. In [38] *quasi-suprabarrelled* spaces were introduced by removing the density requirement of the definition of suprabarrelled space. Quasi-suprabarrelled spaces have been called d spaces by Saxon in [101]. The next example (*cf.* [38]), where here and throughout the entire section $\omega = \mathbb{K}^{\mathbb{N}}$, shows that the space F need not coincide with E.

Example 3.1 (Ferrando and López-Pellicer [38]) In the space $\omega \simeq \omega^{\mathbb{N}}$ consider the sequence $\{E_n : n \in \mathbb{N}\}$ of non-barrelled subspaces $E_n = \omega \times \overset{n)}{\cdots} \times \omega \times \varphi_\omega \times \varphi_\omega \times \cdots$, where φ_ω means φ with the topology of ω. Then $E = \bigcup_{n=1}^{\infty} E_n$ is a dense and barrelled subspace of the Fréchet space ω which is neither quasi-suprabarrelled nor dominated by any (LF)-space.

Example 3.2 (Ferrando and López-Pellicer [40]) Equip $F_n := \omega^n \times \ell_1 \times \ell_1 \times \cdots$ with the product topology τ_n, and define the (LF)-space $(F, \tau) = \varinjlim (F_n, \tau_n)$. Then τ coincides with the relative topology of $F = \bigcup_{n=1}^{\infty} F_n$ as a linear subspace of ω and F is not suprabarrelled. Assuming by induction that there exists a dense barrelled subspace E in ω of class $s-1$ but not of class s, it turns out that $G := \bigcup_{n=1}^{\infty} G_n$ with $G_n = \omega^n \times E \times E \times \cdots$ is a dense barrelled subspace of $\omega^{\mathbb{N}}$ of class s but not of class $s+1$.

In both examples the use of the closed graph theorem for quasi-suprabarrelled or suprabarrelled spaces in the domain class is critical. Now, borrowing a classic result by Eidelheit (*cf.* [66, 31.4 (1)]) that states that each Fréchet space which is not Banach has a quotient isomorphic to ω, it follows that (see also [44, Theorem 3.3.3]).

Theorem 3.1 (Ferrando and López-Pellicer [39, 40]) *Given $n \in \mathbb{N}$, each non-normable Fréchet space contains a dense barrelled subspace of class $n-1$ but not of class n.*

As regards barrelled spaces of class $\aleph_0$, a detailed exposition is given in [44, Chap. 4]. Let us exhibit some separation examples. It was shown by Valdivia and Pérez Carreras in [110] that if E is a TB space which is not UBL and F is a locally convex space, then the projective tensor product $E \otimes_\pi F$ is TB if and only if $\dim F < \aleph_0$. Since $E \otimes_\pi F$ is barrelled of class $\aleph_0$ whenever both E and F are barrelled of class $\aleph_0$ and one of them is metrizable [44, Proposition 4.3.1], if E is an infinite-dimensional Fréchet space and F is a TB but not UBL dense linear subspace of E (see [102]), it turns out that $E \otimes_\pi F$ is a dense barrelled subspace of class $\aleph_0$ of $E \otimes_\pi E$ which is not TB. Particularly, if $E = \omega$ then the Fréchet space $\omega \widehat{\otimes}_\pi \omega \simeq \omega^{\mathbb{N}} \simeq \omega$ contains a dense linear subspace which is barrelled of class $\aleph_0$ but not TB.

Example 3.3 (Ferrando and López-Pellicer [43]) Each non normable infinite-dimensional Fréchet space contains a dense barrelled subspace of class $\aleph_0$ which is not TB.

Example 3.4 (Ferrando and López-Pellicer [43]) If (Ω, Σ, μ) is a nontrivial measure space, $L_p(\mu)$ with $1 \leq p < \infty$ has a dense subspace which is barrelled of class $\aleph_0$ but not TB.

If (Ω, Σ) is a measure space, it was established in [41] by Ferrando and López-Pellicer that the space $\ell_0^\infty(\Sigma)$ of all scalarly-valued Σ-simple functions $f : \Omega \to \mathbb{K}$ equipped with the supremum-norm is barrelled of class $\aleph_0$. Since, according to a result of Arias de Reyna, if Σ is a non trivial σ-algebra the space $\ell_0^\infty(\Sigma)$ is not TB (see [5]), it follows that $\ell_0^\infty(\Sigma)$ is another example of a normed barrelled space of class $\aleph_0$ which is not TB.

Another class of strong barrelled spaces is that of *baireled* spaces, introduced in [46]. A *linear web* of a locally convex space E is a countable family $\left\{E_{n_1 \cdots n_p} : p, n_1, \ldots, n_p \in \mathbb{N}\right\}$ of linear subspaces of E such that $\left\{E_{n_1} : n_1 \in \mathbb{N}\right\}$ is an *increasing* sequence covering E and if $\left(n_1, \ldots, n_{p-1}\right) \in \mathbb{N}^{p-1}$ then $\{E_{n_1 \cdots n_{p-1} n_p} : n_p \in \mathbb{N}\}$ is *increasing* and verifies that $\bigcup_{n_p=1}^{\infty} E_{n_1 \cdots n_{p-1} n_p} = E_{n_1 \cdots n_{p-1}}$. A *baireled* space is a locally convex space E such that each linear web in E contains a *strand* $\left\{E_{m_1 \cdots m_p} : p \in \mathbb{N}\right\}$ of barrelled and dense spaces.

Baireled spaces are strictly located between TB spaces and barrelled spaces of class $\aleph_0$, and bairelednes is transmitted from dense subspaces and inherited by closed quotients, countable-codimensional subspaces and finite products. If E is baireled and metrizable and F is UBL, then $E \otimes_\pi F$ is baireled [46, Proposition 4]. Hence if E is a metrizable TB space which is not UBL, then $E \otimes_\pi \ell_2$ is baireled but not TB. Non-baireled spaces which are barrelled of class $\aleph_0$ are obtained as usual in each non-normable Fréchet space by Eidelheit's quotient theorem after showing that ω contains a dense subspace E of those characteristics. Main result of [74] reveals the strongest barrelledness property known so far enjoyed by the Σ-simple scalarly-valued function space $\ell_0^\infty(\Sigma)$ over a σ-algebra Σ.

Theorem 3.2 (López-Pellicer [74]) *If* (Ω, Σ) *is a measurable space, then* $\ell_0^\infty(\Sigma)$ *is baireled.*

In order to get the proof, professor López-Pellicer introduced the notions of v-web and v-tree, two combinatorial objects that can be defined as follows. Denote by $W(\mathbb{N})$ the language of the infinite alphabet $\mathbb{N}$ without the empty word, i. e., $W(\mathbb{N}) = \bigcup\{\mathbb{N}^k : k \in \mathbb{N}\}$. If $w = (n_1, \ldots, n_i, \ldots, n_q) \in W(\mathbb{N})$, denote by $|w| = q$ the *length* of the word w and set $P_i w := (n_1, \ldots, n_i)$ for $1 \le i \le |w|$. Then, for each $T \subseteq W(\mathbb{N})$ define $P_i T := \{P_i w : w \in T,\ i \le |w|\}$. A non-empty subset of words $T \subseteq W(\mathbb{N})$ is called a *v-web* of $W(\mathbb{N})$ if

1. For each word $w \in T$ and each $1 \le i \le |w|$ there are infinitely many words in T of the same length than w whose first $i - 1$ letters coincide with those of w and whose ith letter is different in each one of these words.
2. For each $w \in T$ there is no longer word v in T such that $P_{|w|-1} v = P_{|w|-1} w$.
3. For each sequence $\{w_n\}_{n=1}^{\infty} \subseteq T$ with $|w_n| \ge n$ for all $n \in \mathbb{N}$ there are two consecutive words w_p and w_{p+1} whose first p letters do not coincide.

I will call *leaves* the words of a v-web T. If T is a v-web of $W(\mathbb{N})$ and $S \subseteq T$ does not contain any v-web, it can be shown that $T \setminus S$ does. Further, if $w \in W(\mathbb{N})$ then $b(w) := \{P_1 w, P_2 w, \ldots, P_{|w|} w\}$ is called the *branch* of w. The set $\mathscr{B}_T = \bigcup_{w \in T} b(w)$ consisting of the branches of the leaves is called the *v-tree* determined by T. One may see $\mathscr{B}_T$ as a tree with infinitely many branches of finite length (finitely many vertices or *knots*), each of them ending in a leaf (a word of T), and a root (the empty word), i. e., an arborescence such that each of his infinitely many branches has finite length and each father vertex $w \notin T$ (knot) of a branch of $\mathscr{B}_T$ has infinitely many sons (w, k) belonging to other branches of $\mathscr{B}_T$, but if a son belongs to T (i. e., if a son is a leaf) then all his siblings belong to T. For the proof of the theorem, one first establishes that $(*)$ if $\{E_w : w \in W(\mathbb{N})\}$ is a linear web in $\ell_0^\infty(\Sigma)$ and T is a v-web, there is some $w \in T$ such that E_w is barrelled. Then proceed by contradiction, assuming that there is a linear web $\{E_w : w \in W(\mathbb{N})\}$ in $\ell_0^\infty(\Sigma)$ none of whose strands is entirely formed by barrelled and dense subspaces. This produces a v-web $T \subseteq W(\mathbb{N})$ of leaves enjoying the property that no E_w with $w \in T$ is both dense and barrelled but each $E_{P_i w}$ with $1 \le i < |w|$ is, condition 3 above being consequence of the fact that there is no strand of dense and barrelled subspaces. That is, we get an v-tree with no leaf w indexing a barrelled and dense subspace. The aforementioned property of T clearly contradicts observation $(*)$.

3.2 On the Nikodým Boundedness Theorem

The Nikodým-Grothendieck theorem assures that each pointwise bounded family $\mathscr{M} = \{\mu_\alpha : \alpha \in A\}$ of scalarly-valued *bounded* finitely additive measures defined on a σ-algebra Σ of subsets of a set Ω is uniformly bounded. In other words, each set $\mathscr{M} \subseteq ba(\Sigma)$ such that $\sup_{\alpha \in A} |\mu_\alpha(E)| = k_E < \infty$ for every $E \in \Sigma$, is uniformly bounded, i.e., such that $\sup_{\alpha \in A} |\mu_\alpha| < \infty$. The norm involved here is the *variation norm* $|\mu| = |\mu|(\Omega) = \|\mu\|_1$, where

$$|\mu|(E) = \sup \sum_{i=1}^{n} |\mu(E_i)|$$

with the supreme over all finite partitions $\{E_1, \ldots, E_n\}$ of E by members of Σ. Another equivalent norm is $\|\mu\|_\infty := \sup\{|\mu(E)| : E \in \Sigma\}$, which satisfies that $\|\cdot\|_\infty \leq |\cdot| \leq 4\|\cdot\|_\infty$. If instead of a σ-algebra of sets, we consider a Boolean algebra $\mathscr{A}$ (or a Boolean ring $\mathscr{R}$) then, in case that $\mathscr{A}$ verifies the Nikodým boundedness theorem, since Schachermayer's [103, 2.4 Definition] such $\mathscr{A}$ is called a *Boolean algebra with property* (N).

Theorem 3.3 (Nikodým [93]) *If* (Ω, Σ) *is a measurable space, each pointwise bounded subset of* $ba(\Sigma)$ *is norm bounded.*

If $\mathscr{A}$ is an algebra of sets, a subclass $\mathscr{N}$ of $\mathscr{A}$ will be called a *Nikodým set* for $ba(\mathscr{A})$ if each set $\{\mu_\alpha : \alpha \in \Lambda\}$ in $ba(\mathscr{A})$ which is pointwise bounded on $\mathscr{N}$ is norm-bounded in $(ba(\mathscr{A}), \|\cdot\|_1)$. With this terminology, the baireledness of the space of Σ-simple functions provides the following extension of the Nikodým boundedness theorem.

Theorem 3.4 (López-Pellicer [74]) *If* $\{\Sigma_w : w \in W(\mathbb{N})\}$ *is an increasing web of subclasses of a* σ*-algebra* Σ *of subsets of a set* Ω*, there exists a strand* $\{\Sigma_{n_1 n_2 \ldots n_i} : i \in \mathbb{N}\}$ *consisting of Nikodým sets for* $ba(\Sigma)$.

Pioneering research on this subject comes from Valdivia's seminal paper [106]. Further research on this matter has been done in [59, 70], although no real improvement of the previous theorem has been achieved (due to [70, Proposition 1]).

Even if Σ is a σ-algebra of subsets of Ω and $\mathrm{Clo}(\mathrm{ult}(B))$ denotes the algebra of clopen subsets of the Stone space $\mathrm{ult}(\Sigma)$ of Σ, it must be pointed out that if $\{E_n : n \in \mathbb{N}\}$ is a sequence of elements of Σ, the union Q of their homologue counterparts $\{K_n : n \in \mathbb{N}\}$ in $\mathrm{Clo}(\mathrm{ult}(B))$ need not be a clopen set, so that it may happen that $Q \notin \mathrm{Clo}(\mathrm{ult}(B))$. This means that in general $\mathrm{Clo}\,\underline{(\mathrm{ult}(B))}$ is not a σ-algebra. The homologue of $\bigcup_{n=1}^{\infty} E_n \in \Sigma$ is not $\bigcup_{n=1}^{\infty} K_n$ but $\overline{\bigcup_{n=1}^{\infty} K_n}$. Which the Stone representation theorem assures is that if $E_n \mapsto K_n$ for each $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} E_n \mapsto \sup\{K_n : n \in \mathbb{N}\} = \overline{\bigcup_{n=1}^{\infty} K_n}$. So it make sense to extend the Nikodým-Grothendrieck boundedness theorem for algebras. Concerning the algebra $\mathscr{J}(K)$ of Jordan measurable subsets of the compact interval $K = \prod_{i=1}^{k} [a_i, b_i]$ of $\mathbb{R}^k$ with $a_i < b_i$ for $1 \leq i \leq k$, it has been showed in [109] that the space $\ell_0^\infty(\mathscr{J}(K))$ is suprabarrelled. This result has been extended in [69] by proving that $\ell_0^\infty(\mathscr{J}(K))$ is a baireled space. It is worthwhile to mention that the algebra $\mathscr{J}(I)$ of Jordan subsets of the interval $I = [0, 1]$ was the first example, due to Schachermayer [103], of a Boolean algebra with property (N) that does not have the so-called property (G). For more information about the research on the Nikodým-Dieudonné theorem, see [47] and references therein.

3.3 Barrelled Spaces of Vector-Valued Functions

Let (Ω, Σ) be a usually nontrivial measurable space and X be a normed space over $\mathbb{K}$. If $B(\Sigma, X)$ denotes the normed space of X-valued functions defined on Ω that are the uniform limit of a sequence of Σ-simple X-valued functions defined on Ω endowed with uniform convergence topology, the research on the barrelledness of locally convex spaces of vector-valued functions starts in 1982 when J. Mendoza shows that $B(\Sigma, X)$ is barrelled if and only if X is barrelled (*cf.* [89]). If K is a compact space and $C(K, X)$ denotes the linear space of all X-valued continuous functions defined on Ω endowed with the compact-open topology, the following result, also due to Mendoza, characterizes the barrelledness of $C(K, X)$ in terms of X.

Theorem 3.5 (Mendoza [90]) *$C(K, X)$ is barrelled if and only if both $C(K)$ and X are.*

If now Ω stands for a locally compact space and $C_0(\Omega, X)$ denotes the space over $\mathbb{K}$ of continuous functions $f : \Omega \to X$ vanishing at infinity (i. e., such that for $\varepsilon > 0$ there is a compact set $K_{f,\varepsilon}$ in Ω such that $\|f(\omega)\| < \varepsilon$ for $\omega \in \Omega \setminus K_{f,\varepsilon}$) equipped with the supremum norm, the following result answers a question raised by J. Horváth.

Theorem 3.6 (Ferrando, Kąkol and López-Pellicer [26]) *If Ω is a normal locally compact space, then $C_0(\Omega, X)$ is barrelled if and only if X is barrelled.*

If $c_0(\Gamma, X)$ denotes the linear space of all X-valued functions defined on Ω such that for each $\varepsilon > 0$ the set $\{\omega \in \Omega : \|f(\omega)\| > \varepsilon\}$ is finite, provided with the supremum norm, using the fact that each compact subset of a discrete topological space (hence locally compact and normal) is finite, it holds that $c_0(\Gamma, X) = C_0(\Gamma, X)$ whenever Γ is endowed with the discrete topology. So we have that $c_0(\Gamma, X)$ is barrelled if and only if X does. In [45] is shown that $c_0(\Gamma, X)$ is ultrabornological or UBL if and only if X enjoys the corresponding property. This research was continued in [85], where it is proved that $c_0(\Gamma, X)$ is suprabarrelled if and only if X is suprabarrelled. Then in [86], where is shown that $c_0(\Gamma, X)$ is suprabarrelled of class p if and only if X barrelled of class p for every $p \in \mathbb{N}$ and, finally, in [63], where among others properties it is shown that $c_0(\Gamma, X)$ is TB if and only if X is TB. These results are summarized in the next theorem.

Theorem 3.7 (López-Pellicer et al. [85, 86]) *Let Ω be a nonempty set, X be a normed space and $p \in \mathbb{N}$. Then $c_0(\Omega, X)$ is barrelled of class p or totally barrelled if and only if X is respectively barrelled of class p or totally barrelled.*

As regards the spaces $L_p(\mu, X)$ the following results come from [18, 20].

Theorem 3.8 (Drewnowski, Florencio and Paúl [20]) *If (Ω, Σ, μ) is atomless finite measure space and X a normed space, then $L_p(\mu, X)$ is barrelled for $1 \le p < \infty$.*

Theorem 3.9 (Díaz, Florencio and Paúl [18]) *If (Ω, Σ, μ) is atomless finite measure space and X a normed space, then $L_\infty(\mu, X)$ is barrelled.*

In [24] we obtained the following generalization of the latter two theorems.

Theorem 3.10 (Ferrando, Ferrer and López-Pellicer [24]) *If (Ω, Σ, μ) is atomless finite measure space and X a normed space, then $L_p(\mu, X)$ is barrelled of class $\aleph_0$ for $1 \leq p \leq \infty$.*

3.4 Metrizability of Precompact Sets

Although we shall define the notion of trans-separability for uniform spaces later on, by now let us recall that a locally convex space E is called *trans-separable* if for every absolutely convex neighborhood of zero U in E there exists a countable subset N_U of E such that $E = N_U + U$. Clearly, a locally convex space E is trans-separable if and only if E is isomorphic to a subspace of a product of separable Banach spaces. Linear subspaces, locally convex products, completions, and linear continuous images of trans-separable locally convex spaces are trans-separable. If E is a locally convex space with topological dual E', then clearly $\left(E, \sigma\left(E, E'\right)\right)$ and $\left(E', \sigma\left(E', E\right)\right)$ are always trans-separable spaces.

A completely regular space X is *quasi-Souslin* (*cf.* [108]) if there is a map φ from $\mathbb{N}^{\mathbb{N}}$ into the family of all (countably compact) subsets of X such that: (i) $\bigcup\left\{\varphi(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\right\} = X$, and (ii) if a sequence $\{\alpha_n\}_{n=1}^{\infty}$ in $\mathbb{N}^{\mathbb{N}}$ (here $\mathbb{N}$ is equipped with the discrete topology and $\mathbb{N}^{\mathbb{N}}$ with the product topology) converges to α and $x_n \in \varphi(\alpha_n)$ for all $n \in \mathbb{N}$, then $\{x_n\}_{n=1}^{\infty}$ has an cluster point in X contained in $\varphi(\alpha)$. Since each metrizable quasi-Suslin locally convex space is separable, it turns out that each quasi-Suslin locally convex space is trans-separable. In paper [28] we get the following applicable result.

Theorem 3.11 (Ferrando, Kąkol, López-Pellicer [28]) *In order for* [*pre*]*compact sets of a locally convex space E to be metrizable, it is both necessary and sufficient that E' endowed with the topology τ_c of uniform convergence on the compact sets of E* [*resp. with the topology τ_{pc} of uniform convergence on the precompact sets of E*] *be trans-separable.*

Since every quasi-Souslin locally convex space is trans-separable, our previous theorem includes Valdivia's [108, 1.4.3 (27)] *if (E', τ_c) is quasi-Souslin, then all compact sets in E are metrizable*. On the other hand, in [12] Cascales and Orihuela introduced a large class $\mathfrak{G}$ of locally convex spaces including (LF)-spaces and (DF)-spaces and proved that *every precompact set of a locally convex space in class $\mathfrak{G}$ is metrizable.* The following result from [35, Theorems 4 and 5] sheds light on this fact.

Theorem 3.12 (Ferrando, Kąkol, Saxon and López-Pellicer [35]) *If $E \in \mathfrak{G}$ then both its weak* dual $\left(E', \sigma\left(E', E\right)\right)$ and its Grothendieck dual $\left(E', \tau_{pc}\right)$, where τ_{pc}*

is the topology of uniform convergence on the precompact sets in E, is a quasi-Suslin space.

So, if $\mathfrak{M}$ denotes the class of locally convex spaces having quasi-Suslin weak* duals, it follows from the previous theorem that $\mathfrak{G} \subseteq \mathfrak{M}$ and that every precompact set of a space in class $\mathfrak{G}$ is metrizable, as stated. Although class $\mathfrak{M}$ is strictly wider than class $\mathfrak{G}$, there is one important case where both classes coincide. Recall that a locally convex space E is *ℓ_∞-barrelled* if every weak* bounded sequence in E' is equicontinuous.

Theorem 3.13 (Ferrando, Kąkol, Saxon and López-Pellicer [35]) *For an ℓ_∞-barrelled space, it happens that $\mathfrak{G} = \mathfrak{M}$.*

Class $\mathfrak{M}$ is the best known where the thesis of classic Kaplansky's theorem holds.

Theorem 3.14 (Ferrando, Kąkol, López-Pellicer and Saxon [35]) *Let E be a locally convex space. If $E \in \mathfrak{M}$, then E* (weak) *has countable tightness* (see below).

3.5 Closed Graph Theorems

There are a number of papers of López-Pellicer that contain a closed graph theorem. Here we shall exhibit three of them which are particularly useful. Recall that a locally convex space E is called quasi-suprabarrelled (*cf.* [38]) if given an increasing sequence of subspaces of E covering E, there is one of them which is barrelled. A locally convex space F is called a *Γ_r-space* (*cf.* [105, Theorem 2]) if every linear map $T : E \to F$ from a barrelled space E into F with closed graph is continuous. Each B_r-complete space is a Γ_r-space, so every Fréchet space is a Γ_r-space. For a definition of B_r-complete space and an account of classic closed graph theorems, see [67, Chap. 7]. The first closed graph theorem of our particular selection comes from [38].

Theorem 3.15 (Ferrando and López-Pellicer [38]) *Assume that E is a quasi-suprabarrelled space and let $\{F_n : n \in \mathbb{N}\}$ be an increasing sequence of linear subspaces of a locally convex space F covering F. Assume that each space F_n is dominated by a Γ_r-space. If T is a linear map from E into F with closed graph, then T is continuous.*

A nonempty set X is said to have a *resolution* if X is covered by a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets such that $A_\alpha \subseteq A_\beta$ for $\alpha \leq \beta$ coordinatewise, i. e., such that $\alpha(i) \leq \beta(i)$ for every $i \in \mathbb{N}$. A topological space (X, τ) is said to have a *relatively countably compact resolution* if X has a resolution consisting of relatively countably compact sets. Since Valdivia's quasi-Suslin spaces have a relatively countably compact resolution, the following closed graph theorem (taken from [30]) extends Valdivia's [108, I.4.2 (11)], and the case $E = F$ (previously considered in [58] in the locally convex setting) extends a classic result of De Wilde and Sunyach that states that each Baire K-analytic locally convex space (a completely regular space X is K-analytic if it is the continuous image of a Čech-complete and Lindelöf space) is a separable Fréchet space (see [108, I.4.3 (21)]).

Theorem 3.16 (Ferrando, Kąkol and López-Pellicer [30]) *Let E and F be topological vector spaces such that E is Baire and F admits a relatively countable compact resolution. If $T : E \to F$ is a linear map with closed graph, then T is continuous. If $E = F$, then E is a separable F-space.*

Recall that a nonempty topological space (X, τ) is called *Fréchet–Urysohn* if for every subset A of X and any point $x \in \overline{A}$, where $\overline{A}$ denotes the closure of A in X, there exits a sequence of points of A converging to x. The following version of the closed graph theorem for topological groups can be found in [36].

Theorem 3.17 (Ferrando, Kąkol, López-Pellicer and Śliwa [36]) *Let X and Y be topological groups such that X is Baire and Fréchet–Urysohn and Y admits a relatively countable compact resolution. If $T : X \to Y$ is a group homomorphism with closed graph, then T is continuous.*

A Fréchet–Urysohn additive topological group G for which every null sequence $\{x_n\}_{n=1}^{\infty}$ is a K-*sequence* (i. e., such that each subsequence $\{y_n\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{z_n\}_{n=1}^{\infty}$ so that $\sum_{n=1}^{\infty} z_n$ converges in G) is a Baire space [11, Theorem 3]. Other results of López-Pellicer related to the closed graph theorem can be found in [49].

4 Research on C_p-Theory

4.1 Bounding Tightness

If X is a topological space, we write $t(X) \leq \aleph_0$ to denote that X is *countably tight*, i. e., if $A \subseteq X$ and each $x \in \overline{A}$ there is a countable set $B \subseteq A$ such that $x \in \overline{B}$. A completely regular space X is said to have [*countable*] *bounding tightness* if $x \in \overline{A} \subseteq X$ implies that there is a topologically bounded [resp. topologically bounded and countable] set $B \subseteq A$ with $x \in \overline{B}$. If X has countable bounding tightness we write $t_b(X) \leq \aleph_0$. It is clear that

$$X \text{ Fréchet–Urysohn} \Rightarrow t_b(X) \leq \aleph_0 \Rightarrow t(X) \leq \aleph_0.$$

Franklin [50] recorded an example of a compact space with countable tightness, hence countable bounding tightness, which is not Fréchet–Urysohn. If X is completely regular, we denote by $C(X)$ the linear space of real-valued functions on X, or by $C_p(X)$ when equipped with the *pointwise* topology τ_p. If X is a k-space, an extension of a result of Grothendieck (see [6, III 4.15]) asserts that each topologically bounded set Y in $C_p(X)$ is relatively compact. In [57] is shown that $t_b\left(C_p[0,1]\right) > \aleph_0$. Since, as is well-known $t\left(C_p[0,1]\right) \leq \aleph_0$, it turns out that $C_p([0,1])$ is a countably tight locally convex space with uncountable bounding tightness. Countable bounding tightness has been used in [57] to characterize some classes of locally convex spaces.

Theorem 4.1 (Kąkol and López-Pellicer [57]) *An (LF)-space is metrizable if and only if has countable bounding tightness.*

If E is a metrizable (LF)-space then E is Fréchet–Urysohn and thus $t_b(E) \leq \aleph_0$. Conversely, if $t_b(E) \leq \aleph_0$ the fact that $t_b(\varphi) > \aleph_0$ assures that E contains no copy of φ. Since E is barrelled, then E is BL by virtue of a deep result of [100]. But each Baire-like (LF)-space is metrizable (see [94]). A classic result of C_p-theory is the following.

Theorem 4.2 (Asanov [7]) *If $C_p(X)$ is Lindelöf, then X^n is countably tight for all $n \in \mathbb{N}$.*

Since $C_p(\varphi)$ is a Lindelöf space, but φ has not countable bounding tightness, we see that the analog to Asanov's theorem for countable bounding tightness does not hold.

Theorem 4.3 (Kąkol and López-Pellicer [57]) *If X is a K-analytic space, then $C_p(X)$ is Fréchet–Urysohn if and only if $C_p(X)$ has bounding tightness.*

If X is K-analytic, there is a Čech-complete and Lindelöf space Y and a continuous map φ from Y onto X. Since the map $T : C_p(X) \to C_p(Y)$ given by $Tf = f \circ \varphi$ is a linear homeomorphism from $C_p(X)$ into $C_p(Y)$, if $f \in \overline{A} \subseteq C_p(X)$ and $C_p(X)$ has bounding tightness there is a topologically bounded set $B \subseteq A$ with $f \in \overline{B}$, so that $Tf \in T\left(\overline{B}\right) \subseteq \overline{T(B)}^{C_p(Y)}$. Since Y is a k-space, Grothendieck's theorem guarantees that $\overline{T(B)}^{C_p(Y)}$ is a compact set of $C_p(Y)$. Using the fact that $C_p(Y)$ is *angelic* (i. e., every [relatively] countably compact set is [relatively] compact and if A is relatively compact and $x \in \overline{A}$ there is a sequence in A that converges to x), we get a sequence $\{f_n\}_{n=1}^{\infty}$ in B such that $Tf_n \to Tf$ in $C_p(Y)$. Consequently $f_n \to f$ in $C_p(X)$.

4.2 Bounded Tightness

When working with topological vector spaces, a more natural property than that of bounding tightness seems to be the following. A locally convex space E is said to have [*countable*] *bounded tightness* if $x \in \overline{A} \subseteq E$ implies the existence of a bounded [resp. countable bounded] set $B \subseteq A$ such that $x \in \overline{B}$ (*cf.* [27]). Recall that a subset of a topological vector space is called *bounded* if it is absorbed by each neighborhood of the origin. Clearly, if a locally convex spaces E has [countable] bounding tightness, then E has [countable] bounded tightness. Moreover, if E is countable tight, then E has bounded tightness if and only if E has countable bounded tightness.

Theorem 4.4 (Ferrando, Kąkol and López-Pellicer [27]) *Each locally convex space E with bounded tightness is bornological.*

So, if E has bounded tightness, each bornivorous absolutely convex set of E is a neighborhood of the origin. As a consequence, each locally convex space E with bounded tightness is b-Baire-like, which means that if $\{A_n : n \in \mathbb{N}\}$ is an increasing sequence of bornivorous absolutely convex sets of E, one of them is a neighborhood of the origin. Other results on bounding and bounded tightness were obtained by Cascales and Raja [14] and Cascales, Kąkol and Saxon [16] (warning: in the latter paper as in many others is called bounded tightness what we have called bounding tightness).

According to the last theorem of the previous subsection, for a K-analytic X space Fréchet–Urysohn and bounding tightness are equivalent properties. Are they equivalent in $C_p(X)$ for any completely regular space X? In [65] an affirmative answer is given to a question raised by Nyikos about whether or not Fréchet–Urysohn and bounded tightness were equivalent properties for $C_p(X)$. Main result of this paper assures that for a linear topological space the two properties are the same. Concretely they show the following.

Theorem 4.5 (Kąkol, López-Pellicer and Todd [65]) *For a topological vector space E the following are equivalent*

1. *E is Fréchet–Urysohn.*
2. *For a subset A of E such that $\mathbf{0} \in \overline{A}$ there is a bounded subset B of A with $\mathbf{0} \in \overline{B}$.*
3. *For any sequence $\{A_n : n \in \mathbb{N}\}$ of subsets of E, each with $\mathbf{0} \in \overline{A_n}$, there is a sequence $B_n \subseteq A_n$ for $n \in \mathbb{N}$, such that $\bigcup_{n=1}^{\infty} B_n$ is bounded and $\mathbf{0} \in \overline{\bigcup_{k=1}^{n} B_k}$ for each $n \in \mathbb{N}$.*

This result implies that for any topological vector space Fréchet–Urysohn, [countable] bounding tightness and [countable] bounded tightness are the same. This fact allows us to state a classic result of C_p-theory as follows.

Theorem 4.6 *For a completely regular space X the following are equivalent*

1. *$C_p(X)$ is a Fréchet–Urysohn space.*
2. *$C_p(X)$ is a sequential space.*
3. *$C_p(X)$ is a k-space.*
4. *$C_p(X)$ has bounding tightness.*
5. *$C_p(X)$ has bounded tightness.*

4.3 Trans-separable Spaces

A uniform space $(X, \mathscr{N})$ is called *trans-separable* if for every vicinity U of $\mathscr{N}$ there is a countable subset Z of X such that $U[Z] = X$, [55, 56]. The term trans-separable was coined by Lech Drewnowski in [19]. Separable uniform spaces and

Lindelöf uniform spaces are trans-separable, and each uniform pseudometrizable trans-separable space is separable. A uniform space is trans-separable if and only if is uniformly isomorphic to a subspace of a uniform product of separable pseudometric spaces. For topological vector spaces E trans-separability means that E is isomorphic to a subspace of a product of metrizable and separable topological vector spaces. If E is a locally convex space with topological dual E', then E is trans-separable provided with the translation-invariant uniformity of the weak topology $\sigma(E, E')$. The class of trans-separable uniform spaces is hereditary, uniformly productive and closed under uniform continuous images. So each uniformly continuous image of a trans-separable space onto a uniform pseudometrizable space is separable. Trans-separable locally convex spaces are closed for linear subspaces, topological products and continuous linear images. Robertson proved that each uniform space $(X, \mathscr{N})$ that is covered by a family $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of precompact sets such that $K_\alpha \subseteq K_\beta$ whenever $\alpha \leq \beta$ is trans-separable, [98]. Moreover, every compact subset of a completely regular space X is metrizable if and only if the space $C_c(X)$ of all continuous functions defined on X equipped with the compact-open topology τ_c is trans-separable. In [29] we characterized the trans-separable uniform spaces as follows, where $\tau_\mathscr{N}$ denotes the uniform topology on X, that is, the topology defined by the uniformity $\mathscr{N}$ for X. Recall that a family $\mathscr{F}$ of functions from a uniform space $(X, \mathscr{N})$ into a uniform space $(Y, \mathscr{M})$ is called *uniformly equicontinuous* if for each $V \in \mathscr{M}$ there is $U \in \mathscr{N}$ such that $(f(x), f(y)) \in V$ whenever $f \in \mathscr{F}$ and $(x, y) \in U$.

Theorem 4.7 (Ferrando, Kąkol and López-Pellicer [29]) *The following are equivalent:*

1. *The uniform space $(X, \mathscr{N})$ is trans-separable.*
2. *Each pointwise bounded uniformly equicontinuous set of functions from $(X, \mathscr{N})$ to $\mathbb{R}$, provided with the usual uniformity, is metrizable in $C_p(X, \tau_\mathscr{N})$.*
3. *Each pointwise bounded uniformly equicontinuous set of functions from $(X, \mathscr{N})$ to $\mathbb{R}$, with the usual uniformity, has countable tightness in $C_p(X, \tau_\mathscr{N})$.*

Since on each equicontinuous family $\mathscr{F} \subseteq C(X, \tau_\mathscr{N})$ both topologies τ_p and τ_c coincide, this theorem can also be stated for $C_c(X, \tau_\mathscr{N})$. On the other hand, a topological space X is said to have the *Discrete Countable Chain Condition* (DCCC) if every discrete family of open sets is countable, which is equivalent to require that each continuous metrizable image of X is separable. Since a topological space X has the DCCC if and only if every pointwise bounded equicontinuous subset of $C(X)$ is τ_p-metrizable (see [13, Theorem 4]), it follows that each uniform space $(X, \mathscr{N})$ such that $(X, \tau_\mathscr{N})$ has the DCCC is trans-separable. If $X = [0, \omega_1)$ where ω_1 is the first ordinal of uncountable cardinality, and for each $\gamma \in X$ we set $U_\gamma := \{(\alpha, \beta) : \alpha = \beta \vee (\alpha \geq \gamma \wedge \beta \geq \gamma)\}$, then $\{U_\gamma : 0 \leq \gamma < \omega_1\}$ is a base of a uniformity $\mathscr{N}$ for X such that $(X, \mathscr{N})$ is trans-separable but $(X, \tau_\mathscr{N})$ has not the DCCC. It is shown in [112, Theorem 3.5] that if $C_p(X)$ is angelic, X has the DCCC.

5 Research on Descriptive Topology

5.1 Tightness and Distinguished Fréchet Spaces

Let us recall that the *tightness* $t(X)$ of a topological space X is the smallest cardinal κ such that for every set $A \subseteq X$ and each $x \in \overline{A}$ there exists a set $B \subseteq A$ with $|B| \leq \kappa$ such that $x \in \overline{B}$. On the other hand, the *character* $\chi(E)$ of a locally convex space E is the smallest cardinal for a base of neighborhoods of the origin. In terms of these two indices, classic Kaplansky's theorem reads as *each locally convex space E satisfies both* $t(E) \leq \chi(E)$ *and* $t\left(E, \sigma\left(E, E'\right)\right) \leq \chi(E)$. Note that a separable locally convex space need not have countable tightness, since $t\left(\mathbb{R}^{\mathbb{R}}\right) = \chi\left(\mathbb{R}^{\mathbb{R}}\right) = \mathfrak{c}$. Recall that a Fréchet space E is called *distinguished* if its strong dual $F = \left(E', \beta\left(E', E\right)\right)$ is barrelled, which always happens if E is a Banach space.

A completely regular space X is *K-analytic* if there exists a map $T : \mathbb{N}^{\mathbb{N}} \to 2^X$ with each $T(\alpha)$ compact such that $\bigcup\left\{T(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\right\} = X$ and if $\{\alpha_n\}_{n=1}^{\infty}$ converges to α in $\mathbb{N}^{\mathbb{N}}$ and $x_n \in T(\alpha_n)$ for every $n \in \mathbb{N}$, then $\{x_n\}_{n=1}^{\infty}$ has a cluster point $x \in T(\alpha)$, i.e., if there is an *upper semi-continuous* map (an *usc* map) $T : \mathbb{N}^{\mathbb{N}} \to 2^X$ such that $\bigcup\left\{T(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\right\} = X$. If T is countably compactly-valued we recover the definition of *quasi-Suslin* space. Valdivia showed [108, pp. 65–66] that *if E is a Fréchet space, the bidual* $E'' = \left(E', \beta\left(E', E\right)\right)'$ *of E equipped with the weak* topology is always quasi-Suslin, but is K-analytic if and only if* $\left(E', \mu\left(E', E''\right)\right)$ *is barrelled*, where $\mu(E, F)$ is the Mackey topology of the dual pair $\langle E, F\rangle$. In [34, Corollary 4] the following is proved.

Theorem 5.1 (Ferrando, Kąkol, López-Pellicer and Saxon [34]) *A Fréchet space E is distinguished if and only if its strong dual F has countable tightness, i. e.,* $t(F) \leq \aleph_0$.

The first example of a nondistinguished Fréchet space was provided by Grothendieck and Köthe (*cf.* [66]). This is the echelon space $\left(\lambda, \nu(\lambda, \lambda^{\times})\right)$ of all numerical double sequences $x = (x_{ij})$ such that $\sum_{i,j=1}^{\infty} |a_{ij}^{(n)} x_{ij}| < \infty$ for each $n \in \mathbb{N}$. The steps $\mathfrak{a}^{(n)} = (a_{ij}^{(n)})$ are defined so that $a_{ij}^{(n)} = j$ for $i \leq n$ and all $j \in \mathbb{N}$ and $a_{ij}^{(n)} = 1$ for $i > n$ and all $j \in \mathbb{N}$. Since λ is echelon space, it is a perfect sequence space and a Fréchet space in its normal topology $\nu(\lambda, \lambda^{\times})$ and $\left(\lambda, \nu(\lambda, \lambda^{\times})\right)' = \lambda^{\times}$. According to the previous theorem, the strong dual $\left(\lambda^{\times}, \beta\left(\lambda^{\times}, \lambda\right)\right)$ of the Grothendieck-Köthe space $\left(\lambda, \nu(\lambda, \lambda^{\times})\right)$ has uncountable tightness. Since according to [34, Example 5] it turns out that $t\left(\lambda^{\times}, \sigma\left(\lambda^{\times}, \lambda''\right)\right) > \aleph_0$, it follows that $\left(\lambda'', \sigma\left(\lambda'', \lambda^{\times}\right)\right)$ is not K-analytic [15, Theorem 4.6]. But according to Valdivia theorem $\left(\lambda'', \sigma\left(\lambda'', \lambda'\right)\right)$ is a quasi-Suslin space. So we get the following additional information about the space $\left(\lambda, \nu(\lambda, \lambda^{\times})\right)$.

Example 5.1 (Ferrando, Kąkol, López-Pellicer and Saxon [34]) The weak* bidual of the Grothendieck-Köthe space is a quasi-Suslin locally convex space which is not K-analytic.

If κ is an infinite cardinal, let us denote by $cf(\kappa)$ the *cofinality* of κ. A last result from [34, Corollary 4] is in order.

Example 5.2 (Ferrando, Kąkol, López-Pellicer and Saxon [34]) If $cf(\kappa) > \aleph_0$ then $C_c([0,\kappa))$, where here κ is regarded as a (limit) ordinal, is quasi-Suslin but not K-analytic.

5.2 Two Counterexamples

Our first example, taken from [36], exhibits a countably compact topological space G whose product $G \times G$ cannot be covered by an ordered family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of relatively countably compact sets. This shows that quasi-Suslin spaces are not productive (see also [48]).

Let X be a discrete space of cardinality $\mathfrak{c}$ and let X_1 and X_2 be two subspaces of X such that (i) $X_1 \cap X_2 = \emptyset$, (ii) $X_1 \cup X_2 = X$, and (iii) $|X_1| = |X_2| = \mathfrak{c}$. By (iii) there exists a bijection σ from X_1 onto X_2 whose Stone-Čech extension σ^β is a homeomorphism from βX_1 onto βX_2. Since X is a discrete space, we have $\overline{X_1}^{\beta X} \cap \overline{X_2}^{\beta X} = \emptyset$ and $\overline{X_1}^{\beta X} \cup \overline{X_2}^{\beta X} = \overline{X}^{\beta X}$. If Y is a subspace of X, then we can identify βY with $\overline{Y}^{\beta X}$. Hence $\beta X_1 \cap \beta X_2 = \emptyset$ and $\beta X_1 \cup \beta X_2 = \beta X$. Moreover, if N is a countable infinite subspace of X then $|\overline{N}^{\beta X}| = |\beta N| = |\beta \mathbb{N}| = 2^{\mathfrak{c}}$.

Now define a homeomorphism $\varphi : \beta X \to \beta X$ by $\varphi(x) = \sigma^\beta(x)$ if $x \in \beta X_1$ and $\varphi(x) = (\sigma^\beta)^{-1}(x)$ if $x \in \beta X_2$. Clearly $\varphi(\varphi(p)) = p$ for every $p \in \beta X$, and $p \in X$ if and only if $\varphi(p) \in X$. Since $\varphi(\beta X_1) = \beta X_2$ and $\varphi(\beta X_2) = \beta X_1$, the map φ does not have fixed points. If $\mathscr{N}$ denotes the family of all countable infinite subsets of X, put $Z := \bigcup\{\overline{N}^{\beta X} : N \in \mathscr{N}\}$ and denote by $\mathscr{M}$ the family of all countable infinite subsets of Z. Since $|\mathscr{N}| = \mathfrak{c}^{\aleph_0}$ then $|Z| = \mathfrak{c}^{\aleph_0} \times 2^{\mathfrak{c}} = 2^{\mathfrak{c}}$ and hence $|\mathscr{M}| = 2^{\mathfrak{c}}$. So, if $\mathfrak{m}$ is the first ordinal of cardinality $2^{\mathfrak{c}}$, one gets that $\mathscr{M} = \{M_\alpha : 0 \le \alpha < \mathfrak{m}\}$. Note that $\alpha < \mathfrak{m}$ implies that $|\alpha| = |[0,\alpha)| < 2^{\mathfrak{c}}$ and X is contained in Z. Moreover, it can be easily seen that if $M \in \mathscr{M}$ then $|\overline{M}^{\beta X}| = 2^{\mathfrak{c}}$. Now it is possible to define inductively a set $\Gamma = \{y_\gamma : 0 \le \gamma < \mathfrak{m}\}$ such that $y_\alpha \in \overline{M_\alpha}^{\beta X} \setminus (M_\alpha \cup \{\varphi(y_\gamma) : 0 \le \gamma < \alpha\})$ for every $0 \le \alpha < \mathfrak{m}$.

Example 5.3 (Ferrando, Kąkol, López-Pellicer and Śliwa [36]) Setting $G := X \cup \Gamma$, due to every countable infinite subset A of G is equal to M_α for some $0 \le \alpha < \mathfrak{m}$, it turns out that G contains a limit point of A. Therefore G is countably compact. On the other hand, the graph $\{(p, \varphi(p)) : p \in \beta X\}$ of the continuous map $\varphi : \beta X \to \beta X$ is closed in $\beta X \times \beta X$, so that $S := \{(x, \varphi(x)) : x \in X\}$ is a *closed subspace of* $G \times G$ homeomorphic to X. So S is uncountable and discrete, which prevents $G \times G$ to be covered by an ordered family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of relatively countably compact sets.

If X is completely regular, our second example characterize those $C_p(X)$ spaces whose Mackey dual is analytic. Let us denote by $L(X)$ the topological dual of $C_p(X)$

and by $L_p(X)$ the weak* dual of $C_p(X)$. The space $L(X)$ consists of the linear span of the vectors of the canonical copy $\delta(X)$ of X in $C_p(C_p(X))$, so that each $x \in X$ is depicted in $L(X)$ by the evaluation map δ_x at x, defined by $\delta_x(f) = f(x)$ for each $f \in C(X)$. This forces X to be represented in $L(X)$ as an algebraic basis. When $X = I = [0, 1]$ the following result is shown in [22].

Theorem 5.2 (Ferrando [22]) *The locally convex space $(L(I), \mu(L(I), C(I)))$ is weakly analytic but not K-analytic.*

This result is complemented in [64] by the following useful characterization.

Theorem 5.3 (Kąkol, López-Pellicer and Śliwa [64]) *For a completely regular space X, the Mackey dual of $C_p(X)$ is analytic if and only if X is countable.*

5.3 Metrizable-Like Topological Groups

In [62] is shown that a locally compact topological group G is metrizable if each compact subgroup K has countable tightness. In [51] is proved that each *cosmic* (i. e., with a countable network) Baire topological group is metrizable (and separable). A $\mathfrak{G}$-*base* of a topological group is a base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of neighborhoods of the neutral element e such that $U_\beta \subseteq U_\alpha$ if $\alpha \leq \beta$. Clearly, every metrizable topological group has a $\mathfrak{G}$-base. Conversely, every Fréchet–Urysohn topological group with a $\mathfrak{G}$-base is metrizable [52, Theorem 1.2]. As shown in [25, Theorem 2], a space $C_c(X)$ has a $\mathfrak{G}$-base of neighborhoods of the origin if and only if X has a covering $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ made up of compact sets with $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ that swallows the compact sets. Combining with Christensen's theorem [17, Theorem 3.3] one gets

Proposition 5.1 *For a metrizable space X the following are equivalent*

1. *X is a Polish space.*
2. *$C_c(X)$ has a $\mathfrak{G}$-base of neighborhoods of the origin.*

In [31] the notion of *Σ-base* is introduced. A topological group G is said to have a Σ-base if for some (pointwise) unbounded and directed subset Σ of $\mathbb{N}^\mathbb{N}$ the neutral element of G has a base of neighborhoods $\{U_\alpha : \alpha \in \Sigma\}$ such that $U_\beta \subseteq U_\alpha$ if $\alpha \leq \beta$ with $\alpha, \beta \in \Sigma$. The requirement for Σ to be directed is not a serious constraint, since if Γ is any unbounded subset of $\mathbb{N}^\mathbb{N}$ and $\mathscr{F}(\Sigma)$ stands for the family of finite subsets of Σ then $\Sigma := \{\sup \digamma : \digamma \in \mathscr{F}(\Gamma)\}$, where $\gamma = \sup \digamma \in \mathbb{N}^\mathbb{N}$ is given by $\gamma(i) = \sup\{\alpha(i) : \alpha \in \digamma\}$ for each $i \in \mathbb{N}$, is unbounded and directed and has the same cardinality as Γ. The following theorems from [31] characterize those $C_c(X)$ spaces that admit a Σ-base.

Theorem 5.4 (Ferrando, Kąkol and López-Pellicer [31]) *For completely regular space X, the following are equivalent*

1. *There is a compact covering* $\{A_\alpha : \alpha \in \Sigma\}$ *of* X*, with* Σ *unbounded and directed, such that* $A_\alpha \subseteq A_\beta$ *whenever* $\alpha \leq \beta$ *in* Σ*, that swallows the compact sets.*
2. $C_c(X)$ *has a* Σ*-base of absolutely convex neighborhoods of the origin.*

Theorem 5.5 (Ferrando, Kąkol and López-Pellicer [31]) *If* X *is a separable and metrizable space that is not a Polish space, then* $C_c(X)$ *admits a* Σ*-base of neighborhoods of the origin but it does not admit any* $\mathfrak{G}$*-base.*

A special class of Σ-bases, called Σ_2-bases, have close properties to those of $\mathfrak{G}$-bases. A subset Σ of $\mathbb{N}^{\mathbb{N}}$ is called *boundedly complete* if each bounded set Δ of Σ has a bound at Σ. If Σ is a boundedly complete subset of $\mathbb{N}^{\mathbb{N}}$ then Σ is itself directed. A Σ-base of neighborhoods of the unit element of a topological group G indexed by a boundedly complete set Σ of $\mathbb{N}^{\mathbb{N}}$ is referred to as a Σ_2-base. It is shown in [31] that (i) if G is a Fréchet–Urysohn topological group with a Σ_2-base then G is metrizable, and (ii) a $C_p(X)$ space has a Σ_2-base if and only if X is countable.

Example 5.4 (Ferrando, Kąkol and López-Pellicer [31]) In any ZFC consistent model for which $\aleph_1 = \mathfrak{d}$ but $\mathfrak{d} < \mathfrak{c}$ there exists a Σ_2-base of absolutely convex neighborhoods of the origin of the space $C_c([0, \omega_1))$ which is not a $\mathfrak{G}$-base.

5.4 ℓ_c-Invariance of Some Topological Properties

Two completely regular spaces X and Y are called ℓ_p*-equivalent* if the corresponding spaces $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Moreover, according to an old result of Nagata, if the topological rings $C_p(X)$ and $C_p(Y)$ are topologically isomorphic (as rings), then X and Y are homeomorphic (*cf.* [92]). A topological property $\mathscr{P}$ is said to be *preserved* by ℓ_p-equivalence if whenever two completely regular spaces X and Y are ℓ_p-equivalent and X has property $\mathscr{P}$, then Y has property $\mathscr{P}$ as well. Properties as metrizability, local compactness, countable weight, normality and paracompactness are not ℓ_p-invariant, whereas hemicompactness and the properties of being an $\aleph_0$-space, a Lindelöf Σ-space, a K-analytic space or an analytic space are all preserved by ℓ_p-equivalence (see [82]).

In [8, Theorem 3.3] Baars, de Groot and Pelant proved that complete metrizability is preserved by ℓ_p-equivalence in the class of metrizable spaces. Later on, Valov proved, that a Čech-complete and first countable space Y is metrizable when it is ℓ_p-equivalent to a metrizable space X (*cf.* [111]). The combination of the these facts assures that property of complete metrizability is preserved by the ℓ_p-equivalence for spaces satisfying the first axiom of countability.

In [60] two completely regular spaces X and Y are said to be ℓ_c*-equivalent* if the spaces $C_c(X)$ and $C_c(Y)$ are linearly homeomorphic. It must be pointed out that if X and Y are ℓ_p-equivalent and either (i) X is a μ-space or X is *Dieudonné complete* (in particular, if X is paracompact or realcompact), then X and Y are also ℓ_c-equivalent. The first statement is essentially consequence of the fact that is X is a μ-space, due to the Nachbin-Shirota theorem, the compact-open topology $\tau_c(X)$ on $C(X)$ coincides with the strong topology $\beta(C(X), L(X))$ of the dual pair $\langle C(X), L(X)\rangle$. Paper [60] investigates whether some properties $\mathscr{P}$ are preserved by ℓ_c-equivalence. Let us recall that a completely regular space X is called a $\aleph_0$-*space* if it has a countable k-network (*cf.* [91]). First countable or locally compact $\aleph_0$-spaces are separable and metrizable. A completely regular space X is said to be of *pointwise countable type* if for each $y \in X$ there exists a compact set K such that $y \in K$ and K has a countable base of neighborhoods in X (*cf.* [6, Chap. 0]). First countable, locally compact or, in general, Čech-complete spaces, are spaces of pointwise countable type. Main results of [60] are the following.

Theorem 5.6 (Kąkol, López-Pellicer and Okunev [60]) *The property of being a $\aleph_0$-space is preserved by ℓ_c-equivalence among the class of completely regular spaces.*

Theorem 5.7 (Kąkol, López-Pellicer and Okunev [60]) *The property of being metrizable and separable is preserved by ℓ_c-equivalence among the class of first countable spaces.*

Theorem 5.8 (Kąkol, López-Pellicer and Okunev [60]) *Second countability and the property of being a Polish space are both preserved by ℓ_c-equivalence among the class of spaces of pointwise countable type.*

Analogous results to those of Sect. 2 of the cited paper [60] can be found in [23]. The interested reader can find more information about ℓ_c-equivalence in the excellent expository paper [82].

5.5 Rainwater Sets and Weak K-Analyticity of $C^b(X)$

A subset Y of the dual closed unit ball B_{E^*} of a Banach space E is called a *Rainwater set* for E if every bounded sequence of E that converges pointwise on Y converges weakly in E (*cf.* [95]). If Y is a Rainwater set for E, then Y separates the points of E. Classic Rainwater's theorem [97] asserts that the set of the extreme points of the closed dual unit ball of E is a Rainwater set for E. In paper [32] López-Pellicer et al. study some topological properties of Rainwater sets for the Banach space $C^b(X)$ of real-valued continuous and bounded functions over a completely regular space X, equipped with the supremum-norm. The following result characterizes the Rainwater sets $Y \subseteq X$ for $C(X)$ with compact X.

Proposition 5.2 (Ferrando, Kąkol and López-Pellicer [32]) *Let X be a compact space and be $Y \subseteq X$. The following are equivalent*

1. *Y is a Rainwater set for $C(X)$.*
2. *Y is G_δ-dense in X.*
3. *Y is a James boundary for $C(X)$.*

If X is completely regular and υX denotes the Hewitt realcompactification of X, then X is *pseudocompact* if $C(X) = C^b(X)$ or alternatively if $\upsilon X = \beta X$. The previous proposition implies that if X is completely regular, then (i) X is a Rainwater set for $C^b(X)$ if and only if X is pseudocompact, and (ii) if $Y \subseteq X$ is a Rainwater set for $C^b(X)$, then X is pseudocompact and Y is G_δ-dense in X. Moreover, if Y is a pseudocompact subset of $B_{C^b(X)^*}$ (weak*) that contains X, then Y is a Rainwater set for $C^b(X)$.

A space X is called *Lindelöf* Σ if there are a set $\Sigma \subseteq \mathbb{N}^{\mathbb{N}}$ and a *usc* map $T : \Sigma \to 2^X$ with $\bigcup_{\alpha \in \Sigma} T(\alpha) = X$ ([6, Chap. II] or [61, Chap. 3]). A Banach space is *weakly countably determined* (WCD for short) or a Vašák space if is a Lindelöf Σ-space in its weak topology. For the definition of *weakly Lindelöf determined* (WLD for short) Banach space, see [61, Sect. 19.12]. Since $C^b(X)$ is WLD if X is pseudocompact [32], denoting by σ_Y the topology on $C^b(X)$ of the pointwise convergence on Y, we get

Theorem 5.9 (Ferrando, Kąkol and López-Pellicer [32]) *Let X be a completely regular space. The following are equivalent*

1. *There exists a Rainwater set Y for $C^b(X)$ such that $\left(C^b(X), \sigma_Y\right)$ is K-analytic (resp. WCD) and $C_p(Y)$ is angelic.*
2. *There exists a Rainwater set Y for $C^b(X)$ such that $\left(C^b(X), \sigma_Y\right)$ is both K-analytic (resp. WCD) and angelic.*
3. *$C^b(X)$ is weakly K-analytic (resp. WCD).*

As a corollary, we get classic Talagrand's result that asserts that if X is pseudocompact, then $C_p(X)$ is K-analytic (resp. Lindelöf Σ-space) if and only if $C(X)$ is weakly K-analytic (resp. WCD). Next theorem characterizes Talagrand and Gul'ko compactness.

Theorem 5.10 (Ferrando, Kąkol and López-Pellicer [32]) *Let X be a compact space and Y be a G_δ-dense subspace. Then X is a Talagrand compact set (resp. Gul'ko compact) if and only if the space $(C(X), \sigma_Y)$ is K-analytic (resp. a Lindelöf Σ-space).*

5.6 Quantitative Descriptive Topology

Some cardinal functions have shown to be useful in descriptive topology. If X is a completely regular space we can mention, among others, the *Lindelöf number* $\ell(X)$, the *density* $d(X)$, the *hereditarily density* $hd(X)$, the *weight* $w(X)$, the *network weight* $nw(X)$, the *tightness* $t(X)$ and the *Hewitt–Nachbin number* $q(X)$. We appeal to Arkhangel'skiĭ's book [6] for the definition of those indices. The *Nagami index* $Nag(X)$ has been introduced in order to generalize the notion of Lindelöf Σ-space. Recalling that the weight $w(X)$ of X is the least cardinality of an open base of X, the Nagami index is reported to be the smallest infinite cardinal number $\mathfrak{m}$ such that there exists a topological space Y of weight $\mathfrak{m}$ and a (compactly-valued) *usc* map $T : Y \to 2^X$ covering X. The Nagami index measures how far a completely regular space X is from being a Lindelöf Σ-space, in the sense that $Nag\,(X) \leq \aleph_0$ if and only X is a Lindelöf Σ-space. Here is useful to mention that if both X and $C_p\,(X)$ are Lindelöf Σ-spaces, one has $d\left(C_p\,(X)\right) = d\left(L_p\,(X)\right)$. Main theorem of [33] reads as follows.

Theorem 5.11 (Ferrando, Kąkol, López-Pellicer and Muñoz [33]) *If X is a topological space and $L \subseteq C_p\,(X)$ there exists a space Y and two completely regular topologies $\tau' \leq \tau$ on Y such that L is embedded in $C_p\,(Y, \tau)$ and (i) $Nag\,(Y, \tau) \leq Nag\,(\upsilon X)$, (ii) $w\left(Y, \tau'\right) \leq d\,(L)$, (iii) $nw\,(Y, \tau) \leq \max\{Nag\,(X), d\,(L)\}$, and (iv) $d\,(L) \leq \max\{Nag\,(L), d\,((Y, \tau)\}$.*

As a consequence, if $C_p\,(X)$ is a Lindelöf Σ-space (which implies that υX is a Lindelöf Σ-space) and $L \subseteq C_p(X)$ is separable, i. e., $d\,(L) \leq \aleph_0$, there is a separable submetrizable (by (iii) and Urysohn'metrizability theorem) Lindelöf Σ-space (Y, τ) (by (ii)) such that L is embedded into $C_p(Y, \tau)$. From this it can be readily shown that $C_p(X)$ is analytic if and only if $C_p(X)$ is separable and admits a compact resolution. A striking consequence of the previous theorem in locally convex space theory is the following result.

Corollary 5.1 (Cascales and Orihuela [13]) *A weakly compact set Y in a locally convex space E in class $\mathfrak{G}$ is weakly metrizable if and only if Y is contained in a weakly separable set.*

In Banach space theory some indices have also been introduced to get quantitative versions of the classic Krein or Eberlein theorems, among others. If H is a subset of a Banach space E, the index $k\,(H) := \sup\{d\,(x^{**}, E) : x^{**} \in \overline{H}^{w^*}\}$, where the closure is in the weak* topology w^* of the bidual E^{**} and $d\,(x^{**}, E) = \inf\{\|x^{**} - x\| : x \in E\}$, is zero if and only if H is weakly relatively compact. The inequality $k\,(\text{co}\,(H)) \leq 2k\,(H)$ for a bounded subset H of a Banach space E (*cf.* [21]) or $k\,(\text{co}\,(H)) \leq 5k\,(H)$ for a bounded subset H of the bidual E^{**} (*cf.* [54]) are quantitative versions of Krein's theorem. On the other hand, the index

$$ck\,(H) = \sup\left\{d\,(\text{Clust}_{E^{**}}\,(s)\,, E) : s \in H^{\mathbb{N}}\right\}$$

where $\text{Clust}_{E^{**}}\,(h)$ designs the set of cluster points of the sequence s in $E^{**}\,(w^*)$ and $d\,(A, B) = \inf\{\|x - y\| : x \in A,\ y \in B\}$, measures how far $H \subseteq E$ is far from being relatively weakly countably compact, in the sense that in the latter case $ck\,(H) = 0$. For a bounded subset H of a Banach space the equivalence of statements (i) $ck\,(H) = k\,(H) = 0$ and (ii) H is weakly relatively compact (*cf.* [2]) is a quantitative version of Eberlein's theorem. In [3] the following version of Eberlein's theorem for Fréchet spaces is shown.

Theorem 5.12 (Angosto, Kąkol and López-Pellicer [3]) *If H is a bounded subset of a Fréchet space E, the following are equivalent (i) $ck\,(H) = 0$, (ii) $k\,(H) = 0$, (iii) H is weakly relatively countably compact, and (iv) H is weakly relatively compact.*

The following version of Krein's theorem for Fréchet space was obtained in [4].

Theorem 5.13 (Angosto, Kąkol, Kubzdela and López-Pellicer [4]) *If H is a bounded set of a Fréchet space the inequality*

$$k\,(\text{co}\ (H)) \le \sqrt{k\,(H)}(3 - 2\sqrt{k\,(H)})$$

holds.

6 Publications on Linear Algebra and Popular Science

There are two beautiful publications of professor López-Pellicer on Linear Algebra, both in 1985 and coauthored by Rafael Bru. In [10] a necessary and sufficient condition for a maximal set M of linearly independent eigenvectors of an endomorphism f of a finite-dimensional vector space E to be extendable to a Jordan basis of E with respect to f is provided. In [10] the authors give a proof of the existence of a Jordan basis of an infinite-dimensional vector space E for each endomorphism f which is a root of some (annihilating) polynomial. As in the finite-dimensional case, they give a necessary and sufficient condition to extend a Jordan basis of an invariant subspace to a Jordan basis of the whole vector space. These two papers have been the seed of further research by professor Bru and his collaborators.

Regarding popular science. Seventeen papers in the *MathSciNet* list of publications of professor López-Pellicer are devoted to this subject. All of them have been published in Spanish journals and written in the tongue of Cervantes. There are many others which are not included in the MathSciNet, as for instance his speech on the occasion of his admission to the Spanish Royal Academy, in 1998, devoted to the history of Functional Analysis, which occupies 106 pages at the journal of the Academy. Many conferences, inside and outside of the Royal Academy have been also dedicated to popular science. The life and work of some universal mathematicians as Euclides, Fermat, Euler, Poincaré, Banach, von Neumann, Russell or Ramanujan, have focused the interest of professor López-Pellicer informative job. Particularly interesting are his articles on some Spanish scientists, engineers and mathematicians, as Jorge Juan (1713–1773), Agustin de Bethencourt (1758–1824), Julio Rey Pastor (1888–1962) and Manuel Valdivia (1928–2014). For the sake of completeness we include references [75–81, 83] on popular science which do not have been referenced in the MathSciNet list.

7 Work as Editor-in-Chief of RACSAM

Professor López-Pellicer is Editor-in-Chief of the Journal of the Royal Academy of Exact, Physical and Natural Sciences, Series A, Mathematics, acronym RACSAM, since 2004 (volume **98**) to the present. From 2004 to 2010 (volume **104**) the magazine

Table 1 Evolution of RACSAM Impact Factor (*Source* InCites *Journal Citation Reports* dataset updated Jun 06, 2018)

Year	Total cites	Citable items	Impact factor	Rank	Quartile	JIF percentile
2011	91	33	0.340	239/289	Q4	17.474
2012	149	28	0.733	84/296	Q2	71.791
2013	171	30	0.689	98/302	Q2	67.715
2014	208	64	0.776	95/312	Q2	69.712
2015	221	52	0.468	223/312	Q3	28.686
2016	295	56	0.690	140/311	Q2	55.145
2017	430	87	1.074	56/309	Q1	82.039

published two annual issues, edited by the Academy itself. During years 2008, 2009 and 2010 the journal underwent evaluation in the JCR (*Journal Citation Reports*), which requires follow-up by the JCR for three consecutive years. In 2011 Professor López-Pellicer was the architect of the signing of a contract with Springer. Currently the journal publishes 4 issues per year. The evolution of the journal since the entry of RACSAM in the JCR list is depicted in Table 1.

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A Note on Nonautonomous Discrete Dynamical Systems

In Honour of Manuel López-Pellicer

Gerardo Acosta and Manuel Sanchis

Abstract A discrete nonautonomous dynamical system (a NDS for short) is a pair $(X, f_{1,\infty})$ where X is a topological space and $f_{1,\infty}$ is a sequence of continuous functions $(f_1, f_2, \ldots)$ from X to itself. The orbit of a point $x \in X$ is defined as the set $\mathcal{O}_{f_{1,\infty}}(x) := \{x, f_1^1(x), f_1^2(x), \ldots, f_1^n(x), \ldots\}$. NDS' were introduced by S. Kolyada and L. Snoha in 1996 and they are related to several mathematical fields; among others the theory of difference equations. Notice that NDS' generalize the *usual* notion of a discrete dynamical system: indeed, if suffices to take $f_{1,\infty}$ as a constant sequence. The aim of this note is twofold. First we analyze several definitions of a periodic point in the framework of NDS'. The interest of this notion comes from the fact that, in the realm of discrete dynamical systems, the third condition of the definition of Devaney's chaos (sensitivity) follows from the first two (transitivity and the set of periodic points is dense). This result need not be true for NDS' and the results in this context depend upon the definition of a periodic point we consider. Secondly, we present several results on transitivity. In contrast to the situation for discrete dynamical systems, there exists second countable, perfect metric NDS' with the Baire property which have transitive points but they are not transitive. Among other things, we study the relationships between these two notions.

Keywords Cantor set · Devaney's chaos · Discrete dynamical systems · Discrete nonautonomous dynamical systems · Equicontinuity · Periodic points · Point transitivity · Sensitive dependence on initial conditions · Transitivity

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1 Introduction

A *discrete dynamical system* (an ADS for short) is a pair (X, f) where X is a topological space and $f: X \to X$ is a continuous function. Discrete dynamical systems can be generalized in the following way: a *nonautonomous discrete dynamical system* (a NDS for short) is a pair $(X, f_{1,\infty})$ where X is a topological space and $f_{1,\infty}$ is a sequence of continuous functions $(f_n: X \to X)_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, the *n-iterate* of a NDS $(X, f_{1,\infty})$ is the composition

$$f_1^n := f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1.$$

The symbol f_1^0 will stand for the identity map from X onto itself.

Notice that an ADS (X, f) coincides with the NDS $(X, f_{1,\infty})$, where $f_n = f$ for each $n \in \mathbb{N}$. Given a NDS $(X, f_{1,\infty})$, the *orbit* of a point $x \in X$ is the set

$$\mathcal{O}_{f_{1,\infty}}(x) := \left\{x, f_1^1(x), f_1^2(x), \ldots, f_1^n(x), \ldots\right\}.$$

NDS' were introduced by S. Kolyada and L. Snoha in [15]. The paper [5] describes some recent developments on the theory of NDS'. Note that NDS' are related to nonautonomous difference equations: indeed, given a compact metric space (X, d) and a sequence of continuous functions $(f_n: X \to X)_{n\in\mathbb{N}}$, if for each $x \in X$ we set

$$\begin{cases} x_0 &= x, \\ x_{n+1} &= f_n(x_n), \end{cases}$$

we obtain a nonautonomous difference equation (see, for example, [23, 25]). Observe that, from the definition of a NDS, the orbit of a point forms a solution of a nonautonomous difference equation. The orbit can be also described by the difference equation $x_1 = x$ and $x_{n+1} = f_1^n(x_n)$ for each $n \in \mathbb{N}$.

An ADS (X, f) is said to be *topologically transitive* (transitive for short) if for any pair U and V of nonempty open sets of X there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. If X is a metric space via a metric d, it is said to have *sensitive dependence on initial conditions* (*sensitive* for short) if there exists a constant $\delta > 0$ such that for any $x \in X$and $\varepsilon > 0$ there exists $y \in Y$ with $d(x, y) < \varepsilon$ such that $d(f^n(x), f^n(y)) \geq \delta$ for some $n \in \mathbb{N}$. If we replace f^n by f_1^n we obtain the corresponding definitions of transitivity and sensitivity for NDS'. Note that if X contains an isolated point, then the NDS $(X, f_{1,\infty})$ is not sensitive.

The reader familiar with the theory of uniform spaces will have noticed that the definition of sensitivity is given by means of the uniformity induced by the metric d. Thus, it can be extended to uniformizable spaces, that is, to Tychonoff spaces.

The paper is organized as follows. In the second section we present and analyze three notions of a periodic point for NDS'. Periodic points are of great interest in the study of Devaney's chaos. In this set-up, we present several results and examples. Section 3 is devoted to transitivity. Among other results we study the relationship between transitive points and transitivity. Several examples are presented. For instance, we show that there exist NDS' with equicontinuous points which are nontransitive points. This is in striking contrast to the case of ADS.

Our notation and terminology are standard. The interested reader might consult [13] for further information about ADS' and [4, 5] for NDS'. Further information in chaos, transitivity and sensitivity for NDS' can be found in [6, 11, 14, 18]. For topological notions and concepts not defined here see [14].

2 Periodic Points

An interesting question in the framework of NDS' is how to obtain a suitable definition of a periodic point. Given a NDS $(X, f_{1,\infty})$ where $f_{1,\infty}$ is the sequence $(f_n : X \to X)_{n\in\mathbb{N}}$, several different definitions are known, each one with its pros and cons (see, for example [20–22, 26]):

(P1) A point $x \in X$ is periodic if there exists $k \in \mathbb{N}$ such that $f_1^k(x) = x$.
(P2) A point $x \in X$ is periodic if there exists $k \in \mathbb{N}$ such that $f_1^{kn}(x) = x$ for all $n \in \mathbb{N}$.
(P3) A point $x \in X$ is periodic if it satisfies the following conditions: (a) there exists $k \in \mathbb{N}$ such that $f_1^{k+n}(x) = f_1^n(x)$ for all $n \in \mathbb{N} \cup \{0\}$, (2) if $k \neq 1$ then k is the smallest natural number such that $f_1^k(x) = x$, and $f_1^1(x), f_1^2(x), \ldots, f_1^{k-1}(x)$ are pairwise different.

It is clear that

$$(\text{P3}) \implies (\text{P2}) \implies (\text{P1})$$

and easy examples show that the converses fail to be true. If in the above notions we have $k = 1$, then we say that x is a *fixed point* in the sense of either (P1), (P2) or (P3). For example $x \in X$ is a fixed point in the sense of (P1) if $f_1(x) = x$. In such case $f_2(x)$ and $f_1(x)$ can be different points of X. Indeed, for the NDS $(\mathbb{I}, f_{1,\infty})$ described in Example 3.1 below, a_1 is a fixed point in the sense of (P1) and $f_n(a_1) \neq a_1$ for every $n \in \mathbb{N} \setminus \{1\}$. Note that x is a fixed point in the sense of (P2) if $f_1^n(x) = x$ for each $n \in \mathbb{N}$. In such situation $\mathcal{O}_{f_{1,\infty}}(x) = \{x\}$ is a finite set.

Theorem 2.1 *Let* $(X, f_{1,\infty})$ *be a* NDS. *Then* $x \in X$ *is a fixed point in the sense of* (P2) *if and only if* $f_n(x) = x$ *for every* $n \in \mathbb{N}$.

Proof Assume first that x is a fixed point in the sense of (P2). Then $f_1^n(x) = x$ for each $n \in \mathbb{N}$. In particular, $f_1(x) = f_1^1(x) = x$ and if $n \geq 2$, then $f_n(x) = f_n(f_1^{n-1}(x)) = f_1^n(x) = x$. This shows that $f_n(x) = x$ for every $n \in \mathbb{N}$.

Now assume that $f_n(x) = x$ for every $n \in \mathbb{N}$. It is straightforward to show that $f_1^n(x) = (f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1)(x) = x$ for each $n \in \mathbb{N}$, so x is a fixed point in the sense of (P2). □

Note that x is a fixed point in the sense of (P3) if $f_1^{n+1}(x) = f_1^n(x)$ for all $n \in \mathbb{N} \cup \{0\}$. Using this equality it is not difficult to prove that x is a fixed point in the sense of (P3) if and only if $f_1^n(x) = x$ for each $n \in \mathbb{N}$. Hence fixed points according to (P2) coincide with the ones according to (P3).

Now we construct a NDS in which every point is periodic in the sense of (P3). Let $f: X \to X$ be a bijective function. Define $f_n = f$ if n is odd, $f_n = f^{-1}$ if n is even and $f_{1,\infty} = (f_n)_{n\in\mathbb{N}}$. Then $f_1^n = f$ if n is odd and f_1^n is the identity function from X to X if n is even. Hence, for each $x \in X$, we have $\mathcal{O}_{f_{1,\infty}}(x) = \{x, f(x)\}$. If $f(x) \neq x$, then $f_1^{2+n}(x) = f_1^n(x)$ for every $n \in \mathbb{N} \cup \{0\}$ and x is a periodic point in the sense of (P3) with $k = 2$. If $f(x) = x$ then $f_1^{n+1}(x) = f_1^n(x)$ for all $n \in \mathbb{N} \cup \{0\}$ and x is a fixed point in the sense of (P3).

Let $x \in X$ be a periodic point in the sense of (P3). Let $k \in \mathbb{N}$ be such that $f_1^{k+n}(x) = f_1^n(x)$ for all $n \in \mathbb{N} \cup \{0\}$. Then $\mathcal{O}_{f_{1,\infty}}(x) = \{x, f_1^1(x), \ldots, f_1^{k-1}(x)\}$ is a finite set. It is not difficult to prove that $f_1^{km+n}(x) = f_1^n(x)$ for each $m \in \mathbb{N} \cup \{0\}$. If $x \in X$ is a periodic point in the sense of either (P1) or (P2) then the set $\mathcal{O}_{f_{1,\infty}}(x)$ is either finite (for example, when x is a fixed point in the sense of (P2)) or infinite (see Theorem 2.3). In any of the three situations, if x is a periodic point and $y \in \mathcal{O}_{f_{1,\infty}}(x)$ then y is not necessarily a periodic point (see Theorem 2.3 and Example 2.1 in this section).

Assume that the sequence $f_{1,\infty} = (f_n)_{n\in\mathbb{N}}$ satisfies the following property:

(P) there exists $k \in \mathbb{N}$ such that for each $i \in \{1, 2, \ldots, k\}$ we have $f_i = f_{nk+i}$ for every $n \in \mathbb{N} \cup \{0\}$.

Then any point $x \in X$ for which $f_1^k(x) = x$ satisfies that $f_1^{k+n}(x) = f_1^n(x)$ for all $n \in \mathbb{N} \cup \{0\}$. However, even under the assumption of (P), a periodic point in the sense of (P1) is not necessarily a periodic point in the sense of (P2). To see this, let $I_3 = \{a, b, c\}$ be a set with three points and the discrete metric. For each $n \in \mathbb{N}$, define $h_n: I_3 \to I_3$ so that $h_i = h_j$ if and only if i and j are congruent mod 3. Then $h_{1,\infty} = (h_n)_{n\in\mathbb{N}}$ satisfies (P) with $k = 3$. Define h_1, h_2 and h_3 as follows: $h_1(a) = b$, $h_2(b) = c$, $h_3(c) = a$, $h_1(b) = a$, $h_2(a) = b$, $h_3(b) = a$, $h_1(c) = a$, $h_3(a) = c$ and $h_2(c) = a$. Then a is a periodic point in the sense of (P3), $\mathcal{O}_{h_{1,\infty}}(a) = \mathcal{O}_{h_{1,\infty}}(b) = \mathcal{O}_{h_{1,\infty}}(c) = I_3$, b and c are periodic point in the sense of (P1) but not in the sense of (P2). Note that the NDS $(I_3, h_{1,\infty})$ is transitive, not sensitive and the set of periodic points in the sense of (P1) is dense in I_3.

Let (X, f) be an ADS where X is an infinite T_1 space. It is known that if (X, f) is transitive, then X does not contain isolated points. This result is not valid for NDS' as [26, Example 2.3] shows. In such example the functions that describe $f_{1,\infty}$ are all constant. In Example 3.4 we present a NDS in which the elements of $f_{1,\infty}$ are not constant functions. In the presence of isolated points we have the following result.

Theorem 2.2 *If $(X, f_{1,\infty})$ is a transitive* NDS *and $x \in X$ is an isolated point of X, then x is a periodic point in the sense of* (P1) *with a dense orbit.*

Proof Since $\{x\}$ is open in X by transitivity there exists $k \in \mathbb{N}$ such that $f_1^k(\{x\}) \cap \{x\} \neq \emptyset$, so $f_1^k(x) = x$ and then x is a periodic point in the sense of (P1). Now if

U is a nonempty open subset of X then, by transitivity, there is $n \in \mathbb{N}$ such that $f_1^n(\{x\}) \cap U \neq \emptyset$, so $f_1^n(x) \in U$ and then the orbit of x is dense in X. □

In order to avoid isolated points in our examples and results, from now on if $(X, f_{1,\infty})$ is a NDS, we will consider that X is a T_1 space without isolated points (this implies that X is infinite). Hence no periodic point in the sense of (P3) has a dense orbit. The following theorem shows that periodic points in the sense of (P2) (so that in the sense of (P1)) can have a dense orbit.

Theorem 2.3 *In the unit interval* $\mathbb{I}$, *with its usual topology, there is a* NDS $(\mathbb{I}, f_{1,\infty})$ *with the following properties: there exist a periodic point* x *in the sense of* (P2) *with a dense orbit and* $y \in \mathcal{O}_{f_{1,\infty}}(x)\backslash\{x\}$ *so that neither* y *is a periodic point nor its orbit is dense. Moreover,* x *is not a periodic point in the sense of (P3).*

Proof Let $\{s_1, s_2, \dots, s_n, \dots\} \subset \mathbb{I} \setminus \{0, 1\}$ be a dense subset of $\mathbb{I}$ such that $1/2 < s_1$. Let $(\mathbb{I}, f_{1,\infty})$ be the NDS defined as follows: for each $n \in \mathbb{N}$, $f_n(0) = 0$, $f_n(1) = 0$, $f_{2n+1}(1/2) = s_{n+1}$, $f_{2(n+1)}(s_{(n+1)}) = 1/2$ and both f_{2n+1} and $f_{2(n+1)}$ are piecewise linear elsewhere. Moreover

$$f_1(x) = \begin{cases} s_1, & \text{if } x = 1/2, \\ 1/2, & \text{if } x = s_1, \end{cases} \qquad \text{and} \qquad f_2(x) = \begin{cases} 1, & \text{if } x = 1/2, \\ 1/2, & \text{if } x = s_1, \end{cases}$$

and both f_1 and f_2 are piecewise linear elsewhere. Note that the orbit of the point $x = 1/2$ is the set

$$\mathcal{O}_{f_{1,\infty}}(x) = \{1/2, s_1, 1/2, s_2, \dots, 1/2, s_n, \dots\}$$

and the orbit of the point $y = s_1$ is the set $\{s_1, 1/2, 1, 0, 0, \dots\}$. Hence, x is a periodic point in the sense of (P2), whose orbit is dense in $\mathbb{I}$, while $y \in \mathcal{O}_{f_{1,\infty}}(x) \setminus \{x\}$ is not a periodic point, and the orbit of y is not dense in $\mathbb{I}$. Since the orbit of a periodic point in the sense of (P3) is a finite set, x is not a periodic point in the sense of (P3). □

Our interest in a suitable definition of a periodic point in the framework of NDS' comes from the fact that they play a central role in the definition of Devaney's chaos. In order to make our description of this fact precise, we give the original definition.

Definition 2.1 ([9]) Let (X, d) be an infinite metric space. An ADS (X, f) is Devaney chaotic if it satisfies the following conditions.

(i) (X, f) is topologically transitive.
(ii) (X, f) has a dense set of periodic points.
(iii) (X, f) has sensitive dependence on initial conditions.

Recall that two ADS' (X, f) and (Y, g) are said to be *conjugate* if there exists a homeomorphism $h\colon X \to Y$ such that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{g} & Y \end{array}$$

commutes. The definition of chaos in the sense of Devaney rises the question whether Devaney's chaos is preserved by conjugation. The reason is apparent: indeed, it is an easy matter to find an example showing that sensitivity is not preserved under conjugation (see, for example, [7]). In the sitting of ADS', Banks et al. [7] answered this question in 1992. The surprisingly neat answer is that Condition (iii) in Devaney's definition follows from the two previous ones.

Remark 2.1 As a matter of definition, sensitivity is preserved under conjugations that are uniform isomorphisms [19]. Since compact spaces have a unique uniformity, this fact implies that sensitivity is preserved when working with compact spaces [7].

In the realm of NDS' the situation is quite different. In 2016, Lan [17, Problem 1] proposed the following question.

In nonautonomous dynamical systems, does transitivity together with density of periodic points imply sensitivity?

By means of additional conditions, a positive answer was given by Zhu et al. [26, Theorems 3.1 and 3.2], using periodic points in the sense of (P3). Later, in the same situation, Miralles et al. [20, Theorem 2.4] provided a positive answer using periodic points in the sense of (P2) and assuming that the sequence $f_{1,\infty}$ which defines the NDS converges uniformly to a function $f: X \to X$. However, in the paper above, Lan showed that the answer is in general negative if we use periodic points in the sense of (P1). Moreover, Sánchez et al. [21, Example 4.4] gave an example on the interval of a transitive NDS with a dense subset of periodic points in the sense of (P1) which is not sensitive.

Nevertheless, in considering periodic points in the sense (P3), Zhu et al. [26, Theorems 3.1 and 4.1] have obtained interesting results. We summarize the most significative ones in the following theorem.

Theorem 2.4 *Let* $(X, f_{1,\infty})$ *be a* NDS. *The following holds:*

(i) *If* X *is an unbounded metric space and* $(X, f_{1,\infty})$ *is transitive and the set of periodic points in the sense of* (P3) *is dense in* X*, then* $(X, f_{1,\infty})$ *is sensitive.*
(ii) *Assume that the sequence* $f_{1,\infty}$ *pointwise converges to a continuous function* $f: X \to X$. *If* $(X, f_{1,\infty})$ *is transitive and the set of periodic points in the sense of* (P3) *is dense in* X*, then* $(X, f_{1,\infty})$ *is sensitive.*

Given an ADS (X, f), each element of a periodic orbit of period k is also periodic of period k. Despite the previous result and Theorem 2.3, this property fails to be true for a periodic point in the sense of (P3). For this, we will use the Cantor set $\mathscr{C}$. By an outstanding result of Brouwer's [8] every nonempty, compact, totally disconnected and metrizable space without isolated points is homeomorphic to the Cantor set. In the setting of ADS', the Cantor set enjoys interesting properties. Among others, we can cite that the Cantor set $\mathscr{C}$ has minimal equicontinuous systems: indeed, they are conjugated to an odometer, that is, they are Kronecker systems on $\mathscr{C}$ (see [16]) and, consequently, they are isometries. Moreover, the Cantor sets are the unique compact subsets of the real line that have chaotic homeomorphisms in the sense of Devaney [1]. Another property of $\mathscr{C}$ is its homogeneity. This means that for every two points x and y in $\mathscr{C}$ there exists a homeomorphism $f: \mathscr{C} \to \mathscr{C}$ such that $f(x) = y$. An easy proof of this fact runs over the following lines. The two-points discrete space $\{0, 1\}$ endowed with the operation of addition mod 2 is a topological group. By Brouwer's theorem, the topological group $\{0, 1\}^{\mathbb{N}}$ is homeomorphic to the Cantor set $\mathscr{C}$ and, consequently, it is homogeneous.

Example 2.1 The points in the orbit of a periodic point in the sense of (P3) need not be periodic. Also, the points in the orbit of a point with dense orbit does not have dense orbit.

Proof By Brouwer's theorem we can write $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2$ with $\mathscr{C}_i \subsetneq \mathscr{C}$ $(i = 1, 2)$ homeomorphic to $\mathscr{C}$ and $\mathscr{C}_1 \cap \mathscr{C}_2 = \emptyset$. Using the homogeneity of both $\mathscr{C}_1$ and $\mathscr{C}_2$, it follows that for every $z_1, z_2 \in \mathscr{C}_1$ and each $y_1, y_2 \in \mathscr{C}_2$ there exist homeomorphisms $h_1: \mathscr{C}_1 \to \mathscr{C}_2$ and $h_2: \mathscr{C}_2 \to \mathscr{C}_1$ such that $h_1(z_1) = y_1$ and $h_2(y_2) = z_2$. Taking into account the results in [2] commented above, from now on in this example, all the functions that we will consider from $\mathscr{C}_i$ onto $\mathscr{C}_j$ $(i, j = 1, 2)$ are homeomorphisms.

Let $f_1: \mathscr{C}_1 \to \mathscr{C}_2$ be a homeomorphism. Pick now $x \in \mathscr{C}_1$ and $y \in \mathscr{C}_2$ with $f_1(x) = y$. Let $f_2 = f_1^{-1}$. Next, for $n \geq 0$, define homeomorphisms $g_{2n+1}: \mathscr{C}_2 \to \mathscr{C}_1$ and $g_{2n+2}: \mathscr{C}_1 \to \mathscr{C}_2$ which satisfy the following property: fix two sequences $(x_{2n+1})_{n\geq 0} \subset \mathscr{C}_2$ and $(x_{2n+2})_{n\geq 0} \subset \mathscr{C}_1$ whose elements are pairwise different and such: (a) $x_1 = y$ and, for each $n \geq 0$, (b) $g_{2n+1}(x_{2n+1}) = x_{2n+2}$, and (c) $g_{2n+2}(x_{2n+2}) = x_{2(n+1)+1}$.

To finish the proof we consider the sequence of homeomorphisms

$$h_{1,\infty} = \{h_1, h_2, \ldots, h_n, \ldots\}$$

where, for $n \geq 0$, $h_{2n+1}: \mathscr{C} \to \mathscr{C}$ and $h_{2n+2}: \mathscr{C} \to \mathscr{C}$ are defined as follows:

$$h_{2n+1}(t) = \begin{cases} f_1(t), & \text{if } t \in \mathscr{C}_1, \\ g_{2n+1}(t), & \text{if } t \in \mathscr{C}_2, \end{cases} \quad \text{and} \quad h_{2n+2}(t) = \begin{cases} f_2(t), & \text{if } t \in \mathscr{C}_2, \\ g_{2n+2}(t), & \text{if } t \in \mathscr{C}_1. \end{cases}$$

Consider now the NDS $(\mathscr{C}, h_{1,\infty})$. It is an easy matter to check that $h_1^{2+n}(x) = h_1^n(x)$ for each $n \in \mathbb{N} \cup \{0\}$, so x is a periodic point in the sense of (P3): indeed its orbit is the set $\{x, y\}$. By construction the orbit of y is infinite. Notice that, being $\mathscr{C}$ separable, the orbit of y can be chosen dense in $\mathscr{C}$. Then y is a point with dense orbit which is no a periodic point in the sense of (P3). Making $x_2 = x$ the point x is in the orbit of y and, since the orbit of x is the set $\{x, y\}$, the orbit of x is not dense. □

3 Transitivity and NDS

In this section we point out some similarities and some differences between transitive ADS' and transitive NDS'. In the realm of NDS', the similarities are in spirit of topological nature. Thus, they run over similar lines to the ADS' case and, in general, the proofs can be applied in more general settings. On the other hand, the differences come from the definition of a NDS. We start with the relationship between point transitivity and transitivity. Recall that an ADS (respectively, a NDS) is called *point topologically transitive* (point transitive for short) if there exists a point, say x_0, whose orbit is dense. The point x_0 is said to be a *transitive point*. If we admit isolated points then, by Theorem 2.2, all the isolated points of a transitive NDS are transitive points which are also periodic points in the sense of (P1).

Recall that a topological space X is called a *Baire space* if each countable intersection of open dense subsets is a dense set. A well-known result says that complete metric spaces are Baire spaces. A useful result for ADS' states the following.

Theorem 3.1 [24, Proposition 1.1] *Let* (X, f) *be an* ADS *with* X *a perfect space. The following hold.*

(i) *If* (X, f) *has a transitive point, then it is transitive.*
(ii) *Suppose that* X *is a second countable Baire space. If* (X, f) *is transitive, then it has a transitive point.*

The previous result can be applied to the most usual situations. However, for NDS' Theorem 3.1 is not valid. Proposition 4.6 and Example 4.7 of [21] shows that (i) of the previous result holds for NDS' but (ii) does not. Therefore, a natural question is to establish a link between these two notions. Our first result is the following one.

Theorem 3.2 (Compare with [3, Theorem 1.1 (c)]) *Let* $(X, f_{1,\infty})$ *be a* NDS *where* X *is* T_3, *second countable and a Baire space. If* $(X, f_{1,\infty})$ *is transitive, then the set of transitive points is dense in* X.

Proof Let $\{U_n : n \in \mathbb{N}\}$ be a countable base for X. Set $S_n = \bigcup_{p\in\mathbb{N}}(f_1^p)^{-1}(U_n)$ for each $n \in \mathbb{N}$. Notice that S_n is open for all $n \in \mathbb{N}$. Moreover, being $(X, f_{1,\infty})$ transitive, S_n is dense in X for all $n \in \mathbb{N}$.

Take now a nonempty open set U of X. Since X is T_3 we can pick a nonempty open set V in X such that $\mathrm{cl}_X V \subset U$. It is apparent that the elements of the family $\{S_n \cap V : n \in \mathbb{N}\}$ are dense open sets of the space $\mathrm{cl}_X V$. Since $\mathrm{cl}_X V$ is a Baire space (see [10, Sect. 10, Exercise 4]), we have

$$\bigcap_{n\in\mathbb{N}} (S_n \cap V) \neq \emptyset.$$

Next choose a point $x_0 \in \bigcap_{n\in\mathbb{N}} (S_n \cap V)$. Fix now $n \in \mathbb{N}$. Since $x_0 \in S_n$, the definition of S_n says us that there exists $m \in \mathbb{N}$ such that $x_0 \in \left(f_1^m\right)^{-1}(U_n)$. This implies that the orbit of x_0 is dense in X. Since $x_0 \in V \subset U$, the transitive points are dense. This completes the proof. □

Assume that X is T_3, second countable, Baire and without isolated points. If $(X, f_{1,\infty})$ is a transitive NDS then, by Theorem 3.2, X contains a dense set of transitive points. Moreover, if x_0 is a transitive point, then x_0 is not a periodic point in the sense of (P3). Can x_0 be a periodic point in the sense of either (P1) or (P2)? To answer this consider the following example, which is a modification of [25, Example 2.3].

Example 3.1 There exist a transitive NDS $(\mathbb{I}, f_{1,\infty})$ and a dense subset D of $\mathbb{I}$ such that each element of D is a transitive point and a periodic point in the sense of (P1).

Proof Let $\{a_n : n \in \mathbb{N}\}$ be the set of rational numbers in $\mathbb{I}$ so that $D = \{a_{2m+1} : m \in \mathbb{N} \cup \{0\}\}$ is dense in $\mathbb{I}$. Given $m \in \mathbb{N} \cup \{0\}$ we define $f_{2m+1}(x) = a_{2m+1}$ and $f_{2m+2}(x) = 1$, for every $x \in \mathbb{I}$. Then $(\mathbb{I}, f_{1,\infty})$ is a NDS. Now consider a point $a_{2m+1} \in D$ and let W be a nonempty open subset of $\mathbb{I}$. Note that $f_1^{2m+1}(a_{2m+1}) = a_{2m+1}$. Since D is dense in $\mathbb{I}$, there exists $n \in \mathbb{N} \cup \{0\}$ so that $a_{2n+1} \in W$. Note that $f_1^{2n+1}(a_{2m+1}) = a_{2n+1} \in W$. This show that any element of the dense set D is a transitive point and a periodic point in the sense of (P1). Now consider two nonempty open subsets U and V of $\mathbb{I}$. By the density of D, there exist $n, m \in \mathbb{N} \cup \{0\}$ so that $a_{2n+1} \in U$ and $a_{2m+1} \in V$. Note that $f_1^{2m+1}(a_{2n+1}) = a_{2m+1}$, so $f_1^{2m+1}(U) \cap V \neq \emptyset$ and then $(\mathbb{I}, f_{1,\infty})$ is transitive. □

For the converse of Theorem 3.2, we have the following result.

Theorem 3.3 *Let $(X, f_{1,\infty})$ be a* NDS *where X is a T_1 space without isolated points. If the set of transitive points is dense in X, then $(X, f_{1,\infty})$ is transitive.*

Proof Let U and V be two nonempty open sets of X. Being the transitive points dense, there exists a transitive point, say x_0, which belongs to U. Since X does not have isolated points, there exists $y_0 \in V$ such that $x_0 \neq y_0$. Let W be an open subset of X such that $y_0 \in W$ and $x_0 \notin W$. Density of $\mathscr{O}_{f_{1,\infty}}(x_0)$ implies that there is $k \geq 0$ such that $f_1^k(x_0) \in V \cap W$. Thus $k \in \mathbb{N}$ and $f_1^k(U) \cap V \neq \emptyset$, so $(X, f_{1,\infty})$ is transitive. □

We can put together the two previous results in order to obtain the promised connection into the existence of transitive points and transitivity.

Corollary 3.1 *Let $(X, f_{1,\infty})$ be a* NDS *where X is T_3, second countable, Baire and without isolated points. Then $(X, f_{1,\infty})$ is transitive if and only if the set of transitive points is dense in X.*

Compact metric spaces without isolated points satisfy the conditions of Corollary 3.1.

The following result is well-known for ADS. Its proof lies in the definition of transitive point. In fact, it is a consequence of this concept. Thus, its proof runs over the similar lines to the ADS case (see the book of Akin [2, Theorem 4.12]).

Theorem 3.4 *Let $(X, f_{1,\infty})$ be a transitive* NDS *with $X := (X, d)$ a separable complete metric space. Then the set T of transitive points is a G_δ-set. Indeed, T is an intersection of countably many open dense sets.*

Proof Let $D = \{s_1, s_2, s_3, \ldots\}$ be a dense subset of X. Given an element $s_k \in D$, a positive rational number ε and a natural number n, we define the set

$$M_{s_k,n,\varepsilon} = \left\{x \in X : d(f_1^j(x), s_k) < \varepsilon \text{ for some } j > n\right\}.$$

The family of all elements of the form $M_{s_k,n,\varepsilon}$ are open and dense (notice that density follows from Therem 3.2). It is clear that the transitive points of $(X, f_{1,\infty})$ is the intersection of all subsets $M_{s_k,n,\varepsilon}$ and the proof is complete. □

We move on to some differences between ADS' and NDS'. Given an ADS (X, f) with $X := (X, d)$ a metric space, we say that a point $x \in X$ is *an equicontinuous point* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(f^n(x), f^n(y)) < \varepsilon$ for all $n \in \mathbb{N}$ provided that $d(x, y) < \delta$. The corresponding definition for NDS' is self-explanatory.

It is known that for transitive NDS' a point is a transitive point whenever it is equicontinuous [20, Theorem 3.1]. For compact metric ADS' the converse is also valid in the case that there exists at least one equicontinuous point ([3, Theorem 2.4]). The following example points out that the situation is quite different in the realm of NDS'. We will take advantage from the fact that, fixing $x \in [0, z]$ with $z < 1$, the sequence $\left\{x^2, x^3, \ldots, x^n, \ldots\right\}$ uniformly converges to the constant function zero and, consequently, it is equicontinuous on $[0, z]$. This outcome can be obtained by an easy calculation or by means of the Dini's theorem ([12, 3.10.F (b)]).

Example 3.2 There exists a minimal NDS $(\mathscr{C}, f_{1,\infty})$ and a point which is not an equicontinuous point. Indeed, let $\mathscr{C}$ be the Cantor set. Choose now a transitive equicontinuous homeomorphism f on $\mathscr{C}$ and consider, for each $n \in \mathbb{N}$, the functions $g_{2n+1}(x) = x^{2n+1}$ for all $x \in \mathscr{C}$. Define now the NDS $(\mathscr{C}, f_{1,\infty})$ as

$$\left\{g_3, (g_3)^{-1}, f, f^{-1}, g_5, (g_5)^{-1} f,^2, f^{-2}, \ldots\right\}.$$

Since $(\mathscr{C}, f)$ is a transitive ADS, every point of $(\mathscr{C}, f)$ is a transitive point. Then $x \in X$ is a transitive point of $(\mathscr{C}, f_{1,\infty})$ for all $x \in \mathscr{C}$. Indeed, we have

$$f_1^1 := g_3, \quad f_1^2 := \text{id}, \quad f_1^3 := f, \quad f_1^4 := id,$$

$$f_1^5 := g_5, \quad f_1^6 := \text{id}, \quad f_1^7 := f^2, \quad f_1^8 := id,$$

and so on and, consequently, $\{f^n : n \in \mathbb{N}\}$ is a subsequence of $f_{1,\infty}$. Thus, every point of X is a transitive point of $(\mathscr{C}, f_{1,\infty})$. In particular, $(\mathscr{C}, f_{1,\infty})$ is minimal.

Next notice that the sequence $\left\{x^{2n+1} : n \in \mathbb{N}\right\}$ converges to zero whenever $0 \leq x < 1$ and to one if $x = 1$. Thus, $(\mathscr{C}, f_{1,\infty})$ is not equicontinuous at the transitive point $x = 1$. Moreover, since the sequence of functions $\left\{x^3, x^5, \ldots, x^{2n+1}, \ldots\right\}$ is equicontinuous on $[0, z]$ for all $z < 1$, the set of points of equicontinuity of $(\mathscr{C}, f_{1,\infty})$ is dense in $\mathscr{C}$: indeed, it coincides with $\mathscr{C} \setminus \{1\}$. □

Let $(X, f_{1,\infty})$ be a NDS with $f_{1,\infty} = (f_n)_{n\in\mathbb{N}}$. We say that $(X, f_{1,\infty})$ is *surjective* if each function f_n is surjective. A well-known result for ADS' (and easy to proof) says that if (X, f) is a transitive ADS with X a compact T_2 space, then f is surjective. For NDS' this result does not hold as Example 3.1 shows. Note that in such example each function f_n is constant. Now we present an example in which the functions that define the sequence $f_{1,\infty}$ are not constant.

Example 3.3 There exists a NDS $(\mathscr{C}, f_{1,\infty})$ such that each f_n is neither surjective nor constant. For this, let $\{\mathscr{C}_1, \mathscr{C}_2\}$ be a partition of the Cantor set $\mathscr{C}$ in two subspaces homeomorphic to $\mathscr{C}$. Consider bases $\mathscr{A} = \{U_n : n \in \mathbb{N}\}$ and $\mathscr{B} = \{V_n : n \in \mathbb{N}\}$ for the topologies of $\mathscr{C}_1$ and $\mathscr{C}_2$, respectively, formed by clopen sets. Consider an enumeration of $\mathscr{A} \times \mathscr{B} \times \mathscr{A} \times \mathscr{B}$, say,

$$\left\{\left(U_{j_{4t+1}}, V_{s_{4t+2}}, U_{r_{4t+3}}, V_{m_{4t+4}}\right) : t = 0, 1, 2, \ldots\right\}.$$

For each $(t, j) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}$ define a continuous function $f_{4t+j} : \mathscr{C} \to \mathscr{C}$ such that the restriction of f_1 to $\mathscr{C}_2$ is the identity and with the following properties:

(1) The restriction of both f_{4t+1} and f_{4t+3} to $\mathscr{C}_1$ is a homeomorphism onto $\mathscr{C}_2$.
(2) The restriction of both f_{4t+2} and f_{4t+4} to $\mathscr{C}_2$ is a homeomorphism onto $\mathscr{C}_1$.
(3) $f_{4t+1}(U_{j_{4t+1}}) = V_{s_{4t+2}}$.
(4) $f_{4t+2}(V_{s_{4t+2}}) = U_{r_{4t+3}}$.
(5) $f_{4t+3}(U_{r_{4t+3}}) = V_{m_{4t+4}}$.
(6) $f_{4t+4}(V_{m_{4t+4}}) = U_{j_{4(t+1)+1}}$.

Let $(\mathscr{C}, f_{1,\infty})$ be the NDS with

$$f_{1,\infty} = \{f_1, f_2, f_3, f_4, f_5, \ldots\}.$$

Since $\mathscr{A}$ and $\mathscr{B}$ are bases of $\mathscr{C}_1$ and $\mathscr{C}_2$, respectively, $\mathscr{A} \cup \mathscr{B}$ is a base for the topology of $\mathscr{C}$. Then, by the properties of the functions f_n, an easy calculation shows that $(\mathscr{C}, f_{1,\infty})$ is transitive. Notice that no function of the sequence $f_{1,\infty}$ is surjective. □

As we mentioned in Sect. 2 a transitive ADS (X, f) has no isolated points if and only if X is infinite. This result fails to be true for NDS as the following example shows. We employs a strategy similar to that underlying the proof of Example 3.3. Therefore, we only give an outline of the proof.

Example 3.4 Consider the space $X = \mathscr{C} \cup \{x\}$ with $x \notin \mathscr{C}$ an isolated point. Let $\{\mathscr{C}_1, \mathscr{C}_2\}$, $\mathscr{A}$ and $\mathscr{B}$ as in the previous example. Let $\mathscr{M}$ be an enumeration of $\{x\} \times \mathscr{A} \times \mathscr{B} \times \mathscr{A} \times \mathscr{B}$, say,

$$\left\{\left(x, U_{j_{4t+1}}, V_{s_{4t+2}}, U_{r_{4t+3}}, V_{m_{4t+4}}\right) : t = 0, 1, 2, \ldots\right\}.$$

For each $(t, j) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}$ consider the function $f_{4t+j} : \mathscr{C} \to \mathscr{C}$ defined in Example 3.3, so that the properties (1), (3), (4) and (5) of such example are satisfied. We redefine f_{4t+j} in X by letting $f_{4t+1}(x) \in U_{j_{4t+1}}$, $f_{4t+4}(V_{m_{4t+4}}) = \{x\}$. Moreover, $f_{4t+2}(x)$, $f_{4t+3}(x)$, $f_{4t+4}(x)$ belong to arbitrary clopen sets different from $V_{s_{4t+2}}$, $U_{r_{4t+3}}$, $V_{m_{4t+4}}$ whose preimage coincide with the set $\{x\}$. $\square$

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Linear Operators on the (LB)-Sequence Spaces $\mathbf{ces(p-)}$, $\mathbf{1 < p \le \infty}$

In Honour of Manuel López-Pellicer

Angela A. Albanese, José Bonet and Werner J. Ricker

Abstract We determine various properties of the regular (LB)-spaces $ces(p-)$, $1 < p \le \infty$, generated by the family of Banach sequence spaces $\{ces(q)\colon 1 < q < p\}$. For instance, $ces(p-)$ is a (DFS)-space which coincides with a countable inductive limit of weighted ℓ_1-spaces; it is also Montel but not nuclear. Moreover, $ces(p-)$ and $ces(q-)$ are isomorphic as locally convex Hausdorff spaces for all choices of $p, q \in (1, \infty]$. In addition, with respect to the coordinatewise order, $ces(p-)$ is also a Dedekind complete, reflexive, locally solid, lc-Riesz space with a Lebesgue topology. A detailed study is also made of various aspects (e.g., the spectrum, continuity, compactness, mean ergodicity, supercyclicity) of the Cesàro operator, multiplication operators and inclusion operators acting on such spaces (and between the spaces ℓ_{r-} and $ces(p-)$).

Keywords (LB)-space · Sequence space $ces(p-)$ · Spectrum · Multiplier operator · Cesàro operator · Mean ergodic operator

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1 Introduction

The process of forming averages is time honoured in mathematics. For example, the symmetric partial sums of the Fourier series of a 2π-periodic function on $\mathbb{R}$ do not behave as well with respect to pointwise convergence (or convergence in L^1) as the sequence of their averages (i.e., their Cesàro means). Or, a consideration of the averages of the sequence of powers of a given continuous linear operator leads to its mean ergodic properties. And so on.

The linear operator C which assigns to a numerical sequence $x = (x_n)_n = (x_1, x_2, \ldots) \in \mathbb{C}^{\mathbb{N}}$ the sequence of its averages

$$\mathsf{C}(x) := (x_1, \frac{x_1 + x_2}{2}, \ldots, \frac{x_1 + x_2 + \cdots + x_n}{n}, \ldots) \in \mathbb{C}^{\mathbb{N}}, \quad x \in \mathbb{C}^{\mathbb{N}}, \tag{1}$$

is called the (discrete) *Cesàro operator*. It maps many sequence spaces $X \subseteq \mathbb{C}^{\mathbb{N}}$ into themselves (e.g. $X = c_0, c, \ell_p$ for $1 < p \leq \infty$) but, also exhibits other features. For instance, the element $x = ((-1)^n)_n \notin c_0$ yet its image $\mathsf{C}(x) \in c_0$. So, given a vector space $X \subseteq \mathbb{C}^{\mathbb{N}}$ one may consider the vector space

$$[\mathsf{C}, X] := \{x \in \mathbb{C}^{\mathbb{N}} \colon \mathsf{C}(x) \in X\} \subseteq \mathbb{C}^{\mathbb{N}} \tag{2}$$

generated by C and X. It was noted above that c_0 is a *proper* subspace of $[\mathsf{C}, c_0]$. For some classical spaces $X \subseteq \mathbb{C}^{\mathbb{N}}$ it can happen that $X \nsubseteq [\mathsf{C}, X]$, eg. if $X = \ell_1$, then $[\mathsf{C}, \ell_1] = \{x \in \mathbb{C}^{\mathbb{N}} \colon \mathsf{C}(x) \in \ell_1\} = \{0\}$. On the other hand, Hardy's inequality, [27, Theorem 326], implies that $\mathsf{C}(\ell_p) \subseteq \ell_p$ for every $1 < p \leq \infty$, and so $\ell_p \subseteq [\mathsf{C}, \ell_p]$. This inclusion is proper, [21]. The corresponding spaces (2) for $X = \ell_p$ were considered in [33].

From the viewpoint of analysis, a desirable property of a sequence space $X \subseteq \mathbb{C}^{\mathbb{N}}$ is that it should be *solid*, that is, if $x \in X$ and $y \in \mathbb{C}^{\mathbb{N}}$ satisfy $|y| \leq |x|$, then also $y \in X$. Here, for $z \in \mathbb{C}^{\mathbb{N}}$, we define $|z| := (|z_n|)_n \in \mathbb{C}^{\mathbb{N}}$ and write $z \geq 0$ if $z = |z|$. Hence, $|y| \leq |x|$ means that $(|x| - |y|) \geq 0$. For this order defining its positive cone, $\mathbb{C}^{\mathbb{N}}$ is a (complex) locally convex Fréchet lattice for the topology of coordinatewise convergence. For example, each space ℓ_p for $1 \leq p \leq \infty$ is solid whereas c is not solid. The spaces $[\mathsf{C}, X]$ given by (2) are typically *not* solid, even if X is solid. Note that C is a positive operator in $\mathbb{C}^{\mathbb{N}}$, that is, $\mathsf{C}(x) \geq 0$ whenever $x \geq 0$ in $\mathbb{C}^{\mathbb{N}}$ (in particular, $\mathsf{C}(|x|) \leq \mathsf{C}(|y|)$ whenever $|x| \leq |y|$ and $|\mathsf{C}(x)| \leq \mathsf{C}(|x|)$). For instance, $X = c_0$ is solid and the element $x := ((-1)^n)_n \in [\mathsf{C}, c_0]$ but, $|x| \notin [\mathsf{C}, c_0]$. Hence, $[\mathsf{C}, c_0]$ is *not* solid. So, given a solid Banach space $X \subseteq \mathbb{C}^{\mathbb{N}}$ (with norm $\|\cdot\|_X$), perhaps more relevant than $[\mathsf{C}, X]$ is its *solid* Banach space counterpart

$$[\mathsf{C}, X]_s := \{x \in \mathbb{C}^{\mathbb{N}} \colon \mathsf{C}(|x|) \in X\},$$

equipped with the norm

$$\|x\|_{[\mathsf{C},X]_s} := \|\mathsf{C}(|x|)\|_X, \quad x \in [\mathsf{C}, X]_s.$$

As pointed out in [21], the space $[\mathsf{C}, X]_s$ is the *largest* amongst all solid Banach spaces $Y \subseteq \mathbb{C}^{\mathbb{N}}$ satisfying $\mathsf{C}(Y) \subseteq X$.

When X is one of the spaces ℓ_p, $1 < p \leq \infty$, equipped with its standard norm $\|\cdot\|_p$, then the solid Banach lattice $[\mathsf{C}, \ell_p]_s$ generated by C and ℓ_p is more traditionally denoted by

$$ces(p) := \{x \in \mathbb{C}^{\mathbb{N}} \colon \mathsf{C}(|x|) \in \ell_p\}$$

with norm

$$\|x\|_{ces(p)} := \|\mathsf{C}(|x|)\|_p = \left(\subseteq \Sigma_{n=1}^{\infty} \left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{1/p}, \quad x \in ces(p). \tag{3}$$

The space $ces(p)$, $1 < p < \infty$, was first introduced in [34, 40] and became prominent in [28], where the description of its (somewhat complicated) dual Banach space $(ces(p))'$ was presented as the solution to a problem posed by the Dutch Mathematical Society in 1968. The first thorough investigation of the Banach spaces $ces(p)$, $1 < p < \infty$, was carried out in [12], where other equivalent norms to (3) were also presented; see p. 25 and p. 54 of [12]. A further equivalent norm, based on a certain weighted block decomposition of the elements of $ces(p)$, is investigated in [25]. Each Banach space $ces(p)$, $1 < p < \infty$, is reflexive, p-concave and the canonical vectors $e_n := (\delta_{nk})_k$, for $n \in \mathbb{N}$, form an unconditional basis, [12, 21]. For every pair $1 < p, q < \infty$ the space $ces(p)$ is *not* isomorphic to ℓ_q, [12, Proposition 15.13]; it is also *not* isomorphic to $ces(q)$ if $p \neq q$, [6, Proposition 3.3]. In view of (3), Hardy's inequality ensures that $\ell_p \subseteq ces(p)$ with $\|x\|_{ces(p)} \leq p'\|x\|_p$ for each $x \in \ell_p$, where the conjugate index p' of p is given by $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, the containment $\ell_p \subseteq ces(p)$ is *proper*, [21, Remark 2.2]. It is routine to verify that C maps $ces(p)$ continuously into ℓ_p. Many more properties of $ces(p)$, $1 < p < \infty$, are known; see, for example, [9, 10, 35], and the references therein.

In recent years there has been a renewed interest in the Banach spaces $ces(p)$, $1 < p < \infty$, and various linear operators acting in them (e.g., the Cesàro operator, multiplication operators, inclusion maps, convolution operators); see, for example, [6, 11, 21, 22, 39]. In [19] a detailed investigation is made of the Banach space of Dirichlet series defined in a fixed right half-plane of $\mathbb{C}$ and whose coefficients come from $ces(p)$, together with the multiplier operators acting in this space.

Non-normable sequence spaces $X \subseteq \mathbb{C}^{\mathbb{N}}$ are also abundant and form an important part of functional analysis; see, for example, [13, 14, 30, 36, 43] and the references therein. The particular family of classical Fréchet spaces $\ell_{p+} := \cap_{p<q}\ell_q$, $1 \leq p < \infty$, is well understood, [23, 37]. Its analogue $ces(p+) := \cap_{p<q} ces(q)$, $1 \leq p < \infty$, was recently introduced and studied in [4]. The Fréchet spaces $ces(p+)$ are *very different* to the spaces ℓ_{p+} that generate them (in the same sense that ℓ_p generates $ces(p)$). Certain aspects of various linear operators (e.g., their spectrum, compactness, mean ergodicity, supercyclicity) acting on the spaces $ces(p+)$,

$1 \le p < \infty$, are investigated in [7]. The aim of this paper is to carry out a similar study for the corresponding class of *(LB)-spaces* given via the inductive limit $ces(p-) := \mathrm{ind}_{1<q<p} ces(q)$, for $1 < p \le \infty$, and operators acting in these spaces. In order to summarize the main features of this paper we first require some notation and preliminaries.

Let X and Y be a locally convex Hausdorff spaces (briefly, lcHs). The identity operator on X is denoted by I and $\mathscr{L}(X, Y)$ denotes the space of all continuous linear operators from X into Y. If $X = Y$, we write $\mathscr{L}(X) = \mathscr{L}(X, Y)$ and if $Y = \mathbb{C}$ we write X' for $\mathscr{L}(X, \mathbb{C})$, i.e., the dual space of X. The dual (or transpose) operator of $T \in \mathscr{L}(X, Y)$ is denoted by $T' \colon Y' \to X'$. When X' is equipped with the strong dual topology β we denote it by X'_β. In this case, $T' \in \mathscr{L}(Y'_\beta, X'_\beta)$. We denote by Γ_X a system of continuous seminorms determining the topology of X. Let $\mathscr{L}_s(X)$ denote $\mathscr{L}(X)$ endowed with the strong operator topology τ_s which is determined by the seminorms $T \to q_x(T) := q(Tx)$, for each $x \in X$ and $q \in \Gamma_X$. Moreover, $\mathscr{L}_b(X)$ denotes $\mathscr{L}(X)$ equipped with the topology τ_b of uniform convergence on bounded subsets of X which is determined by the seminorms $T \to q_B(T) := \sup_{x \in B} q(Tx)$, for each bounded subset $B \subseteq X$ and $q \in \Gamma_X$.

For a lcHs X and $T \in \mathscr{L}(X)$, the *resolvent set* $\rho(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathscr{L}(X)$. The set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T. The *point spectrum* $\sigma_{pt}(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)$ is not injective. If we need to stress the space X, then we write $\sigma(T; X)$, $\sigma_{pt}(T; X)$ and $\rho(T; X)$. Given $\lambda, \mu \in \rho(T)$ the *resolvent identity* $R(\lambda, T) - R(\mu, T) = (\mu - \lambda)R(\lambda, T)R(\mu, T)$ holds. Unlike for Banach spaces, it may happen that $\rho(T) = \emptyset$ or that $\rho(T)$ is not open. This is why some authors (e.g. [42]) prefer the subset $\rho^*(T)$ of $\rho(T)$ consisting of all $\lambda \in \mathbb{C}$ for which there exists $\delta > 0$ such that $B(\lambda, \delta) := \{z \in \mathbb{C} \colon |z - \lambda| < \delta\} \subseteq \rho(T)$ and $\{R(\mu, T) \colon \ \mu \in B(\lambda, \delta)\}$ is equicontinuous in $\mathscr{L}(X)$. Define $\sigma^*(T) := \mathbb{C} \setminus \rho^*(T)$, which is a closed set containing $\sigma(T)$. If $T \in \mathscr{L}(X)$ with X a Banach space, then $\sigma(T) = \sigma^*(T)$.

In Sect. 2 we establish some important properties of the (LB)-spaces $ces(p-)$, $1 < p \le \infty$. A remarkable property of the Banach space $ces(p)$, which is *not* shared by ℓ_p, is that $x \in \mathbb{C}^{\mathbb{N}}$ belongs to $ces(p)$ if and only if $\mathsf{C}(|x|) \in ces(p)$, [12, Theorem 20.31]. This useful property carries over to $ces(p-)$; see Proposition 2.1(i). It is also established (in Proposition 2.1) that $ces(p-)$ is a (DFS)-space which is solid in $\mathbb{C}^{\mathbb{N}}$. Moreover, $ces(p-)$ is generated by ℓ_{p-} in the sense that $[\mathsf{C}, \ell_{p-}]_s = ces(p-)$; see Proposition 2.1(iii). It turns out that $ces(p-)$ coincides as a vector space and topologically with a countable inductive limit of weighted ℓ_1-spaces and hence, that it is the strong dual of a Köthe echelon space $\lambda_0(A)$ for a certain Köthe matrix A. Actually, $\lambda_0(A)$ is precisely the power series Fréchet-Schwartz space $\Lambda^{\infty}_{(1/p')}((\log k)_k)$ of finite type $1/p'$ and order infinity (cf. Theorem 2.1). It follows that the (DFS)-space $ces(p-)$ is *not* nuclear and that it is isomorphic to $ces(\infty-) := \mathrm{ind}_n ces(n+1)$, a space which is independent of p; see Corollary 2.1. In particular, $ces(p-)$ and $ces(q-)$ are isomorphic lcHs' for all choices of $p, q \in (1, \infty]$.

Section 3 is devoted to an analysis of the Cesàro operator $\mathsf{C} \colon ces(p-) \to ces(p-)$ for $1 < p \le \infty$. It is shown that C has no eigenvalues and its spectrum is localized according to

$$\{0\} \cup \left\{\lambda \in \mathbb{C} \colon |\lambda - \frac{p'}{2}| < \frac{p'}{2}\right\} \subseteq \sigma(\mathsf{C}; ces(p-)) \subseteq \left\{\lambda \in \mathbb{C} \colon |\lambda - \frac{p'}{2}| \leq \frac{p'}{2}\right\};$$

see Proposition 3.2. Consequently, C cannot be a compact operator. Moreover, in Proposition 3.3 it is shown that

$$\sigma^*(\mathsf{C}; ces(p-)) = \left\{\lambda \in \mathbb{C} \colon |\lambda - \frac{p'}{2}| \leq \frac{p'}{2}\right\}.$$

This knowledge of the spectra of $\mathsf{C} \in \mathscr{L}(ces(p-))$ implies that C is neither power bounded nor mean ergodic (cf. Remark 3.2, Lemma 3.2 and Proposition 3.4). It is also verified that C fails to be supercyclic in $ces(p-)$; see Proposition 3.5.

Given $a = (a_n)_n \in \mathbb{C}^{\mathbb{N}}$, the multiplication operator $M_a \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is defined by $M_a(x) := (a_n x_n)_n$ for each $x \in \mathbb{C}^{\mathbb{N}}$. Section 4 treats various operator theoretic aspects of M_a when it is restricted to $ces(p-)$ for $1 < p \leq \infty$. The continuity of $M_a \colon ces(p-) \to ces(p-)$ is completely characterized (in terms of $a \in \mathbb{C}^{\mathbb{N}}$ belonging to a certain family of weighted ℓ_∞-spaces) in Proposition 4.1. Furthermore, the compactness of $M_a \in \mathscr{L}(ces(p-))$ is fully determined in Proposition 4.2, namely that the sequence $(\sup_{k \geq n} |a_k|)_n$ generated by $a \in \mathbb{C}^{\mathbb{N}}$ should belong to $\cup_{p' < t} \ell_t$. It turns out that the power boundedness of $M_a \in \mathscr{L}(ces(p-))$ is equivalent to its mean ergodicity which in turn is equivalent to the requirements that $a \in \ell_\infty$ and $\|a\|_\infty \leq 1$; see Proposition 4.3. Finally, the spectra of M_a (cf. Proposition 4.4) are given by $\sigma_{pt}(M_a; ces(p-)) = \{a_n \colon n \in \mathbb{N}\}$ and

$$\overline{\sigma(M_a; ces(p-))} = \sigma^*(M_a; ces(p-)) = \overline{\{a_n \colon n \in \mathbb{N}\}}.$$

As a consequence, whenever $M_a \in \mathscr{L}(ces(p-))$ is compact, then necessarily $a \in c_0$; the converse is not true: For the purpose of comparison the continuous (and compact) multiplication operators $M_a \colon \ell_{p-} \to \ell_{p-}$ are also determined; see Proposition 4.5.

The aim of Sect. 5 is to identify all pairs $1 < p, q \leq \infty$ for which the natural inclusion operator and the Cesàro operator map X into Y, where $X \in \{\ell_{p-}, ces(p-)\}$ and $Y \in \{\ell_{q-}, ces(q-)\}$. By the Closed Graph Theorem these operators are then necessarily continuous. Of particular interest is the boundedness and the compactness of such operators (in the sense of Gröthendieck). The possible values of p, q for which this is the case are precisely determined; see Propositions 5.2 and 5.4.

The final section collects together some relevant properties of the spaces ℓ_{p-} and $ces(p-)$, $1 < p \leq \infty$, when they are viewed as locally solid, lc-Riesz spaces within $\mathbb{C}^{\mathbb{N}}$ (for its coordinatewise order). Differences are to be expected since $ces(p-)$ is a Montel space whereas ℓ_{p-} is not. The fact that they are the strong dual of suitable Fréchet lattices plays an important role. The relevance of such properties is due to the fact the Cesàro operator and inclusion maps are *positive operators* between Riesz spaces, as are the multiplication operators M_a whenever $a \geq 0$.

2 The Space $ces(p-)$

Let $p \in (1, \infty]$. We define

$$ces(p-) := \cup_{1<q<p} ces(q)$$

and endow $ces(p-)$ with the inductive limit topology. The union is *strictly increasing* in the sense that $ces(q_1) \subsetneqq ces(q_2)$ whenever $1 < q_1 < q_2 < p$; see the discussion prior to Proposition 3.3 in [6]. If $1 < p_n < p_{n+1} < p$ with $p_n \to p$ as $n \to \infty$, then $ces(p-) = \text{ind}_n ces(p_n)$ is an (LB)-space, i.e., a countable inductive limit of Banach spaces, [36, pp. 290–291]. Since the inclusion $ces(p) \subseteq ces(q)$, for $p < q$, is *compact*, [6, Proposition 3.4(ii)], the space $ces(p-)$ is a (DFS)-space, i.e., the strong dual of a Fréchet-Schwartz space, [36, Proposition 25.20]. In particular, it is complete, regular and Montel, [13, pp. 61–62]. Since each Banach space $ces(p)$, $1 < p < \infty$, is solid in $\mathbb{C}^{\mathbb{N}}$, the space $ces(p-)$ is also solid in $\mathbb{C}^{\mathbb{N}}$.

Consider the (LB)-space $\ell_{p-} := \text{ind}_n \ell_{r_n}$, where $1 \leq r_n < r_{n+1} < p$ and $r_n \uparrow p$. Then $\ell_{p-} \subseteq ces(p-)$ with a continuous inclusion since $\ell_q \subseteq ces(q)$ for all $q > 1$ with a continuous inclusion (by Hardy's inequality).

The Cesàro operator C maps $ces(p-)$ continuously into ℓ_{p-} since $\mathsf{C}\colon ces(q) \to \ell_q$ is continuous for each $q > 1$; this follows from [36, Proposition 24.7], for example. Since also $\ell_{p-} \subseteq ces(p-)$ continuously, we deduce that $\mathsf{C}\colon ces(p-) \to ces(p-)$ is continuous.

Proposition 2.1 *Let* $1 < p \leq \infty$.

(i) *Let* $x \in \mathbb{C}^{\mathbb{N}}$. *Then* $x \in ces(p-)$ *if and only if* $\mathsf{C}(|x|) \in ces(p-)$.
(ii) $ces(p-)$ *is a (DFS)-space, which is solid in the (Fréchet) lattice* $\mathbb{C}^{\mathbb{N}}$. *Moreover,*

$$\mathsf{C}(ces(p-)) \subseteq \ell_{p-} \subseteq ces(p-).$$

In addition, the canonical vectors $\{e_n\}_{n=1}^{\infty}$ *are a Schauder basis for* $ces(p-)$.
(iii) *Let* X *be any solid lcHs contained in* $\mathbb{C}^{\mathbb{N}}$ *such that* $\mathsf{C}(X) \subseteq \ell_{p-}$. *Then* $X \subseteq ces(p-)$. *Accordingly,* $[\mathsf{C}, \ell_{p-}]_s = ces(p-)$.
(iv) *Let* X *be any solid lcHs contained in* $\mathbb{C}^{\mathbb{N}}$ *such that* $\mathsf{C}(X) \subseteq ces(p-)$. *Then* $X \subseteq ces(p-)$. *Accordingly,* $[\mathsf{C}, ces(p-)]_s = ces(p-)$.

Proof (i) This is a direct consequence of [12, Theorem 20.31] and the definition of $ces(p-)$.

(ii) All claims (except the one about $\{e_n\}_{n=1}^{\infty}$) follow from the discussion prior to Proposition 2.1.

Recalling that $\{e_n\}_{n=1}^{\infty}$ is a basis for each Banach space $ces(q)$, $1 < q < \infty$, and that the natural inclusion $ces(q) \subseteq ces(p-)$ is continuous for each $1 < q < p$, it follows that $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis for $ces(p-)$.

(iii) If $x \in X$, then also $|x| \in X$, as X is solid in $\mathbb{C}^{\mathbb{N}}$. Then $\mathsf{C}(|x|) \in \ell_{p-} \subseteq ces(p-)$. Thus, $x \in ces(p-)$ by part (i). Since C maps $ces(p-)$ continuously into ℓ_{p-}, we can conclude that $[\mathsf{C}, \ell_{p-}]_s = ces(p-)$.

(iv) This is again a consequence of part (i). Indeed, if $x \in X$, then $|x| \in X$ and hence, $\mathsf{C}(|x|) \in ces(p-)$ by assumption. So, $x \in ces(p-)$. Since $\mathsf{C}(ces(p-)) \subseteq ces(p-)$, it follows that $[\mathsf{C}, ces(p-)]_s = ces(p-)$. This completes the proof.

We now show that $ces(p-)$ coincides algebraically and topologically with a countable inductive limit k_1 of weighted ℓ_1-spaces. This co-echelon space is the strong dual of a power series space $\Lambda^{\infty}_{(1/p')}((\log k)_k)$ of finite type $1/p'$ and infinite order. This requires some explanation.

Given $p \in (1, \infty)$, set $p_n := (p - \frac{1}{n})$ for $n \geq n(p)$ so that $1 < p_n < p_{n+1} < p$ for each $n \geq n(p)$, where $n(p)$ is the smallest $n \in \mathbb{N}$ such that $(p - \frac{1}{n}) > 1$. If $p = \infty$, we set $p_n := n + 1$ for each $n \in \mathbb{N}$. In both cases $p_n \uparrow p$ and so $p'_n \downarrow p'$. Consider the sequence of strictly decreasing weights $v_p := (v_n)_n$, where $v_n : \mathbb{N} \to (0, \infty)$ is given by $v_n(k) := k^{-1/p'_n}$ for $k, n \in \mathbb{N}$. Clearly, $v_{n+1}(k) < v_n(k)$ for each $k, n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, consider the weighted ℓ_1-space

$$\ell_1(v_n) := \left\{ x \in \mathbb{C}^{\mathbb{N}} : \|x\|_{\ell_1(v_n)} := \sum_{k=1}^{\infty} v_n(k)|x_k| < \infty \right\},$$

which is a solid Banach space relative to the norm $\|\cdot\|_{\ell_1(v_n)}$. Clearly $\ell_1(v_n) \subseteq \ell_1(v_{n+1})$ with a continuous inclusion. Accordingly, $k_1(v_p) := \mathrm{ind}_n \ell_1(v_n)$ is a countable inductive limit of Banach spaces. It coincides with the strong dual of the Köthe echelon space $\lambda_0(A) := \mathrm{proj}_n c_0(1/v_n)$; see [14] and [36, Chap. 27], where $\lambda_0(A)$ is denoted by $c_0(A)$. For each $n \in \mathbb{N}$, set $a_n(k) := (1/v_n(k)) = k^{1/p'_n}$, for each $k \in \mathbb{N}$, in which case $(a_n)_n$ is a strictly increasing sequence of weights on $\mathbb{N}$. Accordingly, $\lambda_0(A) = \mathrm{proj}_n c_0(a_n)$ coincides with the power series space $\Lambda^{\infty}_{(1/p')}((\log k)_k)$ of finite type $1/p'$ and infinite order given by

$$\Lambda^{\infty}_{(1/p')}((\log k)_k) := \left\{ y \in \mathbb{C}^{\mathbb{N}} : |||y|||_n := \sup_{k \in \mathbb{N}} |y_k| k^{1/p'_n} < \infty \ \forall n \in \mathbb{N} \right\}.$$

This is a Fréchet-Schwartz space; see [36, Propositions 24.18 and 27.10]. Moreover, the canonical vectors $\{e_n\}_{n=1}^{\infty}$ form a Schauder basis in $\Lambda^{\infty}_{(1/p')}((\log k)_k)$. Indeed, for each $k, n \in \mathbb{N}$ we have $k^{1/p'_n} = k^{1/p'_{n+1}} k^{\alpha_n}$ with $\alpha_n := \frac{p'_{n+1} - p'_n}{p'_n p'_{n+1}} < 0$. It follows, for $n \in \mathbb{N}$ and a given $x \in \Lambda^{\infty}_{(1/p')}((\log k)_k)$, that

$$|||x - \sum_{j=1}^{N} x_j e_j|||_n = \sup_{k > N} |x_k| k^{1/p'_{n+1}} k^{\alpha_n} \leq (N+1)^{\alpha_n} |||x|||_{n+1}, \quad N \in \mathbb{N}.$$

Accordingly, $\lim_{N \to \infty} \sum_{j=1}^{N} x_j e_j = x$ in $\Lambda^{\infty}_{(1/p')}((\log k)_k)$, as required.

Theorem 2.1 *Let $1 < p \leq \infty$. The (LB)-space $ces(p-)$ coincides algebraically and topologically with $k_1(v_p) = (\Lambda^{\infty}_{(1/p')}((\log k)_k))'_{\beta}$.*

Proof Fix $1 < p \leq \infty$. It suffices to establish the following two facts, where $p_n := 1 - \frac{1}{n}$ for all large enough n if $p < \infty$ and $p_n := n + 1$ if $p = \infty$. Namely, for each n large enough,

(a) $ces(p_n) \subseteq \ell_1(v_{n+1})$ with a continuous inclusion, and

(b) $\ell_1(v_n) \subseteq ces(p_{n+1})$ with a continuous inclusion, [36, Corollary 24.35].

We first verify the fact (a). Fix $n \in \mathbb{N}$ and $x \in ces(p_n)$. For $q > 1$, Lemma 4.7 of [12] states that $\frac{1}{(q-1)k^{q-1}} < \sum_{j=k}^{\infty} \frac{1}{j^q}$ for $k \in \mathbb{N}$. Setting $q := 1 + \frac{1}{p'_{n+1}}$, in which case $\frac{1}{(q-1)k^{q-1}} = p'_{n+1} k^{-1/p'_{n+1}}$, it follows from this inequality that

$$p'_{n+1} \|x\|_{\ell_1(v_{n+1})} = p'_{n+1} \sum_{k=1}^{\infty} |x_k| k^{-1/p'_{n+1}} \leq \sum_{k=1}^{\infty} |x_k| \left(\sum_{j=k}^{\infty} \frac{1}{j} j^{-1/p'_{n+1}} \right)$$

$$= \sum_{j=1}^{\infty} \left(\frac{1}{k} \sum_{j=1}^{k} |x_j| \right) j^{-1/p'_{n+1}}.$$

Now Hölder's inequality yields

$$p'_{n+1} \|x\|_{\ell_1(v_{n+1})} \leq \left(\sum_{j=1}^{\infty} \left(\frac{1}{k} \sum_{j=1}^{k} |x_j| \right)^{p_n} \right)^{1/p_n} \left(\sum_{j=1}^{\infty} j^{-p'_n/p'_{n+1}} \right)^{1/p'_n}$$

$$\leq \left(\sum_{j=1}^{\infty} j^{-p'_n/p'_{n+1}} \right)^{1/p'_n} \|x\|_{ces(p_n)}.$$

Since $\frac{p'_n}{p'_{n+1}} > 1$, we have $\sum_{j=1}^{\infty} j^{-p'_n/p'_{n+1}} < \infty$ and the proof of (a) is complete.

To establish the fact (b), fix $n \in \mathbb{N}$ and $x \in \ell_1(v_n)$. For $k \in \mathbb{N}$ we have

$$\frac{1}{k} \sum_{j=1}^{k} |x_j| = \frac{1}{k} \sum_{j=1}^{k} \frac{|x_j|}{j^{1/p'_n}} j^{1/p'_n} \leq \frac{1}{k} \|x\|_{\ell_1(v_n)} k^{1/p'_n} = k^{-1/p_n} \|x\|_{\ell_1(v_n)}.$$

This inequality implies that

$$\|x\|_{ces(p_{n+1})}^{p_{n+1}} = \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{j=1}^{k} |x_j| \right)^{p_{n+1}} \leq \|x\|_{\ell_1(v_n)}^{p_{n+1}} \sum_{k=1}^{\infty} \frac{1}{k^{p_{n+1}/p_n}}.$$

But, $K := \sum_{k=1}^{\infty} \frac{1}{k^{p_{n+1}/p_n}} < \infty$ because $p_n < p_{n+1}$ and so

$$\|x\|_{ces(p_{n+1})} \leq K^{1/p_{n+1}} \|x\|_{\ell_1(v_n)}.$$

Fact (b) is thereby established.

Remark 2.1 Since $\Lambda^\infty_{(1/p')}((\log k)_k)$ is a Fréchet-Schwartz space, it equals its bidual. Accordingly, Theorem 2.1 implies that

$$(ces(p-))'_\beta = (k_1(v_p))'_\beta = \Lambda^\infty_{(1/p')}((\log k)_k).$$

Corollary 2.1 *Let* $1 < p \leq \infty$*. Then* $ces(p-)$ *is a (DFS)-space, but it is not nuclear. Moreover, it is not isomorphic to* ℓ_{q-} *for each* $q > 1$*, whereas it is isomorphic to* $ces(\infty-)$*.*

Proof That $ces(p-)$ is a (DFS)-space has been already shown. This implies that it cannot be isomorphic to ℓ_{q-} for each $q > 1$ as ℓ_{q-} is not Montel.

Suppose that $\Lambda^\infty_{(1/p')}((\log k)_k)$ is nuclear. Recall that $\{e_k\}_{k=1}^\infty$ is a Schauder basis of $\Lambda^\infty_{(1/p')}((\log k)_k)$ for the topology given by the norms $\{|||\cdot|||_n : n \in \mathbb{N}\}$. With $n = 1$ it follows from the Gröthendieck–Pietsch Theorem, [36, Theorem 28.15], that there exists $m > 1$ such that

$$\sum_{k=1}^\infty \frac{|||e_k|||_1}{|||e_k|||_m} = \sum_{k=1}^\infty \frac{k^{1/p'_1}}{k^{1/p'_m}} = \sum_{k=1}^\infty \frac{1}{k^{1/p'_m - 1/p'_1}} < \infty.$$

This is impossible as $(\frac{1}{p'_m} - \frac{1}{p'_1}) < \frac{1}{p'_m} \in (0, 1)$. So, $\Lambda^\infty_{(1/p')}((\log k)_k)$ is *not* nuclear. Since $ces(p-) = (\Lambda^\infty_{(1/p')}((\log k)_k))'_\beta$, it follows that $ces(p-)$ cannot be nuclear either, [38, p. 78 Theorem].

All finite type power series spaces $\Lambda^\infty_r(\alpha)$, with α fixed, are diagonally isomorphic; see the argument in [36, p. 358]. This implies that

$$ces(p-) = (\Lambda^\infty_{(1/p')}((\log k)_k))'_\beta$$

is isomorphic to $(\Lambda^\infty_{(1)}((\log k)_k))'_\beta = ces(\infty-)$.

3 The Cesàro Operator on $ces(p-)$

The aim of this section is to make a detailed analysis of the Cesàro operator $\mathsf{C}\colon ces(p-) \to ces(p-)$ for $1 < p \leq \infty$. We first examine its spectrum and then, with this information available, the linear dynamics and mean ergodicity of C can be investigated.

We begin with an abstract result concerning the spectra of operators in (LB)-spaces, [5, Lemma 5.2].

Lemma 3.1 *Let* $E = \mathrm{ind}_n E_n$ *be a Hausdorff inductive limit of Banach spaces. Let* $T \in \mathscr{L}(E)$ *satisfy the following condition:*

(A) For each $n \in \mathbb{N}$ *the restriction* T_n *of* T *to* E_n *maps* E_n *into itself and* $T_n \in \mathscr{L}(E_n)$*.*

Then the following properties are satisfied.

(i) $\sigma_{pt}(T; E) = \cup_{n\in\mathbb{N}}\sigma_{pt}(T_n; E_n)$.
(ii) $\sigma(T; E) \subseteq \cap_{m\in\mathbb{N}} \left(\cup_{n=m}^{\infty}\sigma\big(T_n; E_n\big)\right)$.
(iii) *If* $\cup_{n=m}^{\infty}\sigma(T_n; E_n) \subseteq \overline{\sigma(T; E)}$ *for some* $m \in \mathbb{N}$*, then*

$$\sigma^*(T; E) = \overline{\sigma(T; E)}.$$

We will also require the following fact, [21, Theorem 5.1]

Theorem 3.1 *Let* $p \in (1, \infty)$*. The Cesàro operator* $\mathsf{C} \in \mathscr{L}(ces(p))$ *and* $\|\mathsf{C}\| = p'$*. Moreover,* $\sigma_{pt}(\mathsf{C}; ces(p)) = \emptyset$ *and*

$$\sigma(\mathsf{C}; ces(p)) = \{\lambda \in \mathbb{C}\colon |\lambda - \frac{p'}{2}| \leq \frac{p'}{2}\}$$

with $\overline{\mathrm{Im}(\lambda I - \mathsf{C})} \neq ces(p)$ *whenever* $\lambda \in \mathbb{C}$ *satisfies* $|\lambda - \frac{p'}{2}| < \frac{p'}{2}$.

Since $ces(p-) = \mathrm{ind}_n ces(p_n)$ with $1 < p_n \uparrow p$, we observe that the Cesàro operator $\mathsf{C} \in \mathscr{L}(ces(p-))$ satisfies all the assumptions of Lemma 3.1 with $\mathsf{C}_n := \mathsf{C}|_{ces(p_n)}$ for each $n \in \mathbb{N}$.

Proposition 3.1 *Let* $p \in (1, \infty]$*. Then* $\sigma_{pt}(\mathsf{C}; ces(p-)) = \emptyset$.

Proof This is a direct consequence of Lemma 3.1(i) and Theorem 3.1.

Proposition 3.2 *Let* $p \in (1, \infty]$*. Then*

$$\{0\} \cup \{\lambda \in \mathbb{C}\colon |\lambda - \frac{p'}{2}| < \frac{p'}{2}\} \subseteq \sigma(\mathsf{C}; ces(p-)) \subseteq \{\lambda \in \mathbb{C}\colon |\lambda - \frac{p'}{2}| \leq \frac{p'}{2}\}.$$

Proof We first establish the second inclusion. So, fix $m \in \mathbb{N}$. If $n \geq m$, then $p_n' < p_m'$ and so, by Theorem 3.1, we have

$$\sigma(\mathsf{C}_n; ces(p_n)) = \{\lambda \in \mathbb{C}\colon |\lambda - \frac{p_n'}{2}| \leq \frac{p_n'}{2}\} \subseteq \{\lambda \in \mathbb{C}\colon |\lambda - \frac{p_m'}{2}| \leq \frac{p_m'}{2}\}.$$

Accordingly,

$$\cup_{n=m}^{\infty}\{\lambda \in \mathbb{C}\colon |\lambda - \frac{p_n'}{2}| \leq \frac{p_n'}{2}\} \subseteq \{\lambda \in \mathbb{C}\colon |\lambda - \frac{p_m'}{2}| \leq \frac{p_m'}{2}\}.$$

This implies that

$$\cap_{m=1}^{\infty}\left(\cup_{n=m}^{\infty}\{\lambda \in \mathbb{C}\colon |\lambda - \frac{p_n'}{2}| \leq \frac{p_n'}{2}\}\right) \subseteq \cap_{m=1}^{\infty}\{\lambda \in \mathbb{C}\colon |\lambda - \frac{p_m'}{2}| \leq \frac{p_m'}{2}\}.$$

But, $p_m' \downarrow p'$ and so, by Lemma 3.1(ii), we obtain

$$\sigma(\mathsf{C}; ces(p-)) \subseteq \{\lambda \in \mathbb{C}\colon |\lambda - \frac{p'}{2}| \le \frac{p'}{2}\}.$$

Concerning the first inclusion, observe that C is not surjective on $ces(p-)$. Indeed, $(\frac{1-(-1)^k}{2k})_k \in \ell_{p-} \subseteq ces(p-)$ for each $p > 1$. But,

$$\mathsf{C}^{-1}((\frac{1-(-1)^k}{2k})_k) = ((-1)^{k+1})_k \notin \cup_{q>1}\ell_q,$$

thereby implying that $\mathsf{C}^{-1}\left(\left(\frac{1-(-1)^k}{2k}\right)_k\right) = ((-1)^{k+1})_k \notin ces(p-)$. Since C is an isomorphism on $\mathbb{C}^{\mathbb{N}}$, it follows that C is not surjective on $ces(p-)$ and so $\lambda = 0 \in \sigma(\mathsf{C}; ces(p-))$.

Let $\lambda \in \mathbb{C}\backslash\{0\}$. If $\lambda \in \rho(\mathsf{C}; ces(p-))$, then $(\lambda I - \mathsf{C})(ces(p-)) = ces(p-)$. Since $ces(p-)$ is dense in $ces(p)$, it follows (with the bar denoting the closure in $ces(p)$) that

$$ces(p) = \overline{ces(p-)} = \overline{(\lambda I - \mathsf{C})(ces(p-))} \subseteq \overline{(\lambda I - \mathsf{C})(ces(p))} \subseteq ces(p).$$

Thus, $|\lambda - \frac{p'}{2}| \ge \frac{p'}{2}$ by Theorem 3.1. Accordingly, $|\lambda - \frac{p'}{2}| < \frac{p'}{2}$ implies that $\lambda \in \sigma(\mathsf{C}; ces(p-))$.

Remark 3.1 An operator $T \in \mathscr{L}(X, Y)$ between lcHs' X and Y is called compact if there exists a neighbourhood of 0 in X whose image under T is relatively compact in Y.

(i) Proposition 3.2 implies that $\mathsf{C}\colon ces(p-) \to ces(p-)$ has uncountable spectrum and so it cannot be compact (by Gröthendieck's theorem), [24, Theorem 9.10.2], [26, p. 204].

(ii) Let $1 < p < \infty$. Then for the dual operator C' of $\mathsf{C} \in \mathscr{L}(ces(p-))$ we have

$$\left\{\lambda \in \mathbb{C}\colon |\lambda - \frac{p'}{2}| < \frac{p'}{2}\right\} \subseteq \sigma_{pt}(\mathsf{C}'; (ces(p-))').$$

Indeed, $ces(p-) \subseteq ces(p)$ and so $(ces(p))' \subseteq (ces(p-))' \subseteq \mathbb{C}^{\mathbb{N}}$. Moreover, C' is the "same" operator in all three spaces, namely $\mathsf{C}'(x) = (\sum_{k=n}^{\infty} \frac{x_k}{k})_n$. So,

$$\sigma_{pt}(\mathsf{C}'; (ces(p))') \subseteq \sigma_{pt}(\mathsf{C}'; (ces(p-))').$$

The conclusion follows from the proof of Theorem 3.1 above as given in [21, Theorem 5.1], where it is shown that $\{\lambda \in \mathbb{C}\colon |\lambda - \frac{p'}{2}| < \frac{p'}{2}\} \subseteq \sigma_{pt}(\mathsf{C}'; (ces(p))')$.

Proposition 3.3 *Let* $p \in (1, \infty]$. *Then the spectrum*

$$\sigma^*(\mathsf{C}; ces(p-)) = \{\lambda \in \mathbb{C}\colon |\lambda - \frac{p'}{2}| \le \frac{p'}{2}\}.$$

Proof Fix $p \in (1, \infty]$. In general, $\overline{\sigma(\mathsf{C}; ces(p-))} \subseteq \sigma^*(\mathsf{C}; ces(p-))$ and so, by Proposition 3.2, we have

$$\left\{\lambda \in \mathbb{C} \colon |\lambda - \frac{p'}{2}| \leq \frac{p'}{2}\right\} \subseteq \sigma^*(\mathsf{C}; ces(p-)). \tag{4}$$

Fix $\lambda \in \mathbb{C}$ with $|\lambda - \frac{p'}{2}| > \frac{p'}{2}$. Recall that $ces(p-) = \text{ind}_n ces(p_n)$ with $p'_n \downarrow p'$. So, there exist $m \in \mathbb{N}$ (i.e., $p'_m > p' > 1$) and $\delta > 0$ such that

$$\overline{B(\lambda, \delta)} \cap \left\{\alpha \in \mathbb{N} \colon |\alpha - \frac{p'_m}{2}| \leq \frac{p'_m}{2}\right\} = \emptyset. \tag{5}$$

By Proposition 3.2, for each $\mu \notin \{\alpha \in \mathbb{N} \colon |\alpha - \frac{p'}{2}| \leq \frac{p'}{2}\}$ the inverse operator $R(\mu, \mathsf{C}) = (\mu I - \mathsf{C})^{-1} \in \mathscr{L}(ces(p-))$ exists. So, $H := \{R(\mu, \mathsf{C}) \colon \mu \in B(\lambda, \delta)\} \subseteq \mathscr{L}(ces(p-))$ is well defined. Moreover, (5) implies (as $p'_n \downarrow p'$) that

$$\overline{B(\lambda, \delta)} \cap \left\{\alpha \in \mathbb{N} \colon |\alpha - \frac{p'_n}{2}| \leq \frac{p'_n}{2}\right\} = \emptyset, \quad \forall n \geq m.$$

The claim is that H is equicontinuous in $\mathscr{L}(ces(p-))$. For this, it suffices to show that $\{R(\mu, \mathsf{C})(x) \colon \mu \in B(\lambda, \delta\}$ is a bounded set in $ces(p-)$ for each $x \in ces(p-)$. So, fix $x \in ces(p-)$. Select $n \geq m$ such that $x \in ces(p_n)$. Since $\mathsf{C}_n := \mathsf{C}|_{ces(p_n)} \in \mathscr{L}(ces(p_n))$, the spectrum $\sigma(\mathsf{C}_n; ces(p_n)) = \{\mu \in \mathbb{C} \colon |\mu - \frac{p'_n}{2}| \leq \frac{p'_n}{2}\}$ (cf. Theorem 3.1) and $\overline{B(\lambda, \delta)} \cap \sigma(\mathsf{C}_n; ces(p_n)) = \emptyset$. Moreover, via the spectral theory of Banach space operators we have that $\{R(\mu, \mathsf{C})(x) \colon \mu \in B(\lambda, \delta\}$ is a bounded set in $ces(p_n)$. Hence, it is also a bounded set in $ces(p-)$. This shows that $\lambda \in \rho^*(\mathsf{C}; ces(p-))$ and thereby establishes the reverse containment to that in (4).

Open Problem: It would be interesting to know precisely which non-zero points of the circle $\{\lambda \in \mathbb{C} \colon |\lambda - \frac{p'}{2}| = \frac{p'}{2}\}$ belong to $\sigma(\mathsf{C}; ces(p-))$, for $1 < p \leq \infty$.

An operator $T \in \mathscr{L}(X)$, with X a lcHs, is called *power bounded* if $\{T^n\}_{n=1}^\infty$ is an equicontinuous subset of $\mathscr{L}(X)$. Given $T \in \mathscr{L}(X)$, the averages

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m, \qquad n \in \mathbb{N},$$

are called the Cesàro means of T. They satisfy the identity

$$\frac{T^n}{n} = T_{[n]} - \frac{(n-1)}{n} T_{[n-1]}, \qquad n \in \mathbb{N}, \tag{6}$$

where $T_{[0]} := I$. The operator T is said to be *mean ergodic* (resp., *uniformly mean ergodic*) if $\{T_{[n]}\}_{n=1}^\infty$ is a convergent sequence in $\mathscr{L}_s(X)$ (resp., in $\mathscr{L}_b(X)$); in view of (6) it follows that necessarily $\frac{T^n}{n} \to 0$ in $\mathscr{L}_s(X)$ (resp., in $\mathscr{L}_b(X)$) for $n \to \infty$. A relevant text for mean ergodic operators is [32].

Proposition 3.4 *Let $p \in (1, \infty]$. The Cesàro operator $\mathsf{C} \in \mathscr{L}(ces(p-))$ is not mean ergodic and not power bounded.*

Proof Assume that $\mathsf{C} \in \mathscr{L}(ces(p-))$ is mean ergodic. Since $ces(p-)$ is barrelled, [36, Theorem 24.16], it follows from Theorem 2.4 of [1] that

$$ces(p-) = \mathrm{Ker}(I - \mathsf{C}) \oplus \overline{\mathrm{Im}(I - \mathsf{C})}. \tag{7}$$

It is routine to check that if $(I - \mathsf{C})(x) = 0$ for some $x \in \mathbb{C}^{\mathbb{N}}$, then $x = x_1 u$, where $u := (1, 1, \ldots)$. But, if $x \in ces(p-)$, then $x \in ces(q)$ for some $q \in (1, p)$. Since $\mathsf{C}(ces(q)) \subseteq \ell_q$, it follows that $\mathsf{C}(x) \in \ell_q$, that is, $x \in \ell_q$ (as $\mathsf{C}(x) = x$). But, $x = x_1 u \in \ell_q$ if and only if $x_1 = 0$ (equivalently, $x = 0$). So, $\mathrm{Ker}(I - \mathsf{C}) = \{0\}$ which implies, via (7), that

$$ces(p-) = \overline{\mathrm{Im}(I - \mathsf{C})}. \tag{8}$$

It is straight-forward to verify that $\mathrm{Im}(I - \mathsf{C}) \subseteq Z := \{x \in ces(p-) \colon x_1 = 0\}$. Since Z is also closed, we can conclude that $\overline{\mathrm{Im}(I - \mathsf{C})} \subseteq Z$. This contradicts (8) as Z is a *proper* closed subspace of $ces(p-)$. Accordingly, C is not mean ergodic.

Since $ces(p-)$ is a reflexive (LB)-space, it follows from [1, Corollary 2.7] that C is also *not* power bounded.

Remark 3.2 By Remark 3.1(ii) if $1 < p < \infty$, then $\lambda := \frac{1+p'}{2}$ satisfies $|\lambda| > 1$ and $\lambda \in \sigma_{pt}(\mathsf{C}'; (ces(p-))')$. The following Lemma 3.2 then provides an alternative proof of the fact that $\mathsf{C} \in \mathscr{L}(ces(p-))$ is neither power bounded nor mean ergodic. Since $\sigma(\mathsf{C}; ces(\infty-)) \subseteq \{\lambda \in \mathbb{C} \colon |\lambda - \frac{1}{2}| \le \frac{1}{2}\} \subseteq \mathscr{U}$, with $\mathscr{U} := \{z \in \mathbb{C} \colon |z| \le 1\}$ (cf. Proposition 3.2) the above argument does *not* apply to $p = \infty$ (in which case $\frac{1+p'}{2} = 1$).

Lemma 3.2 *Let X be a lcHs and let $T \in \mathscr{L}(X)$. If*

$$\sigma_{pt}(T'; X') \cap \{\lambda \in \mathbb{C} \colon |\lambda| > 1\} \neq \emptyset, \tag{9}$$

then the sequence $\{T^n/n\}_{n=1}^{\infty}$ does not converge to 0 in $\mathscr{L}_s(X)$. In particular, T is neither power bounded nor mean ergodic.

Proof Let λ belong to (9). Then there exists $u \in X' \setminus \{0\}$ satisfying $T'u = \lambda u$. Select any $x \in X \setminus \{0\}$ such that $\langle x, u \rangle = 1$. Then

$$\left\langle \frac{T^n(x)}{n}, u \right\rangle = \frac{1}{n} \langle x, (T')^n(u) \rangle = \frac{\lambda^n}{n}, \quad n \in \mathbb{N}.$$

Since $|\lambda| > 1$, the sequence $\{\frac{T^n(x)}{n}\}_{n=1}^{\infty}$ fails to converge to 0 in X. As noted prior to Proposition 3.4, T cannot be mean ergodic. Furthermore, if $\{T^n/n\}_{n=1}^{\infty}$ is not a null sequence in $\mathscr{L}_s(X)$, then T cannot be power bounded.

An operator $T \in \mathscr{L}(X)$, with X a separable lcHs, is called *hypercyclic* if there exists $x \in X$ such that its orbit $\{T^n(x) : n \in \mathbb{N}_0\}$ is dense in X, where $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. If, for some $z \in X$, its projective orbit $\{\lambda T^n(z) : \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is dense in X, then T is called *supercyclic*. Clearly, hypercyclicity implies supercyclicity.

Proposition 3.5 *Let $p \in (1, \infty]$. The Cesàro operator $\mathsf{C} \in \mathscr{L}(ces(p-))$ is not supercyclic.*

Proof Since $ces(p-)$ is dense in $\mathbb{C}^{\mathbb{N}}$ and $ces(p-) \subseteq \mathbb{C}^{\mathbb{N}}$ continuously, it follows that C is not supercyclic on $ces(p-)$ because it is not supercyclic on $\mathbb{C}^{\mathbb{N}}$, [3, Proposition 4.3].

Remark 3.3 For $1 < p < \infty$ an alternative proof of Proposition 3.5 is possible. According to Remark 3.1(ii) the dual operator C' of $\mathsf{C} \in \mathscr{L}(ces(p-))$ has at least two linearly independent eigenvectors. Since supercyclicity is the same as 1-supercyclicity in the sense of [18], it follows from Theorem 2.1 of [18] that C is not supercyclic.

4 Multipliers on $ces(p-)$

Given $a = (a_n)_n \in \mathbb{C}^{\mathbb{N}}$, the multiplication (or diagonal) operator $M_a : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is defined by $M_a(x) := (a_n x_n)_n$, for $x \in \mathbb{C}^{\mathbb{N}}$. Such operators on Banach sequence spaces $X \subseteq \mathbb{C}^{\mathbb{N}}$ have been dealt with in detail in [12], for example; see also [6]. For various classical Banach spaces such as ℓ_p, $1 \le p \le \infty$, c_0 and c, we refer to [41, §4.51]. For X the Fréchet space ℓ_{p+} or $ces(p+)$ see [7]. Also [20] is relevant. Our aim in this section is to investigate the case when $X = ces(p-)$ for $1 < p \le \infty$. We begin with the following result; see [12, pp. 69–70], after noting that $cop(p) = ces(p)$, for $p > 1$, [12, p. 26].

Theorem 4.1 (i) *Let $1 < p \le q < \infty$. Then $M_a : ces(p) \to ces(q)$ is continuous if and only if $(a_k k^{(1/q)-(1/p)})_k \in \ell_\infty$.*
(ii) *Let $1 < r < q < \infty$ and define s by $\frac{1}{s} = \frac{1}{r} - \frac{1}{q}$. Then $M_a : ces(q) \to ces(r)$ is continuous if and only if $(\sup_{k \ge n} |a_k|)_n \in \ell_s$.*

An operator $T \in \mathscr{L}(X, Y)$ with X, Y lcHs' is called bounded if there exists a neighbourhood of 0 in X whose image under T is a bounded subset of Y. If $Y = \text{ind}_m Y_m$ is a regular (LB)-space, then a set $B \subseteq Y$ is bounded if and only if there exists $m \in \mathbb{N}$ such that $B \subseteq Y_m$ and B is bounded in the Banach space Y_m. The following result follows from the various definitions involved. For part (i) the Gröthendieck Factorization Theorem is required, [13, Theorem 2, p. 76], [36, Theorem 24.33].

Lemma 4.1 *Let $X = \text{ind}_n X_n$ and $Y = \text{ind}_m Y_m$ be two (LB)-spaces with increasing unions of Banach spaces $X = \cup_{n=1}^\infty X_n$ and $Y = \cup_{m=1}^\infty Y_m$. Let $T : X \to Y$ be a linear map.*

(i) *T is continuous if and only if for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $T(X_n) \subseteq Y_m$ and the restriction $T: X_n \to Y_m$ is continuous.*
(ii) *Assume that Y is a regular (LB)-space. Then T is bounded if and only if there exists $m \in \mathbb{N}$ such that $T(X_n) \subseteq Y_m$ and $T: X_n \to Y_m$ is continuous for all $n \geq m$.*

We can now characterize the multiplication operators in $\mathscr{L}(ces(p-))$.

Proposition 4.1 *Let $a \in \mathbb{C}^{\mathbb{N}}$ and $p \in (1, \infty]$. The operator $M_a: ces(p-) \to ces(p-)$ is continuous if and only if for each $q \in (1, p)$ there exists $r \in [q, p)$ such that $(a_k k^{(1/r)-(1/q)})_k \in \ell_\infty$.*

Proof This is a direct consequence of Theorem 4.1(i) and Lemma 4.1(i).

Proposition 4.1 can be formulated in terms of weighted ℓ_∞-spaces. Namely, given $1 < p \leq \infty$ and $a \in \mathbb{C}^{\mathbb{N}}$, the operator $M_a \in \mathscr{L}(ces(p-))$ if and only if

$$a \in \cap_{1<q<p}(\cup_{q \leq r<p} \ell_\infty(w_{r,q})),$$

where the weight $w_{r,q}(k) := k^{\frac{1}{r}-\frac{1}{q}}$ for $k \in \mathbb{N}$. It follows from this criterion that $M_a \in \mathscr{L}(ces(p-))$ for every $a \in \ell_\infty$. On the other hand, $a := (\log(n))_n \notin \ell_\infty$ but $M_a \in \mathscr{L}(ces(p-))$ for every $1 < p \leq \infty$.

Since $ces(p-)$, $1 < p \leq \infty$, is a Montel space, there is no distinction between the bounded operators and the compact operators in $\mathscr{L}(ces(p-))$.

Proposition 4.2 *Let $a \in \mathbb{C}^{\mathbb{N}}$ and $p \in (1, \infty]$. The operator $M_a \in \mathscr{L}(ces(p-))$ is compact (equivalently, bounded) if and only if there is $t > p'$ such that $(\sup_{k \geq n} |a_k|)_n \in \ell_t$.*

Proof Case (i): $1 < p < \infty$. Assume first that $(\sup_{k \geq n} |a_k|)_n \in \ell_t$ for some $t > p'$. Then $\frac{pt}{p+t} > 1$. Put $\varepsilon := \frac{p^2}{p+t}$ and $r := p - \varepsilon$. Clearly $r < p$. Moreover, $r = p - \varepsilon = p - \frac{p^2}{p+t} = \frac{pt}{p+1} > 1$. So, $1 < r < p$. Given *any* $q \in (r, p) = (p - \varepsilon, p)$ define s by $\frac{1}{s} = \frac{1}{r} - \frac{1}{q}$. Then $\frac{1}{p} < \frac{1}{q}$ and so

$$\frac{1}{s} = \frac{1}{p-\varepsilon} - \frac{1}{q} < \frac{1}{p-\varepsilon} - \frac{1}{p} = \frac{\varepsilon}{(p-\varepsilon)p}.$$

Thus, $s > \frac{(p-\varepsilon)p}{\varepsilon} = t$ and hence, $\ell_t \subseteq \ell_s$. Accordingly, $(\sup_{k \geq n} |a_k|)_n \in \ell_s$ and so $M_a \in \mathscr{L}(ces(q), ces(r))$ via Theorem 4.1(ii). Now Lemma 4.1 ensures that $M_a \in \mathscr{L}(ces(p-))$ is compact.

Conversely, assume that the operator $M_a \in \mathscr{L}(ces(p-))$ is compact. By Lemma 4.1(ii) and Theorem 4.1(ii) there exists $r \in (1, p)$ such that for every $q \in (r, p)$ we have $(\sup_{k \geq n} |a_k|)_n \in \ell_s$, where $\frac{1}{s} = \frac{1}{r} - \frac{1}{q}$. Select any $q \in (r, p)$. For t specified by $\frac{1}{t} = \frac{1}{r} - \frac{1}{q}$ we have $\frac{1}{t} < (1 - \frac{1}{p}) = \frac{1}{p'}$ and so $t > p'$. Accordingly, $(\sup_{k \geq n} |a_k|)_n \in \ell_t$.

Case (ii): $p = \infty$. Assume there exists $t > p' = 1$ with $(\sup_{k\geq n}|a_k|)_n \in \ell_t$. Select any $r > t$ (i.e., $\frac{1}{r} < \frac{1}{t}$). If $q \in (r, \infty)$, then s defined by $\frac{1}{s} = \frac{1}{r} - \frac{1}{q}$ satisfies $\frac{1}{s} < (\frac{1}{t} - \frac{1}{q}) < \frac{1}{t}$, that is, $t < s$. Accordingly, $\ell_t \subseteq \ell_s$ and so $(\sup_{k\geq n}|a_k|)_n \in \ell_s$, which implies via Theorem 4.1(ii) that $M_a \in \mathscr{L}(ces(q), ces(r))$. Now Lemma 4.1 ensures that $M_a \in \mathscr{L}(ces(p-))$ is compact.

The proof of the converse as given in Case (i) also applies for $p = \infty$.

Remark 4.1 For each $1 < p \leq \infty$ Proposition 4.2 can be formulated by saying that $M_a \in \mathscr{L}(ces(p-))$ is compact if and only if $(\sup_{k\geq n}|a_k|)_n \in \ell_{\infty-} = \cup_{q>1}\ell_q$, where $\ell_{\infty-}$ is independent of p.

We now turn our attention to the mean ergodic properties of the operators M_a acting in $ces(p-)$.

Proposition 4.3 *Let $a \in \mathbb{C}^{\mathbb{N}}$ and $p \in (1, \infty]$ satisfy $M_a \in \mathscr{L}(ces(p-))$. The following conditions are equivalent.*

(i) *$a \in \ell_\infty$ and $\|a\|_\infty \leq 1$.*
(ii) *The operator M_a is power bounded.*
(iii) *The operator M_a is uniformly mean ergodic.*
(iv) *The operator M_a is mean ergodic.*

Proof Recall that $ces(p-) = \text{ind}_n ces(p_n)$ with $1 < p_n \uparrow p$.

(i)⇒(ii). Since $a \in \ell_\infty$ and $\|a\|_\infty \leq 1$, we have that $M_a \in \mathscr{L}(ces(p_n))$ and $\|M_a\|_{op} \leq \|a\|_\infty \leq 1$, for each $n \in \mathbb{N}$, [6, Lemma 2.6]. Let $x \in ces(p-)$, in which case $x \in ces(p_n)$ for some $n \in \mathbb{N}$. Then $\|M_a^m(x)\|_{ces(p_n)} \leq \|x\|_{ces(p_n)}$ and so $\{M_a^m(x)\}_{m=1}^\infty$ is contained and bounded in the Banach space $ces(p_n)$. Hence, it is also bounded in $ces(p-)$. Since $ces(p-)$ is barrelled, we conclude that $\{M_a^m\}_{m=1}^\infty$ is equicontinuous.

(ii)⇒(iii). This follows from the fact that the (LB)-space $ces(p-)$ is Montel, [1, Proposition 2.8].

(iii)⇒(iv). Immediate from the definitions.

(iv)⇒(i). Since M_a is mean ergodic, (6) implies (with $T := M_a$) that, for each $x \in ces(p-)$, the sequence $(\frac{1}{k}M_a^k(x))_k$ converges to 0 in $ces(p-)$. Taking $x := e_j$, for $j \in \mathbb{N}$ fixed, we conclude that $\frac{1}{k}a_j^k e_j \to 0$ in $ces(p-)$ as $k \to \infty$. Since $ces(p-) \subseteq \mathbb{C}^{\mathbb{N}}$ continuously, also $\frac{1}{k}a_j^k e_j \to 0$ in $\mathbb{C}^{\mathbb{N}}$ as $k \to \infty$ (i.e. coordinatewise) which implies that $\frac{1}{k}a_j^k \to 0$ in $\mathbb{C}$ as $k \to \infty$. Hence, $|a_j| \leq 1$. Since $j \in \mathbb{N}$ is arbitrary, it follows that $a \in \ell_\infty$ and $\|a\|_\infty \leq 1$.

Proposition 4.4 *Let $a \in \mathbb{C}^{\mathbb{N}}$ and $p \in (1, \infty]$ satisfy $M_a \in \mathscr{L}(ces(p-))$.*

(i) *$\sigma_{pt}(M_a; ces(p-)) = \{a_k : k \in \mathbb{N}\}$.*
(ii) *$\lambda \in \rho(M_a; ces(p-))$ if and only if for each $1 < q < p$ there exist $r \in [q, p)$ and $\varepsilon > 0$ such that $|\lambda - a_k| \geq \varepsilon k^{\frac{1}{r}-\frac{1}{q}}$ for each $k \in \mathbb{N}$. In this case $M_{\lambda-a}^{-1} = M_{(1/(\lambda-a))} \in \mathscr{L}(ces(p-))$.*
(iii) *$\overline{\sigma(M_a; ces(p-))} = \sigma^*(M_a; ces(p-)) = \overline{\{a_k : k \in \mathbb{N}\}}$.*

Proof (i) It is clear from $M_a(e_k) = a_k e_k$, for $k \in \mathbb{N}$, that

$$\{a_k : k \in \mathbb{N}\} \subseteq \sigma_{pt}(M_a; ces(p-)).$$

On the other hand, if there is a non-zero $x \in ces(p-)$ and $\lambda \in \mathbb{C}$ with $M_a(x) = \lambda x$, then $(a_k x_k)_k = M_a(x) = \lambda x = (\lambda x_k)_k$ and so $\lambda \in \{a_k : k \in \mathbb{N}\}$.

(ii) Clearly $\lambda \in \rho(M_a; ces(p-))$ if and only if $\lambda \notin \{a_k : k \in \mathbb{N}\}$ and $\psi_\lambda := (\frac{1}{\lambda - a_k})_k \in \mathbb{C}^{\mathbb{N}}$ satisfies $M_{\psi_\lambda} \in \mathscr{L}(ces(p-))$. The conclusion is then immediate from Proposition 4.1. Clearly $M^{-1}_{\lambda - a} = M_{(1/(\lambda - a))}$.

(iii) Part (i) implies that $\{a_k : k \in \mathbb{N}\} \subseteq \sigma(M_a; ces(p-))$. Therefore,

$$\overline{\{a_k : k \in \mathbb{N}\}} \subseteq \overline{\sigma(M_a; ces(p-))} \subseteq \sigma^*(M_a; ces(p-)).$$

Now take $\lambda \notin \overline{\{a_k : k \in \mathbb{N}\}}$. Then there exists $\varepsilon > 0$ such that $|\lambda - a_k| \geq \varepsilon$ for each $k \in \mathbb{N}$. If $|\mu - \lambda| < \varepsilon/2$, then $|\mu - a_k| \geq \varepsilon/2$ for each $k \in \mathbb{N}$. Let $1 < q < p$. Part (ii) implies (taking $r := q$ there for each $r < p$) that $\mu \in \rho(M_a; ces(p-))$ and $M^{-1}_{\mu - a} = M_{1/(\mu - a)}$. Moreover, $\|M_{1/(\mu-a)}(x)\|_{ces(r)} \leq (2/\varepsilon)\|x\|_{ces(r)}$, for $x \in ces(r) \subseteq ces(p-)$. Since $ces(p-) = \text{ind}_{1<r<p} ces(r)$, we conclude that $\{M^{-1}_{\mu - a} : |\mu - \lambda| < \varepsilon/2\}$ is bounded in $\mathscr{L}_s(ces(p-))$ and hence, equicontinuous in $\mathscr{L}(ces(p-))$. Accordingly, $\lambda \notin \sigma^*(M_a; ces(p-))$.

If $T \in \mathscr{L}(X)$, with X a lcHs, is compact, then $\sigma(T; X)$ is a compact subset of $\mathbb{C}$ and every non-zero point of $\sigma(T; X)$ is isolated, [24, Theorem 9.10.2], [26, p. 204]. For a given $p \in (1, \infty]$ this implies, via Proposition 4.4(iii), that if $M_a \in \mathscr{L}(ces(p-))$ is compact, then necessarily $a \in c_0$. The converse is not true. Indeed, since $(|a_n|)_n \leq (\sup_{k \geq n} |a_k|)_n$, it follows from Proposition 4.2 that if $M_a \in \mathscr{L}(ces(p-))$ is compact, then necessarily $a \in \ell_t$ for some $t > p'$. But $\cup_{1 \leq t < \infty} \ell_t \subseteq c_0$ is a *proper* containment. So, there exist elements $a \in c_0$ such that, for every $p \in (1, \infty]$, the operator $M_a \in \mathscr{L}(ces(p-))$ (cf. the discussion after Proposition 4.1) but M_a is *not* compact.

We end this section by characterizing the multipliers for ℓ_{p-} and the subclass of those which are compact operators. For $1 \leq r < \infty$ define $\mathscr{M}(\ell_{r+}) := \{a \in \mathbb{C}^{\mathbb{N}} : M_a \in \mathscr{L}(\ell_{r+})\}$ and for $1 < p \leq \infty$ define $\mathscr{M}(\ell_{p-}) := \{a \in \mathbb{C}^{\mathbb{N}} : M_a \in \mathscr{L}(\ell_{p-})\}$. The subclass of compact multipliers are defined by

$$\mathscr{M}_c(\ell_{r+}) := \{a \in \mathscr{M}(\ell_{r+}) : M_a \text{ is compact in } \mathscr{L}(\ell_{r+})\}$$

and

$$\mathscr{M}_c(\ell_{p-}) := \{a \in \mathscr{M}(\ell_{p-}) : M_a \text{ is compact in } \mathscr{L}(\ell_{p-})\}.$$

It is known that

$$\mathscr{M}(\ell_{r+}) = \ell_\infty \quad \text{and} \quad \mathscr{M}_c(\ell_{r+}) = \cup_{s>1} \ell_s; \tag{10}$$

see [16, Corollary 5.3] and [7, Proposition 17], respectively.

Proposition 4.5 *Let* $1 < p \leq \infty$. *Then*

$$\mathscr{M}(\ell_{p-}) = \ell_\infty \quad and \quad \mathscr{M}_c(\ell_{p-}) = \cup_{s>1}\ell_s. \tag{11}$$

Proof Recall that $(\ell_{p-})'_\beta = \ell_{p'+}$ and $(\ell_{r+})'_\beta = \ell_{r'-}$ for $1 \leq r < \infty$. Since the dual operator of $M_a \in \mathscr{L}(\ell_{r+})$ is again M_a, but considered as an element of $\mathscr{L}(\ell_{r'-})$, and the dual operator of $M_a \in \mathscr{L}(\ell_{p-})$ is again M_a, but considered as an element of $\mathscr{L}(\ell_{p'+})$, we see that the first equality in (11) follows from the first equality in (10).

Concerning the second equality in (11), let $a \in \cup_{s>1}\ell_s$. It follows from (10) that $M_a \in \mathscr{L}(\ell_{p'+})$ with M_a compact. By Lemma 4.2 below, the dual operator $M'_a \in \mathscr{L}(\ell_{p-})$ is compact, that is, $M_a \in \mathscr{L}(\ell_{p-})$ is compact. Accordingly, $a \in \mathscr{M}_c(\ell_{p-})$.

Conversely, let $a \in \mathscr{M}_c(\ell_{p-})$. Then $T := M'_a \in \mathscr{L}(\ell_{p'+})$. Since $T' = M''_a \in \mathscr{L}((\ell_{p'+})'_\beta) = \mathscr{L}(\ell_{p-})$ is compact (by the assumption on a), it follows from Lemma 4.2 below that $T = M'_a \ (= M_a) \in \mathscr{L}(\ell_{p'+})$ is a compact operator. Then (10) implies that $a \in \cup_{s>1}\ell_s$.

Lemma 4.2 *Let X be a quasinormable Fréchet space. Then $T \in \mathscr{L}(X)$ is compact if and only if $T' \in \mathscr{L}(X'_\beta)$ is compact. Moreover, if X is also reflexive, then $T'' = T$ and $(X'_\beta)'_\beta = X$.*

Proof If $T \in \mathscr{L}(X)$ is compact, then so is $T' \in \mathscr{L}(X'_\beta)$, even if X is not quasinormable, [31, p. 203 Proposition (12)].

Conversely, suppose that $T' \in \mathscr{L}(X'_\beta)$ is compact. Then there exists a closed, balanced, bounded set $B \subseteq X$ such that $T'(B^\circ)$ is relatively compact in X'_β, where B° is the polar of B. Since X is quasinormable (hence, distinguished, [36, Corollary 26.19]), X'_β is a *boundedly retractive* (LB)-space; see, for example, [13, Sect. 3 and Appendix], [15, Theorem], [36, Remark 25.13]. Accordingly, let $\{\mathscr{U}_n\}_{n=1}^\infty$ be a basis of closed, balanced and convex neighbourhoods of 0 in X, so that $X'_\beta = \mathrm{ind}_n (X')_{\mathscr{U}_n^\circ}$. Since compact sets in X'_β are compact in some step, there exists $m \in \mathbb{N}$ such that $T'(B^\circ)$ is relatively compact (hence, precompact) in $(X')_{\mathscr{U}_m^\circ}$. It follows from Gröthendieck's precompactness lemma, [31, p. 203 Proposition (10)], that $T(\mathscr{U}_m^{\circ\circ}) = T(\mathscr{U}_m)$ is precompact in X_B. Since X is complete, $T(\mathscr{U}_m)$ is relatively compact in the Banach space X_B and hence, also in X. So, T is compact.

Under reflexivity, that $T'' = T$ and $(X'_\beta)'_\beta = X$ is well known.

5 Operators from $ces(p-)$ into $ces(q-)$

Consider a pair $1 < p, q \leq \infty$. Denote by $\mathsf{C}_{c(\hat{p}),c(\hat{q})}$ (resp. $\mathsf{C}_{c(\hat{p}),\hat{q}}$; $\mathsf{C}_{\hat{p},c(\hat{q})}$; $\mathsf{C}_{\hat{p},\hat{q}}$) the Cesàro operator C when it acts from $ces(p-)$ into $ces(q-)$ (resp. $ces(p-)$ into ℓ_{q-}; resp. ℓ_{p-} into $ces(q-)$; resp. ℓ_{p-} into ℓ_{q-}), whenever this operator exists. The Closed Graph Theorem for operators between (LB)-spaces, [36, Theorem 24.31 and Remark 24.36], then ensures that this operator is continuous. We use the analogous notation for the natural inclusion maps $i_{c(\hat{p}),c(\hat{q})}$; $i_{c(\hat{p}),\hat{q}}$; $i_{\hat{p},c(\hat{q})}$; $i_{\hat{p},\hat{q}}$ whenever they exist. The aim of this section is to identify all pairs p, q for which these inclusion operators and Cesàro operators *do exist* and, for such pairs, to determine whether or not the operator is bounded and/or compact.

Proposition 5.1 *Let* $1 < p, q \leq \infty$ *be an arbitrary pair.*

(i) *The inclusion map* $i_{\hat{p},\hat{q}} : \ell_{p-} \to \ell_{q-}$ *exists if and only if* $p \leq q$.
(ii) *The inclusion map* $i_{\hat{p},c(\hat{q})} : \ell_{p-} \to ces(q-)$ *exists if and only if* $p \leq q$.
(iii) *The inclusion map* $i_{c(\hat{p}),c(\hat{q})} : ces(p-) \to ces(q-)$ *exists if and only if* $p \leq q$.
(iv) $ces(p-) \nsubseteq \ell_{q-}$ *for all choices of* $p, q \in (1, \infty]$.

All the inclusions, when they exist, are continuous.

Proof Suppose that $1 < p \leq q \leq \infty$. When forming the inductive limits $ces(p-) = \mathrm{ind}_n ces(p_n)$ and $ces(q-) = \mathrm{ind}_n ces(q_n)$ we can select $p_n \leq q_n$ for each $n \in \mathbb{N}$. The sufficiency conclusion in parts (i), (ii) and (iii) then follows from Lemma 4.1(i) and [6, Proposition 3.2].

We now prove the reverse implications in (i)–(iii). So, assume that $q < p$ and choose any $r \in (q, p)$. By [6, Proposition 3.2], there exists $x \in \ell_r \backslash \ell_q$. Hence, $x \in \ell_{p-} \backslash \ell_{q-}$ which covers case (i). Since $ces(r) \nsubseteq \ell_\infty$, there exists $y \in ces(r) \backslash \ell_{q-}$. Accordingly, $y \in ces(p-) \backslash \ell_{q-}$ which covers case (ii). Finally, it was noted in Sect. 2 that there exists $z \in ces(r) \backslash ces(q)$. In particular, $z \in ces(p-) \backslash ces(q-)$ which covers case (iii).

To see that (iv) holds, we apply [21, Remark 2.2(ii)] to find, for each $p \in (1, \infty]$, an element $x \in ces(p-)$ with $x \notin \ell_\infty$. In particular, $x \notin \ell_{q-} \subseteq \ell_\infty$.

Proposition 5.2 *Let* $1 < p \leq q \leq \infty$.

(i) *The inclusion map* $i_{\hat{p},\hat{q}}$ *is bounded if and only if* $p < q$. *However,* $i_{\hat{p},\hat{q}}$ *is not compact.*
(ii) *The inclusion map* $i_{c(\hat{p}),c(\hat{q})}$ *is bounded (equivalently, compact) if and only if* $p < q$.
(iii) *The inclusion map* $i_{\hat{p},c(\hat{q})}$ *is bounded (equivalently, compact) if and only if* $p < q$.

Proof If $p = q$, then the identity operator cannot be bounded, since neither ℓ_{p-} nor $ces(p-)$ is a Banach space. This shows the necessity of $p < q$ in parts (i) and (ii). Concerning the necessity of part (iii), assume that the natural inclusion $\ell_{p-} \subseteq ces(p-)$ is bounded. By Lemma 4.1(ii) there exists $m \in \mathbb{N}$ such that for $n \geq m$, we have $\ell_{p_n} \subseteq ces(p_m)$ with a continuous inclusion. Since $n := m + 1$ implies that $p_n > p_m$, this is impossible by [6, Proposition 3.2(ii)].

Assume now that $1 < p < q \leq \infty$. Fix any $r \in (p, q)$. Then $\ell_{p-} \subseteq \ell_r \subseteq \ell_{q-}$ in part (i), $ces(p-) \subseteq ces(r) \subseteq ces(q-)$ in part (ii) and $\ell_{p-} \subseteq \ell_r \subseteq ces(q-)$ in part (iii), with all inclusions continuous. Since ℓ_r and $ces(r)$ are Banach spaces, the continuous inclusions $\ell_r \subseteq \ell_q$ and $ces(r) \subseteq ces(q-)$ and $\ell_r \subseteq ces(q-)$ are all bounded. Since the bounded operators between lcH's form a 2-sided ideal, it follows that the inclusions $\ell_{p-} \subseteq \ell_{q-}$ (resp. $ces(p-) \subseteq ces(q-)$, resp. $\ell_{p-} \subseteq ces(q-)$) in part (i) (resp. in part (ii), resp. in part (iii)) are bounded.

Finally, if in part (i) the inclusion $\ell_{p-} \subseteq \ell_{q-}$ were compact, then taking any $p_1 \in (1, p)$, the inclusion $\ell_{p_1} \subseteq \ell_q$ would be compact because of the continuous factorization $\ell_{p_1} \subseteq \ell_{p-} \subseteq \ell_{q-} \subseteq \ell_q$. But, this is not the case, [6, Proposition 3.4(i)].

Proposition 5.3 *Let* $1 < p, q \leq \infty$.

(i) *The Cesàro operator* $\mathsf{C}_{\hat{p},\hat{q}}$ *is continuous if and only if* $p \leq q$.
(ii) *The Cesàro operator* $\mathsf{C}_{\hat{p},c(\hat{q})}$ *is continuous if and only if* $p \leq q$.
(iii) *The Cesàro operator* $\mathsf{C}_{c(\hat{p}),c(\hat{q})}$ *is continuous if and only if* $p \leq q$.
(iv) *The Cesàro operator* $\mathsf{C}_{c(\hat{p}),\hat{q}}$ *is continuous if and only if* $p \leq q$.

Proof As explained in the proof of Proposition 5.1 if $p \leq q$, then we may assume that $p_n \leq q_n$ for each $n \in \mathbb{N}$. So, in this case the continuity in parts (i)–(iv) follows from Lemma 4.1(i) and [6, Proposition 3.5].

Suppose now that $p > q$. Select any $r \in (q, p)$ in which case $\ell_r \subseteq \ell_{p-}$ continuously. Suppose that $\mathsf{C}_{\hat{p},\hat{q}}$ is continuous. Then Lemma 4.1(i) guarantees the existence of $q_m \in (1, q)$ such that $\mathsf{C}\colon \ell_r \to \ell_{q_m}$ is continuous. Since $r > q_m$, this contradicts [6, Proposition 3.5(i)]. This completes the proof of part (i).

The other cases (ii)–(iv) can be established in a similar way.

Proposition 5.4 *Let* $1 < p \leq q \leq \infty$.

(i) *The Cesàro operator* $\mathsf{C}_{\hat{p},\hat{q}}$ *is bounded if and only if it is compact if and only if* $p < q$.
(ii) *The Cesàro operator* $\mathsf{C}_{\hat{p},c(\hat{q})}$ *is bounded (equivalently, compact) if and only if* $p < q$.
(iii) *The Cesàro operator* $\mathsf{C}_{c(\hat{p}),c(\hat{q})}$ *is bounded (equivalently, compact) if and only if* $p < q$.
(iv) *The Cesàro operator* $\mathsf{C}_{c(\hat{p}),\hat{q}}$ *is bounded if and only if* $p < q$.

Proof For the case $p > q$, none of the operators in (i)–(iv) exist (cf. Proposition 5.3) and so, $p \leq q$ is a necessary condition. If $p = q$, then C is not bounded in all four cases. For example, in case (i) if $\mathsf{C}_{\hat{p},\hat{q}}$ were bounded, then there exists $m \in \mathbb{N}$ such that $\mathsf{C}\colon \ell_{p_n} \to \ell_{p_m}$ is continuous for all $n > m$; see Lemma 4.1(ii). Taking $n := m + 1$ (i.e. $p_n > p_m$) gives a contradiction to [6, Proposition 3.5(i)]. The other cases (ii)–(iv) can be argued similarly.

Assume now that $p < q$. Fix any $r \in (p, q)$.

(i) Note that $\mathsf{C}_{\hat{p},\hat{q}}$ can be written as the composition operator $\mathsf{C}_{\hat{p},\hat{q}} = B \circ i$, where $i\colon \ell_{p-} \to \ell_r$ is the natural inclusion and $B\colon \ell_r \to \ell_{q-}$ is the Cesàro operator, both of which are continuous. By Gröthendieck's Factorization Theorem, [36, Theorem 24.33], there exists $q_n \in (r, q)$ such that $B(\ell_r) \subseteq \ell_{q_n}$ with $B \in \mathscr{L}(\ell_r, \ell_{q_n})$. Actually, $B \in \mathscr{L}(\ell_r, \ell_{q_n})$ is compact, [6, Proposition 3.6(i)]. Since $\ell_{p-} \subseteq \ell_{r-} \subseteq \ell_r$ with continuous inclusions, it follows that the inclusion $i\colon \ell_{p-} \to \ell_r$ is bounded (cf. Proposition 5.2(i)). We can then conclude from $\mathsf{C}_{\hat{p},\hat{q}} = B \circ i$ that $\mathsf{C}_{\hat{p},\hat{q}}$ is compact. This completes the proof of part (i).

Since the bounded sets of the Montel space $ces(p-)$ are relatively compact, for each $p \in (1, \infty]$, in parts (ii)–(iii) it suffices to show that the Cesàro operator is bounded.

(ii) Note that $\mathsf{C}_{\hat{p},c(\hat{q})} = \mathsf{C}_{\hat{r},c(\hat{q})} \circ i_{\hat{p},\hat{r}}$. The inclusion $i_{\hat{p},\hat{r}}$ is bounded by Proposition 5.2(i) and $\mathsf{C}_{\hat{r},c(\hat{q})}$ is continuous by Proposition 5.3(ii). So, $\mathsf{C}_{\hat{p},c(\hat{q})}$ is bounded.

(iii) In this case $\mathsf{C}_{c(\hat{p}),c(\hat{q})} = \mathsf{C}_{c(\hat{r}),c(\hat{q})} \circ i_{c(\hat{p}),c(\hat{r})}$. Moreover, $i_{c(\hat{p}),c(\hat{r})}$ is bounded by Proposition 5.2(ii) and $\mathsf{C}_{c(\hat{r}),c(\hat{q})}$ is continuous by Proposition 5.3(iii). So, $\mathsf{C}_{c(\hat{p}),c(\hat{q})}$ is bounded.

(iv) Write $\mathsf{C}_{c(\hat{p}),\hat{q}} = \mathsf{C}_{c(\hat{r}),\hat{q}} \circ i_{c(\hat{p}),c(\hat{r})}$ and observe that $i_{c(\hat{p}),c(\hat{r})}$ is bounded by Proposition 5.2(ii) and $\mathsf{C}_{c(\hat{r}),\hat{q}}$ is continuous by Proposition 5.3(iv). So, $\mathsf{C}_{c(\hat{p}),\hat{q}}$ is bounded.

6 Riesz Space Properties of ℓ_{p-} and $ces(p-)$

Let $1 < p \le \infty$. We record here some properties of ℓ_{p-} and $ces(p-)$ as locally solid lc-Riesz spaces. Since these are the complexification of the corresponding real Riesz spaces, i.e., considered in $\mathbb{R}^{\mathbb{N}}$, we will freely use the relevant results from [8], keeping the same notation ℓ_{p-} and $ces(p-)$ as no confusion will occur.

(I) The space ℓ_{p-} is the strong dual $(\ell_{p'+})'_\beta$ of the Fréchet space $\ell_{p'+}$. Since all spaces ℓ_{p-} and $\ell_{p'+}$ are reflexive and the Fréchet spaces ℓ_{r+} and ℓ_{s+} (for $r \neq s$) are not isomorphic, by duality theory also ℓ_{p-} and ℓ_{q-} (for $p \neq q$) are *not* isomorphic.

(II) Given a locally solid lcHs X (see, e.g. [8]) define a positive cone in the dual space X' by

$$x' \le y' \ \text{ if and only if } \ \langle x, x'\rangle \le \langle x, y'\rangle \ \ \forall x \in X^+ := \{x \in X : x \ge 0\}; \tag{12}$$

see [8, Sect. 3]. For the strong dual topology and the order (12) on X' it turns out that X'_β is a Dedekind complete Riesz space, [8, Theorem 5.7]. Moreover, the Riesz space X'_β is a locally solid lcHs, [8, p. 59]. If, in addition, X is semireflexive, then both X and X'_β have a Lebesgue topology, [8, Theorem 22.4]. Locally solid, [8, Theorem 6.1], means that there exists a family $\{\rho_\alpha\}_\alpha$ of *Riesz seminorms* on X which generate the topology of X, that is, each ρ_α satisfies

$$\rho_\alpha(x) \le \rho_\alpha(y), \quad \forall x, y \in X \text{ with } x \le y.$$

We point out that order intervals in X'_β are necessarily topologically complete, [8, Theorem 19.13].

(III) Since the norms generating the topology of the Fréchet space $\ell_{p'+}$, $1 \le p < \infty$, namely

$$r_n(x) := \|x\|_{p_n}, \quad x \in \ell_{p'+} = \cap_{q>p'}\ell_q,$$

for $n \in \mathbb{N}$, where $p_n \downarrow p'$ with $\{p_n\}_{n=1}^\infty \subseteq (p', \infty)$, are clearly Riesz norms, it follows that $\ell_{p'+}$ is a locally solid, metrizable, lc-Riesz space (for the order induced from $\mathbb{C}^{\mathbb{N}}$) which is known to be complete and reflexive.

Consider now $\ell_{p-} := (\ell_{p'+})'_\beta$. From (II) it follows that ℓ_{p-} is a locally solid lcHs for the order induced by the positive cone (12), has a Lebesgue topology and is Dedekind complete. Moreover, ℓ_{p-} is complete and reflexive.

Claim: The order (12) is the same as that induced by the coordinatewise order $\prec$ in $\mathbb{C}^{\mathbb{N}}$, i.e., $x \prec y$ in $\mathbb{C}^{\mathbb{N}}$ if and only if $x_n \le y_n$ for all $n \in \mathbb{N}$.

To see this, suppose that $x' \le y'$ with $x', y' \in (\ell_{p'+})'_\beta = \ell_{p-}$. Then, for a fixed n, noting that the canonical basis vector $e_n \in (\ell_{p'+})^+$, we have $\langle e_n, x'\rangle \le \langle e_n, y'\rangle$, i.e., $x'_n \le y'_n$. Since $n \in \mathbb{N}$ is arbitrary, it follows that $x' \prec y'$.

Conversely, suppose that $x' \prec y'$ with $x', y' \in \ell_{p-} = (\ell_{p'+})'_\beta$, that is, $x'_n \le y'_n$ for $n \in \mathbb{N}$. Let $x \in (\ell_{p'+})^+$, i.e., $x_n \ge 0$ for all $n \in \mathbb{N}$. Then, by duality,

$$\langle x, x'\rangle = \sum_{n=1}^{\infty} x_n x'_n \le \sum_{n=1}^{\infty} x_n y'_n = \langle x, y'\rangle.$$

Hence, by definition, $x' \le y'$ (in the order (12)).

To summarize: For the order induced by $\mathbb{C}^{\mathbb{N}}$ (which is the complexification of $\mathbb{R}^{\mathbb{N}}$), each (LB)-space ℓ_{p-}, $1 < p \le \infty$, is a complete, reflexive, locally solid, lc-Riesz space which is Dedekind complete, has a Lebesgue topology and its order intervals are topologically complete.

Concerning $ces(p-)$ we know from Sect. 2 that is the strong dual of the Fréchet-Schwartz space $\lambda_0(A) = \Lambda^{\infty}_{(1/p')}((\log k)_k)$, which is a locally solid Fréchet lattice (for the order from $\mathbb{C}^{\mathbb{N}}$). Indeed, from Sect. 2, the sequence of norms $\{|||\cdot|||_n : n \in \mathbb{N}\}$ generating the topology of $\lambda_0(A)$ are clearly Riesz norms. Moreover, unlike for ℓ_{p-}, the space $ces(p-)$ is actually *Montel*. Arguing as for ℓ_{p-}, we can summarize as follows. Each (LB)-space $ces(p-)$, $1 < p \le \infty$, is a complete, Montel, locally solid, lc-Riesz space which is Dedekind complete, has a Lebesgue topology and its order intervals are topologically complete.

The spaces ℓ_{p-} and $ces(p-)$, for $1 < p \le \infty$, also have other desirable Riesz space properties. For instance, they cannot contain an isomorphic lattice copy of either ℓ_∞, ℓ_1 or c_0, [17, Theorem 1.2]. Equivalently, they cannot contain any positively complemented lattice copy of ℓ_∞, ℓ_1, c_0, [17, Remark 2.5(i) and Proposition 3.2].

Of course, for the barrelled space $ces(p-)$, which is Montel (see Sect. 2), this also follows from the fact that the closed subspaces of such spaces are semi-Montel, [29, Proposition 4(b), p. 230], and hence, if they are isomorphic to a Banach space, would be finite dimensional. For the regular, reflexive (LB)-space ℓ_{p-} all closed subspaces are semi-reflexive, [29, Proposition 5(a), p. 228], and hence, cannot be isomorphic to ℓ_∞, ℓ_1, c_0.

How does the fact that $ces(p-)$ is a Montel space whereas ℓ_{p-} is not reveal itself in the Riesz space properties of these spaces? First, since $ces(p-)$ (resp. ℓ_{p-}) is Dedekind complete with a Lebesgue topology, its order intervals are weakly compact, [8, Theorem 22.1]. For the complete *Montel* space $ces(p-)$ it follows that its order intervals are even compact. Moreover, sequences from the positive cone $(ces(p-))^+$ of $ces(p-)$ (resp. $(\ell_{p-})^+$ of ℓ_{p-}) which are topologically bounded and disjoint always converge *weakly* to 0, [8, Theorem 22.4]. Now comes another difference. Since $ces(p-)$ is Montel, weak convergence of a sequence in $ces(p-)$ implies its *strong convergence* in $ces(p-) = \lambda_0(A)'_\beta$, [29, p. 230 Proposition 1]. Hence, topologically bounded, disjoint sequences in $(ces(p-))^+$ converge strongly to 0. This feature *fails* for the reflexive, regular (LB)-space ℓ_{p-}. Indeed, consider the sequence of canonical vectors $B := \{e_n\}_{n=1}^\infty \subseteq (\ell_{p-})^+$ which is surely disjoint. Select any $q \in (p, \infty)$. Then also $B \subseteq \ell_q$. Since $\|e_n\|_q = 1$ for $n \in \mathbb{N}$, it is clear that B is bounded in the Banach space ℓ_q and hence, B is also bounded in ℓ_{p-}. Moreover, since $\ell_{p'+}$ is a quasinormable Fréchet space, its strong dual ℓ_{p-} is boundedly retractive, [14, Theorem 3.4]. In particular, the norm topology in ℓ_q coincides on the bounded set $B \subseteq \ell_{p-}$ with the inductive limit topology of ℓ_{p-}, [14, Definition 1.9]. Since $e_n \not\to 0$ in the Banach space ℓ_q, also $e_n \not\to 0$ in ℓ_{p-}.

Fix $1 < p \leq \infty$. A consequence of the fact that not every topologically bounded, disjoint sequence in ℓ_{p-} converges to 0 is that there *must* exist a power bounded operator T belonging to the centre $Z(\ell_{p-}) \subseteq \mathscr{L}(\ell_{p-})$ of ℓ_{p-} which is *not* uniformly mean ergodic, [17, Theorem 5.1]. Recall, $T \in Z(\ell_{p-})$ means that there exists $0 \leq \lambda \in \mathbb{R}$ such that $|T(x)| \leq \lambda |x|$ for all $x \in \ell_{p-}$, [17, p. 914]. An explicit example of such an operator T is the following one. Fix any sequence $a = (a_n)_n$ with $0 < a_n \uparrow 1$ (strictly), in which case $\|a\|_\infty = 1$. The multiplication operator $M_a \colon \ell_{p'+} \to \ell_{p'+}$ is continuous, [16, Corollary 5.3], and hence, its dual operator $T := M'_a$ belongs to $\mathscr{L}(\ell_{p-})$ as $\ell_{p-} = (\ell_{p'+})'_\beta$. For $x \in \ell_{p-}$ we see that $|T(x)| = |ax| = |a| \cdot |x| \leq |x|$ and so $T \in Z(\ell_{p-})$. Given $y \in \ell_{p-}$ there exists $q \in (p, \infty)$ such that $y \in \ell_q$ and hence,

$$\|T^n(y)\|_q = \|a^n y\|_q \leq \|a\|_\infty^n \|y\|_q \leq \|y\|_q, \quad n \in \mathbb{N},$$

where $a^n = a \cdot a \dots a$ (n-terms). Accordingly, $\{T^n(y)\}_{n=1}^\infty$ is bounded in ℓ_q and so is also bounded in ℓ_{p-}. Since $y \in \ell_{p-}$ is arbitrary, $\{T^n\}_{n=1}^\infty$ is bounded in $\mathscr{L}_s(\ell_{p-})$. But, ℓ_{p-} is barrelled as it is the strong dual of the reflexive Fréchet space $\ell_{p'+}$, [36, Proposition 25.12 and Corollary 25.14], and so $\{T^n\}_{n=1}^\infty$ is an equicontinuous subset of $\mathscr{L}(\ell_{p-})$, [36, Proposition 23.27], that is, T is power bounded. It follows that T is mean ergodic, [1, Corollary 2.7]. Assume that T is also uniformly mean ergodic, that is, $T_{[n]} \to S$ in $\mathscr{L}_b(\ell_{p-})$ for some $S \in \mathscr{L}(\ell_{p-})$. Since ℓ_{p-} is barrelled (with $\ell_{p'+} = (\ell_{p-})'_\beta$), the dual operators $(M_a)_{[n]} \to S'$ in $\mathscr{L}_b(\ell_{p'+})$, [2, Lemma 2.1], that is, $M_a \in \mathscr{L}(\ell_{p'+})$ is uniformly mean ergodic. But, this is known *not* to be the case; see the proof of Proposition 2.15 in [1]. Accordingly, the power bounded operator $T \in Z(\ell_{p-})$ is mean ergodic but *not* uniformly mean ergodic.

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Equicontinuity of Arcs in the Pointwise Dual of a Topological Abelian Group

In Honour of Manuel López-Pellicer

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Abstract We introduce, for any topological abelian group G, the property of equicontinuity of arcs of $G_p^\wedge$, the dual group of G endowed with its pointwise topology. We analyze the implications of this property, which we denote by EAP_σ, and we present some representative examples. Furthermore we prove that if G satisfies EAP_σ, every element of the arcwise connected component of $G_p^\wedge$ can be written as $\phi(1)$ for a suitable one-parameter subgroup $\phi : \mathbb{R} \to G_p^\wedge$.

Keywords Topological group · Dual group · Pointwise topology · Equicontinuity · Arcwise connected component · One-parameter subgroup

1 Introduction

The property of equicontinuity of arcs in the Pontryagin dual (EAP for short) is enjoyed by a wide class of topological abelian groups. It was explicitly defined for the first time in [3]. The origin of this property can be traced back to a known result by Nickolas [8]: for any topological abelian group G which is a k-space, every element of the arcwise connected component of the Pontryagin dual group of G (denoted by $G_{\mathrm{co}}^\wedge$) can be written as $\phi(1)$ for a suitable one-parameter subgroup $\phi : \mathbb{R} \to G_{\mathrm{co}}^\wedge$.

We use the symbols $\mathbb{I}$ and $\mathbb{T}$ for the unit interval $[0, 1]$ and the circle group, respectively. From well-known facts in general topology it follows that a continuous arc $\gamma : \mathbb{I} \to G_{\mathrm{co}}^\wedge$ has an equicontinuous image if and only if the mapping $\widetilde{\gamma} : (t, x) \in \mathbb{I} \times G \to \gamma(t)(x) \in \mathbb{T}$ is continuous. This result makes it possible to prove Nickolas' theorem under the weaker condition EAP.

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It is easy to realize that this characterization of equicontinuity of $\gamma(\mathbb{I})$ in terms of joint continuity of the associated mapping $\widetilde{\gamma}$ is actually true for any continuous arc in the dual group $G^\wedge$ endowed with the topology of pointwise convergence $\sigma(G^\wedge, G)$.

This leads naturally to the following, stronger variant of the equicontinuous arc property, which we introduce in this paper: A topological abelian group (G, τ) has the EAP_σ if for every continuous $\gamma : \mathbb{I} \to (G^\wedge, \sigma(G^\wedge, G))$, the set $\gamma(\mathbb{I})$ is τ-equicontinuous.

In the first part of the paper we study the property EAP_σ in relation with similar properties as EAP or g-barrelledness, and provide some relevant examples. We also characterize EAP_σ for groups and topological vector spaces carrying weak topologies.

Nickolas' theorem is one in an interesting collection of results connecting the real characters of a group G with one-parameter subgroups and the arc component of its dual group $G^\wedge$. The question arises to which extent one can prove natural analogues of these results where dual groups carry the pointwise convergence topology instead of the compact-open topology. We explore these ideas in the final part of the paper. Among other results, we prove that $G^\wedge_{co}$ and $(G^\wedge, \sigma(G^\wedge, G))$ have the same one-parameter subgroups (Proposition 2.5), and that for any group G with the EAP_σ, the arc-component of $(G^\wedge, \sigma(G^\wedge, G))$ is the subgroup of $G^\wedge$ formed by all liftable characters of G (Theorem 2.2).

Notation and Terminology

If G is an abelian group, the set of all homomorphism from G to $\mathbb{T}$ will be denoted by $\mathrm{Hom}(G, \mathbb{T})$, where $\mathbb{T}$ is the unit complex circle. The elements of $\mathrm{Hom}(G, \mathbb{T})$ are called *characters*. The set $\mathrm{Hom}(G, \mathbb{T})$ has a group structure with respect to the pointwise operation.

If (G, τ) is a topological abelian group, its dual group $G^\wedge := \mathrm{CHom}(G, \mathbb{T})$ is the set of all continuous characters of G. It is a subgroup of $\mathrm{Hom}(G, \mathbb{T})$. When $G^\wedge$ separates points of G, we say that G is MAP (a shorthand for "maximally almost periodic").

Let $A \subseteq G$, where G is a topological abelian group. The polar set of A is defined by

$$A^\triangleright = \{\chi \in G^\wedge \ : \ \chi(x) \in \mathbb{T}_+ \ \ \forall x \in A\}$$

where $\mathbb{T}_+ := \{\exp(2\pi i t) : t \in [-1/4, 1/4]\}$.

A subset $S \subseteq G^\wedge$ is equicontinuous with respect to τ if and only if $S \subseteq U^\triangleright$ for some neighborhood of zero U in G.

A *group duality* is a triple $(G, H, \langle\cdot,\cdot\rangle)$ where G and H are abelian groups and $\langle\cdot,\cdot\rangle : G \times H \to \mathbb{T}$ is a bicharacter, that is, $\langle x, \cdot\rangle \in \mathrm{Hom}(H, \mathbb{T})$ and $\langle\cdot, y\rangle \in \mathrm{Hom}(G, \mathbb{T})$ for every $x \in G$ and $y \in H$. We will abbreviate the notation $(G, H, \langle\cdot,\cdot\rangle)$ to $\langle G, H\rangle$ in what follows. To every group duality $\langle G, H\rangle$ we can associate in a natural way its inverse duality $\langle H, G\rangle$.

Given a group duality $\langle G, H\rangle$, the weak topology $\sigma(G, H)$ is the initial topology on G with respect to the elements of H. All precompact Hausdorff topologies on an abelian group G have the form $\sigma(G, L)$ where L is a subgroup of $\mathrm{Hom}(G, \mathbb{T})$ which separates the points of G. Moreover $L = (G, \sigma(G, L))^\wedge$ [6]. This result will be quoted in what follows as "Comfort-Ross Theorem".

An abelian topological group (G, τ) gives rise to the natural dualities $\langle G, G^\wedge\rangle$ and $\langle G^\wedge, G\rangle$. In what follows, the dual group of G endowed with the pointwise convergence topology $\sigma(G^\wedge, G)$ will be often denoted by $G^\wedge_p$.

We will use the notation $G^\wedge_{\text{co}}$ for the dual group $G^\wedge$ endowed with the compact-open topology τ_{co}. For a topological abelian group G, we denote by $\alpha_G : G \to (G^\wedge_{\text{co}})^\wedge$ and $\beta_G : G \to (G^\wedge_p)^\wedge$ the natural homomorphisms defined by $\alpha_G(x)(\chi) = \beta_G(x)(\chi) = \chi(x)$.

Recall that by Pontryagin-van Kampen Theorem, α_G is a topological isomorphism from G to $(G^\wedge_{\text{co}})^\wedge_{\text{co}}$ for any locally compact abelian group G. The proof of the next result is immediate from the preceding considerations:

Proposition 1.1 *Let G be a topological abelian group.*

(a) If G is MAP then both α_G and β_G are monomorphisms.
(b) β_G is onto and continuous from G to $(G^\wedge_p)^\wedge_p$.
(c) α_G is continuous from G to $(G^\wedge_{\text{co}})^\wedge_{\text{co}}$ if and only if all compact subsets of $G^\wedge_{\text{co}}$ are equicontinuous.

The following is a well-known result.

Proposition 1.2 ([1, Proposition 3.5]) *Let G be a topological abelian group and U a neighbourhood of zero in G. Then $U^\triangleright$ is compact in $G^\wedge_{\text{co}}$. In particular, $\sigma(G^\wedge, G)$ and τ_{co} induce the same topology on any equicontinuous subset of $G^\wedge$.*

If G and H are topological abelian groups and $\varphi : G \to H$ is a continuous homomorphism, we define the adjoint of φ as the group homomorphims $\varphi^\wedge : H^\wedge \to G^\wedge$ given by $\varphi^\wedge(\chi) = \chi \circ \varphi$. Note that φ is continuous if we endow both $H^\wedge$ and $G^\wedge$ with their pointwise convergence topologies, or their corresponding compact-open topologies.

A topological abelian group G is said to be g-barrelled if every $\sigma(G^\wedge, G)$-compact subset of $G^\wedge$ is equicontinuous. This notion was introduced in [5]. There are many classes of g-barrelled groups: all locally compact (even all Čech complete) abelian groups are g-barrelled. Pseudocompact abelian groups are also g-barrelled. A recent reference on g-barrelled groups is [4].

2 Main Results

Definition 2.1 A topological abelian group (G, τ) is said to satisfy the

- EAP (equicontinuous arc property), if for every continuous map $\gamma : \mathbb{I} \to G^\wedge_{\text{co}}$ the set $\gamma(\mathbb{I})$ is τ-equicontinuous.

- EAP_σ if for every continuous map $\gamma : \mathbb{I} \to (G^\wedge, \sigma(G^\wedge, G))$ the set $\gamma(\mathbb{I})$ is τ-equicontinuous.

Note that in the definitions of the EAP and the EAP_σ, if we replace arcs with arbitrary compact subsets we obtain the definitions of a group G with α_G continuous and a g-barrelled group, respectively. The following diagram might be useful:

$$\begin{array}{ccc} g\text{-barrelled} & \Longrightarrow & \alpha_G \text{ continuous} \\ \Downarrow & & \Downarrow \\ \text{EAP}_\sigma & \Longrightarrow & \text{EAP} \end{array}$$

The next result makes the relationship between EAP and EAP_σ more precise:

Proposition 2.1 *Given a topological abelian group G, the following are equivalent:*

(1) G has the EAP_σ.
(2) G has the EAP and every $\sigma(G^\wedge, G)$-continuous arc in $G^\wedge$ is continuous in $G^\wedge_{\text{co}}$.

Proof (1)⇒(2): Fix a continuous arc $\gamma : \mathbb{I} \to (G^\wedge, \sigma(G^\wedge, G))$. Since G has the EAP_σ, the subset $\gamma(\mathbb{I}) \subseteq G^\wedge$ is equicontinuous. By Proposition 1.2, the topologies $\sigma(G^\wedge, G)$ and τ_{co} induce the same topology on the equicontinuous subset $\gamma(\mathbb{I})$. We deduce that γ is continuous from $\mathbb{I}$ to $G^\wedge_{\text{co}}$.

(2)⇒(1) is clear.

Corollary 2.1 *Let G be a topological group with the EAP_σ. Then the arc-connected components of $G^\wedge_{\text{co}}$ and $G^\wedge_p$ coincide.*

Example 2.1 A precompact group G with the EAP_σ for which α_G is not continuous, hence G is not g-barrelled: Consider the group $G = (X, \sigma(X, X^\wedge))$ where X is a noncompact, locally compact abelian group such that $X^\wedge_{\text{co}}$ is totally disconnected.

Let us see that α_G is not continuous. Fix a compact neighborhood of zero U in $X^\wedge_{\text{co}}$. Since $X^\wedge_{\text{co}}$ is nondiscrete, U is an infinite set. By [3, Lemma 2.6], U is not equicontinuous with respect to $\sigma(X, X^\wedge)$.

Let us see that G has the EAP_σ. Fix a continuous arc $\gamma : \mathbb{I} \to (X^\wedge, \sigma(X^\wedge, X))$. Since locally compact abelian groups have the EAP_σ, we deduce from Proposition 2.1 that γ is also a continuous arc in $X^\wedge_{\text{co}}$. Since $X^\wedge_{\text{co}}$ is totally disconnected by hypothesis, γ is constant.

Example 2.2 A metrizable group with the EAP_σ which is not g-barrelled: Consider the group of rational numbers $\mathbb{Q}$ endowed with the topology induced by $\mathbb{R}$. By [2, Proposition 5.3], $\mathbb{Q}$ is not g-barrelled. Let us see that $\mathbb{Q}$ has the EAP_σ. Let $q : \mathbb{R} \to \mathbb{R}^\wedge$ be the group isomorphism defined by $q(\lambda)(\mu) = \exp(2\pi i \lambda\mu)$. For every character $\chi \in \mathbb{Q}^\wedge$ let $\widetilde{\chi}$ denote the unique continuous character of $\mathbb{R}$ whose restriction to the rationals is χ. It is clear that the mapping $[\chi \in \mathbb{Q}^\wedge \mapsto q^{-1}(\widetilde{\chi}) \in \mathbb{R}]$ is a group isomorphism which carries equicontinuous subsets of $\mathbb{Q}^\wedge$ to bounded subsets of $\mathbb{R}$ and vice versa. In particular the groups $(\mathbb{Q}^\wedge, \sigma(\mathbb{Q}^\wedge, \mathbb{Q}))$ and $(\mathbb{R}, \sigma(\mathbb{R}, \mathbb{Q}))$ are naturally topologically isomorphic. Fix a continuous arc $\gamma : \mathbb{I} \to (\mathbb{R}, \sigma(\mathbb{R}, \mathbb{Q}))$ and let us show that $\gamma(\mathbb{I})$ is bounded. The arc $\exp(2\pi i\gamma) : \mathbb{I} \to \mathbb{T}$ is continuous with respect to the usual topology of $\mathbb{T}$. Let $\Gamma : \mathbb{I} \to \mathbb{R}$ be a continuous lifting of $\exp(2\pi i\gamma)$. Recall that any continuous arc on a Tychonoff space either is constant or has an image of size greater or equal than $\mathfrak{c}$. Since the continuous arc $\gamma - \Gamma$ in $(\mathbb{R}, \sigma(\mathbb{R}, \mathbb{Q}))$ takes only integer values, it must be constant. This clearly implies that $\gamma(\mathbb{I})$ is a bounded subset of $\mathbb{R}$.

We will see in Example 2.3 that in general the EAP is strictly weaker than the EAP_σ, and also that the converse of Corollary 2.1 is false.

It is clear that every topological group G for which the arc component of the group $(G^\wedge, \sigma(G^\wedge, G))$ is trivial has the EAP_σ. This observation motivates the following result.

Proposition 2.2 *Let $\langle G, H\rangle$ be a duality of abelian groups. Assume that G separates the points of H. The following are equivalent:*

(a) $(G, \sigma(G, H))$ has the EAP_σ.
(b) The arc component of the group $(H, \sigma(H, G))$ is trivial.

Proof Note that because of Comfort-Ross theorem, the pointwise dual group of $(G, \sigma(G, H))$ is $(H, \sigma(H, G))$. Recall that a subset of H which is equicontinuous with respect to $\sigma(G, H)$ is necessarily finite [3, Lemma 2.6]. Taking into account that any arc with finite range in a Hausdorff space as $(H, \sigma(H, G))$ is necessarily constant, the result follows.

Topological vector spaces constitute a very important class of well-behaved topological abelian groups. We next characterize the EAP_σ within this class, in terms of vector space dualities.

For simplicity we restrict ourselves to vector spaces over $\mathbb{R}$ in what follows. The notion of a vector space duality is well known. Given any duality $\langle E, F\rangle$ of vector spaces let us denote by $\langle E, F\rangle_g$ the natural associated group duality given by $\langle x, y\rangle_g = \exp(2\pi i\langle x, y\rangle)$, by $\omega(E, F)$ the initial vector space topology on E with respect to the duality $\langle E, F\rangle$, and by $\sigma(E, F)$ the initial group topology on E with respect to the group duality $\langle E, F\rangle_g$. It is clear that $\sigma(E, F) \le \omega(E, F)$.

Lemma 2.1 ([9, Lemma 1.2]) *If F separates the points of E in the vector space duality $\langle E, F\rangle$, then the topologies $\sigma(E, F)$ and $\omega(E, F)$ have the same compact sets.*

Lemma 2.2 *Let E be a topological vector space. Consider the group homomorphism $\Phi_E : E^* \to E^\wedge$ defined by $\Phi_E(f) = \exp(2\pi i f)$.*

(a) [10, Lemma 1] Φ_E is an isomorphism of abelian groups.
(b) [5, Proposition 1.11] Φ_E preserves equicontinuous sets in both directions.

Proposition 2.3 *Let E be a topological vector space. The following are equivalent:*

(a) E has the EAP_σ
(b) Every arc in $(E^*, \omega(E^*, E))$ is equicontinuous.

Proof Consider the mapping $\Phi_E : E^* \to E^\wedge$ defined by $\Phi_E(f) = \exp(2\pi i f)$. Taking into account Lemma 2.2, it suffices to show that a mapping $\gamma : \mathbb{I} \to E^*$ is $\omega(E^*, E)$-continuous if and only if $\Phi_E \circ \gamma$ is $\sigma(E^\wedge, E)$-continuous.

If $\gamma : \mathbb{I} \to E^*$ is $\omega(E^*, E)$-continuous, then clearly $\Phi_E \circ \gamma$ is $\sigma(E^\wedge, E)$-continuous. Conversely, let $\xi : \mathbb{I} \to E^\wedge$ be $\sigma(E^\wedge, E)$-continuous. We need to show that $\Phi_E^{-1} \circ \xi$ is $\omega(E^*, E)$-continuous. It is an immediate consequence of Lemma 2.2(a) that Φ_E defines a topological isomorphism from $(E^*, \sigma(E^*, E))$ to $(E^\wedge, \sigma(E^\wedge, E))$. In particular the set $\Phi_E^{-1}(\xi(\mathbb{I}))$ is $\sigma(E^*, E)$-compact. By Lemma 2.1 it is also $\omega(E^*, E)$-compact. By minimality of Hausdorff compact topologies, $\omega(E^*, E)$ and $\sigma(E^*, E)$ induce the same topology on $\Phi_E^{-1}(\xi(\mathbb{I}))$. This completes the proof.

Proposition 2.4 *Let $\langle E, F\rangle$ be a vector space duality. Assume that E separates the points of F. The following are equivalent:*

(a) $(E, \omega(E, F))$ is g-barrelled.
(b) $(E, \omega(E, F))$ has the EAP_σ.
(c) Every bounded subset of $(F, \omega(F, E))$ is contained in a finite dimensional subspace of F.

Proof (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c): Assume that the topological vector space $(E, \omega(E, F))$ has the EAP_σ. By
Proposition 2.3, every arc in $(F, \omega(F, E))$ is equicontinuous with respect to $\omega(E, F)$. By [3, Lemma 2.6(b)], we deduce that every arc in $(F, \omega(F, E))$ is contained in a finite dimensional subspace of F. By [3, Lemma 2.13], the same is true for any bounded subset of $(F, \omega(F, E))$.

(c) $\Rightarrow$ (a): It is an immediate consequence of Lemma 2.2(a) that for the topological group $G = (E, \omega(E, F))$ one has a natural isomorphism $(G^\wedge, \sigma(G^\wedge, G)) \cong (F, \sigma(F, E))$. Let K be a $\sigma(F, E)$-compact subset of F. We need to show that K is equicontinuous with respect to $\omega(E, F)$. By Lemma 2.1, K is $\omega(F, E)$-compact. By our assumption, K is contained in the absolutely convex hull of a finite subset $\{y_1, \ldots, y_n\} \subseteq F$. This implies that $|\langle x, y\rangle| \leq 1$ for every $y \in K$ and every x in the $\omega(E, F)$-neighborhood of zero $\bigcap_{j=1}^n \{x \in E : |\langle x, y_j\rangle| \leq 1\}$. This completes the proof.

Example 2.3 A topological group X which satisfies the EAP but not the EAP_σ : Let us denote by $\mathbb{R}^{(\mathbb{N})}$ the subspace of the countable product of real lines formed by those sequences with only finitely many nonzero components. Consider the topological vector space $X = \mathbb{R}^{(\mathbb{N})}$ endowed with the induced product topology. Since X is metrizable, it is a k-space and in particular, by Ascoli's theorem, α_X is continuous. Hence X has the EAP.

It was proved in [7, Theorem 3.1] that X is not g-barrelled. Let us see that actually X does not satisfy the EAP_σ. (Note, however, that $X_p^\wedge$ and $X_{\mathrm{co}}^\wedge$ are arc connected, since both topological groups admit a compatible topological vector space structure.)

The space X can be written as $X = (E, \omega(E, F))$ for $E = F = \mathbb{R}^{(\mathbb{N})}$. By Proposition 2.4, in order to prove that X does not satisfy the EAP_σ it is enough to find a bounded subset of $(F, \omega(F, E)) = X$ which generates an infinite-dimensional subspace of $\mathbb{R}^{(\mathbb{N})}$. Since a subset $B \subseteq X$ is bounded if and only if it is contained in a product of bounded intervals, it is trivial to produce such an example.

We devote the remaining of this paper to prove that if the group G has the EAP_σ, the arc-component of $G_p^\wedge$ can be obtained as the subgroup of $G^\wedge$ formed by all liftable characters of G. We will need some previous results on one-parameter subgroups of $G_p^\wedge$ which are of independent interest.

For any topological group G, we denote by G_a its arc-connected component and by $\mathfrak{L}(G)$ the group $\mathrm{CHom}(\mathbb{R}, G)$. The exponential mapping $\exp_G : \mathfrak{L}(G) \to G$ is defined by $\exp_G(\varphi) = \varphi(1)$. It is clear that $\exp_G$ is continuous with respect to the pointwise convergence topology on $\mathfrak{L}(G)$.

Proposition 2.5 *Let G be a topological abelian group. Then $\mathfrak{L}(G_p^\wedge) = \mathfrak{L}(G_{\mathrm{co}}^\wedge)$.*

Proof Fix any continuous homorphism $\varphi : \mathbb{R} \to G_p^\wedge$ and let us show that φ is also continuous from $\mathbb{R}$ to $G_{\mathrm{co}}^\wedge$. An easy comprobation shows that φ can be obtained as $\beta_G^\wedge \circ \varphi^{\wedge\wedge} \circ \beta_{\mathbb{R}}$ where

- $\beta_{\mathbb{R}} : \mathbb{R} \to (\mathbb{R}_p^\wedge)_{\mathrm{co}}^\wedge$ is continuous since $\mathbb{R}$ is g-barrelled [4];
- $\varphi^{\wedge\wedge} : (\mathbb{R}_p^\wedge)_{\mathrm{co}}^\wedge \to ((G_p^\wedge)_p^\wedge)_{\mathrm{co}}^\wedge$ is the τ_{co}-adjoint of the τ_p-adjoint of the continuous homomorphism φ and consequently is continuous itself;
- $\beta_G^\wedge : ((G_p^\wedge)_p^\wedge)_{\mathrm{co}}^\wedge \to G_{\mathrm{co}}^\wedge$ is the τ_{co}-adjoint of the continuous map $\beta_G : G \to (G_p^\wedge)_p^\wedge$, hence continuous itself.

The following result characterizes the "straight arcs" version of the EAP_σ. We denote by $\mathbb{R}_\sigma$ the topological group $(\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge))$, which is topologically isomorphic to $(\mathbb{R}^\wedge, \sigma(\mathbb{R}^\wedge, \mathbb{R}))$ through the mapping $q : \mathbb{R} \to \mathbb{R}^\wedge$ given by $q(\lambda)(\mu) = \exp(2\pi i \lambda \mu)$.

Proposition 2.6 *The following conditions are equivalent for a topological abelian group G:*

(a) *For every $\varphi \in \mathfrak{L}(G_p^\wedge)$ the set $\varphi(\mathbb{I})$ is equicontinuous.*
(b) $\mathrm{CHom}(G, \mathbb{R}_\sigma) = \mathrm{CHom}(G, \mathbb{R})$.

Proof Proposition 3.5 in [3] is the same result with $\mathfrak{L}(G_{\mathrm{co}}^\wedge)$ instead of $\mathfrak{L}(G_p^\wedge)$. Hence it suffices to apply Proposition 2.5.

Definition 2.2 Let G be a topological abelian group. $G_{\mathrm{lift}}^\wedge$ and $G_{\mathrm{lift}\sigma}^\wedge$ denote the group of continuous characters of G that can be lifted over $\mathbb{R}$ and $\mathbb{R}_\sigma$, respectively.

Theorem 2.1 *Let G be a topological abelian group. We have the following chain of inclusions:*

$$G^{\wedge}_{\text{lift}} \leq G^{\wedge}_{\text{lift}\sigma} = \operatorname{im}\exp_{G^{\wedge}_p} \leq (G^{\wedge}_p)_a$$

Proof The inclusion $G^{\wedge}_{\text{lift}} \leq G^{\wedge}_{\text{lift}\sigma}$ is clear. On the other hand, for every topological abelian group H we have $\operatorname{im}\exp_H \leq H_a$ because $\exp_H : \mathfrak{L}(H) \to H$ is continuous with respect to the pointwise convergence topology on $\mathfrak{L}(H)$, and $\mathfrak{L}(H)$ is arc-connected with this topology.

The identity $G^{\wedge}_{\text{lift}\sigma} = \operatorname{im}\exp_{G^{\wedge}_{\text{co}}}$ was proved in [3, Theorem 3.6]. The identity $G^{\wedge}_{\text{lift}\sigma} = \operatorname{im}\exp_{G^{\wedge}_p}$ follows from this one and Proposition 2.5.

Theorem 2.2 *Let G be a topological abelian group with the EAP$_\sigma$. Then*

$$G^{\wedge}_{\text{lift}} = \operatorname{im}\exp_{G^{\wedge}_p} = (G^{\wedge}_p)_a$$

Proof By Theorem 3.11 in [3] we have $G^{\wedge}_{\text{lift}} = (G^{\wedge}_{\text{co}})_a$. Since G has the EAP$_\sigma$, Corollary 2.1 implies that $(G^{\wedge}_{\text{co}})_a = (G^{\wedge}_p)_a$. It only remains to apply Theorem 2.1.

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On Ultrabarrelled Spaces, their Group Analogs and Baire Spaces

In Honour of Manuel López-Pellicer, Loyal Friend and Indefatigable Mathematician

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Abstract Let E and F be topological vector spaces and let G and Y be topological abelian groups. We say that E is *sequentially barrelled with respect to* F if every sequence $(u_n)_{n\in\mathbb{N}}$ of continuous linear maps from E to F which converges pointwise to zero is equicontinuous. We say that G is *barrelled with respect to* F if every set $\mathscr{H}$ of continuous homomorphisms from G to F, for which the set $\mathscr{H}(x)$ is bounded in F for every $x \in E$, is equicontinuous. Finally, we say that G is *g-barrelled with respect to* Y if every $\mathscr{H} \subseteq \mathrm{CHom}(G, Y)$ which is compact in the product topology of Y^G is equicontinuous. We prove that

- a barrelled normed space may not be sequentially barrelled with respect to a complete metrizable locally bounded topological vector space,
- a topological group which is a Baire space is barrelled with respect to any topological vector space,
- a topological group which is a Namioka space is g-barrelled with respect to any metrizable topological group,
- a protodiscrete topological abelian group which is a Baire space may not be g-barrelled (with respect to $\mathbb{R}/\mathbb{Z}$).

We also formulate some open questions.

Keywords Topological vector space · Locally convex space · Equicontinuity · Barrelledness · Ultrabarrelledness · Topological group · Baire space · Namioka space

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1 Main Results

A locally convex space E is said to be barrelled if every closed, absorbing and absolutely convex subset of E is a neighborhood of zero. Barrelled (real) locally convex spaces were introduced by N. Bourbaki in [3] and they have been extensively studied in many references as the monographs [10, 16].

Already in [3] the following characterization of barrelled spaces can be found:

Theorem 1.1 *Let E be a locally convex space. The following properties are equivalent:*

(i) E is barrelled.
(ii) If F is a nontrivial locally convex Hausdorff topological vector space and $\mathscr{H}$ is a set of continuous linear mappings from E to F for which the set

$$\mathscr{H}(x) = \bigcup_{u \in \mathscr{H}} \{u(x)\}$$

is bounded in F for every $x \in E$, then $\mathscr{H}$ is equicontinuous.

Local convexity of F is essential for the validity of implication $(i) \Rightarrow (ii)$ of Theorem 1.1. It seems that this fact was pointed out for the first time in [23].

W. Robertson obtained in [17, Theorem 4] the following characterization: a locally convex space E with topology η is barrelled if and only if the only locally convex vector space topologies with bases of η-closed neighborhoods of the origin are those coarser than η. This motivated the following definition, included in the same reference:

Definition 1.1 Let E be a topological vector space under the topology η. We say that E is *ultrabarrelled* if the only vector space topologies on E, compatible with the algebraic structure of E and in which there is a base of η-closed neighbourhoods of the origin, are those coarser than η.

For ultrabarrelled spaces we have the following nice analogue of Theorem 1.1:

Theorem 1.2 (W. Robertson, L. Waelbroeck) *For a topological vector space E the following properties are equivalent.*

(i) E is ultrabarrelled.
(ii) If F is a topological vector space and $\mathscr{H}$ is a set of continuous linear mappings from E to F, for which the set

$$\mathscr{H}(x) = \bigcup_{u \in \mathscr{H}} \{u(x)\}$$

is bounded in F for every $x \in E$, then $\mathscr{H}$ is equicontinuous.

The implication $(i) \Rightarrow (ii)$ was proved in [17], where the validity of $(ii) \Rightarrow (i)$ was posed as a question as well. It seems that a (rather delicate) proof of the implication $(ii) \Rightarrow (i)$ appeared for the first time in [22, Proposition I.5]. A proof of Theorem 1.2 is presented also in [1, §7.3] (where the term 'barrelled' is used instead of 'ultrabarrelled').

It is clear that any ultrabarrelled locally convex space is barrelled. In [17] an argument based on an idea from [23] was used to show that the converse implication may fail:

Theorem 1.3 ([17, p. 256]) *There is a normed space which is barrelled but not ultrabarrelled.*

To formulate our first theorem we need to introduce the following concept:

Definition 1.2 Let E be a topological vector space and F a Hausdorff topological vector space. We say that E is *sequentially barrelled with respect to* F if every sequence $(u_n)_{n\in\mathbb{N}}$ of continuous linear maps from E to F which converges pointwise to zero is equicontinuous.

Clearly every ultrabarrelled space is sequentially barrelled with respect to any topological vector space. Hence the following result is a refinement of Theorem 1.3:

Theorem 1.4 *A barrelled normed space need not be sequentially barrelled with respect to a complete metrizable, locally bounded topological vector space.*

Question 1.1 Let E be a normed space which is sequentially barrelled with respect to every complete metrizable (locally bounded) topological vector space. Is then E ultrabarrelled?

The following result establishes a natural connection between ultrabarrelledness and the property of being a Baire space:

Theorem 1.5 ([17, Proposition 12]) *Let E be a topological vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If E as a topological space is a Baire space, then E is ultrabarrelled.*

In view of Theorem 1.5 the underlying topological space of a barrelled normed space which is not ultrabarrelled cannot be a Baire space. According to [7], the first example of a normed barrelled space which is not Baire appeared in [9]; see also [8, 20] for more examples of this sort.

Definition 1.3 Let G be a topological group and F be a topological vector space over a nontrivially valued division ring $\mathbb{K}$. We say that G is *barrelled with respect to* F if every set $\mathscr{H}$ of continuous homomorphisms from G to F, for which the set

$$\mathscr{H}(x) = \bigcup_{u\in\mathscr{H}} \{u(x)\}$$

is bounded in F for every $x \in E$, is equicontinuous.

We shall prove the following statement, which generalizes a similar one obtained in [13] for the case of a normed space F.

Theorem 1.6 *Let G be a topological group and F be a topological vector space over $\mathbb{R}$. If G as a topological space is a Baire space, then G is barrelled with respect to F.*

Question 1.2 Let G be a topological group and F be a topological vector space over a nontrivially valued division ring $\mathbb{K}$. If G as a topological space is a Baire space, is then G barrelled with respect to F?

Definition 1.4 Let X be a topological space.

- X is called *a Namioka space* ([6]), or is said to have *the Namioka property*, if for every compact Hausdorff space K, every metrizable space Z and every separately continuous $f : X \times K \to Z$, there exists a dense G_δ-subset A of X such that f is continuous at every point of $A \times K$.
- X is called *a weak Namioka space*, or is said to have *the weak Namioka property*, if for every compact Hausdorff space K, every metrizable space Z and every separately continuous $f : X \times K \to Z$, there exists $a \in X$ such that f is continuous at every point of $\{a\} \times K$.

Proposition 1.1 *(a) Let X be a topological space. Assume that for every compact Hausdorff space K, every metrizable space Z and every separately continuous $f : X \times K \to Z$ there exists a dense subset A of X such that f is continuous at every point of $A \times K$. Then X is a Namioka space.*

(b) (A. Bouziad, oral communication) Let X be a topological space. Assume that every element of X admits a neighborhood which is a Namioka space. Then X is a Namioka space.

Proof (a) This follows from the following known fact (see [15, p. 518]): for a separately continuous $f : X \times K \to Z$ the set

$$A(f) := \{a \in X : f \text{ is continuous at every point of } \{a\} \times K\}$$

is always a G_δ-subset of X.

(b) Let $f : X \times K \to Z$ be a separately continuous map, where Z is a metric space and K is a compact space. Taking into account (a), we only have to show that the set $A(f)$ is dense in X. Let U be a nonempty open subset of X. Choose x in U and let V be a neighborhood of x in X such that V is a Namioka space. Choose also an open subset W of X such that $x \in W$ and $W \subset V$. The mapping $g := f|_{V \times K} \to Z$ is separately continuous; since V is a Namioka space, the set

$$A(g) := \{a \in V : g \text{ is continuous at every point of } \{a\} \times K\}$$

is dense in V. From this, since $U \cap W$ is a nonempty open subset of V, we get that $A(g) \cap (U \cap W) \neq \emptyset$. Fix an element $a \in A(g) \cap (U \cap W)$. Clearly, f is jointly continuous at each point of $\{a\} \times K$.

The next theorem contains several known results about Namioka spaces.

Theorem 1.7 *The following statements hold:*

(a) [15, Theorem 1.2] If X is a strongly countably complete regular space, then X is a Namioka space. In particular, if X is a Čech complete Tychonoff space, then X is a Namioka space.
(b) [19, Théorème 3] If X is a completely regular Namioka space, then X is a Baire space.
(c) [19, Théorème 7] If X is a metrizable Baire space, then X is a Namioka space.
(d) [21, Théorème 2] There exists a completely regular Hausdorff Baire space, which has not the weak Namioka property.
(e) [21, Corollaire 6] If X is a Baire space which contains a dense σ-compact subset, then X is a Namioka space.
(f) [18] If X is a Baire space which contains a dense $\mathscr{K}$-countably determined subset, then X is a Namioka space.
(g) [2, p. 333] If X is a pseudocompact space, then X is a Namioka space.

Proposition 1.2 *If X is a locally pseudocompact space, then X is a Namioka space.*

Proof This follows from Theorem 1.7(g) and Proposition 1.1(b).

Definition 1.5 Let G be a topological group and Y a Hausdorff topological group. We say that

- G is *g-barrelled with respect to Y* if every $\mathscr{H} \subseteq \mathrm{CHom}(G, Y)$ which is compact in the product topology of Y^G is equicontinuous.
- G is *sequentially g-barrelled with respect to Y* if every sequence $\{u_n\}_{n\in\mathbb{N}}$ contained in $\mathrm{CHom}(G, Y)$ which converges pointwise to zero, is equicontinuous.

In the case where Y is the compact group $\mathbb{R}/\mathbb{Z}$ we will drop the reference to Y and use the shorter expression "(sequentially) g-barrelled group".

g-barrelled topological abelian groups were introduced in [5]. Corollary 1.6 in this reference provides some classes of g-barrelled groups. Also, several permanence properties of this class were established in [5], but only recently it was proved that the class of g-barrelled groups is closed with respect to Cartesian products [4].

For our purposes it is convenient to highlight the following results from [5]:

Theorem 1.8 *Let G and Y be topological groups.*

(a) If G as a topological space is a Baire space, then G is sequentially g-barrelled with respect to Y (cf. [5, Proposition 1.4]).
(b) If G and Y are metrizable and all closed separable subgroups of G are Baire spaces, then G is g-barrelled with respect to Y (cf. [5, Theorem 1.5]).

Here we shall prove the following statements:

Theorem 1.9 *Let G be a topological group and Y be a metrizable topological group.*

(a) If G as a topological space is a Namioka space, then G is g-barrelled with respect to Y.
(b) If G as a topological space is locally pseudocompact, then G is g-barrelled with respect to Y.

Remark 1.1 The following particular case of Theorem 1.9(b) was obtained earlier in [11, Proposition 4.4]: every pseudocompact topological abelian group is g-barrelled.

Question 1.3 Let G be a Hausdorff (locally quasi-convex abelian) topological group, which is g-barrelled with respect to every metrizable (abelian) topological group Y. Is then G as a topological space a weak Namioka space?

It was shown in [21] that the Namioka spaces form a proper subclass of the class of Baire spaces. By using a construction of [21], we will show that Theorem 1.9(a) is no longer true if we replace "Namioka space" with "Baire space", thus answering the question posed in [14, Remark 2.2]. We denote by $\mathbb{Z}(2)$ the 2-element abelian group $\mathbb{Z}/2\mathbb{Z}$.

Theorem 1.10 *There exists a protodiscrete (in particular, locally quasi-convex) Hausdorff topological abelian group G with the following properties:*

(a) G as a topological space is a Baire space.
(b) G is not g-barrelled with respect to the discrete group $\mathbb{Z}(2)$. *In particular, G is not g-barrelled.*

There also exists a submetrizable topological abelian group which as a topological space is a Baire space, but which is not g-barrelled (A. Bouziad, personal communication).

Remark 1.2 In [12] it was introduced a notion of a *g-ultrabarrelled* topological group. This class admits the following remarkable characterization: a Hausdorff topological abelian group G is g-ultrabarrelled iff every closed group homomorphism from G into any separable complete metrizable topological group is continuous [12, Theorem 3.1]. In [12] it is noticed also that any topological group which is a Baire space, is g-ultrabarrelled. From this and Theorem 1.10 it follows that a Hausdorff topological abelian protodiscrete (hence, locally quasi-convex) g-ultrabarrelled group may not be g-barrelled.

2 The Proofs

Proof of Theorem 1.4

Fix a number p with $0 < p \leq 1$ and consider the sequence space

$$l_p = \{\mathbf{x} = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}} : \sum_{k=1}^{\infty} |x_k|^p < \infty\}$$

endowed with the p-norm

$$\|\mathbf{x}\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}}, \quad \mathbf{x} \in l_p .$$

Let us write

$$(l_p)_1 := (l_p, \|\cdot\|_1) .$$

Now we can formulate the following statement, which implies Theorem 1.4.

Theorem 2.1 *Let* $0 < p < 1$. *Then*

(a) $(l_p)_1$ *is a barrelled normed space.*
(b) $(l_p)_1$ *is not sequentially barrelled with respect to* l_p.

Proof (a) is proved in [17, 23].
(b) Fix $n \in \mathbb{N}$ and consider the linear mapping $u_n : (l_p)_1 \to l_p$ defined by the equality

$$u_n(\mathbf{x}) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots), \quad \mathbf{x} \in l_p .$$

We have:

$$\|u_n(\mathbf{x})\|_p \le \|\mathbf{x}\|_p, \quad \mathbf{x} \in l_p , \tag{1}$$

and

$$\|u_n\| := \sup\{\|u_n(\mathbf{x})\|_p : \mathbf{x} \in l_p,\ \|\mathbf{x}\|_1 \le 1\} = n^{\frac{1}{p}-1} . \tag{2}$$

Fix now a number r with $0 < r < \frac{1}{p} - 1$ and write

$$v_n = \frac{1}{n^r} u_n .$$

Then we have
(C1) $v_n : (l_p)_1 \to l_p$ is a continuous linear mapping.
(C2) $\lim_n \|v_n(\mathbf{x})\|_p = 0$ for every $\mathbf{x} \in l_p$. This follows from (1).
(C3) The sequence $(v_n)_{n\in\mathbb{N}}$ is not equicontinuous at $0 \in (l_p)_1$. In fact, from (2) we have

$$\|v_n\| = n^{\frac{1}{p}-(r+1)} . \tag{3}$$

The equicontinuity of $(v_n)_{n\in\mathbb{N}}$ at $0 \in (l_p)_1$ would imply that

$$\sup_n \|v_n\| < \infty$$

in contradiction with (3).

Proof of Theorem 1.6

Let $\mathscr{H}$ be a set of continuous homomorphisms from G to F for which $\mathscr{H}(x)$ is bounded in F for every $x \in G$. Fix a zero neighborhood $W \in \mathscr{N}(F)$. We are going to find $O \in \mathscr{N}(G)$ with $u(O) \subset W$ for every $u \in \mathscr{H}$, which means that $\mathscr{H}$ is equicontinuous at $0 \in G$.

Fix a symmetric closed $W_1 \in \mathscr{N}(F)$ with $W_1 + W_1 \subset W$. Write

$$X_n = \bigcap_{u \in \mathscr{H}} u^{-1}(nW_1), \quad n = 1, 2, \ldots$$

The boundedness in F of $\mathscr{H}(x)$ for every $x \in G$ implies

$$G = \bigcup_{n \in \mathbb{N}} X_n . \tag{4}$$

Since the sets $X_n,\ n = 1, 2, \ldots$ are closed and G is a Baire space, we can find and fix $n_0 \in \mathbb{N}$ such that

$$U := \mathrm{Int}(X_{n_0}) \neq \emptyset .$$

Pick $x_0 \in U$. Then

$$V := U - x_0 \in \mathscr{N}(G) .$$

It is easy to check that

$$u(x) = u(x + x_0) - u(x_0) \in n_0 W_1 + n_0 W_1 \subset n_0 W, \quad \forall x \in V,\ \forall u \in \mathscr{H} . \tag{5}$$

Find and fix now $O \in \mathscr{N}(G)$ such that $O + \overset{n_0}{\cdots} + O \subset V$. As

$$x \in O \Rightarrow n_0 x \in V ,$$

from (5) we get

$$n_0 u(x) = u(n_0 x) \in n_0 W \quad \forall x \in O,\ \forall u \in \mathscr{H} .$$

Hence $u(O) \subset W\ \forall u \in \mathscr{H}$, as required.

Proof of Theorem 1.9

We will prove the following stronger version of Theorem 1.9:

Theorem 2.2 *Let G be a topological group and Y be a metrizable topological group.*

(a) If G as a topological space is a weak Namioka space, then G is g-barrelled with respect to Y.

(b) If G as a topological space is locally pseudocompact, then G is g-barrelled with respect to Y.

Proof (a) Fix a set $\mathscr{H}$ of continuous homomorphisms from G to Y which is compact in the product topology of Y^G. Consider the mapping $f : G \times \mathscr{H} \to Y$ defined as follows:

$$f(x, u) = u(x), \quad x \in G, \ u \in \mathscr{H}.$$

Then f is separately continuous. Since G has the weak Namioka property, there exists an element $a \in G$ such that f is continuous at every point of $\{a\} \times Y$. From this, according to [15, Lemma 2.1] we can conclude that the set

$$\{f(\cdot, u) : u \in \mathscr{H}\} = \mathscr{H}$$

is equicontinuous at a. Since $\mathscr{H}$ consists of homomorphisms, we obtain that Y is equicontinuous.
(b) This follows from (a) and Proposition 1.2.

Proof of Theorem 1.10

Let I be a fixed uncountable set. For $f \in \mathbb{Z}(2)^I$ we denote by supp f the support of f, i. e. the set of all $i \in I$ such that $f(i) = 1$. Write

$$G = \{f \in \mathbb{Z}(2)^I : \text{card}(\text{supp } f) \le \aleph_0\}.$$

Consider on G the group topology which admits as a basis of neighborhoods of zero the sets of the form

$$U_J := \{f \in G : f(i) = 0 \ \forall i \in J\}$$

where J runs through all subsets of I with card $(J) \le \aleph_0$. Since U_J is a subgroup of G for every J, this is a protodiscrete group topology.

(1) G is a Baire space [21].

Let $K = \beta I$ be the Stone-Čech compactification of the discrete space I and let $C(K, \mathbb{Z}(2))$ be the the set of all continuous mappings $h : K \to \mathbb{Z}(2)$. Let us identify G with a subset of $C(K, \mathbb{Z}(2))$ as follows: to each $f \in G$ corresponds its unique continuous extension $\tilde{f} : K \to \mathbb{Z}(2)$. Consider the mapping $\Phi : G \times K \to \mathbb{Z}(2)$ defined by the equality

$$\Phi(f, h) = \tilde{f}(h) \quad \forall (f, h) \in G \times K .$$

We have

(2) For a fixed $h \in K$ the mapping $\Phi(\cdot, h)$ is continuous on G [21].
(3) For each $f \in G$ there exists $h \in K$ such that Φ is not continuous at (f, h) [21]. Consequently G is not a weak Namioka space.

Note also that for each $h \in K$ the mapping $\Phi(\cdot, h)$ is a group homomorphism from G to $\mathbb{Z}(2)$ (indeed, this is so when $h = i \in I$ by the definition of the group operation of G; the general case follows from the density of I in K).

Clearly the set of continuous homomorphisms

$$\mathscr{H} = \{u \in \mathbb{Z}(2)^G : \exists h \in K,\ u(\cdot) = \Phi(\cdot, h)\}$$

is pointwise compact, but it is not equicontinuous (as the equicontinuity of $\mathscr{H}$ at 0 would imply that Φ is continuous at each point of $\{0\} \times K$, which is not the case by (3) above).

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Subdirect Products of Finite Abelian Groups

In Honour of Manuel López-Pellicer

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Abstract A subgroup G of a product $\prod_{i\in\mathbb{N}} G_i$ is *rectangular* if there are subgroups H_i of G_i such that $G = \prod_{i\in\mathbb{N}} H_i$. We say that G is *weakly rectangular* if there are finite subsets $F_i \subseteq \mathbb{N}$ and subgroups H_i of $\bigoplus_{j\in F_i} G_j$ that satisfy $G = \prod_{i\in\mathbb{N}} H_i$. In this paper we discuss when a closed subgroup of a product is weakly rectangular. Some possible applications to the theory of group codes are also highlighted.

Keywords Rectangular subgroup · Weakly rectangular subgroup · Subdirect product · Controllable group · Order controllable subgroup · Weakly controllable subgroup

1 Introduction

For a family $\{G_i : i \in \mathbb{N}\}$ of topological groups, let $\bigoplus_{i\in\mathbb{N}} G_i$ denote the subgroup of elements (g_i) in the product $\prod_{i\in\mathbb{N}} G_i$ such that $g_i = e$ for all but finitely many indices $i \in \mathbb{N}$. A subgroup $G \leq \prod_{i\in\mathbb{N}} G_i$ is called *weakly controllable* if $G \cap \bigoplus_{i\in\mathbb{N}} G_i$ is dense in G, that is, if G is generated by its elements of finite support. The group

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G is called *weakly observable* if $G \cap \bigoplus_{i\in\mathbb{N}} G_i = \overline{G} \cap \bigoplus_{i\in\mathbb{N}} G_i$, where $\overline{G}$ stands for the closure of G in $\prod_{i\in\mathbb{N}} G_i$ for the product topology. Although the notion of (weak) controllability was coined by Fagnani earlier in a broader context (cf. [3, 4]), both notions were introduced in the area of of coding theory by Forney and Trott (cf. [8]). They observed that if the groups G_i are locally compact abelian, then controllability and observability are dual properties with respect to the Pontryagin duality: If G is a closed subgroup of $\prod_{i\in\mathbb{N}} G_i$, then it is weakly controllable if and only if its annihilator $G^{\perp} = \{\chi \in \widehat{\prod_{i\in\mathbb{N}} G_i} : \chi(G) = \{1\}\}$ is a weakly observable subgroup of $\prod_{i\in\mathbb{N}} \widehat{G_i}$ (cf. [8, 4.8]).

In connection with the properties described above, the following definitions was introduced in [12].

Definition 1.1 A subgroup G of a product $\prod_{i\in\mathbb{N}} G_i$ is *rectangular* if there are subgroups H_i of G_i such that $G = \prod_{i\in\mathbb{N}} H_i$. We say that G is *weakly rectangular* if there are finite subsets $F_i \subseteq \mathbb{N}$ and subgroups H_i of $\bigoplus_{j\in F_i} G_j$ that satisfy $G = \prod_{i\in\mathbb{N}} H_i$. We say that G is a *subdirect product* of the family $\{G_i\}_{i\in I}$ if G is weakly rectangular and $G \cap \bigoplus_{i\in I} G_i = \bigoplus_{i\in\mathbb{N}} H_i$.

The observations below are easily verified.

1. Weakly rectangular subgroups and rectangular subgroups of $\prod_{i\in\mathbb{N}} G_i$ are weakly controllable.
2. If each G_i is a pro-p_i-group for some prime p_i, and all p_i are distinct, then every closed subgroup of the product $\prod_{i\in\mathbb{N}} G_i$ is rectangular, and thus is a subdirect product.
3. If each G_i is a finite simple non-abelian group, and all G_i are distinct, then every closed normal subgroup of the product $\prod_{i\in\mathbb{N}} G_i$ is rectangular, and thus a subdirect product.

The main goal addressed in this paper is to study to what extent the converse of these observations hold. In particular, we are interested in the following question (cf. [12]):

Problem 1.1 Let $\{G_i : i \in \mathbb{N}\}$ be a family of compact metrizable groups, and G a closed subgroup of the product $\prod_{i\in\mathbb{N}} G_i$. If G is weakly controllable, that is, $G \cap \bigoplus_{i\in\mathbb{N}} G_i$ is dense in G, what can be said about the structure of G? In particular, under what additional conditions is the group G a subdirect product of $\{G_i : i \in \mathbb{N}\}$, that is, weakly rectangular and $G \cap \bigoplus_{i\in I} G_i = \bigoplus_{i\in\mathbb{N}} H_i$, where each H_i is a subgroup of $\bigoplus_{j\in F_i} G_j$ for some finite subset $F_i \subseteq \mathbb{N}$?

In connection with this question, the following result was established in [5].

Theorem 1.1 *Let I be a countable set, $\{G_i : i \in I\}$ be a family of finite abelian groups and $G = \prod_{i \in I} G_i$ be its direct product. Then every closed weakly controllable subgroup H of G is topologically isomorphic to a direct product of finite cyclic groups.*

Unfortunately, this result does not answer Problem 1.1, which remains open to the best of our knowledge. Finally, it is pertinent to mention here that the relevance of these notions stem from coding theory where they appear in connection with the study of convolutional group codes [8, 13]. However, similar notions had been studied in symbolic dynamics previously. Thus, the notions of weak controllability and weak observability are related to the concepts of *irreducible shift* and *shift of finite type*, respectively, that appear in symbolic dynamics. Here, we are concerned with abelian profinite groups and our main interest is to clarify the overall topological and algebraic structure of abelian profinite groups that satisfy any of the properties introduced above. In the last section, we shall also consider some possible applications to the study of group codes.

2 Basic Facts

2.1 Pontryagin-van Kampen Duality

One of the main tools in this research is Kaplan's extension of Pontryagin van-Kampen duality to infinite products of locally compact abelian (LCA) groups. In like manner that Pontryagin-van Kampen duality has proven to be essential in understanding the structure of LCA, the extension accomplished by Kaplan for cartesian products and direct sums [10, 11] (and some other subsequent results) have established duality methods as a powerful tool outside the class of LCA groups and they have been widely used in the study of group codes.

We recall the basic properties of topological abelian groups and the celebrated Pontryagin-van Kampen duality.

Let G be a commutative locally-compact group. A character χ of G is a continuous homomorphism $\chi : G \longrightarrow \mathbb{T}$ where $\mathbb{T}$ is the multiplicative group of complex numbers of modulus 1. The characters form a group $\widehat{G}$, called *dual group*, which is given the topology of uniform convergence on compact subsets of G. It turns out that $\widehat{G}$ is locally compact and there is a canonical evaluation homomorphism

$$\mathscr{E}_G : G \longrightarrow \widehat{\widehat{G}}.$$

Theorem 2.1 *The evaluation homomorphism $\mathscr{E}_G$ is an isomorphism of topological groups.*

Some examples:

$$\widehat{\mathbb{T}} \cong \mathbb{Z},\ \widehat{\mathbb{Z}} \cong \mathbb{T},\ \widehat{\mathbb{R}} \cong \mathbb{R},\ \widehat{(\mathbb{Z}/n)} \cong \mathbb{Z}/n.$$

(Remark that some groups are self dual, such as finite abelian groups or the additive group of the real numbers.)

Pontryagin-van Kampen duality establishes a duality between the subcategories of compact and discrete abelian groups. If G denotes a compact abelian group and Γ denotes its dual group, we have the following equivalences between topological properties of G and algebraic properties of Γ:

1. $weight(G) = |\Gamma|$ (metrizablity $\Leftrightarrow |\Gamma| \leq \omega$);
2. G is connected $\Leftrightarrow \Gamma$ is torsion free;
3. $Dim(G) = 0 \Leftrightarrow \Gamma$ is torsion; and
4. G is monothetic $\Leftrightarrow \Gamma$ is isomorphic to a subgroup of $\mathbb{T}_d$.

In general, it is said that a topological abelian group (G, τ) satisfies *Pontryagin duality* (is *P-reflexive* for short,) if the evaluation map $\mathcal{E}_G$ is a topological isomorphism onto. We refer to the survey by Dikranjan and Stoyanov [2] and the monographs by Dikranjan, Prodanov and Stoyanov [1] and Hofmann, Morris [9] in order to find the basic results about Pontryagin-van Kampen duality.

The following result, due to Kaplan [10, 11] is essential in the applications of duality methods.

Theorem 2.2 (Kaplan) *Let $\{G_i : i \in I\}$ be a family of P-reflexive groups. Then:*

1. *The direct product $\prod\limits_{i\in I} G_i$ is P-reflexive.*
2. *The direct sum $\bigoplus\limits_{i\in I} G_i$ equipped with a suitable topology is P-reflexive.*
3. *It holds:*

$$\left(\prod_{i\in I} G_i\right)^{\widehat{}} \cong \bigoplus_{i\in I} \widehat{G_i}$$

$$\left(\bigoplus_{i\in I} G_i\right)^{\widehat{}} \cong \prod_{i\in I} \widehat{G_i}$$

Kaplan also set the problem of characterizing the class of topological abelian groups for which Pontryagin duality holds.

Let $g \in G$ and $\chi \in \widehat{G}$, it is said that g and χ are *orthogonal* when $\chi(g) = 1$. Given $S \subseteq G$ and $S_1 \subseteq \widehat{G}$ we define *the orthogonal (or annihilator)* of S and S_1 as

$$S^{\perp} = \{\chi \in \widehat{G} \,:\, \chi(g) = 1\ \forall g \in S\}$$

and

$$S_1^{\perp} = \{g \in G \,:\, \chi(g) = 1\ \forall \chi \in S_1\}.$$

Obviously $G^{\perp} = \{e_{\widehat{G}}\}$ and $\widehat{G}^{\perp} = \{e_G\}$.

The following result has also many applications in connection with duality theory.

Theorem 2.3 *Let S and R be subgroups of a LCA group G such that $S \leq R \leq G$. Then we have* $\widehat{R/S} \cong S^{\perp}/R^{\perp}$.

Corollary 2.1 *Let H be a closed subgroup of a LCA group G. Then* $\widehat{G/H} \cong H^{\perp}$ *and* $\widehat{H} \cong \widehat{G}/H^{\perp}$.

2.2 Abelian Profinite Groups

Our main results concern the structure of abelian profinite groups that appear in coding theory. Firstly, we recall some basic definitions and terminology. For every group G let us denote by $(G)_p$ the largest p-subgroup of G and $\mathbb{P}_G = \{p \in \mathbb{P} : G \text{ contains a } p - \text{subgroup}\}$ where $p \in \mathbb{P}_G$ and $\mathbb{P}$ is the set of all prime numbers.

3 Order Controllable Groups

Definition 3.1 Let $\{G_i : i \in \mathbb{N}\}$ be a family of topological groups and $\mathscr{C}$ a subgroup of $\prod_{i \in \mathbb{N}} G_i$. We have the following notions:

$\mathscr{C}$ is *weakly controllable* if $\mathscr{C} \bigcap \bigoplus_{i \in \mathbb{N}} G_i$ is dense in $\mathscr{C}$.

$\mathscr{C}$ is *controllable* if for every $c \in \mathscr{C}$ and $i \in \mathbb{N}$ there are $n_i \in \mathbb{N}$ and $c_1 \in \mathscr{C}$ such that $c_1|_{[1,i]} = c|_{[1,i]}$ and $c_1|_{]n_i,+\infty[} = 0$ (we assume that n_i is the least natural number satisfying this property). Remark that this property implies the existence of $c_2 \in \mathscr{C}$ such that $c = c_1 + c_2$, $supp(c_1) \subseteq [1, n_i]$ and $supp(c_2) \subseteq [i + 1, +\infty[$.

$\mathscr{C}$ is *order-controllable* if for every $c \in \mathscr{C}$ and $i \in \mathbb{N}$ there are $n_i \in \mathbb{N}$ and $c_1 \in \mathscr{C}$ such that $c_1|_{[1,i]} = c|_{[1,i]}$, $c_1|_{]n_i,+\infty[} = 0$, and $\text{order}(c_1) \leq \text{order}(c|_{[1,n_i]})$ (again, we assume that n_i is the least natural number satisfying this property). This property implies the existence of $c_2 \in \mathscr{C}$ such that $c = c_1 + c_2$, $supp(c_1) \subseteq [1, n_i]$, $supp(c_2) \subseteq [i + 1, +\infty[$, and $\text{order}(c_1) \leq \text{order}(c|_{[1,n_i]})$. As a consequence, we also have that $\text{order}(c_2) \leq \text{order}(c)$. Here, the order of c is taken in the usual sense, considering c as an element of the group $\mathscr{C}$.

Every controllable group is weakly controllable and, if the groups G_i are finite, then the notions of controllability and weakly controllability are equivalent (see [5]). The following result partially answers Problem 1.1 for p-groups. The proof can be founded in [6].

Theorem 3.1 *Let $\{G_i : i \in \mathbb{N}\}$ be a family of finite, abelian, p-groups and let $G = \prod_{i \in \mathbb{N}} G_i$. If $\mathscr{C}$ is an infinite closed subgroup of G which is order-controllable, then $\mathscr{C}$*

is weakly rectangular. In particular, there is a sequence $\{y_m : m \in \mathbb{N}\} \subseteq \mathscr{C} \bigcap \bigoplus_{i\in\mathbb{N}} G_i$ *such that* $\mathscr{C}$ *is topologically isomorphic to* $\prod_{m\in\mathbb{N}} < y_m >$.

Let $\mathscr{C}$ be a closed, subgroup of $G = \prod_{i\in\mathbb{N}} G_i$ a countable product of finite abelian groups. Since each group G_i is finite and abelian, by the fundamental structure theorem of finite abelian groups, we have that every group G_i is a finite sum of finite p-groups, that is $G_i = \bigoplus_{p\in\mathbb{P}_i} (G_i)_p$ and $\mathbb{P}_i = \mathbb{P}_{G_i}$ is finite, $i \in \mathbb{N}$. Note that $\mathbb{P}_G = \cup\mathbb{P}_i$. We have

$$\prod_{i\in\mathbb{N}} G_i \cong \prod_{i\in\mathbb{N}} \left(\prod_{p\in\mathbb{P}_i} (G_i)_p\right) \cong \prod_{p\in\mathbb{P}_G} \left(\prod_{i\in\mathbb{N}_p} (G_i)_p\right)$$

where $\mathbb{N}_p = \{i \in \mathbb{N} \,:\, G_i \text{ has a } p - \text{subgroup}\}$.
Thus

$$(G)_p \cong \prod_{i\in\mathbb{N}_p} (G_i)_p.$$

Consider the embedding

$$j : \mathscr{C} \hookrightarrow \prod_{p\in\mathbb{P}_G} \left(\prod_{i\in\mathbb{N}_p} (G_i)_p\right)$$

and the canonical projection

$$\pi_p : \prod_{p\in\mathbb{P}_G} \left(\prod_{i\in\mathbb{N}_p} (G_i)_p\right) \to \prod_{i\in\mathbb{N}_p} (G_i)_p.$$

Set $\mathscr{C}^{(p)} = (\pi_p \circ j)(\mathscr{C})$, that is a compact group. We have

$$(\mathscr{C})_p \cong \mathscr{C}^{(p)}.$$

Now, it is easily seen that $\mathscr{C}$ is order-controllable (resp. weakly rectangular) if and only if $(\mathscr{C})_p$ has the corresponding property for each $p \in \mathbb{P}_{\mathscr{C}}$. Taking this fact into account, we obtain the following result that gives a partial answer to Problem 1.1 for products of finite abelian groups.

Theorem 3.2 *Let* $\mathscr{C}$ *be an order-controllable, closed, subgroup of a countable product* $G = \prod_{i\in\mathbb{N}} G_i$ *of finite abelian groups* G_i. *Then* $\mathscr{C}$ *is weakly rectangular. In particular, there is a sequence* $\{y_m^{(p)} : m \in \mathbb{N}, p \in \mathbb{P}_{\mathscr{C}}\} \subseteq \mathscr{C} \cap (\bigoplus_{i\in\mathbb{N}} G_i)$ *such that* $\{y_m^{(p)} : m \in \mathbb{N}\} \subseteq (\mathscr{C} \cap (\bigoplus_{i\in\mathbb{N}} G_i))_p$ *and* $\mathscr{C}$ *is topologically isomorphic to* $\prod_{\substack{m\in\mathbb{N}\\ p\in\mathbb{P}_G}} \langle y_m^{(p)} \rangle$.

Proof Following with the notation introduced in the previous paragraph and, since $\mathscr{C} \cap (\bigoplus_{i \in \mathbb{N}} G_i)$ is dense in $\mathscr{C}$, we have that

$$(\pi_p \circ j)\left(\mathscr{C} \cap (\bigoplus_{i \in \mathbb{N}} G_i)\right) \subseteq \mathscr{C}^{(p)} \cap \bigoplus_{i \in \mathbb{N}_p} (G_i)_p$$

is dense in $\mathscr{C}^{(p)}$. Applying Theorem 3.1, for each $p \in \mathbb{P}_{\mathscr{C}}$, there is a sequence

$$\{y_m^{(p)} \ : \ m \in \mathbb{N}\} \subseteq \mathscr{C}^{(p)} \cap \bigoplus_{i \in \mathbb{N}_p} (G_i)_p$$

such that $\mathscr{C}^{(p)} \cong \prod_{m \in \mathbb{N}} \langle y_m^{(p)} \rangle$.

Finally, observe that if $p \in \mathbb{P}_{\mathscr{C}}$, then $\mathscr{C}^{(p)} \cap \bigoplus_{i \in \mathbb{N}_p} (G_i)_p \cong (\mathscr{C} \cap (\bigoplus_{i \in \mathbb{N}} G_i))_p$. Thus, using this isomorphism, we may assume with some notational abuse that

$$\{y_m^{(p)} \ : \ m \in \mathbb{N}\} \subseteq (\mathscr{C} \cap (\bigoplus_{i \in \mathbb{N}} G_i))_p$$

Therefore, the sequence $\{y_m^{(p)} \ : \ m \in \mathbb{N}, \ p \in \mathbb{P}_{\mathscr{C}}\}$ verifies the proof.

The notion of rectangular and weakly rectangular subgroup of an infinite product extend canonically to subgroups of infinite direct sums. In this direction, we have:

Theorem 3.3 *Let $\mathscr{C}$ be an order-controllable subgroup of $\bigoplus_{k \in \mathbb{N}} G_k$ such that every group G_k is finite and abelian. Then $\mathscr{C}$ is weakly rectangular. In particular, there is a sequence $(y_n) \subseteq \mathscr{C}$ such that*

$$\mathscr{C} \simeq \bigoplus_{n \in \mathbb{N}} \langle y_n \rangle$$

4 Group Codes

According to Forney and Trott [8], a *group code* is a set of sequences that has a group property under a componentwise group operation. In this general setting, a group code may also be seen as the behavior of a behavioral group system as given by Willens [14, 15]. It is known that many of the fundamental properties of linear codes and systems depend only on their group structure. In fact, Forney and Trott, loc. cit., obtain purely algebraic proofs of many of their results. In this section, we follow this approach in order to apply the results in the preceding sections to obtain further information about the structure of group codes in very general conditions.

Without loss of generality, assume, from here on, that a *group code* is a subgroup of a (sequence) group $\mathscr{W}$, called *Laurent group*, that has the generical form $\mathscr{W} =$

$\mathscr{W}_f \times \mathscr{W}^c$, where $\mathscr{W}_f$ is a direct sum of abelian groups (locally compact in general) and $\mathscr{W}^c$ a direct product. More precisely, let $I \subseteq \mathbb{Z}$ be a countable index set and let $\{G_k : k \in I\}$ be a set of symbol groups, a *product sequence space* is a direct product

$$\mathscr{W}^c = \prod_{k \in I} G_k$$

equipped with the canonical product topology. A *sum sequence space* is a direct sum

$$\mathscr{W}_f = \bigoplus_{k \in I} G_k$$

equipped with the canonical sum (box) topology. Sequence spaces are often defined to be *Laurent sequences*

$$\mathscr{W}_L = \left(\bigoplus_{k<0} G_k\right) \times \left(\prod_{k \geq 0} G_k\right).$$

The character group of $\mathscr{W}_L$ is

$$\mathscr{W}_{aL} = \left(\prod_{k<0} \widehat{G_k}\right) \times \left(\bigoplus_{k \geq 0} \widehat{G_k}\right).$$

Thus, a group code $\mathscr{C}$ is a subgroup of a group sequence space $\mathscr{W}$ and is equipped with the natural subgroup topology. Next we recall some basic facts of this theory (cf. [4, 7, 8]). These notions are used in the study of *convolutional codes* that are well known and used currently in data transmission (cf. [7]).

Let $\mathscr{C}$ be a group code in the product sequence space $\mathscr{W} = \prod_{k \in I} G_k$. According to Fagnani, $\mathscr{C}$ is called *weakly controllable* if it is generated by its finite sequences. In other terms, if

$$\overline{\mathscr{C}} = \overline{\mathscr{C} \cap \mathscr{W}_f}.$$

The group code $\mathscr{C}$ is called *controllable* if for all $w_1, w_2 \in \mathscr{C}$ and $k \in I$, there exist $L(k) \in \mathbb{N}$ and $w \in \mathscr{C}$ with $w(i) = w_1(i)\ \forall i < k$, and $w(i) = w_2(i)\ \forall i \geq k + L(k)$.

With the notation introduced above, let $\mathscr{C}$ be a group code in $\mathscr{W}$. For any $k \in I$ and $L \in \mathbb{N}$, we set

$$\mathscr{C}_k(L) := \{c \in \mathscr{C} : \text{ there exists } w \in \mathscr{C} \text{ with } w(i) = 0\ \forall i < k \text{ and } w(i) = c(i)\ \forall i \geq k + L\}$$

and

$$\mathscr{C}_k := \bigcup_{L \in \mathbb{N}} \mathscr{C}_k(L).$$

Obviously $\mathscr{C}_k(1) \subseteq \mathscr{C}_k(2) \subseteq \cdots \mathscr{C}_k(L) \subseteq \cdots \subseteq \mathscr{C}_k$.

We have the following equivalence, whose verification is left to the reader.

Proposition 4.1 *$\mathscr{C}$ is controllable if and only if $\mathscr{C} = \bigcap_{k\in I} \bigcup_{L\in\mathbb{N}} \mathscr{C}_k(L)$.*

Given a group code $\mathscr{C}$, the subgroup

$$\mathscr{C}_c := \bigcap_{k\in I} \bigcup_{L\in\mathbb{N}} \mathscr{C}_k(L)$$

is called the *controllable subcode* of $\mathscr{C}$. A code $\mathscr{C}$ is called *uniformly controllable* when for every $k \in I$, there is L_k such that $\mathscr{C} = \bigcap_{k\in I} \mathscr{C}_k(L_k)$. If there is some $L \in \mathbb{N}$ such that $\mathscr{C}_k = \mathscr{C}_k(L)$ for all $k \in I$, it is said that $\mathscr{C}$ is *L-controllable*. Finally, $\mathscr{C}$ is *strongly controllable* if it is L-controllable for some L. If $\mathscr{C}$ is uniformly controllable and the sequence (L_k) is bounded, then $\mathscr{C}$ is strongly controllable and the least such L is the *controllability index* (controller memory) of $\mathscr{C}$.

Using the same words as in [8], the core meaning of "controllable" is that any code sequence can be reached from any other code sequence in a finite interval. The following property is clarifying in this regard. In the sequel

$$\mathscr{C}_{:[k,k+L)} = \{w \in \mathscr{C} : w(j) = 0\ \forall j \notin [k, k+L)\}.$$

Proposition 4.2 *$\mathscr{C}$ is controllable if and only if for any $w \in \mathscr{C}$, there is a sequence (L_k) contained in $\mathbb{N}$ such that $w \in \sum_{k\in I} \mathscr{C}_{:[k,k+L_k)}$.*

Proof Let $w \in \mathscr{C}$ and let $k_1 \in I$ be the first index such that $w(k_1) \neq 0$. Then there is $w_1 \in \mathscr{C}$ and $L_1 \subset \mathbb{N}$ such that $w_1(i) = w(i)$ for all $1 \leq i \leq k_1$ and $w_1(i) = 0$ if $i \geq L_1 + k_1$. Take $k_2 = L_1 + k_1 + 1$ and let $w_2 \in \mathscr{C}$ satisfying $w_2(i) = (w - w_1)(i)$ for all $i \leq k_2$ and $w_2(i) = 0$ for all $i \geq k_2 + L_2$. In general we select $w_n \in \mathscr{C}$ such that $w_n(i) = 0$ if $i \leq k_{n-1}$, $w_n(i) = (w - w_1 - \cdots w_{n-1})(i)$ for all $i < k_n$ and $w_n(i) = 0$ if $i \geq k_n + L_n$. We have that $w = \sum_{n\in\mathbb{N}} w_n$ in the product topology and furthermore the sum $\sum_{n\in\mathbb{N}} w_n(i)$ is finite for all $i \in I$.

Analogous notions are defined regarding the observability of a group code. The group code $\mathscr{C}$ is called *weakly observable* if

$$\mathscr{C} \cap \mathscr{W}_f = \overline{\mathscr{C}} \cap \mathscr{W}_f.$$

Let $\mathscr{C}$ a group code in $\mathscr{W}$, we set

$$(\mathscr{C}_f)_k[L] := \{c \in \mathscr{W}_f\ :\ c_{|[k,k+L]} \in \mathscr{C}_{|[k,k+L]}\}.$$

The group code $\mathscr{C}$ is called *pointwise observable* if

$$\mathscr{C}_f = \bigcap_{k\in I} \bigcap_{L\in\mathbb{N}} (\mathscr{C}_f)_k[L].$$

If $\mathscr{C}$ is a group code, then the supergroup

$$\mathscr{C}_{ob} := \langle \mathscr{C} \cup (\bigcap_{k\in I} \bigcap_{L\in\mathbb{N}} (\mathscr{C}_f)_k[L]) \rangle$$

is called the *observable supercode* of $\mathscr{C}$. A code $\mathscr{C}$ is called *uniformly observable* when for every $k \in I$, there is L_k such that $\mathscr{C}_f = \bigcap_{k\in I} (\mathscr{C}_f)_k[L_k]$. If there is some $L \in \mathbb{N}$ such that $\mathscr{C}_f = \bigcap_{k\in I} (\mathscr{C}_f)_k[L]$ for all $k \in I$, it is said that $\mathscr{C}$ is *L-observable*. Finally, $\mathscr{C}$ is *strongly observable* if it is L-observable for some L. Obviously, if $\mathscr{C}$ is uniformly observable and the sequence (L_k) is bounded, then $\mathscr{C}$ is strongly observable and the least such L is the *observability index* (observer memory) of $\mathscr{C}$.

Recently, Pontryagin duality methods have been applied systematically in the study of convolutional abelian group codes. In this approach, a dual code $\mathscr{C}^{\perp}$ is associated to every group code $\mathscr{C}$ using Pontryagin-van Kampen duality in such a way that the properties of $\mathscr{C}$ can be reflected (*dualized*) in $\mathscr{C}^{\perp}$. Along this line, the following duality theorem provides strong justification for the use of duality in convolutional group codes (see [8]).

Theorem 4.1 ([8]) *Given dual group codes $\mathscr{C}$ and $\mathscr{C}^{\perp}$, then $\mathscr{C}$ is (resp. weakly, strongly) controllable if and only if $\mathscr{C}^{\perp}$ is (resp. weakly, strongly) observable, and vice versa.*

Using duality, we obtain the following additional equivalences (cf. [8]).

Proposition 4.3 *For any group code $\mathscr{C}$ we have*

1. $(\mathscr{C}_c)^{\perp} \cong (\mathscr{C}^{\perp})_{ob}$.
2. *$\mathscr{C}$ is controllable if and only if $\mathscr{C}^{\perp}$ is observable.*
3. *$\mathscr{C}$ is uniformly controllable if and only if $\mathscr{C}^{\perp}$ is uniformly observable.*

Therefore, we can put our attention on the controllability of a group code wlog. In this direction, the following result was proved in [5].

Theorem 4.2 ([5]) *Let $\mathscr{C} \leq \prod_{k\in\mathbb{N}} G_k$ be a complete group code such that every group G_k is finite (discrete). Then the following conditions are equivalent:*

1. *$\mathscr{C}$ is weakly controllable.*
2. *$\mathscr{C}$ is controllable.*
3. *$\mathscr{C}$ is uniformly controllable.*

In [4] Fagnani proves that, if $\mathscr{C}$ is a closed, time invariant, group code in $G^{\mathbb{Z}}$, with G being a compact group, then the properties of weak controllability, controllability and strong controllability are equivalent. A different proof of this result follows easily using the ideas introduced above. Indeed, suppose that $\mathscr{C}$ is a weakly controllable, compact group code in $\mathscr{W}$. By Theorem 4.2, we know that $\mathscr{C}$ is controllable and therefore $\mathscr{C} = \bigcap_{k\in I} \bigcup_{L\in\mathbb{N}} \mathscr{C}_k(L)$. Suppose, in addition, that $\mathscr{C}$ is time invariant, then $\mathscr{C}_k(L) = \mathscr{C}_0(L)$ for all $k \in I$. Furthermore, using Baire category theorem and the compactness of $\mathscr{C}$, it follows that there must be some $L \in \mathbb{N}$ such that $\mathscr{C}_0 = \mathscr{C}_0(L)$, which means that $\mathscr{C}$ is strongly controllable.

The results formulated above do not hold in general. In fact, an example of a group H that is weakly controllable but not controllable is provided in [5]. Furthermore, using Theorem 4.1, we obtain that the group $H^\perp$ is is weakly controllable but not controllable.

As a consequence of the preceding results we obtain the following relation between weakly controllable and controllable group codes (cf. [5]).

Theorem 4.3 *If $\mathscr{C}$ is a group code in*

$$\mathscr{W} = \mathscr{W}_f \times \mathscr{W}^c = \left(\bigoplus_{i<0} G_i\right) \times \left(\prod_{i\geq 0} G_i\right).$$

Then the following assertions hold:

(a) If every group G_i is discrete, then $\mathscr{C}$ is controllable if and only if $\mathscr{C}$ is weakly controllable.
(b) If every group G_i is finite (discrete), then $\mathscr{C}$ is weakly controllable if and only if $\mathscr{C}$ is uniformly controllable.
(c) If every group G_i is a fixed compact group G, and $\mathscr{C}$ is a time-invariant, closed subgroup of $\mathscr{W}^c$, then $\mathscr{C}$ is controllable if and only if $\mathscr{C}$ is strongly controllable.

In case the groups in the family $\{G_i : i \in I\}$ are abelian, Theorem 4.1 yields a similar result for observable group codes, using Pontryagin duality.

Theorem 4.4 *If $\mathscr{C}$ is a group code in*

$$\mathscr{W} = \mathscr{W}_f \times \mathscr{W}^c = \left(\bigoplus_{i<0} G_i\right) \times \left(\prod_{i\geq 0} G_i\right).$$

Then the following assertions hold:

(a) If every group G_i is discrete abelian, then $\mathscr{C}$ is observable if and only if $\mathscr{C}$ is weakly observable.
(b) If every group G_i is finite (discrete) abelian, then $\mathscr{C}$ is weakly observable if and only if $\mathscr{C}$ is uniformly observable.
(c) If every group G_i is a fixed discrete abelian group G, and $\mathscr{C}$ is a time-invariant subgroup of $\mathscr{W}_f$, then $\mathscr{C}$ is observable if and only if $\mathscr{C}$ is strongly observable.

A crucial point in the study of group codes is the finding of appropriate *encoders*. Here, given a group code $\mathscr{C} \leq \prod_{i\in I} G_i$, an encoder is a continuous, injective map

$\alpha\colon \prod_{i\in I} F_i \to \mathscr{C}$ that sends a full direct product onto $\mathscr{C}$. Several techniques for the construction of encoder have been considered so far (see [3, 7, 8]). Of special relevance are the group codes that admit non-catastrophic encoders that are group homomorphisms (cf. [3]). In this direction, the results accomplished in the previous section give sufficient conditions for the existence of homomorphic encoders.

Theorem 4.5 *Let $\mathscr{C}$ be an order-controllable, closed, subgroup of a countable product $G = \prod\limits_{i\in\mathbb{N}} G_i$ of finite abelian groups G_i. Then $\mathscr{C}$ is weakly rectangular. In particular, there is a sequence $\{y_m : m \in \mathbb{N}\} \subseteq \mathscr{C} \cap (\bigoplus\limits_{i\in\mathbb{N}} G_i)$ and a topological isomorphism*

$$\alpha\colon \prod_{m\in\mathbb{N}} \langle y_m\rangle \to \mathscr{C}.$$

Furthermore, if $\alpha(\bigoplus\limits_{m\in\mathbb{N}} \langle y_m\rangle)$ is a weakly observable subgroup, then α defines a non-catastrophic encoder that is a group homomorphism.

5 Conclusion

To conclude, let us point out that, so far, the applications of Harmonic Analysis and duality methods to the study of group codes have basically reached the abelian case (via Pontryagin duality and Fourier analysis). The non-commutative case has not yet been fully studied, but it can be expected that the application of duality techniques in the study of non-abelian group codes could provide some results analogous to those already known in the Abelian case (see the work of Forney and Trott, op.cit). However, the nonabelian duality requires more complicated tools such as Kreǐn algebras, von Neumann algebras, operator spaces, etc.). Therefore, it is first necessary to develop an *appropriate* nonabelian duality that can be applied in a similar way to how it is done in the Abelian case.

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Maximally Almost Periodic Groups and Respecting Properties

In Honour of Manuel López-Pellicer

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Abstract A maximally almost periodic topological (*MAP*) group G respects $\mathscr{P}$ if $\mathscr{P}(G) = \mathscr{P}(G^+)$, where G^+ is the group G endowed with the Bohr topology and $\mathscr{P}$ stands for the subsets of G that have the property $\mathscr{P}$. For a Tychonoff space X, we denote by $\mathfrak{P}$ the family of topological properties $\mathscr{P}$ of being a convergent sequence or a compact, sequentially compact, countably compact, pseudocompact and functionally bounded subset of X, respectively. We study relations between different respecting properties from $\mathfrak{P}$ and show that the respecting convergent sequences (=the Schur property) is the weakest one among the properties of $\mathfrak{P}$. We characterize respecting properties from $\mathfrak{P}$ in wide classes of *MAP* topological groups including the class of metrizable *MAP* abelian groups. Every real locally convex space (lcs) is a quotient space of an lcs with the Schur property, and every locally quasi-convex (lqc) abelian group is a quotient group of an lqc abelian group with the Schur property. It is shown that a reflexive group G has the Schur property or respects compactness iff its dual group $G^\wedge$ is c_0-barrelled or g-barrelled, respectively. We prove that an lqc abelian k_ω-group respects all properties $\mathscr{P} \in \mathfrak{P}$. As an application of the obtained results we show that a reflexive abelian group of finite exponent is a Mackey group.

Keyword Schur property · Glicksberg property · k_ω-group · Locally quasi-convex group · Free locally convex space

1 Introduction

Let X be a Tychonoff space. If $\mathscr{P}$ is a topological property, we denote by $\mathscr{P}(X)$ the set of all subspaces of X with $\mathscr{P}$. Denote by $\mathscr{S}$, $\mathscr{C}$, $\mathscr{SC}$, $\mathscr{CC}$, $\mathscr{PC}$ or $\mathscr{FB}$ the property of being a convergent sequence or being a compact, sequentially compact, countably compact, pseudocompact and functionally bounded subset of X, respectively. In what follows we consider the following families of compact-type topological properties

$$\mathfrak{P}_0 := \{\mathscr{S}, \mathscr{C}, \mathscr{SC}, \mathscr{CC}, \mathscr{PC}\} \quad \text{and} \quad \mathfrak{P} := \mathfrak{P}_0 \cup \{\mathscr{FB}\}.$$

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Let G be a maximally almost periodic (*MAP*) topological group G (for all relevant definitions, see Sect. 2). We denote by G^+ the group G endowed with the Bohr topology. Following [59], a *MAP* group G *respects* a topological property $\mathscr{P}$ if $\mathscr{P}(G) = \mathscr{P}(G^+)$.

The famous Glicksberg theorem [36] states that every locally compact abelian (*LCA*) group respects compactness. If a *MAP* group G respects compactness we shall say also that G has the *Glicksberg property*. Trigos-Arrieta [62, 63] proved that countable compactness, pseudocompactness and functional boundedness are respected by *LCA* groups. Banaszczyk and Martín-Peinador [9] generalized these results to all nuclear groups. Nuclear groups were introduced and thoroughly studied by Banaszczyk in [8]. The concept of Schwartz topological abelian groups appeared in [4]. This notion generalizes the well-known notion of a Schwartz locally convex space. All nuclear groups are Schwartz groups [4]. Außenhofer [2] proved that every locally quasi-convex Schwartz group respects compactness. For a general and simple approach to the theory of properties respected by *MAP* topological groups see [25].

Let (E, τ) be a locally convex space (lcs for short), E' the dual space of E and let $\tau_w = \sigma(E, E')$ be the weak topology on E. Set $E_w := (E, \tau_w)$. An lcs E is said to have the *Schur property* if E and E_w have the same convergent sequences, i.e., $\mathscr{S}(E) = \mathscr{S}(E_w)$. Considering E as an additive topological group one can define E^+. If E is a real lcs, it is proved in [58] that E_w and E^+ have the same compact sets and hence the same convergent sequences. By this reason we shall use of the terminology "the Schur property" and "the Glicksberg property" for locally convex spaces and *MAP* groups simultaneously. It is easy to see that if a *MAP* group G has the Glicksberg property, then it has also the Schur property. In general the Schur property does not imply the Glicksberg property, see [65, Example 6 (p. 267)] and [19, Example 19.19], or [30, Proposition 3.5] for a more general assertion. However, it is a classical result that a Banach space E has the Schur property if and only if E has the Glicksberg property.

The aforementioned results motivate us to consider the following two problems. The first problem concerns finding relationships between the properties $\mathscr{P} \in \mathfrak{P}$. In Sect. 3 we show that the Schur property is equivalent to the respecting sequential compactness and is the weakest one among the properties of $\mathfrak{P}_0$, see Proposition 3.3. Under the additional assumption that G^+ is a μ-space, we prove in Theorem 3.4 that the Glicksberg property implies all other properties $\mathscr{P} \in \mathfrak{P}$. In Proposition 3.5 and Theorem 3.6 we consider some natural classes of *MAP* groups in which the Schur property implies the Glicksberg property and the respecting countable compactness. We show also that the Schur property and the Glicksberg property have a natural categorical characterization, see Proposition 3.1.

The second natural problem is the following: Characterize respecting properties in concrete classes of *MAP* groups. As we mentioned above, every locally quasi-convex Schwartz group has the Glicksberg property. However, there are even metrizable reflexive abelian groups which are not Schwartz groups, see [25]. In [38] Hernández, Galindo and Macario proved that a reflexive metrizable abelian group G has the Glicksberg property if and only if every non-precompact subset A of G has an infinite subset B which is discrete and C^*-embedded in the Bohr compactification bG of G.

This result was generalized by Hernández and Macario who showed in [39] that a complete abelian g-group G has the Glicksberg property if and only if G respects functional boundedness if and only if every non-precompact subset A of G has an infinite subset B which is discrete and C-embedded in G^+. Below we generalize this result, see Theorem 3.4. Let us recall (see [39]) that every complete g-group G is semi-reflexive and G^+ is a μ-space. However, Außenhofer found in [1] an example of a metrizable complete locally quasi-convex abelian group which is not semi-reflexive. Thus the above results do not give a characterization of the Glicksberg property even in the class of complete metrizable abelian groups.

We showed in [25] that if a complete *MAP* group respects functional boundedness, then G^+ must be a μ-space. Therefore to obtain respecting properties for a *MAP* group G we should assume that G and G^+ satisfy some completeness type properties. We say that a topological space X is a *countably μ-space* if every countable functionally bounded subset of X has compact closure. Clearly, every μ-space is a countably μ-space, but the converse is not true in general (see Example 2.2 below). We shall say that a *MAP* group G is *Bohr angelic* if G^+ is angelic. In Theorem 3.6 we characterize the Glicksberg property by means of the Bohr topology in the class of Bohr angelic groups G which are countably μ-spaces. The class of Bohr angelic *MAP* groups is sufficiently rich since it contains all *MAP* abelian groups with a $\mathfrak{G}$-base, see Proposition 2.6. The class of topological groups with a $\mathfrak{G}$-base is introduced in [31], it contains all metrizable groups and is closed under taking completions, quotients, countable products and countable direct sums. Using Theorem 3.6 we prove the following general result.

Theorem 1.1 *Let G be a MAP abelian group with a $\mathfrak{G}$-base. If G is a countably μ-space, then the following assertions are equivalent:*

(i) *G has the Schur property;*
(ii) *G has the Glicksberg property;*
(iii) *G respects sequential compactness;*
(iv) *G respects countable compactness;*
(v) *every non-functionally bounded subset of G has an infinite subset which is closed and discrete in G^+.*

If, in addition, G^+ is a μ-space, then (i)–(v) are equivalent to the following assertions:

(vi) *G respects pseudocompactness;*
(vii) *G respects functional boundedness.*

In particular, since every metrizable space is a μ-space and hence a countably μ-space, Theorem 1.1 gives a characterization of the Glicksberg property in the class of metrizable *MAP* abelian groups without the restrictive assumption of being a g-group as in [39].

An important class of locally convex spaces (lcs for short) is the class of free locally convex spaces $L(X)$ over Tychonoff spaces X. In Sect. 4 we prove the following result.

Theorem 1.2 *Let X be a Tychonoff space. Then the free locally convex space $L(X)$ over X respects all properties $\mathscr{P} \in \mathfrak{P}_0$. If $L(X)$ is complete, then $L(X)$ respects all properties $\mathscr{P} \in \mathfrak{P}$.*

It is well known that every Banach space E is a quotient space of a Banach space with the Schur property (more precisely, E is a quotient space of the space $\ell_1(\Gamma)$ for some set Γ). We generalize this result to all real locally convex spaces and all abelian Hausdorff topological groups.

Corollary 1.3 *Every real locally convex space E is a quotient space of an lcs which respects all properties $\mathscr{P} \in \mathfrak{P}_0$.*

Corollary 1.4 *Every Hausdorff abelian topological group G is a quotient group of a locally quasi-convex abelian group which respects all properties $\mathscr{P} \in \mathfrak{P}_0$.*

In Sect. 5 we obtain dual characterizations of the Schur property and the Glicksberg property in the class of reflexive abelian groups by showing that a reflexive group G has the Schur property or the Glicksberg property if and only if its dual group $G^\wedge$ is c_0-barrelled or g-barrelled, respectively (see Proposition 5.3). Another important class of topological groups is the class of abelian k_ω-groups. This class contains all dual groups of metrizable abelian groups, see [1, 13]. In Theorem 5.7 we show that every locally quasi-convex k_ω-group G respects all properties $\mathscr{P} \in \mathfrak{P}$ (note that the condition of being a locally quasi-convex group cannot be omitted, see Example 5.9).

In the last section we define some Glicksberg type properties and apply the obtained results to show that a reflexive abelian group of finite exponent is a Mackey group, see Theorem 6.7.

2 Preliminary Results

Denote by **TG** (**TAG**) the category of all Hausdorff (respectively, abelian) topological groups and continuous homomorphisms. A compact group bX is called the *Bohr compactification* of $(X, \tau) \in$ **TG** if there exists a continuous homomorphism i from X onto a dense subgroup of bX such that the pair (bX, i) satisfies the following *universal property*: If $p : X \to C$ is a continuous homomorphism into a compact group C, then there exists a continuous homomorphism $j^p : bX \to C$ such that $p = j^p \circ i$. Following von Neumann [51], the group X is called *maximally almost periodic* (*MAP*) if the group X^+ is Hausdorff, where $X^+ := (X, \tau^+)$ is the group X endowed with the Bohr topology τ^+ induced from bX. The family **MAP** (**MAPA**) of all *MAP* (respectively, *MAP* abelian) topological groups is a subcategory of **TG** (respectively, **TAG**). Every irreducible representation of a (pre)compact group X is finite-dimensional, see [41, 22.13]. For an $X \in$ **MAP**, we denote by $\widehat{X}$ the set of all (equivalence classes of) finite-dimensional irreducible representations of X. The *Bohr functor* $\mathfrak{B}$ on **MAP** is defined by $\mathfrak{B}(X) := X^+$ for a *MAP* group X and $\mathfrak{B}(T) = T$ if $T : X \to Y$ is a continuous homomorphism. Denote by **PCom** the class of all precompact groups. If a *MAP* group (G, τ) is abelian, then every $\pi \in \widehat{G}$ is one-dimensional (indeed, since the unitary group of a finite-dimensional Hilbert space is compact, by the universal property, π can be extended to $\widehat{\pi} \in \widehat{bG}$ and [41, 22.17] applies), so $\widehat{G}$ coincides with the group of all continuous characters of G denoted also by $\widehat{G}$. In this case $\tau^+ = \sigma(G, \widehat{G})$, where $\sigma(G, \widehat{G})$ is the smallest group topology on G for which the elements of $\widehat{G}$ are continuous.

Denote by $\mathbb{S}$ the unit circle group and set $\mathbb{S}_+ := \{z \in \mathbb{S} : \operatorname{Re}(z) \geq 0\}$. Let G be an abelian topological group. A character $\chi \in \widehat{G}$ is a continuous homomorphism from G into $\mathbb{S}$. A subgroup H of G is called *dually embedded* if every continuous character of H can be extended to a continuous character of G. A subset A of G is called *quasi-convex* if for every $g \in G \setminus A$ there exists $\chi \in \widehat{G}$ such that $\chi(g) \notin \mathbb{S}_+$ and $\chi(A) \subseteq \mathbb{S}_+$. If $A \subseteq G$ and $B \subseteq \widehat{G}$ set

$$A^{\triangleright} := \{\chi \in \widehat{G} : \chi(A) \subseteq \mathbb{S}_+\}, \quad B^{\triangleleft} := \{g \in G : \chi(g) \in \mathbb{S}_+ \ \forall \chi \in B\}.$$

Then A is quasi-convex if and only if $A^{\triangleright\triangleleft} = A$. The set $\operatorname{qc}(A) := \bigcap_{\chi \in A^{\triangleright}} \chi^{-1}(\mathbb{S}_+)$ is called the *quasi-convex hull* of A. An abelian topological group (G, τ) is called *locally quasi-convex* if it admits a neighborhood base at the neutral element 0 consisting of quasi-convex sets. If G is *MAP*, then the sets $\operatorname{qc}(U)$, where U is a neighborhood of zero in G, form a neighborhood base of a locally quasi-convex group topology $\tau_{\rm qc}$, we set $G_{qc} := (G, \tau_{\rm qc})$. The class **LQC** of all abelian locally quasi-convex groups is one of the most important subclasses of the class **MAPA**. Every *LCA* group is locally quasi-convex. More generally, every nuclear group is locally quasi-convex, see [8, Theorem 8.5]. The dual group $\widehat{G}$ of G endowed with the compact-open topology is denoted by $G^{\wedge}$. The homomorphism $\alpha_G : G \to G^{\wedge\wedge}$, $g \mapsto (\chi \mapsto \chi(g))$, is called *the canonical homomorphism*. If α_G is a topological isomorphism the group G is called *reflexive*. In the dual group $\widehat{G}$, we denote by $\sigma(\widehat{G}, G)$ the topology of pointwise convergence. Recall that a subset A of $\widehat{G}$ is called *equicontinuous* if for every $\varepsilon > 0$ there is a neighborhood U of zero in G such that

$$|\chi(x) - 1| < \varepsilon, \quad \forall x \in U, \ \forall \chi \in A.$$

We shall use the following fact, see [53].

Fact 2.1 *Let U be a neighborhood of zero of an abelian topological group G. Then $U^{\triangleright}$ is an equicontinuous quasi-convex compact subset of $G^{\wedge}$. A subset A of $G^{\wedge}$ is equicontinuous if and only if $A \subseteq V^{\triangleright}$ for some neighborhood V of zero.*

Let X and Y be Tychonoff spaces. We denote by $C_p(X, Y)$ the space $C(X, Y)$ of all continuous functions from X to Y endowed with the pointwise topology. If $Y = \mathbb{R}$, set $C(X, \mathbb{R}) := C(X)$.

A subset A of a topological space X is called

- *relatively compact* if its closure $\bar{A}$ is compact;
- *relatively countably compact* if each countably infinite subset in A has a cluster point in X;

- *relatively sequentially compact* if each sequence in A has a subsequence converging to a point of X;
- *functionally bounded in* X if every $f \in C(X)$ is bounded on A.

Recall that a Hausdorff topological space X is called

- a k_ω*-space* if it is the inductive limit of an increasing sequence $\{C_n\}_{n\in\mathbb{N}}$ of its compact subsets;
- a k_ω*-group* if X is a topological group whose underlying space is a k_ω-space;
- a μ*-space* if every functionally bounded subset of X is relatively compact;
- an (E)*-space* if its relatively countably compact subsets are relatively compact [37];
- a Š*mulyan-space* or a $\check{S}$*-space* if its compact subsets are sequentially compact;
- an *angelic space* if (1) every relatively countably compact subset of X is relatively compact, and (2) any compact subspace of X is Fréchet–Urysohn.

Note that any subspace of an angelic space is angelic, and a subset A of an angelic space X is compact if and only if it is countably compact if and only if A is sequentially compact, see Lemma 0.3 of [56]. Note also that if τ and ν are regular topologies on a set X such that $\tau \le \nu$ and the space (X, τ) is angelic, then the space (X, ν) is also angelic, see [56].

We need also the following property stronger than the property of being a countably μ-space. A topological group G is said to have a **cp**-*property* if every separable (or countable) precompact subset of X has compact closure. If a topological group G is complete, the closure $\overline{A}$ of each precompact subset A is compact. So every complete topological group has the **cp**-property. However there is a non-complete group with the **cp**-property which is not a μ-space.

Example 2.2 There is a sequentially compact non-compact abelian group H which has the **cp**-property. Indeed, let $G := X^\kappa$, where X is a metrizable compact abelian group and the cardinal κ is uncountable. For $g = (x_i)_{i\in\kappa} \in G$, set $\mathrm{supp}(g) := \{i \in \kappa : x_i \neq 0\}$ and define

$$H := \{g \in G : \ |\mathrm{supp}(g)| \le \aleph_0\}.$$

Then H with the induced topology is a proper dense subgroup of G. Any countable subset of H is contained in a countable product Y of copies of X. Since Y is a compact and metrizable subgroup of G, we obtain that the group H is sequentially compact with the **cp**-property. It is easy to see that every continuous function on H is bounded. Since H is not compact, it follows that H is not a μ-space. Note also that H is Fréchet–Urysohn by [52]. □

Example 2.3 It is well known that the ordinal space $X = [0, \omega_1)$ is a pseudocompact non-compact space, and hence X is not a μ-space. On the other hand, if A is a countable subset of X and $\alpha = \sup(A)$, then A is contained in the compact set $[0, \alpha]$. Thus X is a countably μ-space. □

We shall use the following result in which (i) is known but hard to locate explicitly stated.

Lemma 2.4 *Let G be a Hausdorff abelian topological group. Then:*

(i) *every functionally bounded subset A of G is precompact;*
(ii) *if G has the* **cp**-*property, then a separable subset B of G is functionally bounded if and only if B is precompact.*

Proof (i) If A is not precompact, Theorem 5 of [7] implies that A has an infinite uniformly discrete subset C, i.e., there is a neighborhood U of zero in G such that $c - c' \notin U$ for every distinct $c, c' \in C$. So C is not functionally bounded by Lemma 2.1 of [25], a contradiction.

(ii) follows from (i) and the **cp**-property. □

Following Orihuela [54], a Hausdorff topological space X is called *web-compact* if there is a nonempty subset Σ of $\mathbb{N}^{\mathbb{N}}$ and a family $\{A_\alpha : \alpha \in \Sigma\}$ of subsets of X such that, if

$$C_{n_1\dots n_k} := \bigcup\{A_\beta : \beta = (m_k) \in \Sigma,\ m_1 = n_1, \dots, m_k = n_k\}, \quad \forall\alpha = (n_k) \in \Sigma,$$

the following two conditions hold:

(i) $\overline{\bigcup\{A_\alpha : \alpha \in \Sigma\}} = X$, and
(ii) if $\alpha = (n_k) \in \Sigma$ and $x_k \in C_{n_1\dots n_k}$ for all $k \in \mathbb{N}$, then the sequence $\{x_k\}_{k\in\mathbb{N}}$ has a cluster point in X.

The class of web-compact spaces is sufficiently rich, see [45, § 4.3]. In particular, every separable space is web-compact. In what follows we shall use repeatedly the following result, see Proposition 4.2 of [45], which follows from a deep result of Orihuela [54].

Fact 2.5 *If X is web-compact, then the group $C_p(X, \mathbb{S})$ is angelic.*

Following [31], a topological group G is said to have a $\mathfrak{G}$-*base* if there is a base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of neighborhoods at the identity such that $U_\beta \subseteq U_\alpha$ whenever $\alpha \le \beta$ for all $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$, where $\alpha = (\alpha(n))_{n\in\mathbb{N}} \le \beta = (\beta(n))_{n\in\mathbb{N}}$ if $\alpha(n) \le \beta(n)$ for all $n \in \mathbb{N}$. Below we give sufficient conditions on a *MAP* abelian group G or its dual group $G^\wedge$ to be Bohr angelic.

Proposition 2.6 *Let (G, τ) be a MAP abelian group.*

(i) *If G is web-compact, then $(G^\wedge)^+$ is angelic.*
(ii) *If G has a $\mathfrak{G}$-base, then G^+ is angelic.*

Proof (i) The group $\big(\widehat{G}, \sigma(\widehat{G}, G)\big)$, being a closed subgroup of the group $C_p(G, \mathbb{S})$, is angelic by Fact 2.5. As $\sigma(\widehat{G}, G) \le \sigma(\widehat{G}, G^{\wedge\wedge})$, we obtain that the group $(G^\wedge)^+ = \big(\widehat{G}, \sigma(\widehat{G}, G^{\wedge\wedge})\big)$ is also angelic.

(ii) Let $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-base at zero in G. Then the family $\{U_\alpha^{\triangleright} : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution in $G^\wedge$ by Theorem 5.1 of [31]. Therefore the group $H := \big(\widehat{G}, \sigma(\widehat{G}, G)\big)$ is web-compact by Example 4.1(1) of [45]. Hence the space $C_p(H, \mathbb{S})$ is angelic by Fact 2.5. So the group $G^+ = \big(G, \sigma(G, \widehat{G})\big)$, being a subgroup of $C_p(H, \mathbb{S})$, is also angelic. □

We shall use the following results.

Proposition 2.7 ([25]) *Let H be a subgroup of a MAP Abelian group X and $\mathscr{P} \in \mathfrak{P}_0$. If X respects $\mathscr{P}$, then H respects $\mathscr{P}$ as well.*

Let E be a real lcs and E' its topological dual space. Then E is a locally quasi-convex abelian group, see [8]. So it is natural to consider relations between the weak topology $\tau_w := \sigma(E, E')$ and the Bohr topology $\tau^+ := \sigma(E, \widehat{E})$ on E. Denote by τ_k the compact-open topology on E'. The polar of a subset A of E is denoted by $A^\circ := \{\chi \in E' : |\chi(x)| \le 1\ \forall x \in A\}$. Define

$$\psi : E' \to \widehat{E},\ \psi(\chi) := e^{2\pi i \chi}, \quad \big(\text{i.e. } \psi(\chi)(x) := e^{2\pi i \chi(x)} \text{ for } x \in E\big).$$

A proof of the next important result can be found in [8, Proposition 2.3].

Fact 2.8 *Let E be a real lcs and let $\psi : E' \to \widehat{E}$, $\psi(\chi) := e^{2\pi i \chi}$. Then:*

(i) *ψ is an algebraic isomorphism;*
(ii) *ψ is a topological isomorphism of (E', τ_k) onto $E^\wedge$.*

We shall say that ψ is the *canonical isomorphism* of E' onto $\widehat{E}$. Fact 2.8 implies that $\tau^+ < \tau_w \le \tau$ and hence

$$\mathscr{P}(E) \subseteq \mathscr{P}(E_w) \subseteq \mathscr{P}(E^+), \quad \text{for every } \mathscr{P} \in \mathfrak{P}. \tag{1}$$

In [58] it is proved that E_w and E^+ have the same compact sets and hence the same convergent sequences. The next proposition generalizes this result.

Proposition 2.9 ([25]) *Let E be a real lcs and let $\mathscr{P} \in \mathfrak{P}_0$. Then $\mathscr{P}(E_w) = \mathscr{P}(E^+)$.*

For an lcs E, we denote by $\mathsf{Bo}(E)$ the family of all bounded subsets of E. The next assertion complements Proposition 2.9.

Proposition 2.10 *If (E, τ) is a real lcs, then every functionally bounded subset A of E^+ is bounded, i.e., $\mathscr{F}\mathscr{B}(E^+) \subseteq \mathsf{Bo}(E)$.*

Proof Since $\mathsf{Bo}(E) = \mathsf{Bo}(E_w)$, it is sufficient to show that A is weakly bounded. Let $U = [F; \varepsilon]$ be a standard weakly open neighborhood of zero in E, where F is a finite subset of $E' \setminus \{0\}$, $\varepsilon > 0$ and

$$[F; \varepsilon] := \{x \in E : |\chi(x)| < \varepsilon \ \ \forall \chi \in F\}.$$

Fix a $\chi \in F$ and take a $z = z_\chi \in E$ such that $\chi(z) = 1$. By Theorem 7.3.5 of [50], we can represent E in the form $E = L_\chi \oplus \ker(\chi)$, where $L_\chi = \operatorname{span}(z)$ and $\ker(\chi)$

is the kernel of χ. Then $E^+ = L_\chi^+ \oplus \ker(\chi)^+$. Since the projection P_χ from E onto L_χ is continuous (in τ and τ^+), $P_\chi(A)$ is a functionally bounded subset of $L_\chi^+ \cong \mathbb{R}^+$. By [63], $P_\chi(A)$ is bounded in L_χ. Therefore there exists a $C_\chi > 0$ such that $|\chi(a)| < C_\chi$ for every $a \in A$. Set $C := \max\{C_\chi : \chi \in F\}$. Then $(\varepsilon/C)A \subseteq U$ since $|\chi\big((\varepsilon/C)a\big)| = (\varepsilon/C)|\chi(a)| < \varepsilon$ for every $a \in A$ and $\chi \in F$. Thus A is bounded. □

We do not know whether there exists a real lcs E such that $\mathscr{FB}(E_w) \subsetneq \mathscr{FB}(E^+)$.

The weak-$*$ topology on the dual space of an lcs E plays a crucial role in the theory of locally convex spaces. The next assertion complements Fact 2.8 and Proposition 2.9 and is used repeatedly in the paper (item (vii) is noticed in Proposition 2.3 of [40]).

Proposition 2.11 *Let E be a real lcs, $\mathscr{P} \in \mathfrak{P}_0$ and let $\psi : E' \to \widehat{E}$ be the canonical isomorphism. Then:*

- (i) *the map $\psi : \big(E', \sigma(E', E)\big)^+ \to \big(\widehat{E}, \sigma(\widehat{E}, E)\big)$ is a topological isomorphism;*
- (ii) *$A \in \mathscr{P}\big(E', \sigma(E', E)\big)$ if and only if $\psi(A) \in \mathscr{P}\big(\widehat{E}, \sigma(\widehat{E}, E)\big)$;*
- (iii) *the map $\psi : \big(E', \sigma\big(E', (E', \tau_k)'\big)\big)^+ \to \big(\widehat{E}, \sigma(\widehat{E}, E^{\wedge\wedge})\big)$ is a topological isomorphism;*
- (iv) *$\mathscr{P}\Big(E', \sigma\big(E', (E', \tau_k)'\big)\Big) = \mathscr{P}\Big(\big(E', \sigma\big(E', (E', \tau_k)'\big)\big)^+\Big)$;*
- (v) *$A \in \mathscr{P}\big(E', \sigma\big(E', (E', \tau_k)'\big)\big)$ if and only if $\psi(A) \in \mathscr{P}\big(\widehat{E}, \sigma(\widehat{E}, E^{\wedge\wedge})\big)$;*
- (vi) *a subset A of E' is equicontinuous if and only if $\psi(A)$ is equicontinuous;*
- (vii) *the canonical map $\alpha_E : E \to E^{\wedge\wedge}$ is continuous if and only if every compact subset of (E', τ_k) is equicontinuous.*

Proof (i) Since $\big(E', \sigma(E', E)\big)' = E$, the Fact 2.8 implies that the dual group of $\big(E', \sigma(E', E)\big)$ can be identified with E under the map $x \mapsto e^{2\pi i x}$ $(x \in E)$. So the sets

$$[F; \varepsilon] := \big\{\chi \in E' : |e^{2\pi i \chi(x)} - 1| < \varepsilon \ \forall x \in F\big\},$$

where F is a finite subset of E and $\varepsilon > 0$, form a base at zero in $\big(E', \sigma(E', E)\big)^+$. The sets

$$V_{F,\varepsilon} := \big\{z \in \widehat{E} : |z(x) - 1| < \varepsilon \ \forall x \in F\big\},$$

where F is a finite subset of E and $\varepsilon > 0$, form a base at zero in $\big(\widehat{E}, \sigma(\widehat{E}, E)\big)$. Taking into account that ψ is an algebraic isomorphism and $\psi(\chi)(x) = e^{2\pi i \chi(x)}$, we obtain that $\psi\big([F; \varepsilon]\big) = V_{F,\varepsilon}$. Thus the canonical isomorphism ψ is also a topological isomorphism.

(ii) Set $F := \big(E', \sigma(E', E)\big)$. Then $F' = E$ and $F_w = F$. Therefore, by Proposition 2.9, $\mathscr{P}(F) = \mathscr{P}(F^+)$ and (i) applies.

(iii) Note that $\big(E', \sigma\big(E', (E', \tau_k)'\big)\big)' = (E', \tau_k)'$. Therefore, by Fact 2.8, the sets

$$[F; \varepsilon] := \big\{\chi \in E' : |e^{2\pi i \xi(\chi)} - 1| < \varepsilon \ \forall \xi \in F\big\},$$

where F is a finite subset of $(E', \tau_k)'$ and $\varepsilon > 0$, form a base at the origin in $\big(E', \sigma\big(E', (E', \tau_k)'\big)\big)^+$. Analogously, the sets

$$W_{F,\varepsilon} := \big\{z \in \widehat{E} : |s(z) - 1| < \varepsilon \ \forall s \in F\big\},$$

where F is a finite subset of $E^{\wedge\wedge}$ and $\varepsilon > 0$, form a base at zero in $\big(\widehat{E}, \sigma(\widehat{E}, E^{\wedge\wedge})\big)$. By (i) of Fact 2.8, the map

$$\widehat{\psi} : (E', \tau_k)' \to (E', \tau_k)^\wedge, \quad \widehat{\psi}(\xi) := e^{2\pi i \xi},$$

is an algebraic isomorphism and, by (ii) of Fact 2.8, the adjoint map ψ^* of ψ

$$\psi^* : E^{\wedge\wedge} \to (E', \tau_k)^\wedge, \quad (\psi^*(\eta), \chi) = \big(\eta, \psi(\chi)\big), \ \eta \in E^{\wedge\wedge}, \chi \in E', \tag{2}$$

is a topological isomorphism. In particular, for $\alpha = \psi^*(\eta)$, (2) implies

$$(\alpha, \chi) = \big((\psi^*)^{-1}(\alpha), \psi(\chi)\big), \quad \forall \alpha \in (E', \tau_k)^\wedge, \ \forall \chi \in E'. \tag{3}$$

So the map $H := (\psi^*)^{-1} \circ \widehat{\psi} : (E', \tau_k)' \to E^{\wedge\wedge}$ is an algebraic isomorphism such that, for every $z = \psi(\chi) \in E^\wedge$ with $\chi \in E'$ and each $\xi \in (E', \tau_k)'$, we have

$$\big(H(\xi), z\big) = \big((\psi^*)^{-1} \circ \widehat{\psi}(\xi), \psi(\chi)\big) \overset{(3)}{=} \big(\widehat{\psi}(\xi), \chi\big) = e^{2\pi i \xi(\chi)} = e^{2\pi i \xi\left(\psi^{-1}(z)\right)}.$$

Therefore, for a finite subset F of $(E', \tau_k)'$ and $\varepsilon > 0$, we obtain

$$\begin{aligned} \psi\big([F; \varepsilon]\big) &= \left\{z \in \widehat{E} : \left|e^{2\pi i \xi\left(\psi^{-1}(z)\right)} - 1\right| < \varepsilon \ \forall \xi \in F\right\} \\ &= \big\{z \in \widehat{E} : \big|\big(H(\xi), z\big) - 1\big| < \varepsilon \ \forall \xi \in F\big\} = W_{H(F),\varepsilon}. \end{aligned}$$

Thus ψ is a topological isomorphism.

(iv) follows from Proposition 2.9 applied to $G = G_w := \big(E', \sigma\big(E', (E', \tau_k)'\big)\big)$, and (v) follows from (iii) and (iv).

(vi) We shall use the following easily checked inequalities

$$\pi|\phi| \le \left|e^{2\pi i \phi} - 1\right| \le 2\pi|\phi|, \quad \phi \in [-1/2, 1/2]. \tag{4}$$

Let $A \subseteq E'$ be equicontinuous. For every $0 < \varepsilon < 0.1$, take a neighborhood U of zero in E such that

$$|a(x)| < \varepsilon, \quad \forall a \in A, \ \forall x \in U. \tag{5}$$

Then (4) and (5) imply

$$|\psi(a)(x) - 1| = \left|e^{2\pi i a(x)} - 1\right| \le 2\pi\varepsilon, \quad \forall a \in A, \ \forall x \in U.$$

Thus $\psi(A)$ is equicontinuous.

Conversely, let $\psi(A)$ be equicontinuous. For every $0 < \varepsilon < 0.1$, take an absolutely convex neighborhood U of zero in E such that

$$\left|e^{2\pi i a(x)} - 1\right| < \varepsilon, \quad \forall a \in A, \ \forall x \in U.$$

If $a(x) = t + m$ with $t \in [-1/2, 1/2]$ and $m \in \mathbb{Z}$, (4) implies $\pi|t| \le |e^{2\pi i t} - 1| = |e^{2\pi i a(x)} - 1| < \varepsilon$, and hence

$$a(x) \in (-\varepsilon/\pi, \varepsilon/\pi) + \mathbb{Z}, \quad \forall a \in A, \ \forall x \in U. \tag{6}$$

Since U is arc-connected, $0 \in U$ and $\varepsilon < 0.1$, (6) implies

$$a(x) \in (-\varepsilon/\pi, \varepsilon/\pi), \quad \forall a \in A, \ \forall x \in U.$$

Thus A is equicontinuous.

(vii) Recall that α_E is continuous if and only if every compact subset of $E^\wedge$ is equicontinuous, see Proposition 5.10 of [1]. By Fact 2.8(ii), ψ is a topological isomorphism of (E', τ_k) onto $E^\wedge$. Now the assertion follows from (vi). □

3 General Results

To show that the Glicksberg property and the Schur property can be naturally defined by two functors in the category **TG** we consider two classes of topological groups introduced by Noble in [52, 53], namely, the classes of k-groups and s-groups.

For every $(X, \tau) \in \mathbf{TG}$ denote by $k_g(\tau)$ the finest group topology for X coinciding on compact sets with τ. In particular, τ and $k_g(\tau)$ have the same family of compact subsets. Clearly, $\tau \le k_g(\tau)$. If $\tau = k_g(\tau)$, the group (X, τ) is called a *k-group* [53]. The group $(X, k_g(\tau))$ is called the *k_g-modification* of X. The assignment $\mathbf{k}_g(X, \tau) := (X, k_g(\tau))$ is a functor from **TG** to the full subcategory **K** of all k-groups. The class **K** contains all topological groups whose underlaying space is a k-space. In particular, the class **LC** (**LCA**) of all locally compact (and abelian, respectively) groups is contained in **K**. Since every metrizable group is a k-space we have $\mathbf{LC} \subsetneqq \mathbf{K}$. The family of all abelian k-groups we denote by **KA**. Denote by $\mathfrak{RC}$ the class of all *MAP* groups which respect compactness.

Similar to k-groups we define s-groups (we follow [22]). Let (X, τ) be a (Hausdorff) topological group and let S be the set of all sequences in (X, τ) converging to the unit $e \in X$. Then there exists the finest Hausdorff group topology τ_S on the underlying group X in which all sequences of S converge to e. If $\tau = \tau_S$, the group X is called an *s-group*. The assignment $\mathbf{s}_g(X, \tau) := (X, \tau_S)$ is a functor from **TG** to the full subcategory **S** of all s-groups. The class **S** contains all sequential groups [22, 1.14]. Note that X and $\mathbf{s}_g(X)$ have the same set of convergent sequences [22, 4.2]. The family of all abelian s-groups we denote by **SA**. Every s-group is also a k-group

[23], so $\mathbf{S} \subseteq \mathbf{K}$ and $\mathbf{SA} \subseteq \mathbf{KA}$. Denote by $\mathfrak{RS}$ the class of all *MAP* groups which have the Schur property.

For a topological group X with the identity e, set

$$c_0(X) := \left\{(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} : \lim_n x_n = e\right\}$$

and let $\mathfrak{u}_0$ be the uniform topology on $c_0(X)$ generated by the sets of the form $V^{\mathbb{N}}$, where V is an open neighborhood of $e \in X$. Set $\mathfrak{F}_0(X) := (c_0(X), \mathfrak{u}_0)$.

In (1) and (3) of the next proposition we give categorical characterizations of the Schur property and the Glikcsberg property, note also that (8) generalizes Theorem 1.2 of [63].

Proposition 3.1 *Let X and Y be MAP topological groups.*

(1) $X \in \mathfrak{RC}$ *if and only if* $(\mathbf{k}_g \circ \mathfrak{B})(X) = \mathbf{k}_g(X)$.
(2) $X \in \mathbf{K} \cap \mathfrak{RC}$ *if and only if* $(\mathbf{k}_g \circ \mathfrak{B})(X) = X$.
(3) $X \in \mathfrak{RS}$ *if and only if* $(\mathbf{s}_g \circ \mathfrak{B})(X) = \mathbf{s}_g(X)$.
(4) $X \in \mathbf{S} \cap \mathfrak{RS}$ *if and only if* $(\mathbf{s}_g \circ \mathfrak{B})(X) = X$.
(5) $\mathbf{PCom} \subsetneqq \mathfrak{RC}$ *and* $\mathbf{LCA} \subsetneqq \mathbf{KA} \cap \mathfrak{RC}$.
(6) $\mathbf{K} \cap \mathfrak{RC} \subsetneqq \mathbf{K}$ *and* $\mathbf{K} \cap \mathfrak{RC} \subsetneqq \mathfrak{RC}$.
(7) $\mathbf{S} \cap \mathfrak{RS} \subsetneqq \mathbf{S}$ *and* $\mathbf{S} \cap \mathfrak{RS} \subsetneqq \mathfrak{RS}$.
(8) *Let* $X \in \mathbf{K}$ *and* $Y \in \mathfrak{RC}$ *and let* $\phi : X \to Y$ *be a homomorphism. If* $\phi^+ : X^+ \to Y^+$, $\phi^+(x) := \phi(x)$, *is continuous, then* ϕ *is continuous.*
(9) *Let* $X \in \mathbf{S}$ *and* $Y \in \mathfrak{RS}$ *and let* $\phi : X \to Y$ *be a homomorphism. If* $\phi^+ : X^+ \to Y^+$, $\phi^+(x) := \phi(x)$, *is continuous, then* ϕ *is continuous.*

Proof (1) If $X \in \mathfrak{RC}$, then $(\mathbf{k}_g \circ \mathfrak{B})(X) = \mathbf{k}_g(X)$ by the definition of the respecting compactness and the definition of $\mathbf{k}_g(X)$. Conversely, let $(\mathbf{k}_g \circ \mathfrak{B})(X) = \mathbf{k}_g(X)$ and let K be compact in $\mathfrak{B}(X)$. Then K is compact in $(\mathbf{k}_g \circ \mathfrak{B})(X)$ by the definition of k_g-modification. So K is compact in $\mathbf{k}_g(X)$. Hence, by the definition of k_g-modification, K is compact in X. Thus $X \in \mathfrak{RC}$.

(2) If $X \in \mathbf{K} \cap \mathfrak{RC}$, then (1) and the definition of k-groups imply $(\mathbf{k}_g \circ \mathfrak{B})(X) = \mathbf{k}_g(X) = X$. Conversely, let $(\mathbf{k}_g \circ \mathfrak{B})(X) = X$. Since $\mathbf{k}_g \circ \mathbf{k}_g = \mathbf{k}_g$, the equalities

$$\mathbf{k}_g(X) = \mathbf{k}_g \circ (\mathbf{k}_g \circ \mathfrak{B}(X)) = (\mathbf{k}_g \circ \mathfrak{B})(X) = X$$

and (1) imply that X is a k-group and $X \in \mathfrak{RC}$.

(3) and (4) can be proved analogously to (1) and (2), respectively.

(5) Since $\mathfrak{B}(K) = K$ for each precompact group K, the first inclusion follows. The second one holds by the Glicksberg theorem. To prove that these inclusions are strict take an arbitrary compact totally disconnected metrizable group X. Then $\mathfrak{F}_0(X)$ is metrizable, and hence it is a k-group. By Theorem 1.3 of [25], $\mathfrak{F}_0(X)$ respects compactness and it is not locally precompact by [24]. Thus the inclusions are strict.

(6)–(7) Being metrizable the group $\mathfrak{F}_0(\mathbb{T})$ belongs to $\mathbf{SA} \subseteq \mathbf{KA}$ (here $\mathbb{T} = \mathbb{R}/\mathbb{Z}$). However, $\mathfrak{F}_0(\mathbb{T})$ does not respect compactness and convergent sequences by Theorem 1.3 of [25]. Thus $\mathbf{K} \cap \mathfrak{RC} \neq \mathbf{K}$ and $\mathbf{S} \cap \mathfrak{RS} \neq \mathbf{S}$.

To prove that the second inclusion is proper it is enough to find a precompact abelian group X which is not a k-group, and hence it is not an s-group. Take an arbitrary non-measurable subgroup H of $\mathbb{T}$ and set $X := (\mathbb{Z}, T_H)$, where T_H is the smallest group topology on $\mathbb{Z}$ for which the elements of H are continuous. Then the precompact group X does not contain non-trivial convergent sequences (see [16]). Since X is countable, we obtain that X also does not have infinite compact subsets by [20, 3.1.21]. This immediately implies that the k_g-modification of X is discrete. Hence $\mathbf{k}_g(X) = \mathbb{Z}_d$ is an infinite discrete LCA group. So $\mathbf{k}_g(X) \neq X$ and X is not a k-group. Thus the second inclusion is proper.

(8) Let $id_X : X \to X^+$ and $id_Y : Y \to Y^+$ be the identity continuous maps. Fix arbitrarily a compact subset K in X. Then $K^+ := \phi^+(id_X(K))$ is compact in Y^+. As $Y \in \mathfrak{RC}$, K^+ is compact in Y. So $id_Y|_{K^+}$ is a homeomorphism. Hence $\phi|_K = (id_Y|_{K^+})^{-1} \circ \phi^+ \circ (id_X|_K)$ is continuous. So ϕ is continuous on any compact subset of X. As X is a k-group, ϕ is continuous (see [53]).

(9) is proved analogously to (8). □

Remark 3.1 The fact that $(G, \mathscr{T}^+)$ is precompact whenever $(G, \mathscr{T})$ is a *MAP* abelian group suggests the following two natural questions posed in [18, 1.2] (see also [63]):

(i) Let $(G, \mathscr{U})$ be an abelian precompact group. Must there exist a topological group topology $\mathscr{T}$ for G such that $(G, \mathscr{T})$ is a LCA group and $\mathscr{U} = \mathscr{T}^+$?

(ii) Let G be an abelian group with topological group topologies $\mathscr{T}$ and $\mathscr{U}$ such that $(G, \mathscr{T})$ is a LCA group, $(G, \mathscr{U})$ is an abelian precompact group, $\mathscr{U} \subseteq \mathscr{T}$, and a subset $A \subseteq G$ is $\mathscr{T}$-compact if and only if A is $\mathscr{U}$-compact. Does it follow that $\mathscr{U} = \mathscr{T}^+$?

In [18], the authors showed that the answer to both these questions is "no". Let us show that the group X in the proof of (6)–(7) of Proposition 3.1 also answers negatively to these questions. Set $G = \mathbb{Z}$ and $\mathscr{U} = T_H$. Since G is countable, every locally compact group topology $\mathscr{T}$ on G must be discrete. So $\mathscr{T}^+ = T_{\mathbb{T}}$ and $\mathscr{U} \subseteq \mathscr{T}$. Further, as it was noticed in the proof of (6)–(7), a subset A of G is $\mathscr{T}$-compact if and only if A is $\mathscr{U}$-compact (if and only if A is finite). However, since $H \neq \mathbb{T}$, we obtain $\mathscr{U} \neq \mathscr{T}^+$ by [17]. □

We note the following assertion.

Proposition 3.2 *Let (G, τ) be a MAP group such that every functionally bounded subset of G^+ has compact closure in G. Then G respects all properties $\mathscr{P} \in \mathfrak{P}$ and G^+ is a μ-space.*

Proof Let $A \in \mathscr{P}(G^+)$. Then A is functionally bounded in G^+. Therefore its τ-closure $\overline{A}$ is compact in G, so the identity map $id : (\overline{A}, \tau|_{\overline{A}}) \to (\overline{A}, \tau^+|_{\overline{A}})$ is a homeomorphism. Hence G^+ is a μ-space and $A \in \mathscr{P}(G)$. Thus G respects $\mathscr{P}$. □

It is clear that the Glicksberg property implies the Schur property, but as we mentioned in the introduction, the converse is not true in general. Some other relations between respecting properties are given in the next proposition, which gives a partial answer to Problem 7.2 in [25].

Proposition 3.3 *Let (G, τ) be a MAP group. Then:*

(i) *G has the Schur property if and only if it respects sequential compactness;*
(ii) *if G respects countable compactness, then G has the Schur property;*
(iii) *if G respects pseudocompactness, then G has the Schur property;*
(iv) *if G is a countably μ-space and respects functional boundedness, then G has the Schur property;*
(v) *if G is complete and respects countable compactness, then G has the Glicksberg property;*
(vi) *if G is complete and respects pseudocompactness, then G respects countable compactness;*
(vii) *if G is complete and respects functional boundedness, then G respects all properties $\mathscr{P} \in \mathfrak{P}$.*

Proof (i) (If G is an abelian group the necessity is proved in Proposition 23 of [12].) Assume that (G, τ) has the Schur property and let A be a sequentially compact subset of G^+. Take a sequence $S = \{a_n\}_{n\in\mathbb{N}}$ in A. Then S has a τ^+-convergent subsequence S'. By the Schur property S' converges in τ. Hence A is τ-sequentially compact. Thus G respects sequential compactness.

Conversely, assume that (G, τ) respects sequential compactness and let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence τ^+-converging to an element $a_0 \in G$. Set $S := \{a_n\}_{n\in\mathbb{N}} \cup \{a_0\}$, so S is τ^+-compact. Being countable S is metrizable and hence τ^+-sequentially compact. So S is sequentially compact in τ. We show that $a_n \to a_0$ in τ. Suppose for a contradiction that there is a τ-neighborhood U of a_0 which does not contain an infinite subsequence S' of S. Then there is a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ of S' which τ-converges to an element $g \in S$. Clearly, $g \neq a_0$ and $a_{n_k} \to g$ in the Bohr topology, and hence $a_n \not\to a_0$ in τ^+, a contradiction. Therefore $a_n \to a_0$ in τ. Thus G has the Schur property.

(ii), (iii) Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence τ^+-converging to an element $a_0 \in G$. Set $S := \{a_n\}_{n\in\mathbb{N}} \cup \{a_0\}$, so S is τ^+-compact. Hence S is τ^+-countably compact. So S is countably compact or pseudocompact in τ, respectively. As any countable space is normal, in both cases S is countably compact in τ. We show that $a_n \to a_0$ in τ. Suppose for a contradiction that there is a τ-neighborhood U of a_0 which does not contain an infinite subsequence S' of S. Then S' has a τ-cluster point $g \in S$ and clearly $g \neq a_0$. Note that g is also a cluster point of S' in the Bohr topology τ^+. Hence $g = a_0$, a contradiction. Therefore $a_n \to a_0$ in τ. Thus G has the Schur property.

(iv) Let $S = \{a_n : n \in \mathbb{N}\} \cup \{a_0\}$ be a sequence in G^+ which τ^+-converges to a_0. Since S is also functional bounded in G^+, we obtain that S is closed and functionally bounded in G. So S is compact in G because G is a countably μ-space. As the identity map $(S, \tau|_S) \to (S, \tau^+|_S)$ is a homeomorphism, $a_n \to a_0$ in G. Thus G has the Schur property.

(v) Let K be a compact subset of G^+. Then K is countably compact in G^+ and hence in G. Since functionally bounded subsets are precompact by Lemma 2.4, the completeness of G and the closeness of K in G imply that K is a compact subset of G. Thus G has the Glicksberg property.

(vi) Let A be a countably compact subset of G^+. Then A is pseudocompact in G^+ and hence in G. The completeness of G and Lemma 2.4 imply that the closure $\overline{A}$ of A in G is compact. As the identity map $(\overline{A}, \tau|_{\overline{A}}) \to (\overline{A}, \tau^+|_{\overline{A}})$ is a homeomorphism, we obtain that A is countably compact in G. Thus G respects countable compactness.

(vii) This is Theorem 1.2 of [25]. □

Proposition 3.3 shows that the Schur property is the weakest one among the properties of $\mathfrak{P}_0$. We do not know whether the completeness of G in (v)–(vii) of Proposition 3.3 can be dropped. Also we do not know an example of a *MAP* group which respects countable compactness but does not have the Glicksberg property (or respects pseudocompactness but does not respect countable compactness, etc.).

Recall that in a complete group the class of precompact sets and the class of functionally bounded sets are coincide. Recall also that if G is a complete g-group, then G and G^+ are μ-spaces, see Theorem 3.2 of [39]. Therefore the next theorem generalizes Theorem 3.3 of [39], cf. also Theorem 1.2 of [25].

Theorem 3.4 *Let (G, τ) be a MAP group such that G^+ is a μ-space. Then the following assertions are equivalent:*

(i) *G respects compactness;*
(ii) *G respects countable compactness and G is a μ-space;*
(iii) *G respects pseudocompactness and G is a μ-space;*
(iv) *G respects functional boundedness and G is a μ-space;*
(v) *G is a μ-space and every non-functionally bounded subset A of G has an infinite subset B which is discrete and C-embedded in G^+.*

If (i)–(v) hold, then every functionally bounded subset in G^+ is relatively compact in G.

Proof (i) ⇒ (ii) Let A be a countably compact subset of G^+. As G^+ is a μ-space, the τ^+-closure $\overline{A}$ of A is compact in G^+. Therefore $\overline{A}$ is compact in G by the Glicksberg property, and hence A is relatively compact in G. Since the identity map $(\overline{A}, \tau|_{\overline{A}}) \to (\overline{A}, \tau^+|_{\overline{A}})$ is a homeomorphism, we obtain that A is countably compact in G. Thus G respects countable compactness. The same proof shows that every functionally bounded subset in G^+ is relatively compact in G, and in particular G is a μ-space.

(ii) ⇒ (iii) Let A be a pseudocompact subset of G^+. Then the closure K of A in G^+ is τ^+-compact because G^+ is a μ-space. Therefore K is countably compact in G. Being closed K also is compact in G since G is a μ-space. Since the identity map $(K, \tau|_K) \to (K, \tau^+|_K)$ is a homeomorphism, we obtain that A is pseudocompact in G. Thus G respects pseudocompactness.

The implication (iii) ⇒ (iv) is proved analogously to (ii) ⇒ (iii).

(iv) ⇒ (v) Let A be a non-functionally bounded subset of G. As G respects functional boundedness it follows that A is not functionally bounded in G^+. Let f be a continuous function on G^+ which is unbounded on A. If we take B as a sequence $\{a_n\}_{n\in\mathbb{N}}$ in A such that $|f(a_{n+1})| > |f(a_n)| + 1$ for all $n \in \mathbb{N}$, then B is discrete and C-embedded in G^+.

(v) $\Rightarrow$ (i) Let K be a compact subset of G^+. Then K must be functionally bounded in G. Since G is a μ-space and K is also closed in G we obtain that K is compact in G. Thus G respects compactness. □

In several important classes of *MAP* groups some of the properties from $\mathfrak{P}_0$ hold simultaneously.

Proposition 3.5 *Let (G, τ) be a complete MAP group.*

(i) *If G^+ is an (E)-space, then G has the Glicksberg property if and only if G respects countable compactness.*
(ii) *If G^+ is a $\check{S}$-space, then G has the Schur property if and only if G has the Glicksberg property.*

Proof (i) Let G have the Glicksberg property and let K be a countably compact subset of G^+. Since G^+ is an (E)-space, K is relatively compact in G^+, and hence its τ^+-closure $\overline{K}$ is compact in G^+. So $\overline{K}$ is compact in G and the identity map $(\overline{K}, \tau|_{\overline{K}}) \to (\overline{K}, \tau^+|_{\overline{K}})$ is a homeomorphism. Therefore K is countably compact in G. Thus G respects countable compactness. The converse assertion follows from (v) of Proposition 3.3.

(ii) Let G have the Schur property and let K be a compact subset of G^+. Then K is sequentially compact in G^+, and hence in G by (i) of Proposition 3.3. As K is precompact and closed in G and G is complete, we obtain that K is compact in G. Thus G has the Glicksberg property. The converse assertion is clear. □

For Bohr angelic groups we obtain the following result.

Theorem 3.6 *Let (G, τ) be a MAP group. If G^+ is angelic, then the following assertions are equivalent:*

(i) *G has the Schur property;*
(ii) *G has the Glicksberg property;*
(iii) *G respects sequential compactness;*
(iv) *G respects countable compactness.*

If, in addition, G is a countably μ-space, then (i)–(iv) are equivalent to

(v) *every non-functionally bounded subset of G has an infinite subset which is closed and discrete in G^+.*

Proof The equivalence (i) $\Leftrightarrow$ (iii) and the implications (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (i) follow from Proposition 3.3.

(iii) $\Rightarrow$ (ii), (iv) Let K be a compact subset or a countably compact subset of G^+. Then K is sequentially compact in G^+ by [56, Lemma 0.3], and hence K is sequentially compact in G. Since G is also angelic, we obtain that K is compact or countably compact in G. Thus G has the Glicksberg property or respects countable compactness, respectively.

(ii) $\Rightarrow$ (v) Suppose for a contradiction that there is a non-functionally bounded subset A in G such that every countably infinite subset is either non-closed in G^+ or is not discrete in G^+. So, in both cases, every countably infinite subset of A has a cluster point in G^+. Therefore A is relatively countably compact in G^+. The angelicity of G^+ implies that the closure $\overline{A}$ of A in G^+ is compact in G^+. Hence $\overline{A}$ is compact in G by the Glicksberg property. Thus A is functionally bounded in G, a contradiction.

(v) $\Rightarrow$ (i) Let $g_n \to e$ in G^+, where e is the identity of G. Set $S := \{g_n\}_{n\in\mathbb{N}} \cup \{e\}$, so S is a compact subset of G^+. Let us show that S is functionally bounded in G. Indeed, otherwise, there would exist a subsequence $\{g_{n_k}\}_{k\in\mathbb{N}}$ of S which is closed and discrete in G^+. Then g_n does not converge to e in G^+, a contradiction. So S is functionally bounded in G. Thus the set S being also countable and closed in G is compact in G (recall that G is a countably μ-space). Therefore the identity map $(S, \tau|_S) \to (S, \tau^+|_S)$ is a homeomorphism. Hence $g_n \to e$ in G. Thus G has the Schur property. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. The equivalences (i) $\Leftrightarrow$ (ii)$\Leftrightarrow$(iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) follow from Theorem 3.6 and (ii) of Proposition 2.6. The implications (vi) $\Rightarrow$ (i) and (vii) $\Rightarrow$ (i) follow from (iii) and (iv) of Proposition 3.3, respectively. Finally, the implications (ii) $\Rightarrow$ (vi) and (ii) $\Rightarrow$ (vii) follow from Theorem 3.4. □

Corollary 3.7 *For a Lindelöf MAP abelian group G with a $\mathfrak{G}$-base the following assertions are equivalent:*

(i) *there is $\mathscr{P} \in \mathfrak{P}$ such that G respects $\mathscr{P}$;*
(ii) *G respects all properties $\mathscr{P} \in \mathfrak{P}$;*
(iii) *every non-functionally bounded subset of G has an infinite subset which is closed and discrete in G^+.*

If, in addition, G has the **cp***-property, then (i)–(iii) are equivalent to*

(iv) *every non-precompact sequence in G has an infinite subsequence which is closed and discrete in G^+.*

Proof Since G is Lindelöf, the group G^+ is also Lindelöf. Therefore G and G^+ are μ-spaces. Now Theorem 1.1 implies the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). If G has the **cp**-property, the equivalence (iii) $\Leftrightarrow$ (iv) follows from Lemma 2.4. □

4 Real Locally Convex Spaces and Respecting Properties

Following [32, 48], the *free locally convex space* $L(X)$ (the *free topological vector space* $\mathbb{V}(X)$) on a Tychonoff space X is a pair consisting of a locally convex space $L(X)$ (a topological vector space $\mathbb{V}(X)$, resp.) and a continuous map $i : X \to L(X)$ ($i :$

$X \to \mathbb{V}(X)$, resp.) such that every continuous map f from X to a locally convex space E (a topological vector space E, resp.) gives rise to a unique continuous linear operator $\bar{f} : L(X) \to E$ ($\bar{f} : \mathbb{V}(X) \to E$) with $f = \bar{f} \circ i$. The free locally convex space $L(X)$ and the free topological vector space $\mathbb{V}(X)$ always exist and are essentially unique. The set X forms a Hamel basis for $L(X)$ and $\mathbb{V}(X)$, and the map i is a topological embedding, see [32, 57, 64].

For a Tychonoff space X, let $C_k(X)$ be the space $C(X)$ endowed with the compact-open topology τ_k. Then the sets of the form

$$[K;\varepsilon] := \{f \in C(X) : |f(x)| < \varepsilon \ \forall x \in K\}, \text{ where } K \text{ is compact and } \varepsilon > 0,$$

form a base of open neighborhoods at zero in τ_k.

Denote by $M_c(X)$ the space of all real regular Borel measures on X with compact support. It is well-known that the dual space of $C_k(X)$ is $M_c(X)$, see [44]. Denote by τ_e the polar topology on $M_c(X)$ defined by the family of all equicontinuous pointwise bounded subsets of $C(X)$. We shall use the following deep result of Uspenskiĭ [64].

Theorem 4.1 ([64]) *Let X be a Tychonoff space and let μX be the Dieudonné completion of X. Then the completion $\overline{L(X)}$ of $L(X)$ is topologically isomorphic to $\big(M_c(\mu X), \tau_e\big)$. Consequently, $L(X)$ is complete if and only if X is Dieudonné complete and does not have infinite compact subsets.*

Corollary 4.2 *Let X be a Dieudonné complete space. Then the topology τ_e on $M_c(X)$ is compatible with the duality $(C_k(X), M_c(X))$.*

Proof It is well-known that $L(X)' = C(X)$, see [57]. Now Theorem 4.1 implies $(M_c(X), \tau_e)' = L(X)' = C(X)$. □

We need also the following fact, see §5.10 in [50].

Proposition 4.3 *Let X be a Tychonoff space and let A be an equicontinuous pointwise bounded subset of $C(X)$. Then the pointwise closure $\bar{A}$ of A is τ_k-compact and equicontinuous.*

Following [6], a Tychonoff space X is called *Ascoli* if every compact subset $\mathscr{K}$ of $C_k(X)$ is equicontinuous. Note that X is Ascoli if and only if the canonical map $L(X) \to C_k(C_k(X))$ is an embedding of locally convex spaces, see [26]. Below we give another characterization of Ascoli spaces. Denote by τ_k^M the compact-open topology on $M_c(X)$.

Proposition 4.4 *Let X be a Tychonoff space. Then:*

(i) $\tau_e \le \tau_k^M$ *on* $M_c(X)$*;*
(ii) $\tau_e = \tau_k^M$ *on* $M_c(X)$ *if and only if* X *is an Ascoli space.*

Proof (i) immediately follows from Proposition 4.3.

(ii) Assume that X is an Ascoli space. By (i) we have to show that $\tau_k^M \le \tau_e$. Take a standard τ_k^M-neighborhood of zero

$$[K;\varepsilon] = \{\nu \in M_c(X) : |\nu(f)| < \varepsilon \ \forall f \in K\},$$

where K is a compact subset of $C_k(X)$ and $\varepsilon > 0$. Since X is Ascoli, K is equicontinuous and clearly pointwise bounded. Therefore $[K;\varepsilon]$ is also a τ_e-neighborhood of zero. Thus $\tau_k^M \le \tau_e$.

Conversely, let $\tau_e = \tau_k^M$ on $M_c(X)$ and let K be a compact subset of $C_k(X)$. Then the polar K° of K is also a τ_e-neighborhood of zero in $M_c(X)$. So there is an absolutely convex, equicontinuous and pointwise bounded subset A of $C(X)$ such that $A^\circ \subseteq K^\circ$. By Proposition 4.3 we can assume that A is pointwise closed. Now the Bipolar theorem implies that $K \subseteq A^{\circ\circ} = A$. So K is equicontinuous. Thus X is an Ascoli space. □

Recall that a locally convex space E is called *semi-Montel* if every bounded subset of E is relatively compact, and E is a *Montel space* if it is a barrelled semi-Montel space. For real semi-Montel spaces, the following result strengthens Proposition 2.4 of [30].

Theorem 4.5 *A real semi-Montel space E respects all properties $\mathscr{P} \in \mathfrak{P}$.*

Proof Let $A \in \mathscr{P}(E^+)$. Then A is a functionally bounded subset of E^+. Hence A is bounded in E by Proposition 2.10. Therefore the closure $\overline{A}$ of A in E is compact and Proposition 3.2 applies. □

Recall (see Theorem 15.2.4 of [50]) that a locally convex space E is semi-reflexive if and only if every bounded subset A of E is relatively weakly compact. Although the first assertion of the next corollary is known, see Corollary 4.15 of [33], we give its simple and short proof.

Corollary 4.6 *Let E be a real semi-reflexive lcs. Then E has the Glicksberg property if and only if E is a semi-Montel space. In this case E respects all properties $\mathscr{P} \in \mathfrak{P}$.*

Proof Assume that E has the Glicksberg property. If A is a bounded subset of E, then the weak closure $\overline{A}^{\tau_w}$ of A is weakly compact, and hence $\overline{A}^{\tau_w}$ is compact also in E by the Glicksberg property and (1). Thus E is semi-Montel. The converse and the last assertions follow from Theorem 4.5. □

Taking into account that any reflexive locally convex space is barrelled, Corollary 4.6 immediately implies the main result of [58].

Corollary 4.7 ([58]) *Let E be a real reflexive lcs. Then E has the Glicksberg property if and only if E is a Montel space.*

An example of a semi-Montel but non-Montel space is given in Corollary 4.11 below.

Proposition 4.8 *Let X be a Dieudonné complete space and let $\mathscr{K}$ be a τ_e-closed subset of $M_c(X)$. Then the following assertions are equivalent:*

(i) *$\mathscr{K}$ is τ_e-compact;*

(ii) *$\mathscr{K}$ is τ_e-bounded;*
(iii) *there is a compact subset C of X and $\varepsilon > 0$ such that $\mathscr{K} \subseteq [C; \varepsilon]^\circ$.*

In particular, the space $(M_c(X), \tau_e)$ is a semi-Montel space.

Proof (i) $\Rightarrow$ (ii) is clear. Let us prove that (ii) $\Rightarrow$ (iii). Since X being a Dieudonné complete space is a μ-space, $C_k(X)$ is barrelled by the Nachbin–Shirota theorem. This fact and Corollary 4.2 imply that $\mathscr{K}$ is equicontinuous. So there is a compact subset C of X and $\varepsilon > 0$ such that $\mathscr{K} \subseteq [C; \varepsilon]^\circ$. To prove (iii) $\Rightarrow$ (i) we note first that $[C; \varepsilon]^\circ$ is equicontinuous and $\sigma\big(M_c(X), C(X)\big)$-compact by the Alaoglu theorem. Therefore $[C; \varepsilon]^\circ$ is compact in the precompact-open topology τ_{pc} on $M_c(X)$ by Proposition 3.9.8 of [42]. By Proposition 4.4, we have $\tau_e \le \tau_k^M \le \tau_{pc}$. Hence $[C; \varepsilon]^\circ$ is τ_e-compact. Thus $\mathscr{K}$ being closed is also τ_e-compact. □

Theorem 4.5 and Proposition 4.8 imply

Corollary 4.9 *If X is a Dieudonné complete space, then $(M_c(X), \tau_e)$ respects all properties $\mathscr{P} \in \mathfrak{P}$.*

Below we describe bounded subsets of $L(X)$ essentially generalizing Lemma 6.3 of [32]. For $\chi = a_1x_1 + \cdots + a_nx_n \in L(X)$ with distinct $x_1, \ldots, x_n \in X$ and nonzero $a_1, \ldots, a_n \in \mathbb{R}$, we set

$$\|\chi\| := |a_1| + \cdots + |a_n|, \quad \text{and} \quad \mathrm{supp}(\chi) := \{x_1, \ldots, x_n\},$$

and recall that

$$f(\chi) = a_1 f(x_1) + \cdots + a_n f(x_n), \quad \text{for every } f \in C(X) = L(X)'.$$

For $\{0\} \neq A \subseteq L(X)$, set $\mathrm{supp}(A) := \bigcup_{\chi \in A} \mathrm{supp}(\chi)$.

Proposition 4.10 *A nonzero subset A of $L(X)$ is bounded if and only if* $\mathrm{supp}(A)$ *has compact closure in the Dieudonné completion μX of X and $C_A := \sup\{\|\chi\| : \chi \in A\}$ is finite.*

Proof Observe that a subset B of an lcs E is bounded if and only if its closure $\overline{B}$ in the completion $\overline{E}$ of E is bounded. Now assume that A is bounded. By Theorem 4.1, we have $\overline{L(X)} = (M_c(\mu X), \tau_e)$ and, by Corollary 4.2, the topology τ_e is compatible with the duality $(C_k(\mu X), M_c(\mu X))$. As μX is a μ-space, the Nachbin–Shirota theorem implies that $C_k(\mu X)$ is barrelled. Therefore A is a bounded subset of $L(X)$ if and only if its completion $\overline{A}$ in $(M_c(\mu X), \tau_e)$ is equicontinuous and hence if and only if there is a compact subset K of μX and $\varepsilon > 0$ such that $A \subseteq [K; \varepsilon]^\circ \cap L(X)$. By the regularity of μX it is easy to see that

$$\chi = a_1x_1 + \cdots + a_nx_n \in [K; \varepsilon]^\circ \cap L(X),$$

where $x_1, \ldots, x_n \in X$ are distinct and $a_1, \ldots, a_n$ are nonzero, if and only if $x_1, \ldots, x_n \in K$ and $\|\chi\| = |a_1| + \cdots + |a_n| \le 1/\varepsilon$. Therefore, if A is bounded, then $\mathrm{supp}(A) \subseteq K$ and $C_A < 1/\varepsilon$.

Conversely, let $\overline{\mathrm{supp}(A)}$ be compact in μX and $C_A < \infty$. Set $B = \left[\overline{\mathrm{supp}(A)}; 1/C_A\right]^\circ$. Then B is equicontinuous and $\sigma\big(M_c(\mu X), C(\mu X)\big)$-compact by the Alaoglu theorem. Therefore B is compact in the precompact-open topology τ_{pc} on $M_c(\mu X)$ by Proposition 3.9.8 of [42]. Since $\tau_e \le \tau_k^M \le \tau_{pc}$ by Proposition 4.4, we obtain that B is a τ_e-compact subset of $M_c(\mu X)$. As $A \subseteq B \cap L(X)$, the above observation implies that A is a bounded subset of $L(X)$. □

Corollary 4.11 *Let X be a Dieudonné complete space whose compact subsets are finite. Then $L(X)$ is a complete semi-Montel space. If, in addition, X is non-discrete, then $L(X)$ is not Montel.*

Proof By Proposition 4.10, every bounded subset A of $L(X)$ is a bounded subset of a finite-dimensional subspace of $L(X)$. Therefore $\overline{A}$ is compact and hence $L(X)$ is a semi-Montel space. The space $L(X)$ is complete by Theorem 4.1. If additionally X is not discrete, then $L(X)$ is not barrelled by Theorem 6.4 of [32]. Thus $L(X)$ is not a Montel space. □

Now we prove the main result of this section.

Proof of Theorem 1.2. By Theorem 4.1, the space $L(X)$ embeds into $(M_c(\mu X), \tau_e)$. Now Proposition 2.7 and Corollary 4.9 imply that $L(X)$ respects all properties $\mathscr{P} \in \mathfrak{P}_0$.

Assume in addition that $L(X)$ is complete. Then X is Dieudonné complete and does not have infinite compact subsets by Theorem 4.1. Hence $L(X)$ is a semi-Montel space by Corollary 4.11. Thus $L(X)$ respects also functional boundedness by Theorem 4.5. □

Proof of Corollary 1.3. By the universal property of the free lcs $L(E)$, the identity map $id : E \to E$ extends to a continuous linear map $\overline{id}$ from $L(E)$ onto E. Since E is a subspace of $L(E)$ it follows that $\overline{id}$ is a quotient map. It remains to note that $L(E)$ respects all properties $\mathscr{P} \in \mathfrak{P}_0$ by Theorem 1.2. □

Following [48], an abelian topological group $A(X)$ is called the *free abelian topological group* over a Tychonoff space X if there is a continuous map $i : X \to A(X)$ such that $i(X)$ algebraically generates $A(X)$, and if $f : X \to G$ is a continuous map to an abelian topological group G, then there exists a continuous homomorphism $\bar{f} : A(X) \to G$ such that $f = \bar{f} \circ i$. The free abelian topological group $A(X)$ always exists and is essentially unique. The identity map $id_X : X \to X$ extends to a canonical homomorphism $id_{A(X)} : A(X) \to L(X)$. Note that $id_{A(X)}$ is an embedding of topological groups, see [61, 64].

It is known (see [33]) that the free abelian topological group $A(X)$ over a Tychonoff space X has the Glicksberg property. The next corollary generalizes this result.

Corollary 4.12 *Let X be a Tychonoff space. Then the free abelian topological group $A(X)$ over X is locally quasi-convex and respects all properties $\mathscr{P} \in \mathfrak{P}_0$.*

Proof Since $A(X)$ is a subgroup of $L(X)$, $A(X)$ is locally quasi-convex. The group $A(X)$ respects all properties $\mathscr{P} \in \mathfrak{P}_0$ by Proposition 2.7 and Theorem 1.2. □

Proof of Corollary 1.4. Note that G is a Tychonoff space. So, by the universal property of $A(G)$, the identity map $id : G \to G$ extends to a continuous homomorphism $\overline{id}$ from $A(G)$ onto G. Clearly, $\overline{id}$ is a quotient map. Now Corollary 4.12 finishes the proof. □

We do not know whether $L(X)$ and $A(X)$ respect also functional boundedness for every Tychonoff space X. We end this section with the following question.

Question 4.13 *Characterize Tychonoff spaces X for which $L(X)$ is a reflexive group.*

5 $\mathscr{P}$-Barrelledness, Reflexivity and Respecting Properties

Let E be a locally convex space. It is well-known that E is *barrelled* if and only if every $\sigma(E', E)$-bounded subset of E' is equicontinuous. Recall that E is called *c_0-barrelled* if every $\sigma(E', E)$-null sequence is equicontinuous. Analogously, if $\mathscr{P}$ is a (topological) property, we shall say that E is a *$\mathscr{P}$-barrelled space* if every $A \in \mathscr{P}\big(E', \sigma(E', E)\big)$ is equicontinuous.

Following [15], a *MAP* abelian group G is called *g-barrelled* if any $\sigma(\widehat{G}, G)$-compact subset of $\widehat{G}$ is equicontinuous. Every real barrelled lcs E is a g-barrelled group, but the converse does not hold in general by [15] (see also Example 5.6 below). Analogously, G is *sequentially barrelled* or *c_0-barrelled* if any $\sigma(\widehat{G}, G)$-convergent sequence of $\widehat{G}$ is equicontinuous, see [49]. More generally, for a property $\mathscr{P}$, we shall say that a *MAP* abelian group G is *$\mathscr{P}$-barrelled* if every $A \in \mathscr{P}\big(\widehat{G}, \sigma(\widehat{G}, G)\big)$ is equicontinuous. Clearly, every g-barrelled group is also c_0-barrelled. In the next proposition item (i) extends Proposition 1.12 of [15] and explains our use of the notion "c_0-barrelled group" also for c_0-barrelled spaces.

Proposition 5.1 *Let E be a real locally convex space and let $\mathscr{P} \in \mathfrak{P}_0$. Then:*

(i) *E is a $\mathscr{P}$-barrelled space if and only if E is a $\mathscr{P}$-barrelled group;*
(ii) *if E is a barrelled space, then E is a $\mathscr{P}$-barrelled group.*

Proof (i) Let E be a $\mathscr{P}$-barrelled space and $A \in \mathscr{P}\big(\widehat{E}, \sigma(\widehat{E}, E)\big)$. Recall that the canonical isomorphism $\psi : E' \to \widehat{E}$ is defined by $\psi(\chi) := e^{2\pi i \chi}$. Then $\psi^{-1}(A) \in \mathscr{P}\big(E', \sigma(E', E)\big)$ by (ii) of Proposition 2.11, and hence $\psi^{-1}(A)$ is equicontinuous. Now (vi) of Proposition 2.11 implies that A is equicontinuous. Thus E is a $\mathscr{P}$-barrelled group. Conversely, let E be a $\mathscr{P}$-barrelled group and $A \in \mathscr{P}\big(E', \sigma(E', E)\big)$. Then $\psi(A) \in \mathscr{P}\big(\widehat{E}, \sigma(\widehat{E}, E)\big)$ by (ii) of Proposition 2.11, and hence $\psi(A)$ is equicontinuous. Applying now (vi) of Proposition 2.11 we obtain that A is equicontinuous. Thus E is a $\mathscr{P}$-barrelled space.

(ii) follows from (i) and the fact that every weak-$*$ functionally bounded subset of E' is weak-$*$ bounded, see Proposition 2.10. □

Corollary 5.2 *For a Tychonoff space X the space $C_p(X)$ is a c_0-barrelled group if and only if $C_p(X)$ is a barrelled space.*

Proof Recall that $C_p(X)$ is a c_0-barrelled space if and only if it is barrelled (see [46]), and Proposition 5.1 applies. □

In what follows, for $\mathscr{P} = \mathscr{C}$ and $\mathscr{P} = \mathscr{S}$, we shall use the standard terminology of being g-barrelled or c_0-barrelled instead of being $\mathscr{C}$-barrelled or $\mathscr{S}$-barrelled, respectively.

Items (i) and (ii) of the next proposition generalizes (a)-(b') of Proposition 2.3 of [49].

Proposition 5.3 *Let G be a MAP abelian group. Then:*

(i) *if G is $\mathscr{P}$-barrelled for $\mathscr{P} \in \mathfrak{P}_0$, then $G^\wedge$ respects $\mathscr{P}$;*
(ii) *if $G^\wedge$ is $\mathscr{P}$-barrelled for $\mathscr{P} \in \mathfrak{P}_0$ and α_G is a topological embedding, then G respects $\mathscr{P}$;*
(iii) *if G is reflexive, then G is c_0-barrelled if and only if $G^\wedge$ has the Schur property;*
(iv) *if G is reflexive, then G is g-barrelled if and only if $G^\wedge$ has the Glicksberg property;*
(v) *if G is reflexive and $G^\wedge$ is angelic, then G is $\mathscr{C}\mathscr{C}$-barrelled if and only if $G^\wedge$ respects $\mathscr{C}\mathscr{C}$.*

Proof (i) Let $A \in \mathscr{P}\big(\widehat{G}, \sigma(\widehat{G}, G^{\wedge\wedge})\big)$. Then $A \in \mathscr{P}\big(\widehat{G}, \sigma(\widehat{G}, G)\big)$ as well, so A is equicontinuous. Hence there is a neighborhood U of zero in G such that $A \subseteq U^\triangleright$ and the set $U^\triangleright$ is a compact subset of $G^\wedge$, see Fact 2.1. So the identity map $\big(U^\triangleright, \tau_k|_{U^\triangleright}\big) \mapsto \big(U^\triangleright, \sigma(\widehat{G}, G^{\wedge\wedge})|_{U^\triangleright}\big)$ is a homeomorphism, where τ_k is the compact-open topology of the dual group $G^\wedge$. Therefore $A \in \mathscr{P}(G^\wedge)$, and hence $G^\wedge$ respects $\mathscr{P}$.

(ii) follows from (i) and Proposition 2.7.

(iii) Assume that $G^\wedge$ has the Schur property and S is a $\sigma(\widehat{G}, G)$-null sequence in $\widehat{G}$. By the reflexivity of G, S is also a $\sigma(\widehat{G}, G^{\wedge\wedge})$-null sequence. Hence S converges to zero in $G^\wedge$ by the Schur property. Therefore $S^\triangleright$ is a neighborhood of zero in $G^{\wedge\wedge}$. By the reflexivity of G, $S^\triangleleft = \alpha_G^{-1}(S^\triangleright)$ is a neighborhood of zero in G. Since $S \subseteq S^{\triangleleft\triangleright}$ we obtain that S is equicontinuous, see Fact 2.1. Thus G is c_0-barrelled. The converse assertion follows from (i).

The proof of (iv) is similar to the proof of (iii).

(v) Assume that $G^\wedge$ respects countable compactness. Let A be a $\sigma(\widehat{G}, G)$-countably compact subset of $\widehat{G}$. By the reflexivity of G, A is also $\sigma(\widehat{G}, G^{\wedge\wedge})$-countably compact. As $G^\wedge$ respects $\mathscr{C}\mathscr{C}$, A is countably compact in $G^\wedge$ and hence A is compact by the angelicity of $G^\wedge$. The rest of the proof repeats (iii) replacing S by A. □

Corollary 5.4 ([15]) *A locally compact abelian group G is g-barrelled.*

Proof Since G is reflexive and $G^\wedge$ is locally compact by the Pontryagin–van Kampen duality theorem, the assertion follows from Glicksberg's theorem and (iv) of Proposition 5.3. □

The condition of being reflexive in Proposition 5.3 is essential as the following example shows.

Example 5.5 The group $G := C_p(\mathbf{s}, 2)$ of all continuous maps from $\mathbf{s}$ to the discrete group $\mathbb{Z}(2)$ with the pointwise topology has the following properties:

(i) G is a countable non-reflexive precompact metrizable group;
(ii) every compact subset of $G^\wedge$ is equicontinuous;
(iii) $G^\wedge$ respect all the properties $\mathscr{P} \in \mathfrak{P}$;
(iv) G is not c_0-barrelled.

Proof (i) Observe that G is a dense proper subgroup of the compact metrizable group $\mathbb{Z}(2)^{\mathbb{N}}$, so G is metrizable. Being non-complete G is not reflexive, see [13]. To show that G is countable, for every $n \in \mathbb{N}$, set $F_n := \{e_1, \dots, e_n\}$ and $U_n := \mathbf{s}\setminus F_n$. If $f \in G$, there is an $n \in \mathbb{N}$ such that $f|_{U_n} = f(e_0) \in \mathbb{Z}(2)$. Therefore f is uniquely defined by its values on $F_n \cup \{e_0\}$. Thus G is countable.

(ii), (iii) By [1, 13], the group $G^\wedge$ is the countable direct sum $\bigoplus_{\mathbb{N}} \mathbb{Z}(2)$ endowed with the discrete topology. So every compact subset of $G^\wedge$ is finite and hence equicontinuous. Since $G^\wedge$ is discrete, it respects all properties $\mathscr{P} \in \mathfrak{P}$ (see Introduction).

(iv) For every $n \in \mathbb{N}$, set

$$\chi_n := \big(\underbrace{0, \dots, 0}_{2n}, 1, 1, 0, \dots\big)$$

Taking into account the description of continuous functions given in (i), we obtain

$$\chi_n(f) = \exp\big\{\pi i\big(f(e_{2n+1}) + f(e_{2n+2})\big)\big\} = 1$$

for all sufficiently large $n \in \mathbb{N}$. Thus $\chi_n \to 0$ in the pointwise topology on $\widehat{G}$. To show that G is not c_0-barrelled it suffices to prove that the sequence $S := \{\chi_n\}_{n\in\mathbb{N}}$ is not equicontinuous. For every $n \in \mathbb{N}$, define $f_n \in G$ by

$$f_n(e_n) := 1, \text{ and } f_n(e_m) = 0 \text{ if } m \neq n.$$

It is clear that $f_n \to 0$ in G. Since $\chi_n(f_{2n+1}) = \exp\{\pi i\} = -1$ we obtain that S is not equicontinuous. □

In Remark 16 of [15], it is stated that for a non-reflexive real Banach space E, the space $\big(E', \mu(E', E)\big)$ is a g-barrelled lcs which is not barrelled (where $\mu(E', E)$ is the Mackey topology on E'). So the converse in (ii) of Proposition 5.1 is not true in general. Below we propose an analogous example of a g-barrelled real lcs E which is not barrelled.

Example 5.6 Let (E, τ) be a real non-semi-reflexive lcs. Assume that E is complete and has the Glicksberg property (for example, $E = \ell_1^\kappa$ for some cardinal κ). Set $F := \big(E', \mu(E', E)\big)$, where $\mu(E', E)$ is the Mackey topology on E'. As E is not semi-reflexive, the space F is not barrelled by Theorem 11.4.1 of [44]. To show that F is a g-barrelled space take arbitrarily a compact subset K of $\big(F', \sigma(F', F)\big) = \big(E, \sigma(E, E')\big)$. Denote by $C := \overline{\text{acx}}(K)$ the closed absolutely convex hull of K. We claim that C is also $\sigma(E, E')$-compact. Indeed, the set $K \cup (-K)$ is τ-compact by the Glicksberg property of E. So C is τ-compact in E by Theorem 4.8.9 of [50]. Thus C is $\sigma(E, E')$-compact as well. Now the definition of the Mackey topology $\mu(E', E)$ and the claim imply that C and hence K are equicontinuous. Therefore F is a g-barrelled space.

Assume additionally that E is a Banach space. Then E is a reflexive group by [60], and Proposition 5.3(iv) implies that $E^\wedge$ is a g-barrelled group. Hence $(E^\wedge)^\wedge = E$ and the group $E^\wedge$ is a Mackey group, see [15] (or Proposition 6.3 below). By Fact 2.8, the dual space E' endowed with the compact-open topology τ_k is topologically isomorphic to $E^\wedge$. Therefore (E', τ_k) is a Mackey space such that $(E', \tau_k)' = E$. Thus $\tau_k = \mu(E', E)$ by the uniqueness of the Mackey space topology. □

Every abelian locally quasi-convex k_ω-group G is a Schwartz group by Corollary 5.5 of [4], and hence G has the Glicksberg property by [2]. Below we essentially generalize this result using a completely different method.

Theorem 5.7 *An abelian locally quasi-convex k_ω-group G respects all properties $\mathscr{P} \in \mathfrak{P}$.*

Proof First we prove the following claim.

Claim. If G is a metrizable abelian group, then $G^\wedge$ respects all properties $\mathscr{P} \in \mathfrak{P}$. Indeed, let $\bar{G}$ be the completion of G. Then $\bar{G}^\wedge = G^\wedge$ by [1, 13] and $\bar{G}$ is g-barrelled by Corollary 1.6 of [15]. Therefore $G^\wedge$ has the Glicksberg property by (i) of Proposition 5.3. By [1, 13], $G^\wedge$ is a k_ω-space, and hence $(G^\wedge)^+$ is a μ-space. Since every k_ω-group is complete by [43], the group $G^\wedge$ respects all properties $\mathscr{P} \in \mathfrak{P}$ by Theorem 1.2 of [25]. The claim is proved.

Note that $G^\wedge$ is metrizable, and hence $G^{\wedge\wedge}$ respects all properties $\mathscr{P} \in \mathfrak{P}$ by the claim. By 5.12 and 6.10 of [1], the canonical homomorphism α_G is an embedding of G into $G^{\wedge\wedge}$. Since G is complete by [43], $\alpha_G(G)$ is a closed subgroup of the k_ω-group $G^{\wedge\wedge}$. Therefore $\alpha_G(G)$ is C-embedded in $G^{\wedge\wedge}$ by [34, 3D.1]. Thus G respects all properties $\mathscr{P} \in \mathfrak{P}$ by Propositions 4.9 of [25] and Proposition 2.7. □

Following [35], a topological group G is called a *locally k_ω-group* if it has an open k_ω-subgroup.

Corollary 5.8 *A locally quasi-convex locally k_ω-group G respects all properties $\mathscr{P} \in \mathfrak{P}_0$.*

Proof The assertion follows from Theorem 5.7 and Proposition 2.7. □

In the next example we show that the condition of being locally quasi-convex cannot be dropped in Theorem 5.7. Denote by $\mathbb{V}(\mathbf{s})$ and $L(\mathbf{s})$ the free topological vector space and the free locally convex space over the convergent sequence $\mathbf{s}$, respectively.

Example 5.9 (i) $\mathbb{V}(\mathbf{s})$ is a non-locally quasi-convex *MAP* k_ω-group, so $\mathbb{V}(\mathbf{s})$ is a Schwartz group;

(ii) $\mathbb{V}(\mathbf{s})$ does not have the Schur property, and hence $\mathbb{V}(\mathbf{s})$ does not respect any $\mathscr{P} \in \mathfrak{P}$.

Proof (i) The space $\mathbb{V}(\mathbf{s})$ is a k_ω-group by Theorem 3.1 of [32], and it is not locally quasi-convex by Proposition 5.13 of [32] and the fact that a topological vector space E is a locally quasi-convex group if and only if E is locally convex (see Proposition 2.4 of [8]). By Proposition 5.1 of [32], the space $\mathbb{V}(\mathbf{s})$ is a *MAP* group. As a k_ω-group, $\mathbb{V}(\mathbf{s})$ is a Schwartz group by Corollary 5.5 of [4].

(ii) Since the spaces $\mathbb{V}(\mathbf{s})$ and $L(\mathbf{s})$ have the same dual space (see Proposition 5.10 of [32]), it is sufficient to find a sequence $\{z_k\}_{k\in\mathbb{N}}$ such that $\{z_k\}_{k\in\mathbb{N}}$ converges in $L(\mathbf{s})$ but it does not converge in $\mathbb{V}(\mathbf{s})$. For every $k \in \mathbb{N}$, set $d_k := 2^k$ and put

$$z_k := \frac{1}{d_{k+1} - d_k}\left(e_{d_k+1} + \cdots + e_{d_{k+1}}\right).$$

Then $z_k \to e_0$ in $L(\mathbf{s})$ because $L(\mathbf{s})$ is locally convex and since $e_n \to e_0$. On the other hand, $z_k \not\to e_0$ in $\mathbb{V}(\mathbf{s})$ by Corollary 3.4 of [32].

Since any k_ω-space is a μ-space, the last assertion follows from Proposition 3.3. □

Recall that a topological group X is said to have a *subgroup topology* or a *linear topology* if it has a base at the identity consisting of subgroups. In the next section we use the following proposition.

Proposition 5.10 (i) *If G is an abelian topological group with a subgroup topology, then G is a locally quasi-convex nuclear group. So G respects all properties $\mathcal{P} \in \mathfrak{P}_0$.*

(ii) *If (G, τ) is a locally quasi-convex abelian group of finite exponent, then (G, τ) and hence also $(G, \tau)^\wedge$ respect all the properties $\mathcal{P} \in \mathfrak{P}_0$.*

Proof (i) By Proposition 2.2 of [5], G embeds into a product of discrete groups. Therefore G is a locally quasi-convex nuclear group by Propositions 7.5 and 7.6 and Theorem 8.5 of [8]. Finally, the group G respects all properties $\mathcal{P} \in \mathfrak{P}_0$ by Corollary 4.7 of [25].

(ii) Propositions 2.1 of [5] implies that the topologies of the groups (G, τ) and $(G, \tau)^\wedge$ are subgroup topologies, and (i) applies. □

Being motivated by [49], we consider below a "compact" version of the Dunford–Pettis property for abelian topological groups. Let X and Y be topological spaces. A map $p : X \to Y$ is called *s-continuous* (*k-continuous*) if the restriction of p onto every convergent sequence (every compact subset, respectively) of X is continuous. Clearly, every k-continuous map is also s-continuous. The next lemma is straightforward.

Lemma 5.11 *For every abelian topological group G the evaluation map $\psi : G^\wedge \times G \to \mathbb{S}$, $\psi(\chi, g) := \chi(g)$, is k-continuous.*

Following [49], an abelian topological group G has the *sequential Bohr continuity property* (*s-BCP*, for short) if the map

$$\psi : \left(\widehat{G}, \sigma(\widehat{G}, G^{\wedge\wedge})\right) \times \left(G, \sigma(G, G^\wedge)\right) \to \mathbb{S}, \quad \psi(\chi, g) := \chi(g), \tag{1}$$

is s-continuous. We shall say that G has the *k-Bohr continuity property* (*k-BCP*, for short) if the map in (1) is k-continuous. Clearly, if G has the *k-BCP* then it has also the *s-BCP*. The next assertion is an analogue of Proposition 2.4 of [49] and generalizes (d) and (c) of this proposition.

Proposition 5.12 *For an abelian topological group G the following assertions hold:*

(i) *if G and $G^\wedge$ have the Glicksberg property, then G has the k-BCP;*
(ii) *if G has the Glicksberg property and is g-barrelled, then G has the k-BCP;*
(iii) *if G is metrizable and has the Glicksberg property, then G has the k-BCP;*
(iv) *if G is locally compact, then G has the k-BCP;*
(v) *if G is a locally quasi-convex almost metrizable Schwartz group, then G has the k-BCP;*
(vi) *if G is reflexive, then G has the k-BCP if and only if $G^\wedge$ has the k-BCP.*

Proof (i) immediately follows from Lemma 5.11, and (ii) follows from (i) and Proposition 5.3. The Claim of Theorem 5.7 and (i) imply (iii). (iv) follows from (i) and the Glicksberg theorem.

(v) By [2], G has the Glicksberg property. The dual group $G^\wedge$ of G is a locally quasi-convex locally k_ω-group by Proposition 7.1 of [35]. Therefore $G^\wedge$ has the Glicksberg property by Corollary 5.8. Now the assertion follows from (i).

(vi) follows from the definition of the evaluation map ψ and the reflexivity of G. □

Let G and H be abelian topological groups. Denote by $\mathrm{CHom}_p(G,H)$ the group of all continuous homomorphisms from G to H endowed with the pointwise topology.

Proposition 5.13 *Let G and H be MAP abelian groups with the Schur property. Then also the group $\mathrm{CHom}_p(G,H)$ has the Schur property.*

Proof Set $Z := \mathrm{CHom}_p(G,H)$. First we prove the following assertion.
Claim. For every $\Phi \in \widehat{H}$ and each $g \in G$, the homomorphism $T_{\Phi,g} : Z \to \mathbb{S}$ defined by

$$T_{\Phi,g}(\chi) := \Phi\big(\chi(g)\big), \quad \chi \in Z,$$

is continuous, i.e. $T_{\Phi,g} \in \widehat{Z}$.

Indeed, fix $\varepsilon > 0$. Choose an open neighborhood V of zero in H such that $|\Phi(h) - 1| < \varepsilon$ for every $h \in V$. Set $\Delta := \{-g, 0, g\}$. Now if

$$\chi \in [\Delta; V] := \{\chi \in Z : \chi(t) \in V \;\; \forall t \in \Delta\},$$

then $|T_{\Phi,g}(\chi) - 1| = |\Phi\big(\chi(g)\big) - 1| < \varepsilon$. Thus $T_{\Phi,g}$ is continuous and the claim is proved.

Now suppose for a contradiction that Z does not have the Schur property. Then there exists a $\sigma(Z,\widehat{Z})$-null sequence $\{\chi_n\}_{n\in\mathbb{N}}$ in Z which does not converge to zero in Z. So there is a finite subset F of G, an open neighborhood U of zero in H and $\varepsilon > 0$ such that

$$\chi_n \notin [F; U] := \{\chi \in Z : \chi(g) \in U \;\; \forall g \in F\}, \tag{2}$$

for infinitely many indices n. Passing to a subsequence if needed, we shall assume that (2) holds for all $n \in \mathbb{N}$. Since F is finite, we can assume also that $F = \{g\}$ for some $g \in G$. Therefore $\chi_n(g) \notin U$ for every $n \in \mathbb{N}$. But this means that the sequence $\chi_n(g)$ is not a null sequence in H. By the Schur property of H, there are $\Phi \in \widehat{H}$, an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ in $\mathbb{N}$, and $\varepsilon > 0$ such that

$$|\Phi\big(\chi_{n_k}(g)\big) - 1| = |T_{\Phi,g}(\chi_{n_k}) - 1| \geq \varepsilon, \quad k \in \mathbb{N}.$$

Hence $T_{\Phi,g}(\chi_{n_k}) \not\to 1$. Since $T_{\Phi,g} \in \widehat{Z}$ by Claim, we obtain that $\{\chi_n\}_{n\in\mathbb{N}}$ is not a $\sigma(Z, \widehat{Z})$-null sequence, a contradiction. □

6 Glicksberg Type Properties and the Property of Being a Mackey Group

Below we define some versions of respecting properties. For a *MAP* abelian group G and a topological property $\mathscr{P}$, we denote by $\mathscr{P}_{qc}(G)$ the set of all *quasi-convex* subsets of G with $\mathscr{P}$. Recall that a locally convex space E has the *Grothendieck* property if every weak-$*$ convergent sequence in the dual space E' is also weakly convergent, i.e., $\mathscr{S}\big(E', \sigma(E', E)\big) = \mathscr{S}\big(E', \sigma(E', (E'_\beta)')\big)$, where E'_β is the strong dual of E. Analogously, we say that E has the *$\mathscr{P}$-Grothendieck* property if $\mathscr{P}\big(E', \sigma(E', E)\big) = \mathscr{P}\big(E', \sigma(E', (E'_\beta)')\big)$.

Definition 6.1 Let (G, τ) be a *MAP* abelian group and $\mathscr{P}$ a topological property. We say that

(i) (G, τ) *respects* $\mathscr{P}_{qc}$ if $\mathscr{P}_{qc}(G) = \mathscr{P}_{qc}\big(G, \sigma(G, \widehat{G})\big)$;
(ii) $(G, \tau)^\wedge$ *respects* $\mathscr{P}^*$ if $\mathscr{P}(G^\wedge) = \mathscr{P}\big(\widehat{G}, \sigma(\widehat{G}, G)\big)$;
(iii) $(G, \tau)^\wedge$ *respects* $\mathscr{P}^*_{qc}$ if $\mathscr{P}_{qc}(G^\wedge) = \mathscr{P}_{qc}\big(\widehat{G}, \sigma(\widehat{G}, G)\big)$;
(iv) (G, τ) has the *$\mathscr{P}$-Pontryagin–Grothendieck property* if

$$\mathscr{P}\big(\widehat{G}, \sigma(\widehat{G}, G)\big) = \mathscr{P}\big(\widehat{G}, \sigma(\widehat{G}, G^{\wedge\wedge})\big).$$

□

In the case $\mathscr{P}$ is the property $\mathscr{C}$ of being a compact space and a *MAP* abelian group (G, τ) $(G^\wedge)$ respects $\mathscr{P}_{qc}$ ($\mathscr{P}^*$ or $\mathscr{P}^*_{qc}$, respectively), we shall say that the group G $(G^\wedge)$ has the *qc-Glicksberg property* (the *weak-$*$ Glicksberg property* or the *weak-$*$ qc-Glicksberg property*, respectively). Clearly, if a *MAP* abelian group (G, τ) has the Glicksberg property, then it also has the *qc*-Glicksberg property, and if $(G, \tau)^\wedge$ has the weak-$*$ Glicksberg property, then it has also the weak-$*$ *qc*-Glicksberg property.

Remark 6.1 Note that, for a *MAP* abelian group G, if $G^\wedge$ has the weak-$*$ Glicksberg property, then it has also the Glicksberg property. But the converse is not true in general. Indeed, let G be a countable dense subgroup of an infinite compact metrizable group X. Then $G^\wedge = X^\wedge$ by [1, 13]. Hence $G^\wedge$ is a discrete countably infinite group, so $G^\wedge$ has the Glicksberg property. On the other hand, the group $H := \big(\widehat{G}, \sigma(\widehat{G}, G)\big)$ is a precompact metrizable group. So H contains infinite compact subsets which are not compact in $G^\wedge$. □

Proposition 6.2 *Let E be a real lcs and $\mathscr{P} \in \mathfrak{P}_0$. If E has the $\mathscr{P}$-Grothendieck property, then E has the $\mathscr{P}$-Pontryagin–Grothendieck property. But the converse is not true in general.*

Proof Let $A \in \mathscr{P}\big(\widehat{E}, \sigma(\widehat{E}, E)\big)$. Then, by (ii) of Proposition 2.11, the set $B := \psi^{-1}(A)$ belongs to $\mathscr{P}\big(E', \sigma(E', E)\big)$. So $B \in \mathscr{P}\big(E', \sigma(E', (E'_\beta)')\big)$. Since the compact-open topology τ_k on E' is weaker than the strong topology, we obtain that $B \in \mathscr{P}\big(E', \sigma(E', (E', \tau_k)')\big)$. Finally, (v) of Proposition 2.11 implies that $A = \psi(B) \in \mathscr{P}\big(\widehat{E}, \sigma(\widehat{E}, E^{\wedge\wedge})\big)$.

To prove the last assertion, let E be a separable non-reflexive Banach space. Being a reflexive group [60], E trivially has the $\mathscr{S}$-Pontryagin–Grothendieck property. However, a separable Banach space with the Grothendieck property must be reflexive, so E does not have the Grothendieck property. □

Below we show that the weak-$*$ qc-Glicksberg property is dually connected with the property being a Mackey group. Let us recall the definition of Mackey spaces and Mackey groups.

Let (E, τ) be a locally convex space. A locally convex vector topology ν on E is called *compatible with* τ if the spaces (E, τ) and (E, ν) have the same topological dual space. The famous Mackey–Arens theorem states that there is a finest locally convex vector space topology μ on E compatible with τ. Moreover, the topology μ is the topology of uniform convergence on absolutely convex weakly-$*$ compact subsets of the topological dual space E' of E. So E is a Mackey space if and only if every absolutely convex $\sigma(E', E)$-compact subset of E' is equicontinuous.

Two topologies τ and ν on an abelian group G are said to be *compatible* if $\widehat{(G, \tau)} = \widehat{(G, \nu)}$. Being motivated by the classical Mackey–Arens theorem the following notion was introduced and studied in [15]: a locally quasi-convex abelian group (G, τ) is called a *Mackey group* if for every compatible locally quasi-convex group topology ν on G it follows that $\nu \le \tau$.

Proposition 6.3 ([15, Theorem 4.2(1)]) (i) *Let (G, τ) be a locally quasi-convex abelian group. If every $A \in \mathscr{C}_{qc}\big(\widehat{G}, \sigma(\widehat{G}, G)\big)$ is equicontinuous (for example, G is g-barrelled), then G is a Mackey group.*

(ii) *If a real lcs E is a g-barrelled group, then E is a Mackey space.*

Proof (i) Let ν be a locally quasi-convex group topology on G compatible with τ and let U be a quasi-convex ν-neighborhood of zero. Then $U^\triangleright$ is $\sigma(\widehat{G}, G)$-compact and quasi-convex by Fact 2.1. So $U^\triangleright$ is equicontinuous (with respect to the original topology τ). Fact 2.1 implies that $U = U^{\triangleright\triangleleft}$ is also a τ-neighborhood of zero. Thus $\nu \le \tau$ and hence G is a Mackey group.

(ii) Proposition 5.1 implies that E is a g-barrelled space, i.e. every $\sigma(E', E)$-compact subset of E' is equicontinuous. Thus E is a Mackey space. □

Remark 6.2 We proved in [28] that the free lcs $L(X)$ is a Mackey group if and only if it is a Mackey space if and only if X is discrete. Therefore, by Proposition 6.3, $L(X)$ is a g-barrelled group if and only if X is discrete. □

Below we obtain another sufficient condition of being a Mackey group.

Proposition 6.4 *Let (G, τ) be a locally quasi-convex group such that the canonical homomorphism α_G is continuous. If $(G, \tau)^\wedge$ has the weak-$*$ qc-Glicksberg property, then (G, τ) is a Mackey group. Consequently, if a reflexive abelian group (G, τ) is such that $(G, \tau)^\wedge$ has the qc-Glicksberg property (in particular, the Glicksberg property), then (G, τ) is a Mackey group.*

Proof Let ν be a locally quasi-convex topology on G compatible with τ and let U be a closed quasi-convex ν-neighborhood of zero. Fact 2.1 implies that the quasi-convex subset $K := U^\triangleright$ of $\widehat{G}$ is $\sigma(\widehat{G}, G)$-compact, and hence K is a compact subset of $G^\wedge$ by the weak-$*$ *qc*-Glicksberg property. Note that, by definition, $K^\triangleright$ is a neighborhood of zero in $G^{\wedge\wedge}$. As α_G is continuous, $U = K^\triangleleft = \alpha_G^{-1}(K^\triangleright)$ is a τ-neighborhood of zero in G. Hence $\nu \le \tau$. Thus (G, τ) is a Mackey group.

The last assertion follows from the fact that the weak-$*$ *qc*-Glicksberg property coincides with the *qc*-Glicksberg property for any reflexive group. □

Since every *LCA* group is reflexive and has the Glicksberg property, Proposition 6.4 implies

Corollary 6.5 ([15]) *Every LCA group is a Mackey group.*

Remark 6.3 In the last assertion of Proposition 6.4 the reflexivity of G is essential. Indeed, let G be a proper dense subgroup of a compact metrizable abelian group X. Then $G^\wedge = X^\wedge$ (see [1, 13]), and hence the discrete group $G^\wedge$ has the Glicksberg property. Denote by $\mathfrak{p}_0$ the product topology on the group $c_0(\mathbb{S}) := \{(z_n) \in \mathbb{S}^\mathbb{N} : z_n \to 1\}$ induced from $\mathbb{S}^\mathbb{N}$. Then, by [21, Theorem 1], there is a locally quasi-convex topology $\mathfrak{u}_0$ on $c_0(\mathbb{S})$ compatible with $\mathfrak{p}_0$ such that $\mathfrak{p}_0 < \mathfrak{u}_0$. Thus the group $G := \big(c_0(\mathbb{S}), \mathfrak{p}_0\big)$ is a precompact arc-connected metrizable group such that $G^\wedge$ has the Glicksberg property, but G is *not* a Mackey group. Consequently, $G^\wedge$ does not have the weak-$*$ *qc*-Glicksberg property. □

Every real barrelled locally convex space is a Mackey group by [15] (this also follows from Propositions 5.1 and 6.3). Since every real reflexive locally convex space E is barrelled by [44, Proposition 11.4.2], we obtain that E is a Mackey group. This result motivates the following problem.

Problem 6.6 Characterize reflexive abelian groups which are Mackey groups.

Not every reflexive group is Mackey, see [14]. Moreover, there exists a reflexive group which does not admit a Mackey group topology, see [3, 27]. However, if a reflexive group G is of finite exponent, it is a Mackey group as the following theorem shows.

Theorem 6.7 *A reflexive abelian group (G, τ) of finite exponent is a Mackey group.*

Proof Since (G, τ) is locally quasi-convex, Proposition 5.10 implies that $(G, \tau)^\wedge$ has the Glicksberg property. Thus (G, τ) is a Mackey group by Proposition 6.4. □

Corollary 6.8 *Let X be a zero-dimensional realcompact k-space and let $\mathbb{F}$ be a finite abelian group. Then $C_k(X, \mathbb{F})$ is a Mackey group.*

Proof The group $C_k(X, \mathbb{F})$ is reflexive by the main result of [55], and Theorem 6.7 applies. □

Remark 6.4 Any metrizable and precompact abelian group of finite exponent is a Mackey group, see [10, Example 4.4]. If G is a metrizable reflexive group, then G must be complete by [13, Corollary 2]. So there are non-reflexive Mackey groups of finite exponent. Moreover, since the group G in Example 5.5 is not c_0-barrelled, we obtain that there are Mackey groups which are not g-barrelled. □

Recall that an lcs E has the compact convex property (*ccp*) if the absolutely convex hull of any compact subset of E is relatively compact in E. Following [11], we say that a locally quasi-convex abelian group G has a *quasi-convex compactness property* (*qcp*) if the quasi-convex hull of any compact subset of G is relatively compact in G. Clearly, every real lcs E with (*ccp*) has also (*qcp*).

Proposition 6.9 *Let G be a locally quasi-convex abelian group. If G is g-barrelled, then the group $\big(\widehat{G}, \sigma(\widehat{G}, G)\big)$ has (qcp), but the converse is not true in general.*

Proof Let K be a compact subset of $H := \big(\widehat{G}, \sigma(\widehat{G}, G)\big)$. Then K is equicontinuous by the g-barrelledness of G. Now Fact 2.1 implies that $K^{\triangleleft\triangleright}$ is $\sigma(\widehat{G}, G)$-compact and quasi-convex. Thus H has (*qcp*). For the last assertion, see Remark 15 of [15]. □

Remark 6.5 Let (G, τ) be a *MAP* abelian group which respects $\mathscr{P} \in \mathfrak{P}$ and let ν be a group topology on G compatible with τ. If $\tau^+ \le \nu \le \tau$, then clearly (G, ν) respects $\mathscr{P}$ as well. But if $\nu > \tau$ it may happen that (G, ν) does not respect $\mathscr{P}$. Indeed, let (G, ν) be a real Banach space without the Schur property and let $\tau = \sigma(E, E')$. Then the space (G, τ) has the Schur property by Proposition 2.9. For a more non-trivial example, consider the free lcs $L(\mathbf{s})$ which respects all properties $\mathscr{P} \in \mathfrak{P}_0$ by Theorem 1.2, however the space $\big(L(\mathbf{s}), \mu(L(\mathbf{s}), C(\mathbf{s}))\big)$ does not have the Schur property, see Step 3 of the proof of Theorem 2.4 in [29]. □

In this paper we considered respecting properties related to compact-type properties. Of course we can consider other properties as separability or Lindelöfness. It has also sense to consider the property of being a subgroup. More precisely, let (G, τ) be a *MAP* group and let H be a closed subgroup G^+. One can ask: When the topologies $\tau^+|_H$ and $\tau|_H$ coincide? If they coincide it is clear that H must be a precompact subgroup of G. Below we give a partial answer to this question.

Proposition 6.10 *Let (G, τ) be an lqc group and let H be a closed subgroup G^+ such that $(H, \tau^+|_H)$ is a Mackey group. Then the following assertions are equivalent:*

(i) $\tau^+|_H = \tau|_H$;
(ii) *H is dually closed and dually embedded in G.*

Proof First we recall the following well known fact: any closed subgroup of a precompact group is dually closed and dually embedded.

(i) ⇒ (ii) H is dually closed in G since H is dually closed in G^+. To show that H is also dually embedded in G, fix arbitrarily a $\chi \in \widehat{(H, \tau|_H)}$. Then, by (i), $\chi \in \widehat{(H, \tau^+|_H)}$ and the above mentioned fact implies that there is $\eta \in \widehat{G^+} = \widehat{G}$ such that $\eta|_H = \chi$. Thus H is dually embedded in G.

(ii) ⇒ (i) Lemma 2.3 of [25] states that H^+ is a (closed) subgroup of G^+, i.e. $\tau^+|_H = (\tau|_H)^+$. Since $\tau^+|_H$ is Mackey and $\tau|_H$ is locally quasi-convex and compatible with $(\tau|_H)^+$ we must have $\tau^+|_H = \tau|_H$. □

Proposition 6.10 and Corollary 6.5 applied to $H = G$ immediately imply

Corollary 6.11 ([15]) *Let G be an lqc group such that G^+ is compact. Then G is a compact group.*

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Forty Years of Fuzzy Metrics

In Honour of Manuel López-Pellicer

Valentín Gregori and Almanzor Sapena

Abstract Kramosil and Michalek gave in 1975 a concept of fuzzy metric M on a set X which extends to the fuzzy setting the concept of probabilistic metric space introduced by K. Menger. After, George and Veeramani (Fuzzy Sets Syst 64: 395–399, 1994) modified the previous concept and gave a new definition of fuzzy metric. In both cases the fuzzy metric M induces a topology τ_M on X which is metrizable. In this paper we survey some results relative to both concepts. In particular, we focus our attention in the completion of fuzzy metrics in the sense of George and Veeramani, since there is a significative difference with respect to the classical metric theory (in fact, there are fuzzy metric spaces, in this sense, which are not completable), and also in fixed point theory in both senses because it is a high activity area.

Keywords Fuzzy metric · Fuzzy ultrametric · Fuzzy metric completion · Fixed point

1 Introduction

In 1965, Zadeh [48] introduced the concept of fuzzy set which transformed and stimulated almost all branches of Science and Engineering including Mathematics. A fuzzy set can be defined by assigning to each element of a set X a value in [0, 1], representing its grade of membership in the fuzzy set. Mathematically, a fuzzy set A of X is a mapping $A : X \to [0, 1]$.

The problem of finding an appropriate concept of fuzzy metric has been investigated by many authors in different ways. Here we make a brief survey of fuzzy metrics defined by means of t-norms which we describe in the following.

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Probabilistic metric spaces (PM-spaces) where introduced by K. Menger [25] who generalized the theory of metric spaces. In the Menger's theory the concept of distance is considered to be statistical or probabilistic, i.e. he proposed to associate a distribution function F_{xy}, with every pair of elements x, y instead of associating a number, and for any real number t, interpreted $F_{xy}(t)$ as the probability that the distance from x to y be less than t. Recall [40] that a distribution function F is a non-decreasing left continuous mapping from the set of real numbers $\mathbb{R}$ into [0, 1] so that $\inf_{t \in \mathbb{R}} F(t) = 0$ and $\sup_{t \in \mathbb{R}} F(t) = 1$. Now, the concept of fuzziness found place in PM-spaces because, in some cases, uncertainty in the distances between two points is due to fuzziness rather than randomness. With this idea, in 1975, Kramosil and Michalek [24] extended the theory of PM-spaces to the fuzzy setting defining a concept of fuzzy metric. In this paper, following a modern terminology, by a KM-fuzzy metric (Definition 2.2) we mean a fuzzy metric in the sense of Kramosil and Michalek, but defined with the help of a continuous t-norm in the way that it was introduced in [7]. Later, [4] George and Veeramani introduced and studied an interesting notion of fuzzy metric which we deal with here, modifying the concept of KM-fuzzy metric. Many concepts given for fuzzy metrics have been, obviously, extended to KM-fuzzy metrics and *vice versa*.

An interesting aspect of this type of fuzzy metrics is that it includes in its definition a parameter t. This feature has been successfully used in Engineering applications such as color image filtering [2] and perceptual color differences [19]. From the mathematical point of view it allows to introduce novel (fuzzy) metric concepts that only have natural sense in this fuzzy metric context. This is the case of several concepts of Cauchyness and convergence, related to sequences, appeared in the literature (see [22]).

If M is a (KM-)fuzzy metric on X, a topology τ_M on X is deduced from M. In [5, 9] the authors showed that the class of topological spaces which are *fuzzy metrizable* agrees with the class of metrizable spaces, and then some classical theorems on metric completeness have been adapted to fuzzy setting. For instance, precompactness [9], uniform continuity [5, 14], Ascoli-Arzelà theorem [5], Hausdorff fuzzy metric on the set of nonempty compact sets of (X, τ_M) [33], fuzzy ultrametrics [17, 30, 38], fuzzy quasi-metrics [12, 37] and its bicompletion [15], a domain-theoretic approach to fuzzy metric space [32], fuzzy uniform structures and quasi-uniformities induced by fuzzy (quasi-) metric [23, 36], ... Also, fuzzy metrics have been extended to the intuitionistic field [1, 16, 31], and some approaches to fuzzy partial metrics have been given [47]. Nevertheless, the theory of fuzzy metric completion is, in this context, very different from the classical theory of (probabilistic) metric completion. In fact, there are fuzzy metric spaces which are not completable [10, 20].

In this paper, by the last reason, we will focus our attention on completion and also on fixed point theory, since it is a high activity area [26, 27, 45].

2 KM-Fuzzy Metric Spaces

We begin recalling the concept of t-norm.

Definition 2.1 ([40]) A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if it satisfies the following conditions:

(i) $*$ is associative and commutative
(ii) $a * 1 = a$ for every $a \in [0, 1]$
(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for $a, b, c, d \in [0, 1]$

If, in addition, $*$ is continuous, then $*$ is called a continuous t-norm.

The three most commonly used continuous t-norms in fuzzy setting are the minimum, denoted by $\wedge$, the usual product, denoted by $\cdot$ and the Lukasievicz t-norm, denoted by $\mathfrak{L}$ ($x\mathfrak{L}y = \max\{0, x + y - 1\}$). They satisfy the following inequalities:

$$x\mathfrak{L}y \leq x \cdot y \leq x \wedge y \text{ and } x * y \leq x \wedge y$$

for each t-norm $*$. The t-norm $*$ is called integral (positive) if $a * b > 0$ whenever $a \neq 0, b \neq 0$. Notice that $\wedge$ and $\cdot$ are integral but $\mathfrak{L}$ is not.

The concept of fuzzy metric space rewritten by Grabiec is the following.

Definition 2.2 [7, 24] The term $(X, M, *)$ is a KM-fuzzy metric space if X is a non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, +\infty[$ satisfying for all $x, y, z \in X,\ t, s > 0$ the following axioms:

(KM1) $M(x, y, 0) = 0$
(KM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$
(KM3) $M(x, y, t) = M(y, x, t)$
(KM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
(KM5) The function $M_{xy} : [0, +\infty[\to [0, 1]$ defined by $M_{xy}(t) = M(x, y, t)$ for all $t \geq 0$, is left continuous

If $(X, M, *)$ is a KM-fuzzy metric space we say that $(M, *)$ (or simply M) is a KM-fuzzy metric on X.

From the above axioms one can show that M_{xy} is a non-decreasing function.

Remark 2.1 In the original concept of fuzzy metric due to Kramosil and Michalek [24] it is included the axiom

$$(KM6)\ \lim_{t \to \infty} M(x, y, t) = 1$$

and in this way any fuzzy metric M defined on X is equivalent to a Menger space ([24] Corollary of Theorem 1) defining $M(x, y, t) = F_{xy}(t)$ for all $x, y \in X,\ t \in [0, +\infty[$. Then, by this formula, since $*$ is continuous, we can deduce from M a topology τ_M in an analogous way to that in Menger spaces. Moreover, if we extend concepts and results relative to completion in Menger spaces to the fuzzy setting we obtain, imitating the Sherwood's proof [34], that every fuzzy metric space in the sense of Kramosil and Michalek has a completion which is unique up to isometry.

Now, mainly because (KM6) has been removed, in this case a KM-fuzzy metric cannot be regarded as a Menger space. Nevertheless, in the same way as in the Menger spaces theory, a topology τ_M deduced from M is defined on X, and the concepts (Definition 4.1) and results relative to completeness in PM-spaces can be translated to this fuzzy theory. In particular, KM-fuzzy metrics are completable.

3 Fuzzy Metric Spaces (in the Sense of George and Veeramani)

In 1994, George and Veeramani introduced the notion of fuzzy metric space by modifying the modern concept of fuzzy metric due to Kramosil and Michalek (Remark 2.1) which we will adopt from now on.

Definition 3.1 ([4]) A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X\times]0, +\infty[$ satisfying the following conditions, for all $x, y, z \in X,\ s, t > 0$:

(GV1) $M(x, y, t) > 0$
(GV2) $M(x, y, t) = 1$ if and only if $x = y$
(GV3) $M(x, y, t) = M(y, x, t)$
(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
(GV5) M_{xy} $:]0, +\infty[\to]0, 1]$ is continuous where $M_{xy}(t) = M(x, y, t)$

The axiom (GV1) is justified by the authors because in the same way that a classical metric does not take the value ∞ then M cannot take the value 0. The axiom (GV2) is equivalent to the following:

$$M(x, x, t) = 1 \text{ for all } x \in X,\ \ t > 0, \text{ and } M(x, y, t) < 1 \text{ for all } x \neq y,\ \ t > 0$$

The axiom (GV2) gives the idea that only when $x = y$ the degree of nearness of x and y is *perfect*, or simply 1, and then $M(x, x, t) = 1$ for each $x \in X$ and for each $t > 0$. In this manner the values 0 and ∞ in the classical theory of metric spaces are identified with 1 and 0, respectively, in this fuzzy theory. Axioms (GV3)-(GV4) coincide with (KM3)-(KM4), respectively. In (GV5) the authors assume that the variable t behaves nicely, that is, they assume that for fixed x and y, the function $t \to M(x, y, t)$ is continuous. Finally, there is not any imposition for M as $t \to \infty$.

Notice that if $(X, M, *)$ is a fuzzy metric space and $\diamond$ is a continuous t-norm such that $a * b \geq a \diamond b$ for each $a, b \in [0, 1]$ (briefly $* \geq \diamond$), then $(X, M, \diamond)$ is a fuzzy metric space but the converse, in general, is false. Consequently, if $(X, M, \wedge)$ is a fuzzy metric space then $(X, M, *)$ is a fuzzy metric space for each continuous t-norm $*$.

If M is a fuzzy metric on X then we can consider that M is a KM-fuzzy metric on X, defining $M(x, y, 0) = 0$ for all $x, y \in X$.

Definition 3.2 A fuzzy metric space $(X, M, *)$ is said to be *stationary* ([11]) if M does not depend on t, i.e. if for each $x, y \in X$, the function M_{xy} is constant.

If $(X, M, *)$ is a stationary fuzzy metric space, we will simply write $M(x, y)$ instead of $M(x, y, t)$.

Definition 3.3 [39] Let $(X, M, *)$ be a fuzzy metric space. The fuzzy metric M (or the fuzzy metric space $(X, M, *)$) is said to be *strong* if it satisfies for each $x, y, z \in X$ and each $t > 0$

$$M(x, z, t) \geq M(x, y, t) * M(y, z, t) \qquad \text{(GV4}')$$

If $(M, \wedge)$ is a strong fuzzy metric then it is called fuzzy ultrametric [38] and so it satisfies $M(x, z, t) \geq M(x, y, t) \wedge M(y, z, t)$.

Example 3.1 (a) Let (X, d) be a metric space. Let M_d be the fuzzy set defined on $X \times X \times]0, +\infty[$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(M_d, \cdot)$ is a fuzzy metric on X called standard fuzzy metric [4]. Further, $(M_d, \wedge)$ is also a fuzzy metric on X. (Notice that M_d is defined explicitly by means of a metric).

(b) Let $X = \mathbb{R}^+$ and $M(x, y, t) = \dfrac{\min\{x, y\} + t}{\max\{x, y\} + t}$
Then $(M, \cdot)$ is a fuzzy metric on X [4] which is non deduced explicitly from a metric.

(c) Let $X = \mathbb{R}^+$ and $M(x, y) = \dfrac{\min\{x, y\}}{\max\{x, y\}}$
Then $(M, \cdot)$ is a stationary fuzzy metric on X [4].

A collection of examples of fuzzy metrics can be found in [18].

4 Metrizability of Fuzzy Metric Spaces

The results of Subsection 4.1 where established for fuzzy metrics but they are also valid for KM-fuzzy metrics.

4.1 Topology in a Fuzzy Metric Space

George and Veeramani proved in [4] that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \varepsilon, t) : x \in X, \varepsilon \in]0, 1[, t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ for all

$x \in X$, $\varepsilon \in]0, 1[$ and $t > 0$. The set $B_M(x, \varepsilon, t)$ is called the open ball centered at x with radius r and parameter t.

A sequence $\{x_n\}$ in X converges to x in τ_M if and only if $\lim_n M(x_n, x, t) = 1$, for all $t > 0$.

A topological space (X, τ) is called fuzzy metrizable if there exists a fuzzy metric M on X such that $\tau_M = \tau$. In this case it is said that M is compatible with τ.

Remark 4.1 If (X, d) is a metric space, then the topology generated by d coincides with the topology τ_{M_d} generated by the fuzzy metric M_d ([4]). Consequently, every metrizable topological space is fuzzy metrizable.

4.2 Uniformity in Fuzzy Metric Spaces

The results of this section where given in [9].

Let $(X, M, *)$ be a fuzzy metric space. For each $n \in \mathbb{N}$ define:

$$U_n = \{(x, y) \in X \times X : M(x, y, \frac{1}{n}) > 1 - \frac{1}{n}\}$$

The (countable) family $\{U_n : n \in \mathbb{N}\}$ is a base for a uniformity $\mathscr{U}_M$ on X such that the topology induced by $\mathscr{U}_M$ agrees with the topology induced by the fuzzy metric M. The uniformity $\mathscr{U}_M$ will be called the uniformity deduced from M or generated by M. Applying the Kelley's metrization lemma the following result holds.

Lemma 4.1 (Metrizability of fuzzy metric spaces) *Let $(X, M, *)$ be a fuzzy metric space. Then, (X, τ_M) is a metrizable topological space.*

Taking into account that every metrizable topological space is fuzzy metrizable (Remark 4.1), we obtain the following corollary.

Corollary 4.1 *A topological space is metrizable if and only if it admits a compatible fuzzy metric (i.e. it is fuzzy metrizable).*

So, some results which had been stated at the beginning of this theory are, really, consequences of the above corollary. For instance: Every separable fuzzy metric space is second countable [6].

The definition of Cauchy sequence in a fuzzy metric space is similar to the given in a Menger space.

Definition 4.1 [4, 42] A sequence $\{x_n\}$ in a $(KM\text{-})$fuzzy metric space $(X, M, *)$ is said to be *M-Cauchy*, or simply Cauchy, if for each $\epsilon \in]0, 1[$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$ or, equivalently, $\lim_{n,m} M(x_n, x_m, t) = 1$ for all $t > 0$. X is said to be *complete* if every Cauchy sequence in X is convergent with respect to τ_M. In such a case M is also said to be complete.

The proof of the next proposition is easy. Now, it is interesting mainly because, in some cases, it is assumed that appropriate concepts in the fuzzy setting should satisfy propositions as the next one which shows the relationship between a metric concept and its corresponding fuzzy metric concept when considering its associated standard fuzzy metric.

Proposition 4.1 *$\{x_n\}$ is a Cauchy sequence in (X, d) if and only if it is a Cauchy sequence in $(X, M_d, *)$. Further, $(X, M_d, \cdot)$ is complete if and only if (X, d) is complete.*

Let us recall that a metrizable topological space (X, τ) is said to be completely metrizable if it admits a complete metric.

Theorem 4.1 *Let $(X, M, *)$ be a complete fuzzy metric space. Then, (X, τ_M) is completely metrizable.*

Jointly this theorem and the above proposition we obtain the next corollary.

Corollary 4.2 *A topological space is completely metrizable if and only it admits a compatible complete fuzzy metric.*

Then again some results in metric spaces can be stated in fuzzy metric spaces without any additional proof. For instance: Every complete fuzzy metric space is Baire [4].

5 Completion of Fuzzy Metric Spaces

A topic which differs essentially with the classical theory of metric spaces is completion. Indeed, Gregori and Romaguera [10] proved that there exist fuzzy metric spaces which are not completable.

Definition 5.1 Let $(X, M, *)$ and $(Y, N, \star)$ be two fuzzy metric spaces. A mapping f from X to Y is called an *isometry* if for each $x, y \in X$ and each $t > 0$, $M(x, y, t) = N(f(x), f(y), t)$.

As in the classical metric case, it is clear that every isometry is one-to-one.

Definition 5.2 Two fuzzy metric spaces $(X, M, *)$ and $(Y, N, \star)$ are called *isometric* if there is an isometry from X onto Y.

Definition 5.3 Let $(X, M, *)$ be a fuzzy metric space. A *fuzzy metric completion* of $(X, M, *)$ is a complete fuzzy metric space $(Y, N, \star)$ such that $(X, M, *)$ is isometric to a dense subspace of Y.

Proposition 5.1 *If a fuzzy metric space has a fuzzy metric completion then, as in the classical case, it is unique up to isometry.*

The following is a positive result.

Proposition 5.2 ([10]) *Let* (X, d) *be a metric space. Then,* $(X, M_d, \cdot)$ *is completable and its fuzzy metric completion is the standard fuzzy metric space of the metric completion of* (X, d).

In [10] the authors gave the following example of a fuzzy metric space that does not admit any fuzzy metric completion in the sense of Definition 5.3.

Example 5.1 Let $\{x_n\}_{n=3}^{\infty}$ and $\{y_n\}_{n=3}^{\infty}$ be two sequences of distinct points such that $A \cap B = \emptyset$, where $A = \{x_n : n \geq 3\}$ and $B = \{y_n : n \geq 3\}$.

Put $X = A \cup B$. Define a real valued function M on $X \times X \times (0, \infty)$ as follows:

$$M(x_n, x_m, t) = M(y_n, y_m, t) = 1 - \left[\frac{1}{n \wedge m} - \frac{1}{n \vee m}\right]$$
$$M(x_n, y_m, t) = M(y_m, x_n, t) = \frac{1}{n} + \frac{1}{m},$$

for all $n, m \geq 3$. Then $(X, M, \mathfrak{L})$ is a non-completable (stationary) fuzzy metric space.

Later, the same authors gave a characterization of those fuzzy metric spaces that are completable. We reformulate that characterization in the following theorem, for better observing the advances of the theory on completion of fuzzy metric spaces.

Theorem 5.1 (Gregori and Romaguera [11]) *A fuzzy metric space* $(X, M, *)$ *is completable if and only if for each pair of Cauchy sequences* $\{a_n\}$ *and* $\{b_n\}$ *in* X *the following three conditions are fulfilled:*

(c1) *The assignment* $t \to \lim_n M(a_n, b_n, t)$ *for each* $t > 0$ *is a continuous function on* $]0, \infty[$, *provided with the usual topology of* $\mathbb{R}$.
(c2) *Each pair of point-equivalent Cauchy sequences is equivalent, i.e.,* $\lim_n M(a_n, b_n, s) = 1$ *for some* $s > 0$ *implies* $\lim_n M(a_n, b_n, t) = 1$ *for all* $t > 0$.
(c3) $\lim_n M(a_n, b_n, t) > 0$ *for all* $t > 0$.

Since then, to find large classes of completable fuzzy metric spaces turned an interesting question.

It is immediate to verify that $(X, M_d, \cdot)$ satisfies conditions (c1)-(c3) of the last theorem. The following proposition is immediate from the same theorem.

Proposition 5.3 *A stationary fuzzy metric space* $(X, M, *)$ *is completable if and only if* $\lim_n M(a_n, b_n) > 0$ *for each pair of Cauchy sequences* $\{a_n\}$, $\{b_n\}$ *in* X.

Recently, it has been proved [21] that conditions (c1)-(c3) constitute an independent system, i.e., two of such conditions do not imply the third one.

The first non-completable fuzzy metric space which appeared in the literature was Example 5.1, which fulfils $(c1) - (c2)$ but not $(c3)$. The second one was ([11], Example 2), which fulfils $(c1)$ and $(c3)$ but not $(c2)$. A non-completable fuzzy metric space fulfilling $(c2) - (c3)$ but not $(c1)$ was given in ([21], Example 3.3).

The t-norm $*$ plays an interesting role in completion theory. In particular, in [17] it was proved the following result.

Proposition 5.4 (Gregori et al. [17]) *Let $(X, M, *)$ be a strong fuzzy metric space and suppose that $*$ is integral. If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and $t > 0$ then $\{M(x_n, y_n, t)\}_n$ converges in* $]0, 1]$.

As a consequence it was obtained the following class of completable fuzzy metrics.

Theorem 5.2 *If $(M, *)$ is a stationary fuzzy metric on X and $*$ is integral then $(X, M, *)$ is completable.*

Corollary 5.1 *Stationary fuzzy ultrametrics are completable.*

With respect to condition (c1) in [21] the authors obtained the following results.

Theorem 5.3 *Let $(X, M, *)$ be a strong fuzzy metric space, and let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X. Then the assignment*

$$t \to \lim_n M(a_n, b_n, t), \text{ for each } t > 0$$

is a continuous function on $]0, \infty[$ provided with the usual topology of $\mathbb{R}$.

Theorem 5.4 *A strong fuzzy metric space $(X, M, *)$ is completable if and only if for each pair of Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X the following conditions are fulfilled:*

(c2) $\lim_n M(a_n, b_n, s) = 1$ *for some* $s > 0$ *implies* $\lim_n M(a_n, b_n, t) = 1$ *for all* $t > 0$.

(c3) $\lim_n M(a_n, b_n, t) > 0$ *for all* $t > 0$.

The following corollaries are immediate.

Corollary 5.2 *Let $(X, M, *)$ be a strong fuzzy metric space and suppose that $*$ is integral. Then $(X, M, *)$ is completable if and only if for each pair of Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X the condition $(c2)$ is satisfied.*

Corollary 5.3 *Let $(X, M, *)$ be a fuzzy ultrametric space. Then $(X, M, *)$ is completable if and only if for each pair of Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X the condition $(c2)$ is satisfied.*

Remark 5.1 The fuzzy metric of ([11], Example 2) is a fuzzy ultrametric, but it is not completable. (Notice that it does not satisfy (c2)). The fuzzy metric of Example 5.1 is strong and satisfy (c2) but it is not completable. (Notice that $*$ is not integral).

Finally, we will show a class of completable fuzzy metrics which fulfils condition (c2).

Definition 5.4 (*Gregori et al.* [21]) Let $(X, M, *)$ be a fuzzy metric space. We will say that $(X, M, *)$ is a stratified fuzzy metric space if it satisfies one of the following equivalent conditions

(i) $M(a, b, s) = M(a', b', s)$ implies $M(a, b, t) = M(a', b', t)$ for all $t > 0$.
(ii) $M(a, b, s) < M(a', b', s)$ implies $M(a, b, t) < M(a', b', t)$ for all $t > 0$.

In this case, we say that $(M, *)$ (or simply M) is a stratified fuzzy metric on X.

Corollary 5.4 *Under the assumption that $*$ is integral, a stratified fuzzy metric space $(X, M, *)$ is completable if and only if $(c1)$ is satisfied.*

The following theorem gives a large class of completable fuzzy metric spaces.

Theorem 5.5 *Let $(M, *)$ be a stratified strong fuzzy metric on X and suppose that $*$ is integral. Then $(X, M, *)$ is completable.*

Corollary 5.5 *Let $(M, \wedge)$ be a stratified fuzzy ultrametric on X. Then $(X, M, \wedge)$ is completable.*

6 Fuzzy Banach Contraction Principle in KM-Fuzzy Metric Spaces

Several extensions of the classical Banach contraction principle have been given in our fuzzy context. The first one, due to Grabiec [7] was stated for KM-spaces as we show in the following.

Definition 6.1 Let $(X, M, *)$ be a KM-fuzzy metric space. A mapping $f : X \to X$ is called a Banach fuzzy contraction (or Sehgal contraction [41]) if there exists $k \in]0, 1[$ such that $M(f(x), f(y), kt) \geq M(x, y, t)$ for each $x, y \in X$ and $t > 0$.

The author also gave the following definition of Cauchy sequence, denoted here G-Cauchy. A sequence $\{x_n\}$ in a KM-fuzzy metric space $(X, M, *)$ is called G-Cauchy if $\lim_n M(x_{n+p}, x_n, t) = 1$ for each $t > 0$, $p > 0$. X is called G-complete if every G-Cauchy sequence in X converges.

Then it was proved the following fuzzy version of the Banach contraction principle.

Theorem 6.1 (Grabiec [7]) *Let $(X, M, *)$ be a G-complete KM-fuzzy metric space in which $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and let $f : X \to X$ a Banach fuzzy contraction. Then f has a unique fixed point.*

This result was generalized by R. Vasuki in [44], proving the existence of a common fixed point for a sequence of mappings under appropriate conditions.

Grabiec also considered a fuzzy version of the Edelstein contractive mapping as follows.

Definition 6.2 A mapping $f : X \to X$ is called Edelstein fuzzy contractive if it satisfies $M(f(x), f(y), t) > M(x, y, t)$ for all $x, y \in X$, $x \neq y$ and all $t > 0$.

Then Grabiec obtained the next theorem for a compact KM-fuzzy metric space [7]. (A (KM-)fuzzy metric space $(X, M, *)$ is called compact if every sequence has a convergent subsequence.)

Theorem 6.2 *Let $(X, M, *)$ be a compact KM-fuzzy metric space and $T : X \to X$ an Edelstein fuzzy contractive mapping. Then T has a unique fixed point.*

In [3] Ciric introduced a notion of an Edelstein fuzzy locally contractive mapping and extended the last theorem. He also extended the first theorem above due to Grabiec and the corresponding due to Vasuki.

Remark 6.1 Notice that in Theorem 6.1 it is assumed that X is G-complete. This assumption is really a strong condition. In fact, a compact (KM)-fuzzy metric space is not necessarily G-complete as it was proved in [43].

Assuming completeness instead of G-completeness for a KM-fuzzy metric, the following result was given.

Theorem 6.3 (D. Mihet [28]) *Let $(X, M, *)$ be a complete strong KM-fuzzy metric space and let $f : X \to X$ be a ψ-contractive mapping (see Definition 7.3 (iii)). If there exists $x \in X$ such that $M(x, f(x), t) > 0$ for all $t > 0$, then f has a fixed point. Further, if $M(x, y, t) > 0$ for all $t > 0$, then the fixed point is unique.*

In the following section we will see other concepts of contractivity and fixed point theorems.

7 Fuzzy Banach Contraction Principle in Fuzzy Metric Spaces

The first fuzzy Banach contraction principle in complete fuzzy metric spaces was given by Gregori and Sapena [13]. To introduce this result we need some previous concepts.

Definition 7.1 Let $(X, M, *)$ be a fuzzy metric space. We will say the mapping $f : X \longrightarrow X$ is GS-fuzzy contractive if there exists $k \in]0, 1[$ such that

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k(\frac{1}{M(x, y, t)} - 1)$$

for each $x, y \in X$ and $t > 0$.(k is called the contractive constant of f).

The above definition is justified by the next proposition.

Proposition 7.1 *Let (X, d) be a metric space. The mapping $f : X \longrightarrow X$ is contractive (a contraction) on the metric space (X, d) with contractive constant k if and only if f is GS-fuzzy contractive, with contractive constant k, for the standard fuzzy metric space induced by d.*

Recall that a sequence $\{x_n\}$ in a metric space (X, d) is said to be contractive if there exists $k \in]0, 1[$ such that $d(x_{n+1}, x_{n+2}) \leq kd(x_n, x_{n+1})$, for all $n \in \mathbb{N}$. Now, we give the following definition (compare with Definition 7.1).

Definition 7.2 Let $(X, M, *)$ be a fuzzy metric space. We will say that the sequence $\{x_n\}$ in X is GS-fuzzy contractive if there exists $k \in]0, 1[$ such that

$$\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1 \leq k(\frac{1}{M(x_n, x_{n+1}, t)} - 1), \text{ for all } t > 0, \ n \in N$$

Proposition 7.2 *Let $(X, M_d, *)$ be the standard fuzzy metric space induced by the metric d on X. The sequence $\{x_n\}$ in X is contractive in (X, d) iff $\{x_n\}$ is GS-fuzzy contractive in $(X, M_d, *)$.*

Next we extend the Banach fixed point theorem for GS-fuzzy contractive mappings of complete fuzzy metric spaces.

Theorem 7.1 (Gregori and Sapena [13]) (Banach Fuzzy contraction Theorem) *Let $(X, M, *)$ be a complete fuzzy metric space in which fuzzy contractive sequences are Cauchy. Let $T : X \longrightarrow X$ be a GS-fuzzy contractive mapping.*
Then T has a unique fixed point.

It is an open problem to know if GS-fuzzy contractive sequences are Cauchy. D. Mihet obtained a negative answer if M is a KM-fuzzy metric. The following corollary can be considered the fuzzy version of the Banach Contraction Principle.

Corollary 7.1 *Let $(X, M_d, *)$ be a complete standard fuzzy metric space and let $T : X \longrightarrow X$ be a GS-fuzzy contractive mapping. Then T has a unique fixed point.*

Many other concepts of contractive self-mapping on X have been given later in the literature. Here we give some of them which are related.

Denote by $\mathscr{H}$ the family of mappings $\eta :]0, 1] \rightarrow [0, +\infty[$ such that η transforms $]0, 1]$ onto $[0, +\infty[$ and η is strictly decreasing. Denote by Ψ the family of mappings $\psi :]0, 1] \rightarrow]0, 1]$ such that ψ is continuous non-decreasing and $\psi(t) > t$ for all $t \in]0, 1]$. Denote by $\mathscr{H}_\omega$ the family of continuous, strictly decreasing mappings $\eta :]0, 1] \rightarrow [0, +\infty[$ with $\eta(1) = 0$.

Definition 7.3 Let $(X, M, *)$ be a fuzzy metric space. A mapping $f : X \rightarrow X$ is called

(i) RT-contractive [35] if there exists $k \in]0, 1[$ such that

$$M(f(x), f(y), t) \geq 1 - k + k \cdot M(x, y, t) \text{ for all } x, y \in X \text{and } t > 0$$

(ii) $\mathscr{H}$-contractive [46] with respect to $\eta \in \mathscr{H}$ if there exists $k \in]0, 1[$ such that

$$\eta(M(f(x), f(y), t)) \leq k \cdot \eta(M(x, y, t) \text{ for all } x, y \in X \text{ and } t > 0$$

(iii) $\mathscr{H}_\omega$-contractive with respect to $\eta \in \mathscr{H}_\omega$ [29] if there exists $k \in]0, 1[$ such that

$$\eta(M(f(x), f(y), t)) \leq k \cdot \eta(M(x, y, t)) \text{ for all } x, y \in X \text{ and } t > 0$$

(iv) ψ-contractive [28] with respect to $\psi \in \Psi$ if there exists $k \in]0, 1[$ such that

$$M(f(x), f(y), t) \geq \psi(M(x, y, t)) \text{ for all } x, y \in X \text{ and } t > 0$$

The following chain of implications, related to these contractive conditions, is satisfied:

$$RT - \text{contractive} \rightarrow GS - \text{contractive} \rightarrow \mathscr{H} - \text{contractive} \rightarrow \mathscr{H}_\omega - \text{contractive} \rightarrow \psi - \text{contractive}$$

For the most two general contractive conditions above, recently, the following fixed point theorems have been given.

Theorem 7.2 (Gregori and Miñana [8]) *Let $(X, M, *)$ be a complete fuzzy metric space and let $f : X \rightarrow X$ be a fuzzy $\mathscr{H}_\omega$-contractive mapping. If $\bigwedge_{t>0} M(x, f(x), t) > 0$ for each $x \in X$, then f has a unique fixed point $x^* \in X$ (and for each $x \in X$ the sequence of iterates $\left\{f^{n)}(x)\right\}_n$ converges to x^*).*

This last theorem generalizes two results due to Wardowski and Mihet (Theorem 3.2 of [46] and Theorem 2.4 of [29], respectively).

Also, for a particular class of fuzzy metric spaces, the following theorem has been given.

Theorem 7.3 (Gregori and Miñana [8]) *Let $(X, M, *)$ be a complete strong fuzzy metric space and let $f : X \rightarrow X$ be a fuzzy ψ-contractive mapping. Then f has a unique fixed point.*

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Differentiability and Norming Subspaces

In Honour of Manuel López-Pellicer

Antonio José Guirao, Aleksei Lissitsin and Vicente Montesinos

Abstract This is a survey around a property (Property $\mathscr{P}$) introduced by M. Fabian, V. Zizler, and the third named author, in terms of differentiability of the norm. Precisely, a Banach space X is said to have property $\mathscr{P}$ if for every norming subspace $N \subset X^*$ there exists an equivalent Gâteaux differentiable norm for which N is 1-norming. Every weakly compactly generated space has property $\mathscr{P}$. Applications to measure theory, the classification of compacta, and some other structural properties of compact and Banach spaces are given. Some open problems are listed, too. It is based on an earlier paper by Fabian, Zizler, and the third named author, and a recent one by the authors of the survey.

Keywords Corson · Eberlein and Valdivia compacta · Gâteaux differentiability · Norming subspace · Lower semicontinuous norm

Mathematics Subject Classification (2010) 46B03 · 46B20

1 Introduction

Every separable Banach space X has an equivalent strictly convex norm (for this and for other definitions in this paragraph see below; for undefined terms see, for example, [13]). This was an early result by J. A. Clarkson [4], proved first by showing

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that the space $C[0, 1]$ has such a norm—a "weighted" ℓ_2-sum of the supremum norm and the evaluations on a countable dense subset of $[0, 1]$—, and then by isometrically embedding X into $C[0, 1]$. M. M. Day [5] observed that the only ingredient in Clarkson's argument was the existence of a countable total (i.e., linearly w^*-dense) subset of S_{X^*}, and that this amounted to the construction of a continuous linear and one-to-one mapping from X into ℓ_2, from where the result followed. Then ([5, Theorem 4]) he went on by showing that every separable space X had an equivalent norm that was simultaneously strictly convex and smooth (i.e., Gâteaux differentiable). He showed that Clarkson's method, when applied to X^*, gave a dual equivalent norm that was strictly convex (and so its predual norm was smooth, by Šmulyan's Lemma). Finally he proved that every space with a smooth norm and having a continuous linear and one-to-one mapping into a space with a norm that was simultaneously smooth and strictly convex could be renormed by a norm that had simultaneously the two properties.

The space $c_0(\Gamma)$ has an equivalent locally uniformly rotund (LUR) norm (nowadays known as the Day norm). Day proved the strict convexity [5], and the LUR property of that norm was the contribution of J. Rainwater [27].

The transfer technique sketched above and the property of the Day norm on $c_0(\Gamma)$ allowed G. Godefroy [15] to provide a relatively simple proof of the celebrated S. Troyanski's result that every weakly compactly generated (WCG) Banach space has an equivalent LUR norm [29]. Behind was the seminal construction of a projectional resolution of the identity on WCG spaces, and the existence of a fundamental result by D. Amir and J. Lindenstrauss [1]. Indeed, Amir and Lindenstrauss showed that if X is a WCG space, then both X and X^* admit a strictly convex norm and, respectively, a dual strictly convex norm (so X has also a Gâteaux norm), thanks to the existence of injective mappings both from X and from X^* into some $c_0(\Gamma)$.

Lindenstrauss conjectured [22], after the previous results, that if X admits a Gâteaux smooth norm, then X must be SWCG, i.e., a subspace of a WCG space. The conjecture had a negative solution, with S. Mercourakis [23] showing that every weakly countably determined (WCD) space X has the property that both X and X^* admit strictly convex norms. To see how fruitful and far-reaching was the original suggestion of Lindenstrauss, we may mention the following result: *A Banach space X has an equivalent uniformly smooth equivalent norm if, and only if, X is a subspace of a Hilbert-generated Banach space* [9, 12]. A Banach space X is said to be **Hilbert generated** if there is a bounded linear operator from a Hilbert space onto a dense subspace of X.

Broadly speaking, smooth renorming is connected with compact generation, while rotund renorming appears to be linked to special coverings of the unit sphere, in particular if no linear transfer technique is available. The previous paragraphs hinted at the first problem. For the second, a recent monograph that presents the advances in LUR renorming is [24], and for a characterization of the existence of a strictly convex renorming, together with a good account of the previous work done, see, e.g., [26].

The present note takes the first road and contributes to show how fruitful was Lindenstrauss' conjecture: If smoothness was thought to imply SWCG—in other words, that the dual unit ball in the w^*-topology is an Eberlein compact space—, our note shows that some smoothness in $C(K)$ ensures additional compactness properties of the compact space K. It builds mainly on two papers [10, 17], and tries to be a short survey on some of the topics treated in these two references. We complete some of the information provided there, streamlining some proofs or modifyng others, and hoping to make the text almost self-contained. At the end we list some open problems related to the topic. Again, we rely on the questions presented in the two aforementioned papers.

1.1 Notation

Throughout the note we consider only real vector spaces. All compact spaces are assumed to be Hausdorff, unless specified otherwise.

As usual, if $(X, \|\cdot\|)$ is a Banach space, B_X denotes its closed unit ball, and S_X its unit sphere. X^* is the topological dual of X, and the canonical dual norm on X^* will be written $\|\cdot\|^*$ (or again $\|\cdot\|$ if there is no risk of misunderstanding). The action of an element $x^* \in X^*$ on an element $x \in X$ is denoted by $x^*(x)$ or, alternatively, by $\langle x, x^*\rangle$. We put span S for the linear span of a set S in a vector space.

We say that $f \in S_{X^*}$ **attains its norm** if there is $x \in S_X$ such that $f(x) = 1$. The Bishop–Phelps theorem [3] asserts that such elements form a dense set in S_{X^*} (cf. e.g. [7, p. 13], or [13]).

Along this survey we shall use the typographical device of bounding the "support material" or the "complementary explanations" by shaded boxes as the present one. They can be skipped for the first approach.

1.2 Some Generalities on Norming Subspaces

The more general statements about duality in locally convex spaces are formulated for an abstract **dual pair** $\langle E, F\rangle$, where E and F are linear spaces, and $\langle\cdot,\cdot\rangle : E \times F \to \mathbb{R}$ denotes a symmetric bilinear form that separates points of E and F. If a subspace G of F is given, $\langle E, G\rangle$ is still a dual pair (endowed with the restriction to $E \times G$ of the original bilinear form) if, and only if, G is $w(F, E)$-dense in F, where $w(F, E)$ is the topology on F of the pointwise convergence on the elements of E (this is equivalent to say that G **separates points of** E). This topology is Hausdorff thanks to the separating property of $\langle\cdot,\cdot\rangle$.

The definition of the topology $w(E, G)$ above can be pushed further by allowing G to be an arbitrary subset of F. Note then that $w(E, G) = w(E, \operatorname{span} G)$. Thus,

$w(E, G)$ is Hausdorff if, and only if, G is $w(F, E)$-**linearly dense** in F, i.e., span G is $w(F, E)$-dense in F.

In several instances along this paper we shall use real locally convex spaces, mostly real Banach spaces endowed with different topologies. By a **topological vector space** $(E, \mathscr{T})$ we understand a real vector space E endowed with a Hausdorff topology $\mathscr{T}$ that makes continuous the vector space operations; a **locally convex space** is a topological vector space where 0 has a basis of convex neighborhoods.

In the context of a Banach space $(X, \|\cdot\|)$, it is obvious that $\langle X, X^*\rangle$ is a dual pair for the bilinear form $\langle x, x^*\rangle := x^*(x)$ for $x \in X$, $x^* \in X^*$. As usual, the weak topology $w(X, X^*)$ and the weak* topology $w(X^*, X)$ will sometimes be denoted by w and w^*, respectively. If $D \subset X^*$ is a $w(X^*, X)$-dense subspace (not necessarily $\|\cdot\|$-closed), we may consider $B_D := B_{X^*} \cap D$ (i.e., its closed unit ball), and the function $\|\cdot\|_D : X \to \mathbb{R}$ given by

$$\|x\|_D := \sup\{\langle x, x^*\rangle : \ x^* \in B_D\}, \ x \in X. \tag{1}$$

Clearly, $\|\cdot\|_D$ is a (not necessarily equivalent) norm on X. Observe that its closed unit ball is

$$B^D := \{x \in X : \ \|x\|_D \le 1\} = (B_D)_\circ, \tag{2}$$

where $A_\circ \subset X$ denotes the polar set of a set $A \subset X^*$. *With respect to the dual pair* $\langle X, D\rangle$, B_D is just B_X°, where $A^\circ \subset D$ is the polar set of $A \subset X$. Hence

$$B^D = (B_X^\circ)_\circ = \overline{B_X}^{w(X,D)}, \tag{3}$$

where the last equality is just the bipolar theorem.

Note, too, that $\|\cdot\|_D$, as the supremum of a collection of $w(X, D)$-continuous (linear) functionals, is $w(X, D)$-lower semicontinuous. Moreover, $\|\cdot\|_D$ is the largest among all convex $w(X, D)$-lower semicontinuous minorants of the norm $\|\cdot\|$. This is why $\|\cdot\|_D$ is called the $w(X, D)$-**lower semicontinuous envelope** of the norm $\|\cdot\|$.

As we mentioned, the norm $\|\cdot\|_D$ defined in (1) is not always an equivalent norm. For a particular example, see [13, Exercise 3.92]. When this is the case, the subspace D of X^* is said to be **norming for** X. More precisely, for some $\alpha > 0$, a subspace $N \subset X^*$ is said to be $(1/\alpha)$-**norming for** X if $\alpha\|x\| \le \|x\|_N$ $(\le \|x\|)$ for all $x \in X$. In particular, N is 1-norming for X if $\|\cdot\|_N = \|\cdot\|$. We shall omit the reference to X if it is understood from the context. Of course, every norming subspace $N \subset X^*$ is $w(X^*, X)$-dense.

Being a norming or 1-norming subspace can be described in a number of ways. We gather such characterizations in the following two lemmata (note that, in all cases, $B_D \subset B_{X^*}$ and $B_X \subset B^D$). The proofs of the equivalences are easy, and we shall omit them (they can be found, e.g., in [13, Exercises 3.87–3.93, 4.34, 4.60, 5.3–5.6, 7.54–7.56, 8.19, 13.35 and 14.37]). For the "moreover" part of Lemma 1.1, see [18].

Lemma 1.1 *Let* $(X, \|\cdot\|)$ *be a Banach space, and let* D *be a* $w(X^*, X)$*-dense subspace of* X^**. The following are equivalent:*

(i) *D is norming.*

(ii) *B^D is bounded.*

(iii) $\overline{B_D}^{w(X^*,X)}$ *has a nonempty interior.*

(iv) *$X + D^\perp$ is closed in X^{**}, where $D^\perp$ is the subspace of X^{**} orthogonal to D.*

(v) *dist* $(S_X, D^\perp) > 0$ *(distance in X^{**}).*

Under some additional assumptions, we can add more equivalent conditions. A topological space T is said to be **angelic** (a property introduced by Fremlin, see, e.g., [14]) if for every relatively countably compact subset A, we have simultaneously the two following properties: (i) A is relatively compact, and (ii) every point in $\overline{A}$ is the limit of a sequence in A. If K is compact, then angelicity of K coincides with the **Fréchet–Urysohn property**, which means that for every subset A of K, (ii) above holds.

In the next statement the concept of a Mazur space appears: A locally convex space E is said to be **Mazur** (or that it has the **Mazur property**) if every sequentially continuous linear form on E is continuous, i.e., an element of its dual space. A linear form f on E is said to be **sequentially continuous** if $f(x_n) \to 0$ whenever (x_n) is a sequence in E that converges to 0.

Lemma 1 continued (see [18]) *Moreover, if* D *is* $\|\cdot\|$*-closed, and* (B_{X^*}, w^*) *is angelic, then* (i) *is equivalent to any of the following:*

(a) $(X, \mu(X, D))$ *is complete, where* $\mu(X, D)$ *is the topology of the uniform convergence on all absolutely convex* $w(D, X)$*-compact subsets of* D*.*

(b) *the space* $(D, w(D, X))$ *is Mazur.*

Remark 1.1 Without (B_{X^*}, w^*) being angelic, the equivalences in the "moreover" part of Lemma 1.1 may fail. See [18] for examples.

Lemma 1.2 *Let* $(X, \|\cdot\|)$ *be a Banach space, and let* D *be a* $w(X^*, X)$*-dense subspace of* X^**. The following are equivalent:*

(i) *D is* 1*-norming.*

(ii) $B^D = B_X$*.*

(iii) $B_{X^*} = \overline{B_D}^{w(X^*,X)}$.

(iv) *B_X is $w(X, D)$-closed.*

(v) *$\|\cdot\|$ is $w(X, D)$-lower semicontinuous.*

(vi) *$X + D^\perp$ is closed in X^{**}, and the associated projection $P : X + D^\perp \to X$ has norm* 1.

(vii) *dist* $(S_X, D^\perp) = 1$ *(distance in X^{**}).*

Using norming (and, in particular, 1-norming) subspaces allows for unifying some results on Banach spaces if we wish to cover the case of a Banach space $(X, \|\cdot\|)$ that, in some instances, may be a dual space. A trivial observation is that X, as a (closed) subspace of X^{**}, is 1-norming for X^*. The space $(X, \|\cdot\|)$ is said to have an **isomorphic** (**isometric**) **predual** if there exists a Banach space $(P, |||\cdot|||)$ such that $(P^*, |||\cdot|||)$ is isomorphic (respectively, isometric) to $(X, \|\cdot\|)$. It is plain that $(P, |||\cdot|||)$ is isomorphic (respectively, isometric) to a (closed) subspace of $(X^*, \|\cdot\|)$, and this subspace is norming (respectively, 1-norming) for X.

Lemma 1.3 shows an interesting permanence property for norming subspaces:

Lemma 1.3 *Let $(Z, \|\cdot\|)$ be a Banach space, X a subspace of Z, and $N \subset X^*$ a norming subspace for X. Let $q : Z^* \to X^*$ be the canonical quotient mapping. Then $q^{-1}(N)$ is a norming subspace for Z.*

Proof We follow an idea in the proof of [16, Corollary 2.2]. Without loss of generality, we may assume that N is a 1-norming subspace of X^*. Indeed, denote again by $\|\cdot\|$ the norm on X induced by $\|\cdot\|$ on Z. Then $\|\cdot\|_N$ on X defined by (1) makes, certainly, N a 1-norming subspace for X. Now, it is a general fact that every equivalent norm on a subspace of a Banach space can be extended to an equivalent norm on the whole space (see, for example, [7, Lemma II.8.1]).

Take $z \in S_Z$. Suppose first that its distance to X is less than $1/4$. Choose $x \in X$ such that $\|z - x\| < 1/4$ and $y^* \in N \cap B_{X^*}$ with $\langle x, y^*\rangle > \|x\| - 1/4$. Select $z^* \in q^{-1}(\{y^*\}) \cap B_{Z^*}$ (such a selection is possible: Choose a sequence (z_n) in $q^{-1}(\{y^*\})$ with $\|z_n\| \to 1$; it has a w^*-cluster point z^* that, by the w^*-lower semicontinuity of the norm in Z^*, belongs to B_{Z^*}). Then

$$\begin{aligned}\langle z, z^*\rangle &= \langle x, z^*\rangle + \langle z - x, z^*\rangle \\ &= \langle x, y^*\rangle + \langle z - x, z^*\rangle > \|x\| - \tfrac{1}{4} - \|z - x\| \\ &> \|z\| - \tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{4} = \tfrac{1}{4}.\end{aligned}$$

It follows that the supremum of z on $q^{-1}(N) \cap B_{Z^*}$ is greater than 1/4. Second, note that the distance $d(z, X)$ from z to X is $\|Q(z)\|$, where $Q : Z \to Z/X$ is the canonical quotient mapping. Since the dual of Z/X is linearly isometric to $X^\perp$, if $d(z, X) \geq 1/4$ we may choose $z^* \in S_{X^\perp}$ such that $\langle z, z^*\rangle = 1/4$. Note that $q(z^*) = 0$. Thus, the supremum of z on $q^{-1}(N) \cap B_{Z^*}$ is greater or equal than 1/4. This proves that

$$\tfrac{1}{4} B_{Z^*} \subset \overline{q^{-1}(N) \cap B_{Z^*}}^{w^*}.$$

Hence, by (iii) in Lemma 1.1, $q^{-1}(N)$ is norming. □

Another result that will be used below is the following:

Proposition 1.1 *Let X be a Banach space. Let $F \in X^{**} \setminus X$. Then $F^{-1}(0)$ is a norming subspace of X^* for X.*

Proof For a proof see, e.g., [13, Exercise 3.88]. It is based on the so-called Parallel Hyperplane Lemma (see, e.g., [13, Exercise 2.13]). A related, although different argument, uses (iv) in Lemma 1.1: If $N := F^{-1}(0)$, then $N^{\perp} = \text{span}\{F\}$, a one-dimensional subspace of X^{**}, so $X + N^{\perp}$ is closed in X^{**}, a simple consequence of the Separation Theorem. □

Apparently, the following lemma is not directly related to norming subspaces. It states a very simple, although useful, sequential property of the dual unit ball of a Banach space when some smoothness of the norm of X is present. The application to 1-norming subspaces is made explicit in Corollary 1.1.

Proposition 1.2 *Let* $(X, \|\cdot\|)$ *be a Banach space, and let* $x_0^* \in NA(X) \cap S_{X^*}$. *Assume that* $\|\cdot\|$ *is Gâteaux differentiable at some* $x_0 \in S_X$ *such that* $\langle x_0, x_0^* \rangle = 1$. *If* $S \subset B_{X^*}$ *satisfies* $x_0^* \in \overline{S}^{w^*}$, *then there exists a sequence* (s_n^*) *in* S *that* w^**-converges to* x_0^*. *If* $\|\cdot\|$ *is Fréchet differentiable at* x_0, *then* (s_n^*) *is* $\|\cdot\|$*-convergent to* x_0^*.

Proof Find a sequence (s_n^*) in S such that $\langle x_0, s_n^* \rangle \to 1$. Let s^* be an arbitrary w^*-cluster point of (s_n^*). Clearly, $\langle x_0, s^* \rangle = 1$, and $s^* \in B_{X^*}$. Since $\|\cdot\|$ is Gâteaux differentiable at x_0, the Šmulyan lemma gives $s^* = x_0^*$. Thus, (s_n^*) is w^*-convergent to x_0^*. If $\|\cdot\|$ is Fréchet differentiable at x_0, again the Šmulyan lemma gives convergence, now in the norm. □

Corollary 1.1 *Let* $(X, \|\cdot\|)$ *be a Banach space. Let* $N \subset X^*$ *be a* 1*-norming subspace of* X^*. *If* $\|\cdot\|$ *is Gâteaux differentiable, then every* $x^* \in S_{X^*}$ *is in the* w^**-closure of a countable subset of* B_N.

Proof Put $S := B_N$. By (iii) in Lemma 1.2, $\overline{S}^{w^*} = B_{X^*}$. Let $x^* \in S_{X^*}$. If $x^* \in NA(X, \|\cdot\|)$ then in fact x^* is the w^*-limit of a sequence in S, as it follows from Proposition 1.2. Otherwise, x^* is the $\|\cdot\|$-limit of a sequence in $NA(X, \|\cdot\|) \cap S_{X^*}$, by the Bishop–Phelps theorem, and we can apply the first part to conclude the proof. □

Given a subset S of a topological space, the **countable closure** of S is, by definition, the union of the closures of all at most countable subsets of S. The notions of a **countably closed** or **countably dense** set S are also defined as expected — respectively, if S coincides with its countable closure or if its countable closure is the whole space.

Thus, Corollary 1.1 says that S_{X^*} is in the countable w^*-closure of B_N. Note that in this case N is countably w^*-dense in X^*.

2 Property $\mathscr{P}$

A Banach space X is said to be **weakly compactly generated** (briefly, WCG) if it contains a weakly compact and linearly dense subset. The acronym SWCG denotes the class of subspaces of WCG spaces. Certainly, every separable space is WCG, as well as every reflexive space. For any index set Γ, the space $c_0(\Gamma)$ is WCG. If μ is a σ-finite measure, then $L_1(\mu)$ is also WCG (see, e.g., [13, Chap. 13]).

The origin of Theorem 2.2 below can be traced back to several sources, all of them inspired by the early conjecture of J. Lindenstrauss connecting smoothness and weak compactness that was mentioned at the Introduction. One is the following result (the definition of M-smoothness is given below). The argument in the proof of (i) in Theorem 2.1 has its origin in [9, Lemma 1]. This applies, too, to Theorem 2.2 and Remark 2.4 below.

Theorem 2.1 (Fabian, Montesinos, Zizler [11], Theorem 1) (i) *Let M be a bounded subset in a Banach space X. Then M is w-relatively compact if, and only if, for every norming subspace N of X^*, there is an equivalent N-lower semicontinuous norm on X that is M-smooth.*

(ii) *Let M be a bounded subset in the dual space X^*. Then M is w-relatively compact if, and only if, there is an equivalent dual norm on X^* which is M-smooth.*

If M is a bounded subset of a Banach space $(X, \|\cdot\|)$, we say that $\|\cdot\|$ is ***M*-smooth** if

$$\sup\{\|x+th\|+\|x-th\|-2\|x\|:\ h\in M\}=o(t),\ \text{for } t>0,\ 0\neq x\in X.$$

In particular, if $(X, \|\cdot\|)$ is WCG, $M\subset X$ is a linearly dense and w-compact set, and $N\subset X^*$ is a norming subspace for $(X, \|\cdot\|)$, then (i) in Theorem 2.1 gives an equivalent M-smooth and N-lower semicontinuous norm $|||\cdot|||$. Due to the fact that M is linearly dense, it is easy to show that $|||\cdot|||$ is Gâteaux smooth.

The following result is, regarding Gâteaux smoothness, a little bit more precise than the necessary condition in (i) in Theorem 2.1. See also Remark 2.4 below.

Theorem 2.2 (Fabian, Montesinos, Zizler [10]) *Let $(X, \|\cdot\|)$ be a WCG Banach space. Then X admits an equivalent norm $|||\cdot|||$ whose N-lower semicontinuous envelope is Gâteaux differentiable for every norming subspace N of X^*.*

Proof According to the well-known factorization theorem of Davis, Figiel, Johnson, and Pełczyński (see, e.g., [13, Theorem 13.33]), there are a reflexive space R and a linear bounded one-to-one operator $T: R\to X$ with dense range. Let $|\cdot|$ be an equivalent norm on R such that the corresponding dual norm $|\cdot|^*$ on R^* is strictly convex (see, e.g., [13, Theorem 13.25]). Put

$$D:=B_{(X,\|\cdot\|)}+T\big(B_{(R,|\cdot|)}\big).$$

As $T\big(B_{(R,|\cdot|)}\big)$ is weakly compact, D is a bounded convex symmetric closed set with 0 in its interior. Let $|||\cdot|||$ be the Minkowski functional of D. This is an equivalent norm on X, and $B_{(X,|||\cdot|||)}=D$. We shall show that $|||\cdot|||$ has the desired property.

To this end, let $N\subset X^*$ be a norming subspace. For the N-lower semicontinuous envelope $|||\cdot|||_N$ of $|||\cdot|||$ we have, by (3),

$$B_{(X,|||\cdot|||_N)} = \overline{D}^{w(X,N)} = \overline{B_{(X,\|\cdot\|)} + T\left(B_{(R,|\cdot|)}\right)}^{w(X,N)}.$$

As $T\left(B_{(R,|\cdot|)}\right)$ is weakly compact (and thus $w(X,N)$ compact), we get

$$B_{(X,|||\cdot|||_N)} = \overline{B_{(X,\|\cdot\|)}}^{w(X,N)} + T\left(B_{(R,|\cdot|)}\right) = B_{(X,\|\cdot\|_N)} + T\left(B_{(R,|\cdot|)}\right).$$

Thus, for $x^* \in X^*$ we have

$$|||x^*|||_N^* = \|x^*\|_N^* + \sup\left\langle T\left(B_{(R,|\cdot|)}\right), x^*\right\rangle = \|x^*\|_N^* + |T^*x^*|^*.$$

The norm $|\cdot|^*$ on R^* is strictly convex and T^* is injective. We claim that $|||\cdot|||_N^*$ on X^* is strictly convex. Indeed, consider two points $x^*, y^* \in X^*$ such that

$$2(|||x^*|||_N^*)^2 + 2(|||y^*|||_N^*)^2 - (|||x^* + y^*|||_N^*)^2 = 0.$$

Then, a simple argument of convexity gives

$$2(|T^*x^*|^*)^2 + 2(|T^*y^*|^*)^2 - (|T^*x^* + T^*y^*|^*)^2 = 0.$$

The strict convexity of $|\cdot|^*$ gives $T^*x^* = T^*y^*$, and the fact that T has a dense range shows that $x^* = y^*$. This proves the claim (see, for more details, [7, pp. 43, 73] or [13, Ex. Chap. 8]). Finally, $|\cdot|_N$ on X is Gâteaux differentiable, as it follows from the Šmulyan's Lemma (see, e.g., [7, Proposition II.1.6]). □

Remark 2.3 A byproduct of the previous proof is the following observation: If $NA(X, \|\cdot\|) \subset X^*$ denotes the set of all norm-attaining functionals for $(X, \|\cdot\|)$, then $NA(X, \|\cdot\|) = NA(X, |||\cdot|||)$. Indeed, it is clear that the supremum of a linear functional on the algebraic sum of two sets is the sum of the suprema on each of them. To show the claim it is enough to observe that every element in X^* attains its supremum on the w-compact set $T(B_{(R,|\cdot|)})$. Let us give the details. To simplify the notation, put $\sup x^*(A) := \sup\{\langle a, x^*\rangle : a \in A\}$, for $x^* \in X^*$ and $A \subset X$. If $x_0^* \in NA(X, \|\cdot\|)$, then there exists $x_0 \in B_{(X,\|\cdot\|)}$ such that $\sup x_0^*(B_{(X,\|\cdot\|)}) = \langle x_0, x_0^*\rangle$. There exists $r_0 \in B_{(R,|\cdot|)}$ such that $\sup x_0^*(TB_{(R,|\cdot|)}) = \langle Tr_0, x_0^*\rangle$. It is obvious that $z_0 := x_0 + Tr_0 \in B_{(X,|||\cdot|||)}$ satisfies $\sup x_0^*(B_{(X,|||\cdot|||)}) = \langle z_0, x_0^*\rangle$, so $x_0^* \in NA(X, |||\cdot|||)$. Conversely, if $x_0^* \in NA(X, |||\cdot|||)$, there exists $z_0 \in B_{(X,|||\cdot|||)}$ such that

$$\begin{aligned}\langle z_0, x_0^*\rangle &= \sup x_0^*(B_{(X,|||\cdot|||)})\\ &(= \sup x_0^*(B_{(X,\|\cdot\|)}) + \sup x_0^*(TB_{(R,|\cdot|)}) = \sup x_0^*(B_{(X,\|\cdot\|)}) + \langle Tr_0, x_0^*\rangle).\end{aligned}$$

Thus, $\sup x_0^*(B_{(X,\|\cdot\|)}) = \langle z_0, x_0^*\rangle$. If $z_0 = x_0 + Tr_0$, where $x_0 \in B_{(X,\|\cdot\|)}$ and $r_0 \in B_{(R,|\cdot|)}$, then clearly $x_0(B_{(X,\|\cdot\|)}) = \langle x_0, x_0^*\rangle$, hence $x_0^* \in NA(X, \|\cdot\|)$. Ⓡ

Remark 2.4 Following [11], it is possible to improve Theorem 2.2 by showing the following: *Assume that M is a (convex and balanced) linearly dense and w-compact* *subset of* $(X, \|\cdot\|)$. *Then there exists an equivalent norm* $|||\cdot|||$ *on* X *such that, for every norming subspace* $N \subset X^*$ *for* $(X, \|\cdot\|)$, *the* N*-lower semicontinuous envelope* $|||\cdot|||_N$ *is* M*-smooth (in particular, Gâteaux smooth).*

Proof (Sketch) We slightly modify the proof of Theorem 2.2 along the following lines: First, in the construction of the classical factorization theorem, the mapping $T : R \to X$ can be defined in such a way that $M \subset T(B_R)$. Second, the norm $|\cdot|$ in R^* can be taken to be LUR (Troyanski, see, e.g., [13, Theorem 13.25]). Put now $D := \bigcup\{\alpha B_{(X,\|\cdot\|)} + \beta T(B_{(R,|\cdot|)}) : \alpha \geq 0,\ \beta \geq 0,\ \alpha^2 + \beta^2 \leq 1\}$. It is easy to see that D is w-closed. Let $|||\cdot|||$ be the Minkowski functional of D. This is an equivalent norm on X, and $B_{(X,|||\cdot|||)} = D$. Given a norming subspace N for $(X, \|\cdot\|)$, we have

$$\begin{aligned} B_{(X,|||\cdot|||_N)} &= \overline{B_{(X,|||\cdot|||)}}^{w(X,N)} = \overline{D}^{w(X,N)} \\ &= \bigcup\{\alpha B_{(X,|||\cdot|||_N)} + \beta T(B_{(R,|\cdot|)}) : \alpha \geq 0,\ \beta \geq 0,\ \alpha^2 + \beta^2 \leq 1\}. \end{aligned}$$

Thus,

$$|||x^*|||_N^2 = \|x^*\|_N^2 + |T^*x^*|^2.$$

To show that $|||\cdot|||_N$ is M-LUR, take $x^*, x_n^* \in X^*$ such that

$$2|||x^*|||_N^2 + 2|||x^*|||_N^2 - |||x^* + x_n^*|||_N^2 \to 0 \text{ as } n \to \infty.$$

Using the convexity, we get

$$2|T^*x^*|^2 + 2|T^*x_n^*|^2 - |T^*x^* + T^*x_n^*|^2 \to 0 \text{ as } n \to \infty.$$

Since $|\cdot|$ on $\mathbb{R}^*$ is LUR, we conclude that $|T^*x_n^* - T^*x^*| \to 0$ or, in other words, $\sup\{\langle x, x_n^* - x^*\rangle : x \in T(B_R)\} \to 0$ as $n \to \infty$. Since $M \subset T(B_R)$, a usual version of the Šmulyan's lemma gives the conclusion. □

Theorem 2.2 above suggests the following definition of a property of Banach spaces in terms of renorming, that we call "property $\mathscr{P}$" following the notation already used in [10]. According to this result, every WCG Banach space has a property formally stronger than $\mathscr{P}$ (we call it "strong property $\mathscr{P}$", again after [10]). We shall see, for example in Proposition 2.1 below, that there are other spaces —for example, the subspaces of the WCG space, a class denoted in short by SWCG— that share the same strong property. That this class is larger than the class of WCG spaces is a consequence of H. P. Rosenthal's celebrated example [28].

Definition 2.1 (Fabian, Montesinos, Zizler [10]) We say that a Banach space X has **property** $\mathscr{P}$ if for every norming subspace $N \subset X^*$ there is an equivalent Gâteaux differentiable norm on X that is N-lower semicontinuous; and X is said to have the **strong property** $\mathscr{P}$ if there exists an equivalent norm $|||\cdot|||$ on X such that $|||\cdot|||_N$ is Gâteaux differentiable for every norming subspace N of X^*.

Remark 2.5 By the equivalence (i)$\Leftrightarrow$(v) in Lemma 1.2, the N-lower semicontinuity property of the equivalent norm in Definition 2.1 can be substituted by the fact that, for the new norm, N is 1-norming.

Proposition 2.1 (Fabian, Montesinos, Zizler [10]) *Every subspace of a Banach space with property* $\mathscr{P}$ *(strong property* $\mathscr{P}$*) has property* $\mathscr{P}$ *(respectively, strong property* $\mathscr{P}$*).*

Proof Let Z be a Banach space with property $\mathscr{P}$, let X be a closed subspace of Z and let q be the canonical mapping from Z^* onto X^*. By Lemma 1.3, $q^{-1}(N)$ is norming for every norming subspace N of X^*. Property $\mathscr{P}$ of Z provides an equivalent Gâteaux differentiable norm $|\cdot|$ on Z that is $q^{-1}(N)$-lower semicontinuous. Then the restriction of $|\cdot|$ to X is the required norm. The proof for the strong property $\mathscr{P}$ is similar. □

3 Some Applications

In this section we present some applications of the concept introduced above. Each subsection starts with a list of some of the definitions needed. For other undefined terms, see, e.g., [13].

Sections 3.1 and 3.2 below deal with spaces $C(K)$ for a compact space K. Section 3.4 contains an application to Mazur's property.

3.1 Compact Spaces and Properties $\mathscr{P}$ and (M)

If K is a compact topological space, $C(K)$ is assumed, if nothing is said on the contrary, to be endowed with the supremum norm $\|\cdot\|_\infty$. Note that the mapping $\delta : K \to C(K)^*$ given by $\langle f, \delta(k)\rangle = f(k)$ for $k \in K$ and $f \in C(K)$ is a topological homeomorphism into when $C(K)^*$ carries the w^*-topology.

Property $\mathscr{P}$ (Definition 2.1 above) was defined for a Banach space. The reader will notice that in many of the arguments related to this, only some special norming subspaces do really play a role. This is specially so when dealing with $C(K)$ spaces. The norming subspaces needed reduce to just the linear hull of dense subsets of $\delta(K)$. This is why it is convenient to introduce the compact-version (and the strong compact-version) of the property $\mathscr{P}$, as it was done in [10]. We hope that there will be no misunderstanding derived from the same name used in the Banach-space and the compact settings.

Definition 3.1 (Fabian, Montesinos, Zizler [10]) We say that a compact space K has **property** $\mathscr{P}$ if for every dense subset D of K there exists an equivalent Gâteaux differentiable $w(C(K), \delta(D))$-lower semicontinuous norm on $C(K)$.

A compact space K has **strong property** $\mathscr{P}$ if there exists an equivalent Gâteaux differentiable norm that is $w(C(K), \delta(D))$-lower semicontinuous for every dense subset D of K.

Of course, if $C(K)$ has (strong) property $\mathscr{P}$, then K has (respectively, strong) property $\mathscr{P}$. Some examples of compact spaces with (strong) property $\mathscr{P}$ will be provided below.

Compact spaces can always be seen as topological subspaces of some product $\mathbb{R}^\Gamma$ of lines endowed with its product topology $\mathscr{T}_p$. If nothing is said on the contrary, we shall always consider $\mathbb{R}^\Gamma$ endowed with this topology.

A particular subset of $\mathbb{R}^\Gamma$ that will play an important role in the rest of the paper is $\Sigma(\Gamma){:=}\{x \in \mathbb{R}^\Gamma : |\operatorname{supp} x| \leq \aleph_0\}$, where $\operatorname{supp} x{:=}\{\gamma \in \Gamma : x(\gamma) \neq 0\}$. Another one is $c_0(\Gamma) := \{x \in \mathbb{R}^\Gamma : |\{\gamma \in \Gamma : |x(\gamma)| > \varepsilon\}| < \aleph_0$ for every $\varepsilon > 0\}$. Obviously, $c_0(\Gamma) \subset \Sigma(\Gamma)$.

The following concept —whose origin can be traced to the work of H. H. Corson and S. P. Gul'ko, see also [20] and the references therein— will help to simplify some of the expressions below:

Definition 3.2 Let K be a compact space. A subset S of K is said to be a **Σ-subset** if there exists a homeomorphism h from K into some $\mathbb{R}^\Gamma$ such that $h(S) = h(K) \cap \Sigma(\Gamma)$.

It is easy to observe that any $\Sigma(\Gamma)$ is always countably closed in $\mathbb{R}^\Gamma$, and hence so is any Σ-subset of a compact space.

The way a compact space sits in some $\mathbb{R}^\Gamma$ allows for its classification.

For example, K is called a **Corson compact space** if it is homeomorphic to a subset of $\Sigma(\Gamma)$ for some set Γ (in other words, if K is a Σ-subset of itself). Note that *every Corson compact space K is angelic*.

This has an easy proof: We may consider K as a subset of $\mathbb{R}^\Gamma$, and each element k in K has a countable support supp (k). Let A be a nonempty subset of K, and let $k_0 \in \overline{A}$. If supp $(k) = \{\gamma_{0,1}, \gamma_{0,2} \ldots\}$, find $a_1 \in A$ with $|a_1(\gamma_{0,1}) - k_0(\gamma_{0,1})| < 1$. Let supp $a_1 = \{\gamma_{1,1}, \gamma_{1,2}, \ldots\}$. Find $a_2 \in A$ such that $|a_2(\gamma_{i,j}) - k_0(\gamma_{i,j})| < 1/2, i, j = 0, 1$. Continue this way to get a sequence (a_n) in A that pointwise converges to k_0.

A Banach space X is called a **weakly Lindelöf determined space** (briefly, WLD) if the dual unit ball of X^* in its weak* topology is a Corson compact space (see, e.g., [8] or [19] for properties and references on WLD spaces).

A compact space K is said to be an **Eberlein compact space** if it is homeomorphic to a weakly compact subset of a Banach space endowed with the restriction of its weak topology. A compact space K is Eberlein compact if, and only if, it is homeomorphic to a compact subset of $c_0(\Gamma)$ endowed with the restriction of the pointwise topology, for some set Γ (thus every Eberlein compact space is Corson compact). This is a consequence of the work done by D. Amir and J. Lindenstrauss on WCG Banach spaces [1]. As it follows from this, K is an Eberlein compact space if, and only if, $C(K)$ is WCG. Then, according to Theorem 2.2 above, every Eberlein compact space has strong property $\mathscr{P}$.

Thus, the following theorem (a slight extension of one contained in an unpublished version of [10]) contains, as a particular case, the result of A. Grothendieck on the separability of supports of measures on Eberlein compact spaces. Recall that if K is a compact space and μ is a regular Borel measure on it, then the **support** of μ is defined as the complement of the set of all the points of K that have a neighborhood with measure 0.

Theorem 3.1 *Let K be a compact space with property $\mathscr{P}$. Assume that μ is a regular Borel measure on K. Then μ has a separable support.*

Proof Put

$$N := \overline{\text{span}}^{\|\cdot\|_\infty^*}\{\delta(k) : k \in K\} \ (\subset C(K)^*).$$

Obviously, N is 1-norming for $(C(K), \|\cdot\|_\infty)$. Since K has property $\mathscr{P}$, there exists an equivalent Gâteaux and $w(C(K), \delta(K))$-lower semicontinuous norm $|||\cdot|||$ on $C(K)$, hence $w(C(K), N)$-lower semicontinuous. Thus, N is also 1-norming for $(C(K), |||\cdot|||)$. By Corollary 1.1 it follows that μ is in the w^*-closure of a countable subset of N. Thus the support of μ is separable. □

Remark 3.2 The property of the compact space K exhibited in Theorem 3.1 is recorded as **property (M)**. This property was used by S. Argyros, S. Mercourakis, and S. Negrepontis [2] (see also [19, Thm. 5.57] to show that *for a compact space K it is equivalent* (i) *to be Corson and enjoy property (M), and* (ii) *the space $C(K)$ to be WLD*. It is worth noticing that under CH, there is a Corson compact space failing property (M). The first such example was given by K. Kunen in 1975 and published in [21]. Other examples can be found in [25] and in [2] (see also [19, Sect. 5.5], where the example in the last reference is presented).

Thus, Theorem 3.1 above says that *every compact space K with property $\mathscr{P}$ enjoys property (M).* ®

3.2 *Property $\mathscr{P}$, Corson and Valdivia Compacta*

The following concept was introduced by M. Valdivia [30]. A compact space K is called a **Valdivia compact space** if it contains a dense Σ-subset. Obviously, every Corson compact space is Valdivia compact. On the other hand, there are Valdivia compacta that are not Corson compacta.

An example —in a sense canonical, see below— is the interval $[0, \omega_1]$ endowed with the order topology. This is a Valdivia compact space (note that $[0, \omega_1)$ is dense). It is not angelic, since ω_1 is not the limit of a sequence in $[0, \omega_1)$, hence it is not Corson (see the proof at the beginning of Sect. 3.1). We mention that $[0, \omega_1]$ is "canonical" because of the following result, due to Deville and Godefroy [6]: *A Valdivia compact space is Corson compact if, and only if, it does not contain an isomorphic copy of* $[0, \omega_1]$.

A **Markushevich basis** for a Banach space X is, by definition, a biorthogonal system $\{x_\gamma, f_\gamma\}_{\gamma \in \Gamma}$ in $X \times X^*$ such that the closed linear hull of $\{x_\gamma : \gamma \in \Gamma\}$ equals to X and $\{f_\gamma : \gamma \in \Gamma\}$ separates the points of X (cf. e.g. [8] or [13]).

The **density character** dens X of a Banach space X is defined as the least cardinal $\aleph$ such that X admits a dense set of cardinality $\aleph$. The least uncountable ordinal number is denoted by ω_1 and the least uncountable cardinal by $\aleph_1$.

The following result shows a connection between differentiability in $C(K)$ and the quality of the compact space K of being Corson or Valdivia compact.

Theorem 3.3 (Guirao, Lissitsin, Montesinos [17]) *If K is a Valdivia compact space having property $\mathscr{P}$, then $C(K)$ is WLD (in particular, K is a Corson compact space).*

This result was preceded by a restricted version of the same theorem (Theorem 3.4 below), stated and proved only for the case, when the density character of $C(K)$ is $\aleph_1$ [10]. The proof there was based on the existence of a particular projectional resolution of the identity in spaces $C(K)$ for K a Valdivia compact space (due to Valdivia himself [30], see also, e.g., [7, Theorem 7.6]) and a "subordinated" Markushevich basis.

The proof of the general case is, in a sense, easier —at least for showing that K is a Corson compact space—, since it uses nothing but the ingredients in the definition of the property $\mathscr{P}$ and the concepts of Corson and Valdivia compacta. Still, we think that the proof of the particular case is worth to reproduce, as it uses projectional resolutions of the identity and the almost ubiquitous concept of a Markushevich basis. The "plus" of the proof of Theorem 3.4 is that the WLD character of $C(K)$ is obtained without appealing to any measure-theoretical result or the use of Σ-subspaces of the dual space, in contrast with the proof of Theorem 3.3.

Theorem 3.4 (Fabian, Montesinos, Zizler [10]) *Assume that K is a Valdivia compact space having property $\mathscr{P}$ and that the density character of the space $(C(K), \|\cdot\|_\infty)$ is $\aleph_1$. Then $C(K)$ is WLD (in particular, K is a Corson compact space).*

Proof We can assume that $K \subset [-1, 1]^\Gamma$ for some set Γ and that $K \cap \Sigma(\Gamma)$ is dense in K. Let $\{f_\alpha : \alpha < \omega_1\}$ be a dense set in $C(K)$. Using the Urysohn lemma, we can easily see that the family $\{k \in K : f_\alpha(k) > \frac{3}{4}\}$, $\alpha < \omega_1$, is a base for the topology on K. For every $\alpha < \omega_1$ find $k_\alpha \in K \cap \Sigma(\Gamma)$ such that $f_\alpha(k_\alpha) > \frac{3}{4}$. Put then $S = \{k_\alpha : \alpha < \omega_1\}$. Note that this set is dense in K and has cardinality at most $\aleph_1$.

By [30] (see also [7, Lemma VI.7.5]), we have that there exists an increasing transfinite sequence $\{\Gamma_\alpha : \omega_0 \le \alpha \le \omega_1\}$ of subsets of Γ with the following properties: For every $\alpha \in [\omega_0, \omega_1]$,

(i) $\operatorname{card} \Gamma_\alpha \le \operatorname{card} \alpha$,

(ii) $\bigcup_{\beta+1<\alpha} \Gamma_\beta = \Gamma_\alpha$,

(iii) $\Gamma_{\omega_1} = \Gamma$,

and such that $R_{\Gamma_\alpha}(K) \subset K$, where $R_J : [0,1]^\Gamma \to [0,1]^\Gamma$ is defined for $J \subset \Gamma$ and $x \in [0,1]^\Gamma$ by

$$R_J(x)(i) = \begin{cases} x(i), & \text{if } i \in J \\ 0, & \text{if } i \notin J. \end{cases}$$

The corresponding projectional resolution of identity $\{P_\alpha : \omega_0 \le \alpha \le \omega_1\}$ on $C(K)$ (see [7, Theorem VI.7.6]) is defined by $P_\alpha(f) := f|_{K_\alpha} \circ R_\alpha$, $\omega_0 \le \alpha \le \omega_1$, where $K_\alpha := R_\alpha(K) \subset K$ and R_α is the restriction of R_{Γ_α} to K. Now, every $(P_{\alpha+1} - P_\alpha)(C(K))$ is separable, so it has a countable Markushevich basis $\{x_{\alpha,n}, x^*_{\alpha,n}\}_{n=1,2,\ldots}$ (see, e.g., [13, Theorem 4.59]). Following [8, Proposition 6.2.4], we get a Markushevich basis $\{x_{\alpha,n}, x^*_{\alpha,n}\}_{\omega_0\le\alpha<\omega_1,\ n=1,2,\ldots}$ on $C(K)$. Now, every element $s \in S$ has a countable support $N_s \subset \Gamma$. Hence there exists $\alpha_s \in \Gamma$ such that $N_s \subset \Gamma_\alpha$, for all $\alpha_0 \le \alpha \le \omega_1$. For the corresponding Dirac measure δ_s we then have $P^*_\alpha \delta_s = \delta_s$ for all these α. Therefore every element of $Y := \overline{\operatorname{span}}^{\|\cdot\|}\{\delta_s : s \in S\}$ has at most countable support on the set $\{x_{\alpha,n} : \omega_0 \le \alpha < \omega_1,\ n = 1, 2, \ldots\}$.

It remains to show that every element of $C(K)^*$ has a countable support on this set. Let $|\cdot|$ be an equivalent Gâteaux differentiable and Y-lower semicontinuous norm on $C(K)$ (its existence is guaranteed by the property $\mathscr{P}$). According to the Bishop-Phelps theorem, it is enough to consider norm-attaining elements of $(C(K), |\cdot|)^*$. Fix any $x^* \in S_{(C(K),|\cdot|)^*}$ that attains its $\|\cdot\|$ norm. By Proposition 1.2, there exists a sequence (x^*_n) in $Y \cap B_{(C(K)^*,|\cdot|)}$ that weak*-converges to x^*. Therefore x^* has a countable support on $\{x_{\alpha,n} : \omega_0 \le \alpha < \omega_1,\ n = 1, 2, \ldots\}$. This means that $C(K)$ is WLD. □

Let us split the proof of Theorem 3.3 above into the following steps, which may be of some interest on their own.

The first one easily follows from Corollary 1.1 but highlights the essential assumption that we will use.

Proposition 3.1 *Let a compact space K enjoy property $\mathscr{P}$. Then K satisfies the property: For every dense $S \subset K$ the subspace* $\operatorname{span} \delta(S)$ *is countably w^*-dense in $C(K)^*$.*

Proof Property $\mathscr{P}$ gives an equivalent Gâteaux differentiable norm $|||\cdot|||$ on $C(K)$ for which $\operatorname{span} \delta(S)$ is (still) 1-norming. Corollary 1.1 implies the claim. □

Proposition 3.2 *A Valdivia compact space K is Corson compact if and only if it satisfies the property: For every dense $S \subset K$ the subspace* $\operatorname{span} \delta(S)$ *is countably w^*-dense in* $\operatorname{span} \delta(K)$.

Proof The necessity is obvious. It follows from the angelicity, in particular from countable tightness, of Corson compacta.

For the sufficiency, consider K as a subset of $\mathbb{R}^{\Gamma}$ for some index set Γ in such a way that $S := K \cap \Sigma(\Gamma)$ is dense in K. For $\gamma \in \Gamma$, let $p_\gamma \in C(K)$ be the γ-th coordinate function, i.e., $p_\gamma(x) = x(\gamma)$ for $x \in K$. Note that the topology w^* of $C(K)^*$ coincides on $\delta(K)$ with the topology of the pointwise convergence on the set $\{p_\gamma : \gamma \in \Gamma\}$ ($\subset C(K)$). Fix $\delta(k) \in \delta(K)$. By assumption there exists a countable subset N_k of $N := \operatorname{span} \delta(S)$ such that $\delta(k) \in \overline{N_k}^{w^*}$. Observe now that every $\delta(s) \in \delta(S)$ (and so every $\mu \in N$) has a countable support on $\{p_\gamma : \gamma \in \Gamma\}$. This shows that $\delta(k)$ is also countably supported on $\{p_\gamma : \gamma \in \Gamma\}$. Since K was homeomorphic to a subset of $\mathbb{R}^{\{p_\gamma :\, \gamma \in \Gamma\}}$ in its pointwise topology, we get that K is a Corson compact space. □

Proposition 3.3 *Given a Corson compact space K, the space $C(K)$ is WLD if and only if K satisfies the property:* span $\delta(K)$ *is countably w^*-dense in $C(K)^*$.*

Proof Again, the necessity follows from the angelicity of $(B_{C(K)^*}, w^*)$.

For the sufficiency, we can proceed in two slightly different ways.

Path 1. Observe that by the same proof as in Theorem 3.1, the assumption implies property (M). So the claim follows from Remark 3.2.

Path 2. We shall use a result of O. Kalenda [20, Proposition 5.1]. In fact, we need only part of the statement there. It can be reproduced as follows:

Lemma 3.1 *Let K be a compact space and let h be a homeomorphism from K into $\mathbb{R}^{\Gamma}$. Put $S := h^{-1}(\Sigma(\Gamma))$. Then there exists an index set $\widetilde{\Gamma}$ and a linear w^*-continuous injection $\widetilde{h}$ from $C(K)^*$ into $\mathbb{R}^{\widetilde{\Gamma}}$ such that $\delta(S) \subset \widetilde{h}^{-1}(\Sigma(\widetilde{\Gamma}))$.*

Proof The reader will understand that the construction in the proof of Proposition 3.2 above is here pushed further: As before, put $p_\gamma := \pi_\gamma \circ h$, where $\pi_\gamma : \mathbb{R}^{\Gamma} \to \mathbb{R}$ is the γ-th projection for $\gamma \in \Gamma$. The family $\{p_\gamma : \gamma \in \Gamma\}$ ($\subset C(K)$) separates points of $\delta(K)$, and

$$S = \{k \in K : |\{\gamma \in \Gamma : p_\gamma(k) \neq 0\}| \leq \aleph_0\}.$$

Let $\widetilde{\Gamma}$ the set of all (possibly empty) finite sequences of elements in Γ. Given $\widetilde{\gamma} \in \widetilde{\Gamma}$, let us define the element $f_{\widetilde{\gamma}} \in C(K)$ as

$$f_{\widetilde{\gamma}} := \begin{cases} 1_K & \text{if } \widetilde{\gamma} = \emptyset, \\ p_{\gamma_1} \dots p_{\gamma_n} & \text{if } \widetilde{\gamma} = (\gamma_1, \dots, \gamma_n). \end{cases}$$

It follows from the Stone–Weierstrass theorem that

$$\overline{\operatorname{span}}^{\|\cdot\|_\infty}(\{f_{\widetilde{\gamma}} : \widetilde{\gamma} \in \widetilde{\Gamma}\}) = C(K).$$

Define the mapping $\widetilde{h} : C(K)^* \to \mathbb{R}^{\widetilde{\Gamma}}$ by the formula

$$\widetilde{h}(\mu)(\widetilde{\gamma}) := \langle f_{\widetilde{\gamma}}, \mu \rangle, \ \mu \in C(K)^*, \ \widetilde{\gamma} \in \widetilde{\Gamma}.$$

Then $\widetilde{h}$ is a w^*-continuous one-to-one linear mapping. The last assertion of the statement should be clear from the definition of S. □

Path 2 continued. Since K is Corson, $K \subset S$, so $\widetilde{h}(\delta(K)) \subset \Sigma(\widetilde{\Gamma})$. But $\Sigma(\widetilde{\Gamma})$ is a countably closed linear subspace of $\mathbb{R}^{\widetilde{\Gamma}}$, so the assumption implies that $\widetilde{h}(C(K)^*) \subset \Sigma(\widetilde{\Gamma})$, and in particular, $\widetilde{h}(B_{C(K)^*}) \subset \Sigma(\widetilde{\Gamma})$. Thus, $(B_{C(K)^*}, w^*)$ is Corson and $C(K)$ is WLD. □

Proof of Theorem 3.3. This is now just a successive application of Propositions 3.1–3.3. □

Remark 3.5 Observe that the argument in Path 2 can be adapted to obtain the claim directly from the property captured in Proposition 3.1 skipping the (more elementary) proof of Proposition 3.2.

In fact, this argument is also behind the scenes in Path 1. Indeed, Lemma 3.1 is the first step for an alternative proof of the aforementioned theorem of Argyros, Mercourakis, and Negrepontis involving property (M) (see, e.g., [13, Theorem 5.57]) Ⓡ

The space $C[0, \omega_1]$ admits an equivalent Fréchet differentiable norm by a result of M. Talagrand (cf., e.g., [7, p. 313]). As $[0, \omega_1]$ is a Valdivia compact space but not Corson compact (see the proof at the beginning of this subsection), it follows from Theorem 3.3 that $C[0, \omega_1]$ does not have property $\mathscr{P}$ (Theorem 3.4 can also be applied, since $\operatorname{dens} C[0, \omega_1] = \aleph_1$). However, in this case we can state a more precise result. Namely, we get the following proposition:

Proposition 3.4 (Fabian, Montesinos, Zizler [10]) *The space of continuous functions* $C[0, \omega_1]$ *on the interval* $[0, \omega_1]$ *does not admit any equivalent Gâteaux differentiable norm that is* $[0, \omega_1)$*-lower semicontinuous.*

Proof Assume that $\|\cdot\|$ is an equivalent Gâteaux differentiable norm on $C[0, \omega_1]$ that is $[0, \omega_1)$-lower semicontinuous. Put

$$N := \overline{\operatorname{span}}^{\|\cdot\|}\{\delta_\alpha : \alpha < \omega_1\}.$$

The space N is obviously 1-norming for $(C[0, \omega_1], \|\cdot\|_\infty)$. By the assumption, it is also 1-norming for $(C[0, \omega_1], \|\cdot\|)$. It follows from Corollary 1.1 that any point of $C[0, \omega_1]^*$, in particular δ_{ω_1}, lies in the weak*-closure of the linear hull S of a countable set $\{\delta_{\alpha_i}\}, \alpha_i < \omega_1$ for all i. Let $\beta = \sup\{\alpha_i\}_i$. Then $\beta < \omega_1$. Let $f \in C[0, \omega_1]$ be such that $f(\alpha_i) = 0$ for all i and $f(\omega_1) = 1$. Then f equals to zero on S and equal to 1 at δ_{ω_1}. This shows that δ_{ω_1} is not in the weak* closure of S. This contradiction finishes the proof. □

For Fréchet differentiable norms the situation is different. The following Proposition slightly improves a result in [10]. Here, the class of norming subspaces to be considered reduces to the closed hyperplanes of X^*.

Proposition 3.5 *Assume that X is a Banach space having property $\mathscr{P}$ for closed norming hyperplanes and for Fréchet differentiability. Then X is reflexive.*

Proof Let $x^{**} \in X^{**} \setminus X$ and denote by N the kernel of x^{**} in X^*; by Proposition 1.1, this is a norming hyperplane of X^*. Let $\|\cdot\|$ be an equivalent Fréchet differentiable norm on X that is N-lower semicontinuous. It follows from Proposition 1.2 that every element of $S_{(X^*,\|\cdot\|)}$ that attains the norm $\|\cdot\|$ is in N. By the Bishop-Phelps Theorem we get $N = X^*$, a contradiction. □

3.3 Plichko Spaces with Property $\mathscr{P}$

The effect Lemma 3.1 has on a $C(K)$ space, when K is Valdivia, can also be desribed in other Banach spaces. Let us recall the necessary notions (see, e.g., [20, Definition 4.11]). Let X be a Banach space. A subspace $S \subset X^*$ is called a **Σ-subspace** of X^* if for some Γ there is a linear one-to-one w^*-continuous mapping $T : X^* \to \mathbb{R}^\Gamma$ such that $S = T^{-1}(\Sigma(\Gamma))$. The space X is called **Plichko** if X^* contains a norming Σ-subspace, and X is called **1-Plichko** if X^* contains a 1-norming Σ-subspace. For any Plichko space X, the set Γ can be chosen to be a subset of X. Those classes were introduced by A. Plichko himself (see, e.g., [19] and references therein).

Apart from $C(K)$ spaces for K Valdivia, the class of 1-Plichko spaces encompasses, e.g., order-continuous Banach lattices—in particular, abstract L-spaces (see [20, Examples 6.9–6.12]).

Obviously, X is WLD whenever X^* is a Σ-subspace of itself. The reverse implication also holds (see, e.g., [20, Definition 4.11 and Theorem 4.17]).

The same reasoning as in the proof of Theorem 3.3 can be applied to Plichko spaces enjoying property $\mathscr{P}$. The following observations are from [17].

Proposition 3.6 *Let X be a Banach space. The following are equivalent:*

(i) *X is WLD.*
(ii) *X is 1-Plichko and every 1-norming subspace of X^* is countably w^*-dense.*
(iii) *X is Plichko and every norming subspace of X^* is countably w^*-dense.*

Proof (i)⇒(ii) and (i)⇒(iii): Since X^* is a Σ-subspace of itself, X is 1-Plichko. The rest follows from the angelicity of (B_{X^*}, w^*).

(ii)⇒(i) and (iii)⇒(i): Since X is Plichko, there is a norming subspace $N \subset X^*$ that is a Σ-subspace of X^*. Being a Σ-subspace, it is countably w^*-closed but by assumption also countably w^*-dense in X^*, so it coincides with X^*. □

Combining Corollary 1.1 and Proposition 3.6 gives the following variant of Theorem 3.3 above.

Corollary 3.1 *Let X be a Plichko Banach space with property $\mathscr{P}$. Alternatively, let X be a 1-Plichko Banach space such that for every 1-norming subspace $N \subset X^*$ there exists an equivalent Gâteaux differentiable $w(X, N)$-lower semicontinuous norm on X. Then X is WLD.*

3.4 Properties $\mathscr{P}$ and Mazur's

Again, the following theorem slightly improves a result in [10] by reducing the number of norming subspaces to be considered. Recall that (X^*, w^*) is said to have the **Mazur property** whenever every sequentially continuous linear form on (X^*, w^*) is continuous, i.e., an element in X. Note the following: Every sequentially continuous linear form f on (X^*, w^*) is already an element in X^{**}. Indeed, if $f \neq 0$ and $K := \ker(f) \subset X^*$, then K is a w^*-sequentially closed proper hyperplane in X^*. Let $x^* \in \overline{K}^{\|\cdot\|}$. Find a sequence (x_n^*) in K that $\|\cdot\|$-converges to x^*. It is also w^*-convergent, so $(0 =)\ f(x_n^*) \to f(x^*)$, hence $x^* \in K$. Since K is $\|\cdot\|$-closed, f is $\|\cdot\|$-continuous, i.e., $f \in X^{**}$.

Theorem 3.6 *Assume that a Banach space X has property $\mathscr{P}$ for closed norming hyperplanes of X^*. Then (X^*, w^*) has the Mazur property.*

Proof Assume there is $x^{**} \in S_{X^{**}} \setminus X$ that is weak* sequentially continuous. Let N be the kernel of x^{**} in X^*. By Proposition 1.1, N is a norming hyperplane of X^* which is, moreover, weak*-sequentially closed. Assume that an equivalent norm $\|\cdot\|$ on X is Gâteaux differentiable and N-lower semicontinuous. Then, by Proposition 1.2, every element of $S_{(X^*, \|\cdot\|)}$ that attains its $\|\cdot\|$ norm belongs to N. By the Bishop-Phelps theorem, $N = X^*$, a contradiction. □

Remark 3.7 Note that in case that (B_{X^*}, w^*) is angelic, the conclusion of Theorem 3.6 is true without any other hypothesis on the space X. This can be seen as a consequence of the last statement in Lemma 1.1. Indeed, $(X, \mu(X, X^*)$ is complete —note that this topology is just the norm topology—, so under the angelicity of (B_{X^*}, w^*) the space (X^*, w^*) is already Mazur. An alternative argument is the following: Let $x^{**} \in X^{**}$ be w^*-sequentially continuous. Assume that $x^{**} \notin X$. Let $N := \ker x^{**}$, a closed and norming hyperplane of X^* (see Proposition 1.1). If $x^* \in \overline{B_N}^{w^*}$, there exists, by the angelicity hypothesis, a sequence (x_n^*) in B_N that w^*-converges to x^*. This shows that $\langle x^{**}, x^* \rangle = 0$, hence $x^* \in B_N$. Thus, $B_N = \overline{B_N}^{w^*}$. The Krein–Šmulyan theorem (see, e.g., [13, Corollary 3.94]) shows that $x^{**} \in X$, a contradiction. However, it is not necessary to rely on this theorem. Note, simply, that the contradiction is reached because in this case we know that $(B_N =)\ \overline{B_N}^{w^*}$ contains a multiple of B_{X^*}. Ⓡ

4 Some Open Problems

1. We proved (Proposition 2.1) that every SWCG enjoys the strong property $\mathscr{P}$. Does the converse hold?
2. Along this paper, Property $\mathscr{P}$ has been substituted in some of the statements by the formally weaker one defined in terms of the family of closed norming hyperplanes. We do not know whether this requirement is really weaker. So we formulate: Assume that for every $x^{**} \in X^{**} \setminus X$ the space X admits an equivalent Gâteaux differentiable norm that is Y-lower semicontinuous, where Y is the kernel of x^{**} in X^*. Does X necessarily have property $\mathscr{P}$?
3. Are properties $\mathscr{P}$ and strong $\mathscr{P}$ (see Definition 2.1) the same?
4. Is it possible to prove the result of Deville and Godefroy in [6] quoted in Sect. 3.2 by showing that in case K is a Valdivia compact space that does not contain a copy of $[0, \omega_1]$, then $C(K)$ enjoys property $\mathscr{P}$? Recall that here it is proved (see the paragraph after the proof of Theorem 3.3) that $C[0, \omega_1]$ does not have property $\mathscr{P}$.
5. Theorem 3.6 and Remark 3.7 suggest that property $\mathscr{P}$ may have a description in terms of topological properties of (B_{X^*}, w^*).

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Separable (and Metrizable) Infinite Dimensional Quotients of $C_p(X)$ and $C_c(X)$ Spaces

In Honour of Manuel López-Pellicer

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Abstract The famous Rosenthal-Lacey theorem states that for each infinite compact set K the Banach space $C(K)$ of continuous real-valued functions on a compact space K admits a quotient which is either an isomorphic copy of c or ℓ_2. Whether $C(K)$ admits an infinite dimensional separable (or even metrizable) Hausdorff quotient when the uniform topology of $C(K)$ is replaced by the pointwise topology remains as an open question. The present survey paper gathers several results concerning this question for the space $C_p(K)$ of continuous real-valued functions endowed with the pointwise topology. Among others, that $C_p(K)$ has an infinite dimensional separable quotient for any compact space K containing a copy of $\beta\mathbb{N}$. Consequently, this result reduces the above question to the case when K is a *Efimov space* (i.e. K is an infinite compact space that contains neither a non-trivial convergent sequence nor a copy of $\beta\mathbb{N}$). On the other hand, although it is unknown if Efimov spaces exist in ZFC, we note under $\diamondsuit$ (applying some result due to R. de la Vega), that for some Efimov space K the space $C_p(K)$ has an infinite dimensional (even metrizable) separable quotient. The last part discusses the so-called Josefson–Nissenzweig property for spaces $C_p(K)$, introduced recently in [3], and its relation with the separable quotient problem for spaces $C_p(K)$.

Keywords The separable quotient problem · Spaces of continuous functions · Quotient spaces · The Josefson-Nissenzweig theorem · Efimov space

1 Introduction

Let X be a Tychonoff space. By $C_p(X)$ and $C_c(X)$ we denote the space of real-valued continuous functions on X endowed with the pointwise and the compact-open topology, respectively.

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The classic Rosenthal-Lacey theorem, see [21, 27, 30], asserts that the Banach space $C(K)$ of continuous real-valued functions on an infinite compact space K has a quotient isomorphic to c or ℓ_2, or equivalently, there exists a continuous linear (and open; by the open mapping Banach theorem) map from $C(K)$ onto c or ℓ_2.

The proof of this remarkable result splits into two cases:

(i) *K is scattered*. Then K contains a convergent sequence (x_n) of distinct points. The linear map $T : C(K) \to c$, $f \mapsto (f(x_n))$ is a continuous surjection. Hence the quotient $C(K)/T^{-1}(0)$ is isomorphic to c.

(ii) *K is not scattered*. Then K is continuously mapped onto $[0, 1]$. The space ℓ_2 is isomorphic to a closed subspace of $L_1[0, 1]$ and $l_1[0, 1]$ is isomorphic to a closed subspace $L_1(K, \mathfrak{B}_K, \mu)$, where μ is some nonnegative finite regular Borel measure on K. The latter space is isomorphic to a closed subspace of the norm dual Y of $C(K)$. Therefore the reflexive space ℓ_2 is a subspace of Y that is weak*-closed, and then a quotient of $C(K)$ is isomorphic to ℓ_2, see [30, Corollary 1.6, Proposition 1.2] for details.

This theorem motivates the following natural question (first discussed in [24]):

Problem 1.1 (*Kąkol, Śliwa*) Does $C_p(K)$ admit an infinite-dimensional separable quotient (SQ in short) for any infinite compact space K?

Note the following simple

Fact: *Each metrizable (linear) quotient $C_p(X)/Z$ of $C_p(X)$ by a closed vector subspace Z of $C_p(X)$ is separable.*

Indeed, this follows from the separability of metizable spaces of countable cellularity and the fact that $C_p(X)$ has countable cellularity, being a dense subspace of $\mathbb{R}^X$, see [2].

We can also argue as follows: It is well-known that the space $C_p(X)$ carries its weak locally convex topology. As the weak topology is inherited by each Hausdorff quotient, and if additionally this quotient is metrizable, then it is a topological subspace of $\mathbb{R}^{\mathbb{N}}$; for a discussion of most relevant facts concerning the weak topology we refer the reader to [20].

The above Rosenthal-Lacey theorem motivates also the following (particular) question related with Problem 1.1 for spaces $C_p(X)$.

Problem 1.2 (*Banakh, Kąkol, Śliwa*) For which compact spaces K any of the following equivalent conditions holds:

1. The space $C_p(K)$ has an infinite dimensional metrizable quotient.
2. The space $C_p(K)$ has an infinite dimensional metrizable separable quotient.
3. The space $C_p(K)$ has a quotient isomorphic to a dense subspace of $\mathbb{R}^{\mathbb{N}}$.

The present paper gathers (with possible discussion) several recently obtained results concerning Problems 1 and 2.

2 First Motivations and the Case $C_p(\beta\mathbb{N})$

The closed ideals of $C_p(X)$ are precisely the spaces

$$\mathfrak{F}_A = \{f \in C(X) : f(x) = 0 \ \forall x \in A\},$$

where A ranges over the closed subsets of X. These are also the closed ideals of $C_c(X)$. An algebra quotient of $C_c(X)$ or $C_p(X)$ is one by a closed ideal, thus preserving vector multiplication. In Rosenthal's Case (i) the quotient is an algebra quotient, since the kernel of T is $\mathfrak{F}_A$ with $A := \{x_0, x_1, \dots\}$.

In [23] we proved the following general

Theorem 2.1 (Kąkol, Saxon) *For a Tychonoff space X the following assertions are equivalent:*

1. *X contains a closed infinite countable subset D.*
2. *$C_c(X)$ admits an infinite dimensional separable quotient algebra.*
3. *$C_p(X)$ admits an infinite dimensional separable quotient algebra.*

We need the following

Lemma 2.1 *If X has no infinite countable closed subset, then $C_p(X)$ is not separable.*

Proof Let $f_1, f_2, \dots \in C(X)$ be arbitrary. We desire $y_1 \neq y_2$ in X with

$$|f_n(y_1) - f_n(y_2)| \leq 1, \quad n \in \mathbb{N}.$$

By our assumption, every infinite countable set in X has more than one cluster point in X. Fix a cluster point y_1 in X. By continuity we can choose a strictly decreasing sequence of closed neighborhoods V_n of y_1 such that each $f_n(V_n)$ has diameter no larger than 1.

We choose $x_n \in V_n \setminus V_{n+1}$ and let y_2 be a cluster point of $\{x_n\}_n$ distinct from y_1. Since all but finitely many of the x_k are in a selected V_n, this closed set contains the cluster point y_2. In fact, then, the displayed inequality holds for each n. One shows that there exists $h \in C(X)$ with $h(y_1) = 5$ and $h(y_2) = 9$. If we assume some $f_n \in h + [\{y_1, y_2\}, 1]$, we have

$$|f_n(y_1) - f_n(y_2)| \geq (9 - 5) - 1 - 1 = 2,$$

a contradiction. Thus the arbitrary sequence in $C(X)$ is not dense in $C_p(X)$.

Proof (*Skech of the proof of Theorem* 2.1) [(1) ⇒ (2)]. If D admits a compact infinite subset, the sequence space c is a (separable) algebra quotient of $C_c(X)$. If D has no such subset, we may assume D has no cluster points, so that $C_c(X)/\mathfrak{F}_D = C_p(X)/\mathfrak{F}_D$, which is isomorphic to a dense subspace of the metrizable separable $\mathbb{R}^D$. Hence the algebra quotient is separable.

[(2) $\Rightarrow$ (3)]. If $C_c(X)/\mathfrak{F}_A$ is separable, so is $C_p(X)/\mathfrak{F}_A$.

[(3) $\Rightarrow$ (1)]. If A is closed in X and $C_p(X)/\mathfrak{F}_A$ is separable, then so is $C_p(A)$. Since A is infinite, (the contrapositive of) Lemma 2.1 shows that A has a closed countable infinite subset D. Thus D is closed in X, and (1) holds.

This theorem shows, for example, that the space $C_p(\beta\mathbb{N})$ fails to have an infinite dimensional separable quotient algebra. Motivated by this fact one can ask

Problem 2.1 (*Kąkol, Śliwa*) Does the space $C_p(\beta\mathbb{N})$ admit an infinite dimensional separable (even metrizable) quotient?

If K contains a non-trivial convergent sequence, say $x_n \to x_0$, then for $A := \{x_n : n \in \mathbb{N}\} \cup \{x_0\}$, the space $C_p(K)$ has a quotient isomorphic to the infinite-dimensional separable and metrizable space $C_p(A)$. Many compact spaces contain non-trivial convergent sequences; particularly Valdivia compact spaces, by Kalenda's result [26]. They are plentiful, indeed:

$$metrizable\, compact \Rightarrow Eberlein\, compact \Rightarrow Talagrand\, compact \Rightarrow Gulko\, compact$$
$$\Rightarrow Corson\, compact \Rightarrow Valdivia\, compact.$$

Let K be an infinite compact space. If K is scattered, then it contains a non-trivial convergent sequence. If K is not scattered, then there exists a continuous map from K onto [0, 1] but this property seems to be not so helpful for $C_p(K)$.

Nevertheless, we will see that a stronger condition

$$(+)\ K\, is\, continuously\, mapped\, onto\, [0, 1]^{\mathfrak{c}},$$

implies that $C_p(K)$ has SQ, see Theorem 2.2 below. Recall that the condition (+) is equivalent to the fact that K contains a copy of $\beta\mathbb{N}$, see [31].

On the other hand, we have the following easy fact.

Proposition 2.1 *For any infinite compact K the space $C_p(K)$ can be mapped onto an infinite-dimensional separable metrizable locally convex space by a continuous linear map.*

Proof If K is separable, $C_p(K)$ has countable pseudocharacter [2, Theorem 1.1.4]. Hence $C_p(K)$ admits a weaker metrizable and separable locally convex topology, see [16, Lemma 3.2]. If K is arbitrary, choose a compact separable infinite subset L and apply the previous case using the restriction surjective map $C_p(K) \to C_p(L)$.

Clearly Problem 1.1 is motivated by Rosenthal-Lacey theorem, but one can provide more specific motivations. Let X be a Tychonoff space.

1. *If X is of pointwise countable type, then $C_c(X)$ has a quotient isomorphic to either $\mathbb{R}^{\mathbb{N}}$ or c or ℓ_2*, see [23, Corollary 22].
2. *$C_c(X)$ has SQ provided $C_c(X)$ is barrelled*, i.e. every closed absolutely convex absorbing set in $C_c(X)$ is a neighborhood of zero, see [22]. Indeed, X is

pseudocompact if and only if the set $B := \{f \in C(X) : |f(x)| \leq 1, x \in X\}$ is a barrel in $C_c(X)$. If X is pseudocompact and $C_c(X)$ is barrelled, then B is a neighborhood of zero; equivalently X is compact and $C_c(X)$ is a Banach space and the Rosenthal-Lacey theorem applies. If X is not pseudocompact we apply Proposition 2.2 below.

3. *If X is an infinite product of completely regular Hausdorff spaces of cardinality at least two, then $C_p(X)$ has SQ*. Indeed, as X contains a compact metrizable infinite subset Y, the space $C_p(X)$ has a quotient isomorphic to $C_p(Y)$.

Hence, for example, $C_c(X)$ *has SQ whenever X is metrizable or hemicompact.* Recall that all first countable spaces and all locally compact spaces are of pointwise countable type, see [12].

In [23, Corollary 11] we proved that for a fixed Tychonoff space X, if $C_p(X)$ has SQ, then also $C_c(X)$ has SQ. Conversely, if $C_c(X)$ has SQ and for every infinite compact $K \subset X$ the space $C_p(K)$ has SQ, then $C_p(X)$ has also an infinite dimensional separable quotient. Indeed, two cases are possible.

1. *Every compact subset of X is finite.* Then $C_p(X) = C_c(X)$ and hence $C_p(X)$ has SQ.
2. *X contains an infinite compact subset K*. Then $C_p(K)$ has SQ (by assumption). Since the restriction map $f \to f|K$, $f \in C(X)$ is a continuous open surjection from $C_p(X)$ onto $C_p(K)$, the desired conclusion holds.

Problem 1.1 has a simple solution for Tychonoff spaces which are not pseudocompact. We recall that a Tychonoff space X is *pseudocompact* if each continuous real-valued function on X is bounded.

Proposition 2.2 *For a Tychonoff space X the following conditions are equivalent:*

1. *X is not pseudocompact;*
2. *$C_p(X)$ has a subspace, isomorphic to $\mathbb{R}^{\mathbb{N}}$;*
3. *$C_p(X)$ has a quotient space, isomorphic to $\mathbb{R}^{\mathbb{N}}$;*
4. *$C_p(X)$ admits a linear continuous map onto $\mathbb{R}^{\mathbb{N}}$.*

Proof The implication (1) $\Rightarrow$ (2) is proved in Theorem 14 of [23] and (2) $\Rightarrow$ (3) follows from the complementability of $\mathbb{R}^{\mathbb{N}}$ in any locally convex space containing it, see [29, Corollary 2.6.5]. The implication (3) $\Rightarrow$ (4) is trivial. To see that (4) $\Rightarrow$ (1), observe that for a pseudocompact space X the function space $C_p(X)$ is σ-bounded, since it can be written as the countable union

$$C_p(X) = \bigcup_{n=1}^{\infty} \{f \in C_p(X) : \sup_{x \in X} |f(x)| \leq n\}$$

of bounded subsets. Then the image of $C_p(X)$ under any linear continuous operator also is σ-bounded. On the other hand, the Baire Theorem ensures that the space $\mathbb{R}^{\mathbb{N}}$ is not σ-bounded.

The following theorem (from [24, Theorem 4]) shows that $C_p(K)$ has SQ for any compact space K containing a copy of $\beta\mathbb{N}$.

Theorem 2.2 (Kąkol, Śliwa) *Let X be a Tychonoff space with a sequence (K_n) of non-empty compact subsets such that for any $n \geq 1$ the set K_n contains two disjoint subsets homeomorphic to K_{n+1}. Then $C_p(X)$ has SQ. Consequently, if K is a compact space which contains a copy of $\beta\mathbb{N}$, then $C_p(K)$ has SQ.*

For the proof we need to show that there exists a sequence (ξ_n) of non-zero continuous linear functionals on $C_p(X)$ such that

$$E := \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \ker \xi_n \subset C_p(X)$$

is dense in $C_p(X)$, and then apply the main result of [25, Proposition 1] or [24, Lemma 7].

Consequently, the above theorem reduces Problem 1.1 to the case when K is a *Efimov space* (i.e. K is an infinite compact space that contains neither a non-trivial convergent sequence nor a copy of $\beta\mathbb{N}$). Indeed, if X contains a non-trivial convergent sequence, then (as we have already noticed) $C_p(X)$ contains a complemented metrizable (hence separable) subspace. If X contains a copy of $\beta\mathbb{N}$, then $C_p(X)$ (by Theorem 2.1) has SQ (even metrizable by Corollary 2.3).

Although, it is unknown if Efimov spaces exist in ZFC (see [8–11, 14, 15, 17, 19]) we showed in [24] that under $\diamondsuit$ for some Efimov spaces K the function space $C_p(K)$ has SQ.

Theorem 2.1 implies the following

Corollary 2.1 *Let X be a normal topological space with a sequence (S_n) of non-empty closed subsets such that for any $n \geq 1$ the set S_n contains two disjoint closed subsets S_n' and S_n'' that are homeomorphic to S_{n+1}. Then $C_p(\beta X)$ has SQ.*

Proof Let $n \geq 1$. Denote by K_n, K_n', K_n'' and K_{n+1} the closures in βX of the sets S_n, S_n', S_n'' and S_{n+1}, respectively. Then K_n' and K_n'' are compact and disjoint subsets of K_n that are homeomorphic to K_{n+1} by [12, Corollaries 3.6.4 and 3.6.8]. Using the last theorem, we infer that $C_p(\beta X)$ has SQ.

Corollary 2.2 *If K is an infinite compact space and every infinite closed set in K contains two infinite disjoint homeomorphic closed sets, then $C_p(K)$ has SQ.*

It is natural to ask if every infinite compact space K satisfies the assumption of the above corollary. The answer (in the negative) follows from the following Example 2.1 below.

Recall that a compact space K is a *Koszmider space*, see [13], if all operators on $C(K)$ have the form $gI + S$, where $g \in C(K)$ and S is weakly compact. *If K is a connected Koszmider space then $C(K)$ is indecomposable, i.e. there are no infinite-dimensional closed subspaces Y and Z such that $C(K) = Y \oplus Z$*, see [13, Lemma 2.6].

Example 2.1 Under $\diamondsuit$ there exists a separable Efimov space F such that F is a Koszmider space and does not admit two disjoint homeomorphic infinite closed subsets.

Proof Let K be the compact connected space as in [13, Theorem 5.2]. Let F be an infinite separable closed subset of K. Assume that F contains two closed infinite disjoint homeomorphic subsets L_1 and L_2. Put $L := L_1 \cup L_2$. This generates a homeomorphism $\phi : L \to L$ which is not the identity. Then the composition operator

$$C_\phi : C(L) \to C(L), \quad C_\phi(g) := g \circ \phi,$$

provides an operator which contradicts [13, Theorem 5.3]. Then F does not contain $\beta\mathbb{N}$. Moreover, F does not contain non-trivial convergent sequences. Indeed, otherwise $C(K)$ is not a Grothendieck space, so $C(K)$ contains a complemented copy of c_0, see [5, Corollary 2], so that $C(K)$ is not indecomposable, a contradiction with the above remark.

Remark 2.3 As every separable compact space is a continuous image of $\beta\mathbb{N}$, the space F from above Example 2.1 enjoys this property. Therefore we conclude that the construction provided by Theorem 2.1 (which applies to $\beta\mathbb{N}$) is not inherited by continuous surjections.

Having in mind the above results one can also formulate the following

Problem 2.2 Let K be an infinite compact space and assume that $C_p(K)$ has an infinite-dimensional separable quotient. Does $C_p(K)$ admit an infinite-dimensional metrizable quotient?

The main results of the paper [4] are the following two Theorems 2.4 and 2.5 below giving partial answers to Problems 1.2 and 2.2, the proof of the first theorem will be provided below. A subspace A of a topological space X is called *C^*-embedded* if each bounded continuous function $f : A \to \mathbb{R}$ has a continuous extension $\bar{f} : X \to \mathbb{R}$.

Theorem 2.4 (Banakh, Kąkol, Śliwa) *If a pseudocompact Tychonoff space X contains an infinite discrete C^*-embedded subspace D, then the function space $C_p(X)$ has an infinite-dimensional metrizable quotient. More precisely, for any sequence $(F_n)_{n=1}^{\infty}$ of non-empty, finite and pairwise disjoint subsets of D with $\lim_n |F_n| = \infty$ and the closed linear subspace*

$$Z = \bigcap_{n=1}^{\infty} \{f \in C_p(X) : \sum_{x \in F_n} f(x) = 0\}$$

the quotient space $C_p(X)/Z$ is isomorphic to the subspace $\ell_\infty = \{(x_n) \in \mathbb{R}^{\mathbb{N}} : \sup_n |x_n| < \infty\}$ of $\mathbb{R}^{\mathbb{N}}$.

If a Tychonoff space X is compact, then X contains an infinite discrete C^*-embedded subspace if and only if X contains a copy of $\beta\mathbb{N}$. On the other hand, the space ω_1 is pseudocompact noncompact which does not contain C^*-embedded infinite discrete subspaces. Moreover, the space $\Lambda := \beta\mathbb{R} \setminus (\beta\mathbb{N} \setminus \mathbb{N})$ discussed in [18, 6P, p. 97] is pseudocompact, noncompact and contains $\mathbb{N}$ as a closed discrete C^*-embedded set.

We note the following extension of the second part of Theorem 2.1.

Corollary 2.3 *For any infinite discrete space D the space $C_p(\beta D)$ has a quotient space, isomorphic to the subspace ℓ_∞ of $\mathbb{R}^{\mathbb{N}}$.*

Theorem 2.4 and Proposition 2.2 yield immediately

Corollary 2.4 *For each Tychonoff space X containing a C^*-embedded infinite discrete subspace, the function space $C_p(X)$ has an infinite-dimensional metrizable quotient, isomorphic to $\mathbb{R}^{\mathbb{N}}$ or ℓ_∞.*

Besides the subspace ℓ_∞ of $\mathbb{R}^{\mathbb{N}}$, the following corollary of Theorem 2.4 involves also the subspace $c_0 := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} x_n = 0\}$ of $\mathbb{R}^{\mathbb{N}}$.

Corollary 2.5 *If a compact Hausdorff space X is not Efimov, then the function space $C_p(X)$ has a quotient space, isomorphic to the subspaces ℓ_∞ or c_0 in $\mathbb{R}^{\mathbb{N}}$.*

Proof The space X, being non-Efimov, contains either an infinite converging sequence or a copy of $\beta\mathbb{N}$. In the latter case X contains an infinite discrete C^*-embedded subspace and Theorem 2.4 implies that $C_p(X)$ has a quotient space, isomorphic to $\ell_\infty \subset \mathbb{R}^{\mathbb{N}}$. If X contains a sequence $(x_n)_{n\in\mathbb{N}}$ of pairwise distinct points that converges to a point $x \in X$, then for the compact subset $K := \{x\} \cup \{x_n\}_{n\in\mathbb{N}}$ of X the space $C_p(K)$ is isomorphic to $c_0 \subset \mathbb{R}^{\mathbb{N}}$ and is isomorphic to a complemented subspace in $C_p(X)$, see [1, Theorem 1, p.130, Proposition 2, p.128].

The following theorem (proved in [4]) extends the previous Theorem 2.1; its proof is much more technical and complicated than the corresponding one for Theorem 2.1, and the reader is referred to [4] for details.

Theorem 2.5 *For a Tychonoff space X the space $C_p(X)$ has a metrizable infinite-dimensional quotient if there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of non-empty compact subsets of X such that for every $n \in \mathbb{N}$ the compact set K_n contains two disjoint topological copies of K_{n+1}.*

Note however that, it seems to be not clear if in Theorem 2.5 the obtained quotient is isomorphic to ℓ_∞ or c_0. Example 2.2 below provides an Efimov space K (under $\Diamond$) for which Theorem 2.5 applies.

Example 2.2 Under $\Diamond$ there exists an Efimov space K whose function space $C_p(K)$ has a metrizable infinite-dimensional quotient.

Proof De la Vega [33, Theorem 3.22] (we refer also to [32]) constructed under $\Diamond$ a compact zero-dimensional hereditary separable space K (hence not containing a copy of $\beta\mathbb{N}$) such that:

(i) K does not contain non-trivial convergent sequences.
(ii) K has a base of clopen pairwise hemeomorphic sets.

Observe that K admits a sequence (K_n) of infinite compact subsets such that each K_n contains two disjoint subsets homeomorphic to K_{n+1}; therefore by Theorem 2.5 the space $C_p(K)$ has the desired property.

As the space K from Example 2.2 does not contain $\beta\mathbb{N}$, the assumption of Theorem 2.4 is not satisfied. Note that in Example 2.1 we provided (again under $\diamondsuit$) a Efimov space K for which Theorem 2.5 cannot be applied.

3 Proof of Theorem 2.4

Let D be an infinite discrete C^*-embedded subspace of a pseudocompact Tychonoff space X. Choose any sequence $(F_n)_{n\in\mathbb{N}}$ of non-empty, finite and pairwise disjoint subsets of D with

$$\lim_{n\to\infty} |F_n| = \infty.$$

For every $n \in \mathbb{N}$ consider the probability measure

$$\mu_n = \frac{1}{|F_n|}\sum_{x\in F_n} \delta_x,$$

where δ_x is the Dirac measure concentrated at x.

The pseudocompactness of the space X enables us to derive that the linear continuous operator

$$T : C_p(X) \to \ell_\infty \subset \mathbb{R}^{\mathbb{N}}, \quad T : f \mapsto (\mu_n(f))_{n\in\mathbb{N}}$$

is well-defined.

CLAIM: *The operator T is open*. Indeed, take a neighborhood $U \subset C_p(X)$ of zero. We need to show that $T(U)$ is a neighborhood of zero in ℓ_∞. We can assume that U is of the basic form

$$U := \{f \in C_p(X) : \max_{x\in E} |f(x)| < \varepsilon\}$$

for some finite set $E \subset X$ and some $\varepsilon > 0$.

Choose a number $m \in \mathbb{N}$ such that

$$\inf_{k>m} |F_k| \geq 2(|E| + 1).$$

We claim that $T(U)$ contains the open neighborhood

$$V := \{(y_k)_{k=1}^{\infty} \in \ell_\infty : \max_{k \le m} |x_k| < \varepsilon\}$$

of zero in $\ell_\infty \subset \mathbb{R}^{\mathbb{N}}$.

Fix any sequence $(y_k)_{k=1}^{\infty} \in V$. Choose any partition $\mathscr{P}$ of the set $D \setminus \bigcup_{k=1}^{m} F_k$ into $|\mathscr{P}| = |E| + 1$ pairwise disjoint sets such that for every $P \in \mathscr{P}$ and $k > m$ the intersection $P \cap F_k$ has cardinality

$$|P \cap F_k| \ge \frac{|F_k|}{|E|+1} - 1 \ge 1.$$

Since the discrete subspace D is C^*-embedded in X, the sets in the partition $\mathscr{P}$ have pairwise disjoint closures in X. Since $|\mathscr{P}| > |E|$, we find a set $P \in \mathscr{P}$ whose closure $\bar{P}$ is disjoint with E.

Consider the function $f : D \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} y_k & x \in F_k \text{ for some } k \le m \\ y_k \cdot \frac{|F_k|}{|F_k \cap P|} & \text{if } x \in P \cap F_k \text{for some } k > m \\ 0 & \text{otherwise} \end{cases}$$

The function f is bounded because $\sup_{k \in \mathbb{N}} |y_k| < \infty$ and

$$\sup_{k>m} \frac{|F_k|}{|F_k \cap P|} \le \sup_{k>m} \frac{|F_k|}{\frac{|F_k|}{|E|+1} - 1} = \sup_{k>m} \frac{1}{\frac{1}{|E|+1} - \frac{1}{|F_k|}} \le \frac{1}{\frac{1}{|E|+1} - \frac{1}{2(|E|+1)}} = 2(|E|+1) < \infty$$

Since the space D is discrete and C^*-embedded into X, the bounded function f has a continuous extension $\bar{f} : X \to \mathbb{R}$. As the space X is Tychonoff, there exists a continuous function $\lambda : X \to [0, 1]$ such that $\lambda(\bar{D}) = \{1\}$ and $\lambda(x) = 0$ for all $x \in E \setminus \bar{D}$. Replacing $\bar{f}$ by the product $\bar{f} \cdot \lambda$, we can assume that $\bar{f}(x) = 0$ for all $x \in E \cap \bar{D}$. We claim that $\bar{f} \in U$.

If $x \in E$ we should prove that $|\bar{f}(x)| < \varepsilon$. This holds if $x \notin \bar{D}$. If $x \in F_k$ for some $k \le m$, then $|\bar{f}(x)| = |y_k| < \varepsilon$ as $y \in V$. If

$$x \in \bar{D} \setminus \bigcup_{k=1}^{m} F_k,$$

then $x \in \bar{Q}$ for some set $Q \in \mathscr{P} \setminus \{P\}$. By definition of the function f we note that $f|Q \equiv 0$ and then $|\bar{f}(x)| = 0 < \varepsilon$. This completes the proof of the inclusion $\bar{f} \in U$.

Also by definition of the function f we note that $\mu_k(\bar{f}) = y_k$ for all $k \in \mathbb{N}$. So,

$$(y_k)_{k=1}^{\infty} = T(\bar{f}) \in T(U)$$

and $V \subset T(U)$. This completes the proof of the openness of the operator

$$T : C_p(X) \to \ell_\infty \subset \mathbb{R}^{\mathbb{N}}.$$

Since the kernel of the open operator T equals to

$$Z = \bigcap_{n=1}^{\infty} \{f \in C_p(X) : \mu_n(f) = 0\},$$

the quotient space $C_p(X)/Z$ is isomorphic to the subspace ℓ_∞ of $\mathbb{R}^{\mathbb{N}}$.

4 The Josefson–Nissenzweig Property for C_p-Spaces, Metrizable Quotients

The famous *Josefson–Nissenzweig theorem* asserts that for each infinite dimensional Banach space E there exists a null sequence in the weak*-topology of the topological dual E^* of E and which is of norm one in the dual norm, see for example [7]. We need some extra notations and definitions.

For a Tychonoff space X and a point $x \in X$ let $\delta_x : C_p(X) \to \mathbb{R}, \quad \delta_x : f \mapsto f(x)$, be the Dirac measure concentrated at x. The linear hull $L_p(X)$ of the set $\{\delta_x : x \in X\}$ in $\mathbb{R}^{C_p(X)}$ can be identified with the dual space of $C_p(X)$.

Elements of the space $L_p(X)$ will be called *finitely supported sign-measures* (or simply *sign-measures*) on X.

Each $\mu \in L_p(X)$ can be uniquely written as a linear combination of Dirac measures $\mu = \sum_{x \in F} \alpha_x \delta_x$ for some finite set $F \subset X$ and some non-zero real numbers α_x. The set F is called the *support* of the sign-measure μ and is denoted by $\mathrm{supp}(\mu)$. The measure $\sum_{x \in F} |\alpha_x| \delta_x$ will be denoted by $|\mu|$ and the real number $\|\mu\| = \sum_{x \in F} |\alpha_x|$ coincides with the *norm* of μ (in the dual Banach space $C(\beta X)^*$).

The sign-measure $\mu = \sum_{x \in F} \alpha_x \delta_x$ determines the function $\mu : 2^X \to \mathbb{R}$ defined on the power-set of X and assigning to each subset $A \subset X$ the real number $\sum_{x \in A \cap F} \alpha_x$. So, a finitely supported sign-measure will be considered both as a linear functional on $C_p(X)$ and an additive function on the power-set 2^X.

We propose the following corresponding property for spaces $C_p(X)$, see [3].

Definition 4.1 (*Banakh, Kąkol, Śliwa*) For a Tychonoff space X the space $C_p(X)$ satisfies the *Josefson–Nissenzweig property* (JNP in short) if there exists a sequence (μ_n) of finitely supported sign-measures on X such that $\|\mu_n\| = 1$ for all $n \in \mathbb{N}$, and $\mu_n(f) \to_n 0$ for each $f \in C_p(X)$.

Let's start with the following few observations concerning the JNP:

1. *If a compact space K contains a non-trivial convergent sequence, say $x_n \to x$, then $C_p(K)$ satisfies the JNP.* This is witnessed by the weak* null sequence (μ_n) of sign-measures
$$\mu_n := \frac{1}{2}(\delta_{x_n} - \delta_x), \ \ n \in \mathbb{N}.$$

2. *The space $C_p(\beta\mathbb{N})$ does not satisfy the JNP.* This follows directly from the Grothendieck theorem, see [6, Corollary 4.5.8].
3. *There exists a compact space K containing a copy of $\beta\mathbb{N}$ but without non-trivial convergent sequences such that $C_p(K)$ satisfies the JNP,* see Example in [3].

If a compact space K contains a closed subset Z that is metrizable, then, as easily seen, $C_p(K)$ has a complemented subspace isomorphic to $C_p(Z)$. Therefore, if the compact K contains a non-trivial convergent sequence, then $C_p(K)$ has a complemented subspace isomorphic to c_0. On the other hand, for every infinite compact K the space $C_p(K)$ contains a subspace isomorphic to c_0 but not necessary complemented in $C_p(K)$. Nevertheless, there exists a compact space K without infinite convergent sequences and such that $C_p(K)$ enjoys the JNP and so contains a complemented subspace isomorphic to c_0, this follows from Example 1 in [3].

We propose the following

Problem 4.1 Characterize those compact spaces K for which the space $C_p(K)$ has the JNP.

The following theorem from [3] characterizes those spaces $C_p(X)$ over pseudocompact spaces X to have the JNP.

Theorem 4.1 (Banakh, Kąkol, Śliwa) *Let X be a pseudocompact space. The following assertions are equivalent:*

1. *$C_p(X)$ satisfies the JNP;*
2. *$C_p(X)$ contains a complemented subspace isomorphic to c_0;*
3. *$C_p(X)$ has a quotient isomorphic to c_0;*
4. *$C_p(X)$ admits a linear continuous map onto c_0;*
5. *$C_p(X)$ contains a complemented infinite-dimensional metrizable subspace;*
6. *$C_p(X)$ contains a complemented infinite-dimensional separable subspace;*
7. *$C_p(X)$ has an infinite-dimensional Polishable quotient.*

Recall here that a locally convex space X is *Polishable* if X admits a stronger Fréchet locally convex topology. It is easy to see that the subspace c_0 of $\mathbb{R}^{\mathbb{N}}$ is Polishable.

Last result applies to gather some interesting facts concerning the space $C_p(\beta\mathbb{N})$; the item (1) follows from Corollary 2.3.

Corollary 4.1 *The space $C_p(\beta\mathbb{N})$*

1. *has a quotient isomorphic to ℓ_∞;*
2. *contains a subspace isomorphic to c_0;*
3. *does not admit a continuous linear map onto c_0;*
4. *has no Polishable infinite-dimensional quotients;*
5. *contains no complemented separable infinite-dimensional subspaces.*

Recall that in Theorem 4.1 the JNP describes spaces $C_p(X)$ over pseudocompact X for which $C_p(X)$ contains a complemented metrizable subspace which is Polishable. This fact may suggest a question when $C_p(X)$ is Polishable itself. In [3] we proved the following interesting

Theorem 4.2 *For a Tychonoff space X the following conditions are equivalent:*

1. *$C_p(X)$ is Polishable;*
2. *$C_c(X)$ is Polishable;*
3. *$C_c(X)$ is a separable Fréchet space;*
4. *X is a submetrizable hemicompact k-space.*

Proof The implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Assume that the space $C_p(X)$ is Polishable and fix a stronger separable Fréchet locally convex topology τ on $C_p(X)$. By $C_\tau(X)$ we denote the separable Fréchet space $(C_p(X), \tau)$. By τ_c we denote the compact open topology of $C_c(X)$.

Since $C_p(X)$ is a continuous image of the separable Fréchet locally convex space $C_\tau(X)$, we note that $C_p(X)$ has countable network, and by [2, I.1.3], the space X has countable network, too. Hence X is Lindelöf. By the normality (and the Lindelöf property) of X, each closed bounded set in X is countably compact (and hence is compact). So X is a μ-space.

By [29, Theorem 10.1.20] the space $C_c(X)$ is *barrelled*. The continuity of the identity maps $C_c(X) \to C_p(X)$ and $C_\tau(X) \to C_p(X)$ implies that the identity map $C_c(X) \to C_\tau(X)$ has closed graph. Since $C_c(X)$ is barrelled and $C_\tau(X)$ is Fréchet, we can apply the Closed Graph Theorem [29, Theorem 4.1.10] and conclude that the identity map $C_c(X) \to C_\tau(X)$ is continuous.

Now we prove that the identity map $C_\tau(X) \to C_c(X)$ is continuous. Fix arbitrary compact set $K \subset X$ and an $\varepsilon > 0$. We show that there exists a neighborhood of zero $U \subset C_\tau(X)$ such that

$$U \subset \{f \in C(X) : |f(x)| < \varepsilon, x \in K\}.$$

The continuity of the restriction operator $R : C_p(X) \to C_p(K)$, $R : f \mapsto f|K$, and the continuity of the identity map $C_\tau(X) \to C_p(X)$ imply that the restriction operator $R : C_\tau(X) \to C_p(K)$ is continuous and hence has closed graph. The continuity of the identity map $C_c(K) \to C_p(K)$ implies that R seen as an operator $R : C_\tau(X) \to C_c(K)$ still has closed graph.

Since the spaces $C_\tau(X)$ and $C_c(K)$ are Fréchet, we apply the Closed Graph Theorem [29, Theorem 1.2.19] to show that the restriction map

$$R : C_\tau(X) \to C_c(K)$$

is continuous. Hence there exists a neighborhood of zero $U \subset C_\tau(X)$ such that

$$R(U) \subset \{f \in C_k(K) : |f(x)| < \varepsilon, x \in K\}.$$

Then

$$U \subset \{f \in C(X) : |f(x)| < \varepsilon, x \in K\}.$$

This shows our claim. Consequently, we proved that $\tau = \tau_c$ is a Fréchet locally convex topology as claimed.

The implication (3) ⇒ (1) is obvious.

(3) ⇒ (4) If $C_c(X)$ is a separable Fréchet space, we apply [28, Theorem 4.2] to deduce that X is a hemicompact k-space. By the separability of $C_c(X)$ the space X is submetrizable.

(4) ⇒ (3) If X is a submetrizable and hemicompact k-space, then clearly $C_c(X)$ is a separable Fréchet space.

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Multiple Tensor Norms of Michor's Type and Associated Operator Ideals

In Honour of Manuel López-Pellicer

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Abstract We study the $(n+1)$-tensor norms of Michor type and characterize the n-linear mappings of the components of its associated operator ideals.

Keywords Multiple tensor norms · Operator ideals · Ultraproducts

1 Introduction

The goal of this talk is to present some results about a new class of multiple tensor norms and its associated multi-linear operator ideals in the sense of Defant–Floret–Hunfeld [3, 6]. The detailed proofs of the main results are in [16] and here we only add some precisions and complete proofs of some aspects in that paper which deserves clarification.

We consider only linear spaces over the set $\mathbb{R}$ of real numbers. The notation is standard in general. $J_F : F \longrightarrow F''$ will denote the canonical inclusion of the normed space F into its bidual F''. The band generated by an element x in a lattice E will be represented by $B(x)$. Given measure spaces $(\Omega_i, \mathscr{A}_i, \mu_i), i = 1, 2$ and Lebesgue-Bochner spaces $L^{q_1}[\Omega_1, \mathscr{A}_1, \mu_1, L^{q_2}(\Omega_2, \mathscr{A}_2, \mu_2)]$, $0 < q_i \leq \infty, i = 1, 2$ sometimes we will write $L^{q_1}[\Omega_1, \mu_1, L^{q_2}(\Omega_2, \mu_2)]$ simply if there is no risk of confusion (or indeed $L^{q_1}[\Omega_1, L^{q_2}(\Omega_2)]$). In a space $\ell^p[\ell^q]$ the element $((x_{ij})_{j=1}^{\infty})_{i=1}^{\infty}$ such that $x_{km} = 1$ and $x_{ij} = 0$ if $i \neq k$ or $j \neq m$ will be written $\mathbf{e}_{km}$. If F is a quasi-Banach space we shall denote by $\mathscr{L}^n\big(\prod_{j=1}^n E_j, F\big)$ the space of all $n-$linear continuous maps from the product $\prod_{j=1}^n E_j$ of the normed spaces $E_j, 1 \leq j \leq n$, into F. Given $A_j \in \mathscr{L}(E_j, F_j)$ between normed spaces E_j and F_j, $1 \leq j \leq n$, we write $(A_j)_{j=1}^n := (A_1, A_2, ..., A_n) : \prod_{j=1}^n E_j \longrightarrow \prod_{j=1}^n F_j$ to denote the *continuous linear* map sending every $(x_j)_{j=1}^n \in \prod_{j=1}^n E_j$ to $\big(A_1(x_1), A_2(x_2), ..., A_n(x_n)\big) \in \prod_{j=1}^n F_j$. More specific notations will be introduced when they be needed.

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1.1 (n + 1)-Tensor Norms

The first systematical studies of tensor norms on tensor products $E \otimes F$ of two normed or Banach spaces are due to Schatten [25] in the forties of the past century, but the more important impulse was made in 1956 by Grothendieck in his famous Résumé [7]. Roughly speaking, in subsequent years almost all the work in the study of tensor norms was the full development of Grothendieck's program, such as it appears in the book of Defant and Floret [3] in 1993. Around those dates, once the theory has been grown to reach a mature state of knowledge, the study of its natural extension to tensor products $\otimes_{j=1}^{n+1} E_j$ of several spaces began, enhanced by its applications to infinite holomorphy.

We will denote indistinctly by **CAT** the class **NOR** of all normed spaces or the class **BAN** of all Banach spaces. **FIN** will be the class of all finite dimensional linear spaces, **FIN**(E) and **COFIN**(E) the set of finite dimensional and finite codimensional subspaces of $E \in$ **CAT** respectively and **Norms**(F) the set of all norms in a linear space F. If E is a linear subspace of $F \in$ **CAT** we denote by Q_E^F the quotient map from F onto F/E and by I_E^F the inclusion $E \subset F$.

Definition 1.1 An $(n+1)$-tensor norm, $n \geq 1$, in the class **CAT** is an assignment

$$\alpha : (E_i)_{i=1}^{n+1} \in \mathbf{CAT}^{n+1} \longrightarrow \alpha(., \otimes_{i=1}^{n+1} E_i) \in \mathbf{Norms}(\otimes_{i=1}^{n+1} E_i)$$

such that, denoting by $(\otimes_{i=1}^{n+1} E_j; \alpha)$ the tensor product $\otimes_{i=1}^{n+1} E_j$ endowed with the norm $\alpha(., \otimes_{i=1}^{n+1} E_i)$, the following conditions are verified:

(1) $\alpha\big(\otimes_{j=1}^{n+1} x_j, \otimes_{i=1}^{n+1} E_i\big) \leq \prod_{j=1}^{n+1} \|x_j\|_{E_i} \quad \forall\ x_j \in E_j \in \mathbf{CAT},\ 1 \leq j \leq n+1.$

(2) $\|\otimes_{j=1}^{n+1} x'_j\|_{(\otimes_{i=1}^{n+1} E_j;\alpha)'} \leq \prod_{j=1}^{n+1} \|x'_j\|_{E'_i} \quad \forall\ x'_j \in E'_j,\ \ E_j \in \mathbf{CAT},\ 1 \leq j \leq n+1.$

(3) (*The metric mappings property*) For every $T_j \in \mathscr{L}(E_j, F_j)$, $(E_j, F_j) \in \mathbf{CAT}^2$, $1 \leq j \leq n+1$, the tensor product map $\otimes_{j=1}^{n+1} T_j$ which sends every tensor $\sum_{k=1}^m \otimes_{j=1}^{n+1} x_j^k \in (\otimes_{j=1}^{n+1} E_j, \alpha)$ to $\sum_{k=1}^m \otimes_{j=1}^{n+1} T_j(x_j^k) \in (\otimes_{j=1}^{n+1} F_j, \alpha)$ has norm $\|\otimes_{j=1}^{n+1} T_j\| \leq \prod_{j=1}^{n+1} \|T_j\|$.

The completion of the space $(\otimes_{j=1}^{n+1} E_j; \alpha)$ will be denoted by $(\widehat{\otimes}_{j=1}^{n+1} E_j; \alpha)$. Before to present some concrete examples of $(n+1)$-tensor norms and for a better appreciation of the scope of its applications we set another two general definitions:

Definition 1.2 An $(n+1)$-tensor norm in **CAT** is said to be finitely generated if for all $(E_j)_{j=1}^{n+1} \in \mathbf{CAT}^{n+1}$ and all $z \in (\otimes_{j=1}^{n+1} E_j; \alpha)$ one has

$$\alpha(z,\ \otimes_{j=1}^{n+1} E_j) = \inf\left\{\alpha(z, \otimes_{j=1}^{n+1} F_j)\ \middle|\ z \in \otimes_{j=1}^{n+1} F_j,\ F_j \in \mathbf{FIN}(E_j),\ 1 \leq j \leq n+1\right\}.$$

Noting that for every tensor norm α we have algebraically

$$\forall\ (E_j)_{j=1}^{n+1} \in \mathbf{FIN}^{n+1} \quad \left(\otimes_{j=1}^{n+1} E_j'; \alpha\right)' = \otimes_{j=1}^{n+1} E_j,$$

we can define the dual tensor norm α' of α as next finitely generated tensor norm:

Definition 1.3 Dual $(n+1)$-tensor norm α' of an $(n+1)$-tensor norm α

$$\forall\ (E_j)_{j=1}^{n+1} \in \mathbf{CAT}^{n+1} \quad \forall\ z \in \otimes_{j=1}^{n+1} E_j \quad \alpha'\big(z, \otimes_{j=1}^{n+1} E_j\big) =:$$

$$= \inf\left\{ \|z\|_{(\otimes_{j=1}^{n+1} F_j'; \alpha)'} \ \middle|\ z \in \otimes_{j=1}^{n+1} F_j, \quad F_j \in \mathbf{FIN}(E_j),\ 1 \le j \le n+1 \right\}.$$

The most immediate examples of $(n+1)$-tensor norm are the natural extensions to multiple tensor products of the projective and injective tensor norms π and ε, but we are interested in some more elaborated and less direct ones. All of them will be finitely generated $(n+1)$-tensor norms.

Example 1.1 (*Iterations of Saphar-Chevet* 2*-tensor norms*)
Given $1 < p < \infty$ the classical 2-tensor norm g_p of Saphar and Chevet is defined for every $(E_1, E_2) \in \mathbf{NOR}^2$ and every $z \in E_1 \otimes E_2$ as

$$g_p(z, E_1 \otimes F_2) = \inf\left\{ \Big(\sum_{k=1}^{h} \|x_k\|_{E_1}^p\Big)^{\frac{1}{p}} \sup_{\|y'\|_{E_2'} \le 1} \Big(\sum_{k=1}^{h} |\langle y_k, y'\rangle|^{p'}\Big)^{\frac{1}{p'}} \ \middle|\ z = \sum_{k=1}^{h} x_k \otimes y_k \right\},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. To simplify we will write $E_1 \otimes_{g_p} E_2 := (E_1 \otimes E_2; g_p)$. Then, given $(E_j)_{j=1}^{n+1} \in \mathbf{CAT}^{n+1}$ we define the $(n+1)$-tensor norm α_p as the n times iteration of the 2-tensor norm g_p:

$$\left(\otimes_{j=1}^{n+1} E_j; \alpha_p\right) = E_1 \otimes_{g_p} \left(E_2 \otimes_{g_p} \left(.. \otimes_{g_p} \left(E_n \otimes_{g_p} E_{n+1}\right)\right)\right).$$

The tensor norm α_p is important by its applications to the representation of certain Sobolev spaces on $\mathbb{R}^{n+1}$. We recall that for $1 < p < \infty$ the Sobolev space W_p^s over $\mathbb{R}$ of order $s \in \mathbb{R}$ can be defined with help of Fourier transform $\mathfrak{F}$ as

Definition 1.4 (*Sobolev spaces of order s*)

$$W_p^s(\mathbb{R}) = \left\{ f \in \mathscr{S}'(\mathbb{R}) \ \middle|\ \|f\|_{W_p^s(\mathbb{R})} := \left\| \mathfrak{F}^{-1}\big[(1+|x|^2)^{\frac{s}{2}} (\mathfrak{F} f)(x)\big](.)\right\|_{L^p(\mathbb{R})} < \infty \right\}.$$

This definition can be extended to distributions on $\mathbb{R}^{n+1}$ taking an $(n+1)$-tuple $\mathbf{r} = (r_j)_{j=1}^{n+1}$ of real numbers and defining

Definition 1.5 (*The Sobolev space $S_p^{\mathbf{r}} W(\mathbb{R}^{n+1})$ with dominating mixed smoothness $\mathbf{r} = (r_j)_{j=1}^{n+1} \in \mathbb{R}^{n+1}$*) We set

$$S_p^{\mathbf{r}} W(\mathbb{R}^{n+1}) := \Big\{ f \in \mathscr{S}'(\mathbb{R}^{n+1}) \;\Big|$$

$$\|f\|_{S_p^{\mathbf{r}} W(\mathbb{R}^{n+1})} := \Big\| \mathfrak{F}^{-1} \Big[\prod_{j=1}^{n+1} (1 + |\xi_j|^2)^{\frac{r_j}{2}} (\mathfrak{F} f)(\xi) \Big](.) \Big\|_{L^p(\mathbb{R}^{n+1})} < \infty \Big\}.$$

This space can be represented as the completion of an α_p-tensor product:

Theorem 1.1 (Sickel and Ulrich [27, 2009]) *For $1 < p < \infty$ and $\mathbf{r} = (r_1, r_2, ..., r_{n+1}) \in \mathbb{R}^{n+1}$ one has isometrically $S_p^{\mathbf{r}} W(\mathbb{R}^{n+1}) = \big(\widehat{\otimes}_{j=1}^{n+1} W_p^{r_j}(\mathbb{R}); \alpha_p\big)$.*

Example 1.2 (*$(n+1)$-tensor norms of Lapresté's type*)
Let $\mathbf{r} = (r_1, ..., r_{n+1}) \in]1, \infty[^{n+1}$ and $r_0 \in]1, \infty]$ be such that

$$1 = \frac{1}{r_0} + \frac{1}{r_1'} + \frac{1}{r_2'} + ... + \frac{1}{r_{n+1}'}. \tag{1}$$

The $(n+1)$-tensor norm of Lapresté is the $(n+1)$-tensor norm $\alpha_{\mathbf{r}}$ defined for every $(E_j)_{j=1}^{n+1} \in \mathbf{CAT}^{n+1}$ as

$$\forall\, z \in \otimes_{j=1}^{n+1} E_j \quad \alpha_{\mathbf{r}}(z,\ \otimes_{j=1}^{n+1} E_j) := \inf \Big\{ \|(\lambda_k)_{k=1}^h\|_{\ell^{r_0}} \prod_{j=1}^{n+1} \sup_{\|x_j'\|_{E_j'} \le 1} \Big(\sum_{k=1}^h |\langle x_k^j, x_j' \rangle| \Big)^{\frac{1}{r_j'}} \;\Big|$$

$$z = \sum_{k=1}^h \lambda_k \big(\otimes_{j=1}^{n+1} x_k^j\big),\ (\lambda_k)_{k=1}^h \in \mathbb{R}^h \Big\}.$$

If $n = 1$ and $\mathbf{r} = (r_1, r_2)$ we recover the 2-tensor norm $\alpha_{r_1 r_2}$ which was studied by Lapresté in [13, 1976]. The particular case α_{22} has found a few years ago interesting applications in quantum information theory. We do not intend a precise definition for the main concepts in that theory since we use it simply as an example about the scope of applications of tensor norms. We refer the interested reader to the beautiful paper [5] for detailed information on this topic. Here, the so called XOR two-prover games are identified with tensors $G \in \ell_m^1[\ell_2^\infty] \otimes \ell_k^1[\ell_2^\infty]$, $m, k \in \mathbb{N}$ and the so called entangled strategies are represented by elements $G' \in \ell_m^\infty[\ell_2^1] \otimes \ell_k^\infty[\ell_2^1]$. Then it turns out that

Theorem 1.2 (Dukaric, [5], 2011) *The entangled value of an XOR game G is $\alpha'_{22}(G)$.*

Next example is the most important in this talk since gives us the $(n+1)$-tensor norms we shall be concerned with.

Example 1.3 (*$(n+1)$-tensor norms of Michor's type*)

What happens if the sum in (1) is different from 1? This question was considered first time by Michor in [21, 1978], in the case $n = 1$ as a natural extension of the family of 2-tensor norms of Lapresté and without further more deep development related to the general theory of tensor norms. By this reason, the $(n+1)$-tensor norms we are going to study will be named Michor's $(n+1)$-tensor norms.

We will consider only the case of an $(n+2)$-tuple $\mathbf{r} = (r_0, r_1, ..., r_{n+1}) \in]1, \infty[^{n+1}$ (this notation is fixed from now on). Define $s_\mathbf{r} > 0$ verifying the equality

$$\frac{1}{s_\mathbf{r}} := \frac{1}{r_0} + \frac{1}{r'_1} + \frac{1}{r'_2} + \cdots + \frac{1}{r'_{n+1}}. \tag{2}$$

Michor pointed out that if $s_\mathbf{r} > 1$ we always have $\alpha_\mathbf{r}\big(z, \otimes_{j=1}^{n+1} E_j\big) = 0$ for every tensor z in any tensor product of normed spaces E_j, $1 \le j \le n+1$, and so this case is not interesting. However, if $s_\mathbf{r} < 1$ the function $\alpha_\mathbf{r}\big(z, \otimes_{j=1}^{n+1} E_j\big)$ turns out to be an $s_\mathbf{r}$-norm instead of a norm. Then the Minkowski's functional of the absolutely convex cover of the set

$$\left\{ z \in \otimes_{j=1}^{n+1} E_j \;\middle|\; \alpha_\mathbf{r}\big(z, \otimes_{j=1}^{n+1} E_j\big) \le 1 \right\}$$

is a norm $\alpha_\mathbf{r}^C$ on $\otimes_{j=1}^{n+1} E_j$ which can be explicitly computed for each $z \in \otimes_{j=1}^{n+1} E_j$ as

$$\alpha_\mathbf{r}^C(z, \otimes_{j=1}^{n+1} E_j) = \inf\left\{ \sum_{m=1}^{s} \big\| (\lambda_{mk})_{k=1}^h \big\|_{\ell^{r_0}} \prod_{j=1}^{n+1} \sup_{\|x'_j\|_{E'_j} \le 1} \Big(\sum_{k=1}^{h} \big|\langle x_{mk}^j, x'_j \rangle\big| \Big)^{\frac{1}{r'_j}} \;\middle|\; z = \sum_{m=1}^{s} \sum_{k=1}^{h} \lambda_{mk} \, \big(\otimes_{j=1}^{n+1} x_{mk}^j\big) \right\}.$$

The main difference between Lapresté and Michor $(n+1)$-tensor norms is the presence of a double sum in the computation of the norm of tensors in the latter case. This fact will be source of many technical complications. As we shall see later on, the number $t_\mathbf{r}$ defined by the equality

$$\frac{1}{t_{\mathbf{r}}} := \frac{1}{s_{\mathbf{r}}} - \frac{1}{r'_{n+1}} = \frac{1}{r_0} + \sum_{j=1}^{n} \frac{1}{r'_j} \quad \geq \frac{1}{r_{n+1}} \vee \left(\bigvee_{j=1}^{n} \frac{1}{r'_j} \right) \tag{3}$$

and the inequalities

$$r_0 > t_{\mathbf{r}}, \qquad t_{\mathbf{r}} \leq r_{n+1} \qquad \text{and} \qquad t_{\mathbf{r}} \leq r'_j, \ \ 1 \leq j \leq n \tag{4}$$

will be important in our search.

1.2 n-Linear Operator Ideals

Let α be an $(n+1)$-tensor norm in **CAT** and let φ an element in the topological dual of a tensor product $(\otimes_{j=1}^{n+1} E_j; \alpha)$. Since this dual space coincides with $(\otimes_{j=1}^{n+1} \widehat{E}_j; \alpha)'$ we can assume that **CAT** = **BAN**. Then φ defines canonically an n-linear map T_φ from $\prod_{j=1}^n E_j$ into E'_{n+1} setting that for every $(x_j)_{j=1}^n \in \prod_{j=1}^n E_j$ and every $x_{n+1} \in E_{n+1}$ the map T_φ be given by the equality $\left\langle T_\varphi\big((x_j)_{j=1}^n\big), x_{n+1}\right\rangle = \varphi\big(\otimes_{j=1}^{n+1} x_j\big)$. Actually the problem of finding the topological dual $(\otimes_{j=1}^{n+1} E_j; \alpha)'$ is equivalent to the characterization of all the maps T_φ obtained in this way. This leads us to the concept of n-linear operators ideals, an idea which was systematically developed in the case of linear operators by Pietsch and his school since 1968 but with a complete independence of tensor products. The culmination of these studies was Piestch's book [22, 1978]. In the Leipzig Conference [23] of 1983 Pietsch proposed the jump to the multi-linear case introducing next definition:

Definition 1.6 An n-operator ideal $\mathfrak{A}$ in the class **BAN** is a method to simultaneously assign a linear subspace $\mathfrak{A}(E_1, ..., E_n; E_{n+1}) \subset \mathscr{L}^n\big(\prod_{j=1}^n E_j, E_{n+1}\big)$ to every $(n+1)$-tuple of Banach spaces E_j, $1 \leq j \leq n+1$, verifying the following conditions:

(1) For all $(E_j)_{j=1}^{n+1} \in \mathbf{BAN}^{n+1}$ one has $(\otimes_{j=1}^n E'_j) \otimes E_{n+1} \subset \mathfrak{A}(E_1, ..., E_n; E_{n+1})$.

(2) Given $(G_j, E_j) \in \mathbf{BAN}^2$, $1 \leq j \leq n+1$, for every $A \in \mathfrak{A}(E_1, ..., E_n; E_{n+1})$, for each $U_j \in \mathscr{L}(G_j, E_j)$, $1 \leq j \leq n$, and for every $T \in \mathscr{L}(E_{n+1}, G_{n+1})$ it turns out that $T \circ A \circ (U_1, ..., U_n) \in \mathfrak{A}(G_1, ..., G_n; G_{n+1})$.

The set $\mathfrak{A}(E_1, ..., E_n; E_{n+1})$ is called the component of the n-operator ideal $\mathfrak{A}$ corresponding to the spaces $(E_j)_{j=1}^{n+1}$.

Definition 1.7 A Banach n-linear operator ideal $(\mathfrak{A}, \mathbf{A})$ is an n-linear operator ideal $\mathfrak{A}$ endowed with an *ideal norm* $\mathbf{A}$, i. e. a method $\mathbf{A}$ to assign a norm

$\mathbf{A}\big(\prod_{j=1}^{n} E_j;\, E_{n+1}\big)$ (in short $\mathbf{A}$ if there is no risk of confusion) to every component $\mathfrak{A}\big(E_1, ..., E_n;\, E_{n+1}\big)$ verifying the following conditions:

(1) For every $(E_j)_{j=1}^{n+1} \in \mathbf{BAN}^{n+1}$ the normed space $\big(\mathfrak{A}\big(E_1, ..., E_n;\, E_{n+1}\big), \mathbf{A}\big)$ is a Banach space.

(2) If $P : (\lambda_j)_{j=1}^{n} \in \mathbb{R}^n \longrightarrow \prod_{j=1}^{n} \lambda_j \in \mathbb{R}$ one has $\mathbf{A}(P) = 1$.

(3) (*The ideal property for norms*) For every $A \in \mathfrak{A}(E_1, ..., E_n;\, E_{n+1})$ and every $U_j \in \mathscr{L}(G_j, E_j)$, $1 \le j \le n$, and every $T \in \mathscr{L}(E_{n+1}, H)$ one has

$$\big\| T \circ A \circ (U_1, ..., U_n)\big\| \le \big\| T \big\| \, \mathbf{A}\Big(\prod_{j=1}^{n} E_j;\, E_{n+1}\Big)(A) \, \prod_{j=1}^{n} \big\| U_j \big\|.$$

As a consequence of the properties of a tensor norm it turns out that every topological dual $(\otimes_{j=1}^{n+1} E_j;\ \alpha)'$ is always the component of certain n-operator ideal corresponding to the spaces $(E_1, ..., E_n, E'_{n+1})$.

Definition 1.8 The maximal hull $(\mathfrak{A}^{max}, \mathbf{A}^{\max})$ of a Banach n-linear operator ideal $(\mathfrak{A}, \mathbf{A})$ is the Banach operator ideal $(\mathfrak{A}^{max}, \mathbf{A}^{max})$ whose components are

$$\mathfrak{A}^{max}\big(E_1, ..., E_n;\, E_{n+1}\big) :$$

$$= \Big\{ T \in \mathscr{L}^n\Big(\prod_{j=1}^{n} E_j, E_{n+1}\Big) \;\Big|\; \mathbf{A}^{max}(T) := \sup\Big\{ \mathbf{A}\big(Q_F^{E_{n+1}} \circ T \circ (I_{M_j}^{E_j})_{j=1}^{n}\big) \;\Big|$$

$$F \in \mathbf{COFIN}(E_{n+1}),\, M_j \in \mathbf{FIN}(E_j),\ 1 \le j \le n \Big\} < \infty,\, (E_j)_{j=1}^{n+1} \in \mathbf{BAN}^{n+1} \Big\}.$$

The Banach n-linear operator ideal $(\mathfrak{A}, \mathbf{A})$ is called maximal if $(\mathfrak{A}^{max}, \mathbf{A}^{\max}) = (\mathfrak{A}, \mathbf{A})$.

There is a close relation (but non trivial in any way) between maximal Banach n-linear operator ideals and finitely generated $(n+1)$-tensor norms, given in next representation theorem which was proved by Lotz in 1973 for the case $n = 1$.

Theorem 1.3 (Representation theorem for Banach maximal n-linear operator ideals, Lotz [17], Floret and Hunfeld [6]) *A Banach n-linear operator ideal* $(\mathfrak{A}, \mathbf{A})$ *is maximal if and only if there is a finitely generated* $(n+1)$*-tensor norm* α *in* $\mathbf{BAN}$ *such that for every* $(E_j)_{j=1}^{n+1} \in \mathbf{BAN}^{n+1}$ *one has isometrically*

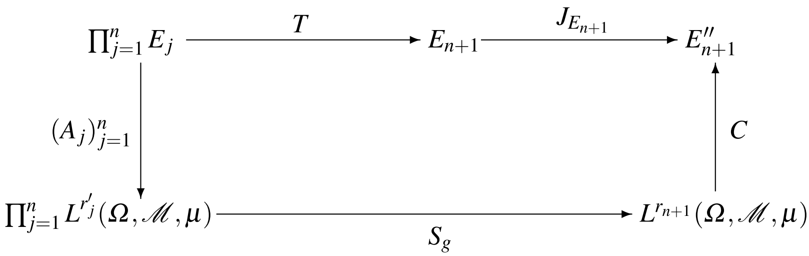
$$\Big(\mathfrak{A}(E_1, ..., E_n;\, E_{n+1}), \mathbf{A}\Big) = \Big(\big(\otimes_{j=1}^{n} E_j\big) \otimes E'_{n+1};\ \alpha'\Big)' \bigcap \mathscr{L}^n\Big(\prod_{j=1}^{n} E_j, E_{n+1}\Big).$$

In such a case the ideal $(\mathfrak{A}, \mathbf{A})$ and the tensor norm α are said to be associated ($\alpha \sim \mathfrak{A}$ in symbols) and the operators in $\mathfrak{A}$ are called n-linear α-integral operators. To emphasize this relation we will write $\mathfrak{A}_\alpha$ and $\mathbf{I}_\alpha$ instead of $\mathfrak{A}$ and $\mathbf{A}$. Moreover, for every $(E_j)_{j=1}^{n+1} \in \mathbf{BAN}^{n+1}$ one has the isometric equality

$$\left(\mathfrak{A}_\alpha\left(E_1, ..., E_n; E'_{n+1}\right), \mathbf{I}_\alpha\right) = \left(\otimes_{j=1}^{n+1} E_j;\ \alpha'\right)'. \tag{5}$$

It is clear the importance of the application of previous abstract concepts to concrete examples of tensor norms to know in detail its properties. In particular it is important to characterize the α-integral operators associated to usual $(n+1)$-tensor norms, but this research area is not quite developed still. As a contribution to fill this gap we have studied in [15] the $\alpha_{\mathbf{r}}$-integral operators obtaining next result:

Theorem 1.4 ([15, Theorem 14]) *Let E_j, $1 \le j \le n+1$ be Banach spaces. An operator $T \in \mathscr{L}^n(\prod_{j=1}^n E_j,\ E_{n+1})$ is $\alpha_{\mathbf{r}}$-integral if and only if $J_{E_{n+1}} \circ T$ can be factorized in the way*



where $A_j \in \mathscr{L}\big(E_j, L^{r'_j}(\Omega, \mathscr{M}, \mu)\big),\ 1 \le j \le n,\ C \in \mathscr{L}\big(L^{r_{n+1}}(\Omega, \mathscr{M}, \mu), F''\big)$ *and S_g is the diagonal operator corresponding to a function $g \in L^{r_0}(\Omega, \mathscr{M}, \mu)$. Moreover* $\mathbf{I}_{\mathbf{r}}(T) = \inf \|D_g\|\ \|C\|\ \prod_{j=1}^n \|A_j\|$ *taking the* inf *over all factorizations as in the previous diagram.*

Our main goal in this talk is to present a characterization of the $\alpha_{\mathbf{r}}^C$-integral operators given in [16].

1.3 Some Preliminary Results

Before to start the study of the established main problem we present some results necessary for our future work. Probably they are well known but we have not been able to find a precise reference in the literature.

Associated with an $(n+2)$-tuple $\mathbf{r}$ as in (2), given measure spaces $(\Omega_i, \mathscr{A}_i, \mu_i)$, $i = 1, 2$, and $g(t,x) \in L^1[\Omega_1, L^{r_0}(\Omega_2)]$ we have a well defined canonical continuous diagonal n-linear map

$$S_g : (f_j(t,x))_{j=1}^n \in \prod_{j=1}^n L^\infty[\Omega_1, L^{r'_j}(\Omega_2)] \longrightarrow g(t,x) \prod_{j=1}^n f_j(t,x) \in L^1[\Omega_1, L^{t_{\mathbf{r}}}(\Omega_2)]$$

verifying $\|S_g\| \leq \|g\|_{L^1[\Omega_1, L^{r_0}(\Omega_2)]}$ as an easy consequence of generalized Hölder's inequality and (3).

Lemma 1.1 *If* $0 < q < \infty$, $0 \leq v(t,x) \in L^1[\Omega_1, \mu_1, L^q(\Omega_2, \mu_2)]$ *and* $D_v^\infty := \{(t,x) \in \Omega_1 \times \Omega_2 \mid v(t,x) = \infty\}$ *one has* $(\mu_1 \times \mu_2)(D_v^\infty) = 0$.

Proof Let $D_v^\infty(t) := \{x \in \Omega_2 \mid (t,x) \in D_v^\infty\}$, $t \in \Omega_1$. Since

$$\int_{\Omega_1} \Big(\int_{D_v^\infty(t)} \chi_{D_v^\infty}(t,x) v(t,x)^q \, d\mu_2 \Big)^{\frac{1}{q}} \, d\mu_1 \leq \|v\|_{L^1[\Omega_1, L^q(\Omega_2)]} < \infty$$

necessarily we have $\mu_2(D_v^\infty(t)) = 0$ almost everywhere on Ω_1 and so

$$(\mu_1 \times \mu_2)(D_v^\infty) = \int_{\Omega_1} \Big(\int_{D_v^\infty(t)} d\mu_2 \Big) \, d\mu_1 = 0.$$

□

Let $\mathscr{M}(\Omega)$ be the set of all measurable real functions defined on a measure space $(\Omega, \mathscr{A}, \mu)$. Given a set $E \subset \mathscr{M}(\Omega)$ we define the α-dual $E^\times$ as the set of functions $h \in \mathscr{M}(\Omega)$ such that $f\,h \in L^1(\Omega, \mathscr{A}, \mu)$ for every $f \in E$. A set $E \subset \mathscr{M}(\Omega)$ is called perfect if $E^{\times\times} := (E^\times)^\times = E$. It is well known that if $(\Omega, \mathscr{A}, \mu)$ is σ-finite one has $L^p(\Omega)^\times = L^{p'}(\Omega)$ for each $1 \leq p \leq \infty$ ([29, Chap. 12, §50, lemma γ, part b)]).

Lemma 1.2 *Let* $(\Omega_j, \mathscr{A}_j, \mu_j)$, $j = 1, 2$, *be* σ*-finite measure spaces. Then if* $1 < p < \infty$ *one has* $(L^\infty[\Omega_1, L^p(\Omega_2)])^\times = L^1[\Omega_1, L^{p'}(\Omega_2)]$.

Proof The inclusion $L^1[\Omega_1, L^{p'}(\Omega_2)] \subset (L^\infty[\Omega_1, L^p(\Omega_2)])^\times$ is trivial by Hölder's inequality and Fubini's theorem. Conversely, let $g(t,x) \in (L^\infty[\Omega_1, L^p(\Omega_2)])^\times$. Consider for each $j = 1, 2$ an increasing sequence $\{A_m^j\}_{m=1}^\infty$ of sets in $\mathscr{A}_j$ such that $\Omega_j = \bigcup_{m=1}^\infty A_m^j$ and $\mu_j(A_m^j) < \infty$ for each $m \in \mathbb{N}$. Let g_k be the function defined as $g_k(t,x) = \min(|g(t,x)|, k)$ if $(t,x) \in A_k^1 \times A_k^2$ and $g_k(t,x) = 0$ if $(t,x) \notin A_k^1 \times A_k^2$. It is clear that $g_k(t,x) \in L^1[\Omega_1, L^{p'}(\Omega_2])$ and $\lim_{k\to\infty} g_k(t,x) = g(t,x)$ pointwise on $\Omega_1 \times \Omega_2$. As $\big(L^1[\Omega_1, L^{p'}[\Omega_2]]\big)' = L^\infty[\Omega_1, L^p(\Omega_2)]$ by the monotone convergence theorem we obtain for every $0 \leq f(t,x) \in L^\infty[\Omega_1, L^p(\Omega_2)]$

$$\lim_{k\to\infty} \int_{\Omega_1 \times \Omega_2} f(t,x) g_k(t,x) \, d\mu_1 d\mu_2 = \int_{\Omega_1 \times \Omega_2} f(t,x) \big| g(t,x) \big| \, d\mu_1 d\mu_2 < \infty \tag{6}$$

and since $h(t,x) = h(t,x)^+ - h(t,x)^-$ for every $h(t,x) \in L^\infty[\Omega_1, L^p(\Omega_2)]$ it turns out that $\{g_k(t,x)\}_{k=1}^\infty$ is a weakly Cauchy sequence in $L^1[\Omega_1, L^{p'}(\Omega_2)] = L^1(\Omega_1) \widehat{\otimes}_\pi L^{p'}(\Omega_2) = L^{p'}(\Omega_2) \widehat{\otimes}_\pi L^1(\Omega_1)$ (Grothendieck's theorem) which is weakly sequentially complete by Lewis's result [14, Corollary 11]. So the weak limit $\lim_{k\to\infty} g_k(t,x) = v(t,x) \in L^1[\Omega_1, L^{p'}(\Omega_2)]$ exists. Consider for every $m \in \mathbb{N}$ the set $N_m := \{(t,x) \in \Omega_1 \times \Omega_2 \mid |g(t,x)| \le m\}$. Clearly $\chi_{N_m}(t,x)|g(t,x)| \in L^1[\Omega_1, L^{p'}(\Omega_2)]$ for every $m \in \mathbb{N}$ and as an easy consequence of (6) and from the definition of $v(t,x)$ one has $\chi_{N_m}(t,x)|g(t,x)| = \lim_{k\to\infty} \chi_{N_m}(t,x)g_k(t,x) = \chi_{N_m}(t,x)v(t,x)$ weakly in $L^1[\Omega_1, L^{p'}(\Omega_2)]$ for every $m \in \mathbb{N}$. Then by Lemma 1.1 $v(t,x) = |g(t,x)|$ on $\Omega_1 \times \Omega_2$ and $\big|g(t,x)\big| \in L^1[\Omega_1, L^{p'}(\Omega_2)]$. □

Given sets $E_j \subset \mathscr{M}(\Omega)$, $j = 1,2$ we define E_1E_2 as the linear span of the set $\{f_1 f_2 \mid f_j \in E_j, j = 1,2\}$. If we have spaces $E_j \subset \mathscr{M}(\Omega)$, $1 \le j \le n+1$ we are interested in determining the set

$$D\Big(\prod_{j=1}^{n} E_j; E_{n+1}\Big) := \Big\{ f \in \mathscr{M}(\Omega) \;\Big|\; f \prod_{j=1}^{n} f_j \in E_{n+1} \;\; \forall (f_j)_{j=1}^{n} \in \prod_{j=1}^{n} E_j \Big\}.$$

Lemma 1.3 *If E_{n+1} is perfect one has $D\big(\prod_{j=1}^{n} E_j; E_{n+1}\big) = \Big(E_1E_2...E_nE_{n+1}^{\times}\Big)^{\times}$.*

Proof Clearly if $f \in D\big(\prod_{j=1}^{n} E_j; E_{n+1}\big)$, $f_j \in E_j$, $1 \le j \le n$ and $g \in E_{n+1}^{\times}$, one has $f(\prod_{j=1}^{n} f_j)g \in L^1(\Omega, \mathscr{M}, \mu)$. That means that $f \in \Big(E_1E_2...E_nE_{n+1}^{\times}\Big)^{\times}$. Conversely, if $f \in \Big(E_1E_2...E_nE_{n+1}^{\times}\Big)^{\times}$, $f_j \in E_j$, $1 \le j \le n$ and $g \in E_{n+1}^{\times}$ it turns out that $f(\prod_{j=1}^{n} f_j)g \in L^1(\Omega, \mathscr{M}, \mu)$ and so $f(\prod_{j=1}^{n} f_j) \in E_{n+1}^{\times\times} = E_{n+1}$, that is $f \in D\big(\prod_{j=1}^{n} E_j; E_{n+1}\big)$ and the lemma is proved. □

Proposition 1.9 *Let* $\mathbf{r}$ *as in (2) such that* $t_{\mathbf{r}} \ge 1$ *and let* $(\Omega_i, \mathscr{A}_i, \mu_i)$, $i = 1,2$ σ*-finite measure spaces. Then*

(1) $L^{\infty}[\Omega_1, L^{r_1'}(\Omega_2)]...L^{\infty}\big[\Omega_1, L^{r_n'}(\Omega_2)]L^{\infty}[\Omega_1, L^{t_{\mathbf{r}}'}(\Omega_2)] = L^{\infty}[\Omega_1, L^{r_0'}(\Omega_2)]$.

(2) $D\big(\prod_{j=1}^{n} L^{\infty}\big[\Omega_1, L^{r_j'}(\Omega_2)]; L^1[\Omega_1, L^{t_{\mathbf{r}}}(\Omega_2)]\big) = L^1[\Omega_1, L^{r_0}(\Omega_2)]$.

Proof (1) It follows from (3) that

$$\frac{1}{r_0'} = \frac{1}{t_{\mathbf{r}}'} + \sum_{j=1}^{n} \frac{1}{r_j'}. \tag{7}$$

If $t_{\mathbf{r}} > 1$ and $f(t,x) \in L^{\infty}\big[\Omega_1, L^{r_0'}(\Omega_2)]$ one has $\big|f(t,x)\big| = \big|f(t,x)\big|^{\frac{r_0'}{t_{\mathbf{r}}'}} \prod_{j=1}^{n} \big|f(t,x)\big|^{\frac{r_0'}{r_j'}}$. Then $\big|f(t,x)\big|^{\frac{r_0'}{t_{\mathbf{r}}'}} \in L^{\infty}[\Omega_1, L^{t_{\mathbf{r}}'}(\Omega_2)]$ and $\big|f(t,x)\big|^{\frac{r_0'}{r_j'}} \in L^{\infty}[\Omega_1, L^{r_j'}(\Omega_2)]$, $1 \le j \le n$, showing that the right side in (1) is contained in the left side. The reverse inclusion follows from Hölder's inequality and (7). If $t_{\mathbf{r}} = 1$ the previous argumentation shows that $L^\infty[\Omega_1, L^{r'_1}(\Omega_2)]...L^\infty\big[\Omega_1, L^{r'_n}(\Omega_2)] = L^\infty[\Omega_1, L^{r'_0}(\Omega_2)]$ and, since we have trivially $L^\infty[\Omega_1, L^{r'_0}(\Omega_2)]L^\infty[\Omega_1, L^\infty(\Omega_2)] = L^\infty[\Omega_1, L^{r'_0}(\Omega_2)]$ the result is true for $t_{\mathbf{r}} = 1$ too.

(2) If $g(t,x) \in D\big(\prod_{j=1}^n L^\infty[\Omega_1, L^{r'_j}(\Omega_2)]; L^1[\Omega_1, L^{t_{\mathbf{r}}}(\Omega_2)]\big)$ by lemmata 1.2, 1.3 and the first part of this proposition

$$g(t,x) \in \big(L^\infty[\Omega_1, L^{r'_0}(\Omega_2)]\big)^\times = L^1[\Omega_1, L^{r_0}(\Omega_2)]$$

and $D\big(\prod_{j=1}^n L^\infty[\Omega_1, L^{r'_j}(\Omega_2)]; L^1[\Omega_1, L^{t_{\mathbf{r}}}(\Omega_2)]\big) \subset L^1[\Omega_1, L^{r_0}(\Omega_2)]$. The reverse inclusion is immediate from the definition of S_g for every $g \in L^1[\Omega_1, L^{r_0}(\Omega_2)]$. □

2 $\alpha_{\mathbf{r}}^C$-Integral Operators

This long section is devoted to explain almost in full detail how to prove a necessary condition for an operator $T \in \mathscr{L}^n\big(\prod_{j=1}^n E_j,\ E_{n+1}\big)$ be $\alpha_{\mathbf{r}}^C$-integral since this is the most hard part of the study [16]. Hence the section will be subdivided in suitable subsections and sub-subsections. Assume that $T \in \mathfrak{A}_{\alpha_{\mathbf{r}}^C}\big(E_1, ..., E_n; E_{n+1}\big)$. By (5) one has

$$J_{E_{n+1}} \circ T \in \mathfrak{A}_{\alpha_{\mathbf{r}}^C}\big(E_1, ..., E_n; E''_{n+1}\big) = \Big((\otimes_{j=1}^n E_j) \otimes E'_{n+1}; (\alpha_{\mathbf{r}}^C)'\Big)'. \tag{1}$$

As $(\alpha_{\mathbf{r}}^C)'$ is a finitely generated $(n+1)$-tensor norm we start our search looking at the behavior of the linear form defined by $J_{E_{n+1}} \circ T$ on the tensor products of finite dimensional subspaces of E_j, $1 \le j \le n$, and E'_{n+1} when they are endowed with the $(\alpha_{\mathbf{r}}^C)'$-norm. This is a particular case of the following general situation:

Definition 2.1 Let $\alpha \sim \mathfrak{A}$. The Banach n-linear operator ideal $(\mathfrak{N}_\alpha, \mathbf{N}_\alpha)$ of the n-linear α-nuclear operators is the smallest Banach n-linear operator ideal whose finite dimensional components are isometric with the finite dimensional components of $(\mathfrak{A}, \mathbf{A})$.

One of the main theorem of the theory of $(n+1)$-tensor norms is next theorem giving a more concrete description of the α-nuclear operators and its norm. The proof for $n = 1$ can be found in [3, §22.2] and can easily be extended to the general case.

Theorem 2.1 Representation theorem of the α-nuclear operators) *Let* $(\mathfrak{A}, \mathbf{A})$ *be a Banach* n*-linear operator ideal and let* $\alpha \sim (\mathfrak{A}^{max}, \mathbf{A}^{max})$. *Given* $(E_j)_{j=1}^{n+1} \in \mathbf{BAN}^{n+1}$ *let* $\Psi \in \mathscr{L}\Big(\big((\widehat{\otimes}_{j=1}^n E'_j)\widehat{\otimes} E_{n+1}; \alpha\big), \mathscr{L}^n\big(\prod_{j=1}^n E_j, E_{n+1}\big)$ *be the canonical linear map. Then* $\Psi : \big((\widehat{\otimes}_{j=1}^n E'_j)\widehat{\otimes} E_{n+1}; \alpha\big) \longrightarrow \big(\mathfrak{N}_\alpha\big(\prod_{j=1}^n E_j;\ E_{n+1}\big), \mathbf{N}_\alpha\big)$ *is a metric surjection.*

By application of Theorem 2.1 to our $(n+1)$-tensor norm $\alpha_{\mathbf{r}}^C$ we obtain next characterization of the $\alpha_{\mathbf{r}}^C-$nuclear operators:

Theorem 2.2 ([16, Theorem 3], Characterization of the $\alpha_{\mathbf{r}}^C$-nuclear operators) *Let E_i, $1 \leq i \leq n+1$, be Banach spaces. A map $T \in \mathfrak{N}_{\alpha_{\mathbf{r}}^C}\left(\prod_{i=1}^n E_i; E_{n+1}\right)$ if and only if T factorizes in the way*

$$\begin{array}{ccccc}
\prod_{j=1}^n E_j & & \xrightarrow{\qquad\qquad T \qquad\qquad} & & E_{n+1}\\
\Big\downarrow {\scriptstyle (A_j)_{j=1}^n} & & & & \Big\uparrow {\scriptstyle C}\\
\prod_{j=1}^n \ell^\infty[\ell^{r'_j}] & \xrightarrow[S_\lambda]{\qquad} & \ell^1[\ell^{t_{\mathbf{r}}}] & \xrightarrow[I_{1,1}\otimes I_{t_{\mathbf{r}},r_{n+1}}]{\qquad} & \ell^1[\ell^{r_{n+1}}]
\end{array}$$

where S_λ is a diagonal map defined by $\lambda = ((\lambda_{mk})_{k=1}^\infty)_{m=1}^\infty) \in \ell^1[\ell^{r_0}]$. Moreover, $\mathbf{N}_{\alpha_{\mathbf{r}}^C}(T) = \inf \left(\prod_{j=1}^n \|A_j\|\right) \|S_\lambda\| \, \|C\|$ taking the infimum over all factorizations of this type.

In the previous diagram $I_{1,1} \otimes I_{t_{\mathbf{r}},r_{n+1}}$ denotes the tensor product map of the inclusions $I_{1,1} : \ell^1 \subset \ell^1$ and $I_{t_{\mathbf{r}},r_{n+1}} : \ell^{t_{\mathbf{r}}} \subset \ell^{r_{n+1}}$ (remember (4)).

Taking into account the characterizations of $\alpha_{\mathbf{r}}$-integral operators the natural conjecture in order that an n-linear map be $\alpha_{\mathbf{r}}^C$-integral is

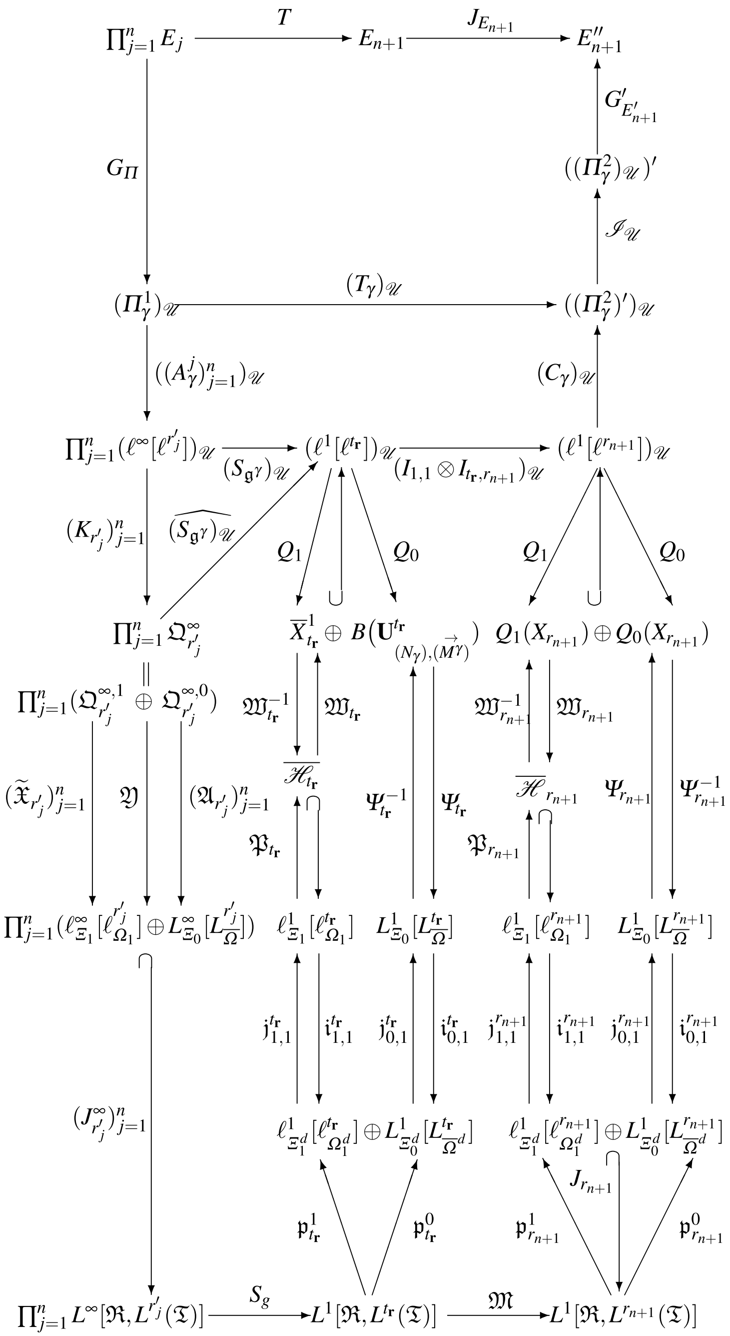
Main conjecture: *Let E_j, $1 \leq j \leq n+1$, be Banach spaces and assume that $T \in \mathfrak{I}_{\alpha_{\mathbf{r}}^C}(\prod_{j=1}^n E_j; E_{n+1})$. Then there are measurable spaces $(\mathfrak{R}, \mathfrak{H}, \nu)$ and $(\mathfrak{T}, \mathfrak{V}, \tau)$, a function $g \in L^1[\mathfrak{R}, L^{r_0}(\mathfrak{T})]$ and a map*

$$\mathfrak{M} \in \mathscr{L}\big(L^1[\mathfrak{R}, L^{t_{\mathbf{r}}}(\mathfrak{T})], L^1[\mathfrak{R}, L^{r_{n+1}}(\mathfrak{T})]\big)$$

such that $J_{E_{n+1}} \circ T$ can be factorized as

$$\begin{array}{ccccc}
\prod_{j=1}^n E_j & \xrightarrow{\qquad T \qquad} & E_{n+1} & \xrightarrow{\qquad J_{E_{n+1}} \qquad} & E''_{n+1}\\
\Big\downarrow {\scriptstyle (A_j)_{j=1}^n} & & & & \Big\uparrow {\scriptstyle C}\\
\prod_{j=1}^n L^\infty[\mathfrak{R}, L^{r'_j}(\mathfrak{T})] & \xrightarrow[S_g]{\qquad} & L^1[\mathfrak{R}, L^{t_{\mathbf{r}}}(\mathfrak{T})] & \xrightarrow[\mathfrak{M}]{\qquad} & L^1[\mathfrak{R}, L^{r_{n+1}}(\mathfrak{T})]
\end{array} \tag{2}$$

Along the long proof of the conjecture we will precise the properties of the map $\mathfrak{M}$. Next global diagram shows the ambient we are moving on and the work we will do to find the desired factorization. After a comparison between this total diagram


$\prod_{j=1}^n E_j$
T
E_{n+1}
$J_{E_{n+1}}$
E''_{n+1}
$G'_{E'_{n+1}}$
G_Π
$((\Pi^2_\gamma)_\mathscr{U})'$
$\mathscr{I}_\mathscr{U}$
$(\Pi^1_\gamma)_\mathscr{U}$
$(T_\gamma)_\mathscr{U}$
$((\Pi^2_\gamma)')_\mathscr{U}$
$((A^j_\gamma)_{j=1}^n)_\mathscr{U}$
$(C_\gamma)_\mathscr{U}$
$\prod_{j=1}^n(\ell^\infty[\ell^{r'_j}])_\mathscr{U}$
$(S_{\mathfrak{g}^\gamma})_\mathscr{U}$
$(\ell^1[\ell^{t_\mathbf{r}}])_\mathscr{U}$
$(I_{1,1}\otimes I_{t_\mathbf{r},r_{n+1}})_\mathscr{U}$
$(\ell^1[\ell^{r_{n+1}}])_\mathscr{U}$
$(K_{r'_j})_{j=1}^n$
$\widehat{(S_{\mathfrak{g}^\gamma})_\mathscr{U}}$
Q_1
Q_0
$\prod_{j=1}^n \mathfrak{Q}^\infty_{r'_j}$
$\overline{X}^1_{t_\mathbf{r}} \oplus B(\mathbf{U}^{t_\mathbf{r}}_{(N_\gamma),(\vec{M}^\gamma)})$
$Q_1(X_{r_{n+1}}) \oplus Q_0(X_{r_{n+1}})$
$\prod_{j=1}^n(\mathfrak{Q}^{\infty,1}_{r'_j} \oplus \mathfrak{Q}^{\infty,0}_{r'_j})$
$\mathfrak{W}^{-1}_{t_\mathbf{r}}$
$\mathfrak{W}_{t_\mathbf{r}}$
$\mathfrak{W}^{-1}_{r_{n+1}}$
$\mathfrak{W}_{r_{n+1}}$
$(\widetilde{\mathfrak{X}}_{r'_j})_{j=1}^n$
$\mathfrak{Y}$
$(\mathfrak{A}_{r'_j})_{j=1}^n$
$\overline{\mathscr{H}_{t_\mathbf{r}}}$
$\Psi^{-1}_{t_\mathbf{r}}$
$\Psi_{t_\mathbf{r}}$
$\overline{\mathscr{H}}_{r_{n+1}}$
$\Psi_{r_{n+1}}$
$\Psi^{-1}_{r_{n+1}}$
$\mathfrak{P}_{t_\mathbf{r}}$
$\mathfrak{P}_{r_{n+1}}$
$\prod_{j=1}^n(\ell^\infty_{\Xi_1}[\ell^{r'_j}_{\Omega_1}] \oplus L^\infty_{\Xi_0}[L^{r'_j}_{\overline{\Omega}}])$
$\ell^1_{\Xi_1}[\ell^{t_\mathbf{r}}_{\Omega_1}]$
$L^1_{\Xi_0}[L^{t_\mathbf{r}}_{\overline{\Omega}}]$
$\ell^1_{\Xi_1}[\ell^{r_{n+1}}_{\Omega_1}]$
$L^1_{\Xi_0}[L^{r_{n+1}}_{\overline{\Omega}}]$
$\mathfrak{j}^{t_\mathbf{r}}_{1,1}$
$\mathfrak{i}^{t_\mathbf{r}}_{1,1}$
$\mathfrak{j}^{t_\mathbf{r}}_{0,1}$
$\mathfrak{i}^{t_\mathbf{r}}_{0,1}$
$\mathfrak{j}^{r_{n+1}}_{1,1}$
$\mathfrak{i}^{r_{n+1}}_{1,1}$
$\mathfrak{j}^{r_{n+1}}_{0,1}$
$\mathfrak{i}^{r_{n+1}}_{0,1}$
$(J^\infty_{r'_j})_{j=1}^n$
$\ell^1_{\Xi_1^d}[\ell^{t_\mathbf{r}}_{\Omega_1^d}] \oplus L^1_{\Xi_0^d}[L^{t_\mathbf{r}}_{\overline{\Omega}^d}]$
$\ell^1_{\Xi_1^d}[\ell^{r_{n+1}}_{\Omega_1^d}] \oplus L^1_{\Xi_0^d}[L^{r_{n+1}}_{\overline{\Omega}^d}]$
$J_{r_{n+1}}$
$\mathfrak{p}^1_{t_\mathbf{r}}$
$\mathfrak{p}^0_{t_\mathbf{r}}$
$\mathfrak{p}^1_{r_{n+1}}$
$\mathfrak{p}^0_{r_{n+1}}$
$\prod_{j=1}^n L^\infty[\mathfrak{R}, L^{r'_j}(\mathfrak{T})]$
S_g
$L^1[\mathfrak{R}, L^{t_\mathbf{r}}(\mathfrak{T})]$
$\mathfrak{M}$
$L^1[\mathfrak{R}, L^{r_{n+1}}(\mathfrak{T})]$

and (2) it is clear that the map $(A_{r'_j})_{j=1}^n$ (resp. C) will be the composition of the applications of the left side (resp. of the right side) of the full diagram. So we only have to define the spaces and operators appearing on it and check its commutativity.

According with the previous ideas consider

$$\mathscr{F} := \Big(\prod_{j=1}^n \mathbf{FIN}(E_j)\Big) \times \mathbf{FIN}(E'_{n+1}),$$

denote its elements in the way $\gamma := (\vec{F}; G) := \big(\prod_{j=1}^n F_j\big) \times G \in \mathscr{F}$ and endow the tensor product of its factor spaces with the tensor norm $(\alpha_{\mathbf{r}}^C)'$. As a consequence of the metric mapping property, for every $\gamma = \big(\prod_{j=1}^n F_j\big) \times G \in \mathscr{F}$ the restriction to $\big((\otimes_{j=1}^n F_j) \otimes G, (\alpha_{\mathbf{r}}^C)'\big)$ of the linear form defined by $J_{E_{n+1}} \circ T$ must be continuous. That means that the restriction $(J_{E_{n+1}} \circ T)_{|_\gamma}$ to $\prod_{j=1}^n F_j$ of the linear map $J_{E_{n+1}} \circ T$ must be an $\alpha_{\mathbf{r}}^C$-integral map from $\prod_{j=1}^n F_j$ into G' and by finite dimensionality it must be $\alpha_{\mathbf{r}}^C$-nuclear. Then given $\varepsilon > 0$, by Theorem 2.2 $(J_{E_{n+1}} \circ T)_{|_\gamma}$ will have a factorization

$$(J_{E_{n+1}} \circ T)_{|_\gamma} = C_\gamma \circ (I_{1,1} \otimes I_{t_{\mathbf{r}}, r_{n+1}}) \circ S_{\mathfrak{g}^\gamma} \circ (A_\gamma^j)_{j=1}^n \tag{3}$$

of the type of Theorem 2.2 where the diagonal operator $S_{\mathfrak{g}^\gamma}$ is defined by a double sequence $\mathfrak{g}^\gamma = \big((g_{mk}^\gamma)_{k=1}^\infty\big)_{m=1}^\infty \in \ell^1[\ell^{r_0}]$ such that $\sup_{\gamma \in \mathscr{F}} \|\mathfrak{g}^\gamma\|_{\ell^1[\ell^{r_0}]} \leq \mathbf{I}_{\alpha_{\mathbf{r}}^C}(T) + \varepsilon$. Of course, these ideas imply that if all the spaces E_j, $1 \leq j \leq n+1$, would be finite dimensional the conjecture would be trivially true. So from now on *we shall assume that at least one of the spaces* E_j, $1 \leq j \leq n+1$, *is infinite dimensional.*

2.1 Ultrapowers of Spaces $\ell^p[\ell^q]$

To joint previous knowledge of the properties of the finite-dimensional restrictions $(J_{E_{n+1}} \circ T)_{|_\gamma}$ in order to obtain some information about the global map $J_{E_{n+1}} \circ T$ we need use ultraproducts. For this topic our main references are [9, 26]. The ultraproduct of a family $\{E_\beta,\ \beta \in \mathscr{G}\}$ of quasi-Banach spaces by a non trivial ultrafilter $\mathscr{U}$ in an index set $\mathscr{G}$ is denoted by $(E_\beta)_{\mathscr{U}}$ (or simply by $(E)_{\mathscr{U}}$ if we deal with ultrapowers, i. e. $E_\beta = E$ for every $\beta \in \mathscr{G}$) and provided with its natural quasi-norm. Analogously, $(x_\beta)_{\mathscr{U}}$ will be the class in $(E_\beta)_{\mathscr{U}}$ of the element $(x_\beta)_{\beta \in \mathscr{G}} \in \prod_{\beta \in \mathscr{G}} E_\beta$ verifying $\sup_{\beta \in \mathscr{G}} \|x_\beta\|_{E_\beta} < \infty$. If every E_β, $\beta \in \mathscr{G}$ is a Banach lattice, the ultraproduct $(E_\beta)_{\mathscr{U}}$ is a Banach lattice with the canonical order defined by the relation $(x_\beta)_{\mathscr{U}} \leq (y_\beta)_{\mathscr{U}}$ if and only if there exist $(\overline{x}_\beta) \in (x_\beta)_{\mathscr{U}}$ and $(\overline{y}_\beta) \in (y_\beta)_{\mathscr{U}}$ such that $\overline{x}_\beta \leq \overline{y}_\beta$ for every $\beta \in \mathscr{G}$. Moreover, one has $(x_\beta)_{\mathscr{U}} \wedge (y_\beta)_{\mathscr{U}} = (x_\beta \wedge y_\beta)_{\mathscr{U}}$ and $(x_\beta)_{\mathscr{U}} \vee (y_\beta)_{\mathscr{U}} = (x_\beta \vee y_\beta)_{\mathscr{U}}$.

Turning to our main problem we consider in $\mathscr{F}$ the order relation given by

$$\gamma_1 := (\prod_{j=1}^{n} N_j^1) \times G_1 \preccurlyeq \gamma_2 := (\prod_{j=1}^{n} N_j^2) \times G_2 \Leftrightarrow G_1 \subset G_2, N_j^1 \subset N_j^2,\ 1 \le j \le n$$

and introduce the notations $\Pi_\gamma^1 := \vec{F} = \prod_{j=1}^n F_j$, $\Pi_\gamma^2 := G$ for every $\gamma := (\vec{F}, G) \in \mathscr{F}$.

By our hypothesis about the dimensionality of the spaces E_j, $1 \le j \le n+1$, the family $\mathscr{B} := \left\{ \mathscr{B}_\gamma := \left\{ \delta \in \mathscr{F} \;\middle|\; \gamma \preccurlyeq \delta \right\} \;\middle|\; \gamma \in \mathscr{F} \right\}$ is a non trivial filter basis in $\mathscr{F}$. Fixing an ultrafilter $\mathscr{U}$ finer than the filter generated by $\mathscr{B}$ we can form the ultraproducts $(\Pi_\gamma^1)_{\mathscr{U}}$ and $(\Pi_\gamma^2)_{\mathscr{U}}$. Note that $\mathscr{U}$ is a countable incomplete ultrafilter, that is, there is a non increasing sequence $\{U_k\}_{k=1}^\infty$ of elements of $\mathscr{U}$ such that $\bigcap_{k=1}^\infty U_k = \emptyset$.

Since there are general canonical isometric inclusions

$$G_\Pi : \prod_{j=1}^{n} E_j \longrightarrow (\Pi_\gamma^1)_{\mathscr{U}}, \quad G_{E'_{n+1}} : E'_{n+1} \longrightarrow (\Pi_\gamma^2)_{\mathscr{U}}, \quad \mathscr{I}_{\mathscr{U}} : \big((\Pi_\gamma^2)'\big)_{\mathscr{U}} \longrightarrow \big((\Pi_\gamma^2)_{\mathscr{U}}\big)'$$

(see [9, Proof of Proposition 6.2 and Sects. 2 and 7]) it turns out that top rectangle in the global diagram is commutative. Making the ultraproduct by $\mathscr{U}$ of the factorizations (3) we obtain the commutativity of its second upper rectangle and we arrive to the main and more difficult problem: to find a factorization throughout "concrete" Lebesgue-Bochner function spaces of the ultraproduct map $\mathfrak{S} := (I_{1,1} \otimes I_{t_{\mathfrak{r}}, r_{n+1}})_{\mathscr{U}} \circ (S_{\mathfrak{g}^\gamma})_{\mathscr{U}}$.

All the ultrapowers

$$\forall\, p \in]0, \infty],\ \forall\, q > 0\ \ \big(\ell^p[\ell^q]\big)_{\mathscr{U}},\quad X_q^\infty := \big(\ell^\infty[\ell^q]\big)_{\mathscr{U}},\quad X_q := \big(\ell^1[\ell^q]\big)_{\mathscr{U}},\quad Y_q := \big(\ell^q[\ell^q]\big)_{\mathscr{U}}$$

are quasi-Banach lattices when provided with its canonical order. We are interested in knowing the structure of the order components of its elements $0 \le \mathbf{W} = \big((w_{mk}^\gamma)\big)_{\mathscr{U}} \in \big(\ell^p[\ell^q]\big)_{\mathscr{U}}$ (an order component of an element $u \ge 0$ in a lattice E is an element $x \in E$ such that $x \wedge (u - x) = 0$).

For this goal we need some new notations. Assume that we have a family $\{G_\gamma \mid \gamma \in \mathscr{F}\} \subset \mathscr{P}(\mathbb{N})$ indexed by $\gamma \in \mathscr{F}$ and that for every set G_γ of this family we have another set $\vec{F^\gamma} := \{F_m^\gamma \mid m \in G_\gamma\} \subset \mathscr{P}(\mathbb{N})$ indexed by the elements of G_γ. We denote by $\mathbf{W}^\gamma_{G_\gamma, \vec{F^\gamma}} = \big((z_{mk}^\gamma)\big) \in \ell^p[\ell^q]$ the "section" of the double sequence $\big((w_{mk}^\gamma)\big)$ defined as $z_{mk}^\gamma = 0$ if $m \notin G_\gamma$, $z_{mk}^\gamma = 0$ if $m \in G_\gamma$ but $k \notin F_m^\gamma$ and $z_{mk}^\gamma = w_{mk}^\gamma$ if $m \in G_\gamma$ and $k \in F_m^\gamma$ whatever be $\gamma \in \mathfrak{F}$ and $(m, k) \in \mathbb{N}^2$. The ultraproduct of these sections is denoted by $\mathbf{W}_{(G_\gamma), (\vec{F^\gamma})}$. If every set F_m^γ, $m \in G_\gamma$ is the same set F independent of m and γ we shall write $\mathbf{W}_{(G_\gamma),(F)}$ instead of $\mathbf{W}_{(G_\gamma), (\vec{F^\gamma})}$. Clearly the map $P_{(G_\gamma), (\vec{F^\gamma})} : \mathbf{W} \in \big(\ell^p[\ell^q]\big)_{\mathscr{U}} \longrightarrow \mathbf{W}_{(G_\gamma), (\vec{F^\gamma})}$ is a projection in $\big(\ell^p[\ell^q]\big)_{\mathscr{U}}$. In particular, if we have two elements $\mathbf{k} = (k_\gamma)_{\mathscr{U}}$ and $\mathbf{h} = (h_\gamma)_{\mathscr{U}}$ of the set theoretic ultrapower $\mathbf{N} = (\mathbb{N})_{\mathscr{U}}$ of the set of natural numbers we put $\mathbf{W_{k,h}}$ for the element $\mathbf{W}_{(G_\gamma),(\vec{F^\gamma})}$ where $G_\gamma = \{k_\gamma\}$ and $\vec{F}_\gamma = \big\{\{h_\gamma\}\big\},\ \gamma \in \mathscr{F}$. We will denote by $\mathfrak{C}(\mathbf{W})$ the set of all the order components of $\mathbf{W} \in \big(\ell^p[\ell^q]\big)_{\mathscr{U}}$. It can be shown

Proposition 2.1 ([16, Lemma 4]) *Let* $p \in]0,\infty]$, $q \in]0,\infty[$ *and* $0 \le \mathbf{W} \in \big(\ell^p[\ell^q]\big)_{\mathscr{U}}$. *Then*

$$\mathfrak{C}(\mathbf{W}) = \Big\{\mathbf{W}_{(G_\gamma),(\vec{F^\gamma})} \;\Big|\; G^\gamma \subset \mathbb{N},\quad \vec{F^\gamma} = \big\{F_m^\gamma \subset \mathbb{N} \,\big|\, m \in G_\gamma\big\}, \gamma \in \mathscr{F}\Big\}.$$

Define $\mathfrak{g}_m^\gamma := (g_{mk}^\gamma)_{k=1}^\infty \in \ell^{r_0}$ for every $\gamma \in \mathscr{F}$ and each $m \in \mathbb{N}$. Clearly one has $\sup_{\gamma\in\mathscr{F}} \sup_{m\in\mathbb{N}} \big\|\mathfrak{g}_m^\gamma\big\|_{\ell^{r_0}} \le \mathbf{I}_{\alpha_{\mathbf{r}}^C}(T) + \varepsilon$. Now we consider the following special elements

$$\forall\, q > 0 \qquad \mathbf{A}^q := \Big(\Big(\Big(\big\|\mathfrak{g}_m^\gamma\big\|_{\ell^{r_0}}^{-\frac{r_0}{q}}\ \big|g_{mk}^\gamma\big|^{\frac{r_0}{q}}\Big)_{k=1}^\infty\Big)_{m=1}^\infty\Big)_{\mathscr{U}} \in X_q^\infty, \tag{4}$$

$$\forall\, q > 0 \qquad \mathbf{U}^q := \Big(\Big(\Big(\big\|\mathfrak{g}_m^\gamma\big\|_{\ell^{r_0}}^{1-\frac{r_0}{q}} \big|g_{mk}^\gamma\big|^{\frac{r_0}{q}}\Big)_{k=1}^\infty\Big)_{m=1}^\infty\Big)_{\mathscr{U}} \in X_q, \tag{5}$$

$$\forall\, q > 0 \qquad \mathbf{V}^q := \Big(\Big(\Big(\big\|\mathfrak{g}_m^\gamma\big\|_{\ell^{r_0}}^{\frac{1-r_0}{q}} \big|g_{mk}^\gamma\big|^{\frac{r_0}{q}}\Big)_{k=1}^\infty\Big)_{m=1}^\infty\Big)_{\mathscr{U}} \in Y_q. \tag{6}$$

These elements are important for several reasons. First of all, if $q \ge 1$, by [8, Proposition 4.6], and [4, Theorem 11.12], the Banach lattices X_q and Y_q are order continuous (and Dedekind order complete), but not necessarily X_q^∞. So, we look for a factorization of $(S_{\mathfrak{g}^\gamma})_{\mathscr{U}}$ through a Banach lattice with good order properties replacing X_q^∞. This can be done for $1 < q < \infty$ with help of $\mathbf{A}^q$ and $\mathbf{U}^{q'}$ considering in a first step the quotient Banach lattice $\mathfrak{Q}_q^\infty := (\ell^\infty[\ell^q])_{\mathscr{U}} \big/ (\mathbf{A}^q)^\perp$ (remember that if Y is a subset of a lattice X we put $Y^\perp = \big\{x \in X \mid |x| \wedge |y| = 0\ \forall\, y \in Y\big\}$ and so $(\mathbf{A}^q)^\perp$ is a solid set). We denote by $K_q : X_q^\infty \longrightarrow \mathfrak{Q}_q^\infty$ the canonical quotient map. It can be shown

Lemma 2.1 ([16, Lemma 6]) *If* $q \in]1,\infty[$ *one has*

$$\mathfrak{C}\big(K_q(\mathbf{A}^q)\big) = \big\{K_q(\mathbf{X}), \mathbf{X} \in \mathfrak{C}(\mathbf{A}^q)\big\}.$$

In a second step, by the order continuity of X_q and Freudenthal's spectral theorem [1, Theorem 6.8], we check that $(\mathbf{U}^{q'})^\circ = \big((\mathbf{U}^{q'})^{\perp\perp}\big)^\circ$. It follows that $(\mathbf{U}^{q'})^\circ$ is a solid set (by [1, Page 166]) and the quotient $\mathscr{D}_{q'} := (X_{q'})'/(\mathbf{U}^{q'})^\circ$ is a well defined Banach lattice. Let $\mathfrak{q}_{q'} : (X_{q'})' \longrightarrow \mathscr{D}_{q'}$ be the canonical quotient map. Using the main result of Kürsten in [12] we obtain next important result containing in particular the desired factorization of $(S_{\mathfrak{g}^\gamma})_{\mathscr{U}}$:

Proposition 2.2 ([16, Lemmata 7 and 8 and Corollary 9]) *Let* $1 < q < \infty$. *Then* $\mathfrak{Q}_q^\infty$ *is a Dedekind complete Banach lattice and*

(1) *The map* $\iota_q : K_q(\mathbf{X}) \in \mathfrak{Q}_q^\infty \longrightarrow \mathfrak{q}_{q'}(\mathbf{X}) \in \mathscr{D}_{q'}$ *is a surjective topological order isomorphism.*

(2) $K_q(\mathbf{A}^q)$ *is a weak order unit in* $\mathfrak{Q}_q^\infty$ *and the linear span* $\langle \mathfrak{C}(K_q(\mathbf{A}^q)) \rangle$ *is dense in* $\mathfrak{Q}_q^\infty$.

(3) *The canonical map* $\widehat{(S_{\mathfrak{g}^\gamma})_\mathscr{U}} \in \mathscr{L}^n\left(\prod_{j=1}^n \mathfrak{Q}_{r'_j}^\infty, X_{t_\mathbf{r}}\right)$ *factorizing* $(S_{\mathfrak{g}^\gamma})_\mathscr{U}$ *through* $\prod_{j=1}^n \mathfrak{Q}_{r'_j}^\infty$ *is well defined and* $\widehat{(S_{\mathfrak{g}^\gamma})_\mathscr{U}}\left(\prod_{j=1}^n \mathfrak{Q}_{r'_j}^\infty\right) \subset B(\mathbf{U}^{t_\mathbf{r}})$.

On the other hand, by (3) we have for components defined by arbitrary families $(G_\gamma), (\overrightarrow{F^\gamma})$

$$(S_{\mathfrak{g}^\gamma})_\mathscr{U}\left((\mathbf{A}^{r'_j}_{(G_\gamma),(\overrightarrow{F^\gamma})})_{j=1}^n\right) = \widehat{(S_{\mathfrak{g}^\gamma})_\mathscr{U}}\left((\mathbf{A}^{r'_j}_{(G_\gamma),(\overrightarrow{F^\gamma})})_{j=1}^n\right)$$

$$= \left(\left(\left\|\mathfrak{g}_m^\gamma\right\|_{\ell^{r_0}}^{-r_0\left(\frac{1}{t_\mathbf{r}}-\frac{1}{r_0}\right)}\left(\left|g_{mk}^\gamma\right|^{r_0\left(\frac{1}{t_\mathbf{r}}-\frac{1}{r_0}\right)+1}\right)_{k\in F_m^\gamma}\right)_{m\in G_\gamma}\right)_\mathscr{U} = \mathbf{U}^{t_\mathbf{r}}_{(G_\gamma),(\overrightarrow{F^\gamma})} \leq \mathbf{U}^{r_{n+1}}_{(G_\gamma),(\overrightarrow{F^\gamma})}. \tag{7}$$

2.2 Building the Measure Spaces

The elements $\mathbf{V}^q$ are used to start the search of the desired measure spaces $\mathfrak{R}$ and $\mathfrak{T}$. The set $\mathfrak{C}(\mathbf{V}^1)$ of all the order components of $\mathbf{V}^1$ is a Boolean algebra under the operations $\vee$, $\wedge$ and $\mathbf{V}^1 - \mathbf{X}$, $\mathbf{X} \in \mathfrak{C}(\mathbf{V}^1)$. As Y_1 is order Dedekind complete, by the known result [1, Theorem 3.15], $\mathfrak{C}(\mathbf{V}^1)$ is a Dedekind complete boolean algebra. By Stone's representation theorem there are an extremely disconnected compact topological space Ω and a boolean algebra isomorphism φ_1 from $\mathfrak{C}(\mathbf{V}^1)$ onto the Boole algebra $\mathscr{B}$ of the clopen sets of Ω which, by Proposition 2.1, generates an onto isomorphism of boolean algebras $\varphi_q : \mathbf{V}^q_{(G_\gamma),(\overrightarrow{F^\gamma})} \in \mathfrak{C}(\mathbf{V}^q) \longrightarrow \varphi_1\left(\mathbf{V}^1_{(G_\gamma),(\overrightarrow{F^\gamma})}\right) \in \mathscr{B}$ from the boolean algebra of components of $\mathbf{V}^q$ onto $\mathscr{B}$. Then, as in the proof of Kakutani-Bohnenblust-Nakano's Theorem [1, Theorem 12.26], it can be shown that the real function μ on $\mathscr{B}$

$$\forall\, 0 < q < \infty \quad \forall\, A \in \mathscr{B} \quad \mu(A) = \left\|\varphi_1^{-1}(A)\right\|_{Y_1} = \left\|\varphi_q^{-1}(A)\right\|_{Y_q}^q,$$

is a measure on $(\Omega, \mathscr{B})$ that can be extended by Caratheodory's method to a measure (again denoted by μ) defined on the σ-algebra $\mathscr{M}$ of μ-measurable sets in Ω in such a way that the quasi-Banach spaces $L^p(\Omega, \mathscr{B}, \mu)$ and $L^p(\Omega, \mathscr{M}, \mu)$, $0 < p < \infty$, are isometric (because for every $\mathscr{M}$-measurable set A there is a $\mathscr{B}$-measurable set $B \supset A$ such that $\mu(B \backslash A) = 0$). Furthermore, it can be shown that the linear map $\mathfrak{J}_q : \mathbf{w} \in \mathfrak{C}(\mathbf{V}^q) \longrightarrow \mathfrak{J}_q(\mathbf{w}) = \chi_{\varphi_q(\mathbf{w})}$ can be extended to a surjective isometry (again denoted by $\mathfrak{J}_q$) $\mathfrak{J}_q : B(\mathbf{V}^q) \longrightarrow L^q(\Omega, \mathscr{B}, \mu)$ from the band $B(\mathbf{V}^q)$ generated by $\mathbf{V}^q$ in Y_q onto the space $L^q(\Omega, \mathscr{B}, \mu)$. It follows that $\mu(\Omega) < \infty$.

Let Ω_1 be the set of atoms of $(\Omega, \mathscr{B}, \mu)$ and let $\Omega_0 := \Omega \backslash \Omega_1$ be its purely non atomic part. As $\mu(\Omega) < \infty$, by the result [28, Theorem 3.2], of Wnuk and

Wiatrowski about discrete elements in ultrapowers of Banach latices the set of atoms of $(\Omega, \mathscr{B}, \mu)$ must be a numerable set of elements of type

$$\Omega_1 = \left\{ \bar{\mathbf{e}}_{\mathbf{k}^m, \mathbf{h}^k} := \varphi_1\left(\mathbf{V}^1_{\mathbf{k}^m, \mathbf{h}^k}\right) \;\middle|\; (m,k) \in \mathbb{P} \subset \mathbb{N}^2 \right\}.$$

Put $\mathfrak{g} := (\mathfrak{g}^\gamma)_{\mathscr{U}} \in X_{r_0}$ and $\mathfrak{g}_{\mathbf{k}} := \mathfrak{g}_{(k_\gamma),(\mathbb{N})}$ for every $\mathbf{k} = (k_\gamma)_{\mathscr{U}} \in \mathbb{N}$. Since for every $(m,k) \in \mathbb{P}$ one has $\mathbf{V}^1_{\mathbf{k}^m, \mathbf{h}^k} \neq 0$ and

$$\mu(\bar{\mathbf{e}}_{\mathbf{k}^m, \mathbf{h}^k}) = \left\| \mathbf{V}^1_{\mathbf{k}^m, \mathbf{h}^k} \right\|_{Y_1} = \lim_{\gamma, \mathscr{U}} \left\| \mathfrak{g}^\gamma_{k^m_\gamma} \right\|^{1-r_0}_{\ell^{r_0}} \left| g^\gamma_{k^m_\gamma h^k_\gamma} \right|^{r_0} = \left(\lim_{\gamma, \mathscr{U}} \left| g^\gamma_{k^m_\gamma h^k_\gamma} \right|^{r_0} \right) \left\| \mathfrak{g}_{\mathbf{k}^m} \right\|^{1-r_0}_{X_{r_0}} \neq 0, \tag{8}$$

noting that $\left| g^\gamma_{k^m_\gamma h^k_\gamma} \right| \left\| \mathfrak{g}^\gamma_{k^m_\gamma} \right\|^{-1}_{\ell^{r_0}} \leq 1$ for every $\gamma \in \mathscr{F}$, we have necessarily

$$\forall\ (m,k) \in \mathbb{P} \quad \left\| \mathfrak{g}_{\mathbf{k}^m} \right\|_{X_{r_0}} = \lim_{\gamma, \mathscr{U}} \left\| \mathfrak{g}^\gamma_{k^m_\gamma} \right\|_{\ell^{r_0}} \neq 0 \quad \text{and} \quad \lim_{\gamma, \mathscr{U}} \left| g^\gamma_{k^m_\gamma h^k_\gamma} \right| \left\| \mathfrak{g}^\gamma_{k^m_\gamma} \right\|^{-1}_{\ell^{r_0}} \neq 0. \tag{9}$$

If follows that $\mathbf{U}^q_{\mathbf{k}^m, \mathbf{h}^k} \neq 0$ for $0 < q < \infty$ and $(m,k) \in \mathbb{P}$.

On the other hand, consider the element $\mathbf{Z} := \left(\left(\left\| \mathfrak{g}^\gamma_m \right\|_{\ell^{r_0}} \right)_{m=1}^{\infty} \right)_{\mathscr{U}} \in (\ell^1)_{\mathscr{U}}$ which is well defined by (3) and its subsequent comments. As in the case of $\mathbf{V}^1$ it can be shown that the set of components of $\mathbf{Z}$ is $\mathfrak{C}(\mathbf{Z}) = \left\{ \mathbf{Z}_{(G_\gamma)} \;\middle|\; G_\gamma \subset \mathbb{N} \ \forall\, \gamma \in \mathscr{F} \right\}$, (the notation is self explanatory, keeping in mind the definitions for the components of $\mathbf{V}^1$). As above $\mathfrak{C}(\mathbf{Z})$ is a Dedekind complete boolean algebra ($(\ell^1)_{\mathscr{U}}$ is order continuous by [8, Proposition 4.6]) and there are an extremely disconnected compact topological space Ξ and a Boolean algebra isomorphism ψ from $\mathfrak{C}(\mathbf{Z})$ onto the Boolean algebra $\mathscr{C}$ of the clopen sets of Ξ. Moreover, the real function

$$\beta : A \in \mathscr{C} \longrightarrow \beta(A) = \left\| \psi^{-1}(A) \right\|_{(\ell^1)_{\mathscr{U}}}$$

is a measure on $(\Xi, \mathscr{C})$ which can be extended to a measure, again denoted by β, defined on the σ-algebra $\mathscr{A}$ of ν-measurable sets of Ξ, the spaces $L^1(\Xi, \mathscr{C}, \beta)$ and $L^1(\Xi, \mathscr{A}, \beta)$ are isometric and the map $\mathfrak{I} : \mathbf{w} \in \mathfrak{C}(\mathbf{Z}) \longrightarrow \mathfrak{I}(\mathbf{w}) = \chi_{\psi(\mathbf{w})}$ can be extended to a surjective isometry (again denoted by $\mathfrak{I}$) $\mathfrak{I} : B(\mathbf{Z}) \longrightarrow L^1(\Xi, \mathscr{C}, \beta)$ from the band $B(\mathbf{Z})$ generated by $\mathbf{Z}$ in $(\ell^1)_{\mathscr{U}}$ onto the space $L^1(\Xi, \mathscr{C}, \beta)$. Note that $\beta(\Xi) < \infty$. With a method analogous to the used one in the case of the atoms of Ω it can be shown that the set of the atoms of (Ξ, β) is a numerable set

$$\Xi_a := \left\{ \bar{\mathbf{e}}_{\mathbf{k}^m} := \psi(\mathbf{Z}_{\mathbf{k}^m}) \;\middle|\; \mathbf{k}^m = (k^m_\gamma)_{\mathscr{U}} \in \mathbf{N},\ m \in \mathbb{A} \subset \mathbb{N} \right\}.$$

Lemma 2.2 *Define* $\Xi_\pi := \left\{ \bar{\mathbf{e}}_{\mathbf{k}^m} \in \Xi_a \;\middle|\; \exists\, \mathbf{h}^\nu \in \mathbf{N} \,/\, \bar{\mathbf{e}}_{\mathbf{k}^m, \mathbf{h}^\nu} \in \Omega_1 \right\}$. *There is a representative* $\mathbf{V}^1_{(P_\gamma), (\vec{Q}_\gamma)} \in \mathfrak{C}(\mathbf{V}^1)$ *such that* $\Omega_1 = \varphi_1(\mathbf{V}^1_{(P_\gamma), (\vec{Q}_\gamma)})$ *and* $\psi(\mathbf{Z}_{(P_\gamma)}) = \Xi_\pi$.

Proof Let $\mathbb{A}_1 := \{m \in \mathbb{A} \mid \bar{\mathbf{e}}_{\mathbf{k}^m} \in \Xi_\pi\}$. As ψ is a boolean algebra isomorphism there is $\mathbf{Z}_{(A_\gamma)} \in \mathfrak{C}(\mathbf{Z})$ such that

$$\Xi_\pi = \psi(\mathbf{Z}_{(A_\gamma)}) = \bigcup_{m\in\mathbb{A}_1} \bar{\mathbf{e}}_{\mathbf{k}^m} \quad \text{and} \quad \mathbf{Z}_{(A_\gamma)} = \bigvee_{m\in\mathbb{A}_1} \mathbf{Z}_{\mathbf{k}^m}. \tag{10}$$

Analogously, φ_1 being a boolean algebra isomorphism there is $\mathbf{V}^1_{(P_\gamma),(\vec{Q}_\gamma)} \in \mathfrak{C}(\mathbf{V}^1)$ such that

$$\varphi_1^{-1}(\Omega_1) = \mathbf{V}^1_{(P_\gamma),(\vec{Q}_\gamma)} = \bigvee_{(m,k)\in\mathbb{P}} \mathbf{V}^1_{\mathbf{k}^m,\mathbf{h}^k}. \tag{11}$$

First of all we show that

$$\exists\, U \in \mathscr{U} \ \text{ such that } \ A_\gamma \subset P_\gamma \quad \forall\, \gamma \in U. \tag{12}$$

In fact, if $m \in \mathbb{A}_1$ there is $\bar{\mathbf{e}}_{\mathbf{k}^m,\mathbf{h}^v} \in \Omega_1$ and φ_1 being a boolean algebra isomorphism $\mathbf{V}^1_{\mathbf{k}^m,\mathbf{h}^v} \le \mathbf{V}^1_{(P_\gamma),(\vec{Q}^\gamma)}$ holds and so

$$\mathbf{V}^1_{\mathbf{k}^m,\mathbf{h}^v} = \mathbf{V}^1_{\mathbf{k}^m,\mathbf{h}^v} \wedge \mathbf{V}^1_{(P_\gamma),(\vec{Q}^\gamma)} = \mathbf{V}^1_{(k^m_\gamma),(\{h^v_\gamma\})} \wedge \mathbf{V}^1_{(P_\gamma),(\vec{Q}^\gamma)} = \mathbf{V}^1_{(\{k^m_\gamma\}\cap P_\gamma),\overrightarrow{(\{h^v_\gamma\}\cap Q^\gamma_{k^m_\gamma})}}.$$

We deduce that there is $U \in \mathscr{U}$ such that $\{k^m_\gamma\} = \{k^m_\gamma\} \cap P_\gamma$, that is $k^m_\gamma \in P_\gamma$, for each $\gamma \in U$. Then $\psi(\mathbf{Z}_{\mathbf{k}^m}) \subset \psi(\mathbf{Z}_{(P_\gamma)})$ and by (10) $\Xi_\pi = \bigcup_{m\in\mathbb{A}_1} \psi(\mathbf{Z}_{\mathbf{k}^m}) \subset \psi(\mathbf{Z}_{(P_\gamma)})$ and taking images by ψ^{-1} we obtain $\mathbf{Z}_{(A_\gamma)} \le \mathbf{Z}_{(P_\gamma)}$. In consequence it turns out that $\mathbf{Z}_{(A_\gamma)} = \mathbf{Z}_{(A_\gamma)} \wedge \mathbf{Z}_{(P_\gamma)} = \mathbf{Z}_{(A_\gamma\cap P_\gamma)}$ and that means (12).

In a second step we show that $\mathbf{V}^1_{(A_\gamma),(\vec{Q}^\gamma)} = \mathbf{V}^1_{(P_\gamma),(\vec{Q}^\gamma)} = \varphi_1^{-1}(\Omega_1)$ and this fact will prove our lemma choosing the family (A_γ) instead of (P_γ). The element $\mathbf{V}^1_{(A_\gamma),(\vec{Q}^\gamma)}$ is well defined understanding that we consider only the sets Q^γ_a corresponding to elements $a \in A_\gamma \subset P_\gamma$. Trivially $\mathbf{V}^1_{(A_\gamma),(\vec{Q}^\gamma)} \le \mathbf{V}^1_{(P_\gamma),(\vec{Q}^\gamma)}$. On the other hand, for every $(m, v) \in \mathbb{P}$ one has $\bar{\mathbf{e}}_{\mathbf{k}^m,\mathbf{h}^v} \subset \Omega_1$ and so $\psi(\mathbf{Z}_{\mathbf{k}^m}) \subset \Xi_\pi$. It follows that $\mathbf{Z}_{\mathbf{k}^m} \le \mathbf{Z}_{(A_\gamma)}$ and hence $\mathbf{Z}_{\mathbf{k}^m} = \mathbf{Z}_{\mathbf{k}^m} \wedge \mathbf{Z}_{(A_\gamma)} = \mathbf{Z}_{(\{k^m_\gamma\}\cap A_\gamma)}$. Then there is $U \in \mathscr{U}$ such that $\{k^m_\gamma\} = \{k^m_\gamma\} \cap A_\gamma$ for every $\gamma \in U$, i. e. $k_\gamma \in A_\gamma$ for every $\gamma \in U$. This implies that $\mathbf{V}^1_{\mathbf{k}^m,\mathbf{h}^v} \le \mathbf{V}^1_{(A_\gamma),(\vec{Q}^\gamma)}$ and by (11) we deduce $\mathbf{V}^1_{(P_\gamma),(\vec{Q}^\gamma)} \le \mathbf{V}^1_{(A_\gamma),(\vec{Q}^\gamma)}$. Then $\mathbf{V}^1_{(P_\gamma),(\vec{Q}^\gamma)} = \mathbf{V}^1_{(A_\gamma),(\vec{Q}^\gamma)}$. □

On the other hand, for $0 < q < \infty$ let $\overline{X^1_q}$ be the closure in X_q of the linear span $\langle \mathbf{U}^q_{\mathbf{k}^m,\mathbf{h}^k}\rangle_{(m,k)\in\mathbb{P}}$. Let $\mathbb{M} := \{(m, v) \mid \bar{\mathbf{e}}_{\mathbf{k}^m,\mathbf{h}^v} \subset \varphi_q(\mathbf{V}^q_{(E_\gamma),(\vec{F}^\gamma)})\}$ for a given fixed $\mathbf{U}^q_{(E_\gamma),(\vec{F}^\gamma)} \in \mathfrak{C}(\mathbf{U}^q_{(P_\gamma),(\vec{Q}^\gamma)})$. If $1 \le q < \infty$ the element

$$\mathbf{S}^q := \bigvee_{(m,k)\in\mathbb{M}} \mathbf{U}^q_{\mathbf{k}^m,\mathbf{h}^k} \in \overline{X^1_q} \subset B\big(\mathbf{U}^q_{(P_\gamma),(\vec{Q^\gamma})}\big)$$

exists because X_q is order continuous and Dedekind complete.

Lemma 2.3 *The equalities* $\mathbf{S}_q = \mathbf{U}^q_{(E_\gamma),(\vec{F^\gamma})}$ *and* $B\big(\mathbf{U}^q_{(E_\gamma),(\vec{F^\gamma})}\big) = P_{(E_\gamma),(\vec{F^\gamma})}\big(B(\mathbf{U}^q)\big)$ *hold for each* $1 \le q < \infty$. *In particular* $B\big(\mathbf{U}^q_{(P_\gamma),(\vec{Q^\gamma})}\big) = P_{(P_\gamma),(\vec{Q^\gamma})}\big(B(\mathbf{U}^q)\big) = \overline{X^1_q}$.

Proof Let $0 \le \mathbf{T} \in X_q$ such that $\mathbf{U}^q_{\mathbf{k}^m,\mathbf{h}^v} \le \mathbf{T} \le \mathbf{U}^q_{(E_\gamma),(\vec{F^\gamma})}$ for every $(m, v) \in \mathbb{M}$. From the definition of the order in ultraproducts we can choose a representative $\mathbf{T} = \big((t^\gamma_{mk})\big)_{\mathscr{U}}$ such that for every $\gamma \in \mathscr{F}$ we have $t^\gamma_{mk} \le \|\mathfrak{g}^\gamma_m\|^{1-\frac{r_0}{q}}_{\ell^{r_0}} |g^\gamma_{mk}|^{\frac{r_0}{q}}$ for every $k \in F^\gamma_m$, $m \in E_\gamma$ and $t^\gamma_{mk} = 0$ in other case. Analogously, fixed $(m, v) \in \mathbb{M}$, there is $\big((\alpha^\gamma_{ik})\big) \in \ell^1[\ell^q]^{\mathscr{F}}$ such that $\lim_{\gamma,\mathscr{U}} \big\|\big((\alpha^\gamma_{ik})\big)\big\|_{\ell^1[\ell^q]} = 0$ and for every $\gamma \in \mathscr{F}$, with respect to the order in $\ell^1[\ell^q]$ one has

$$\|\mathfrak{g}^\gamma_{k^m_\gamma}\|^{1-\frac{r_0}{q}}_{\ell^{r_0}} |g^\gamma_{k^m_\gamma h^v_\gamma}|^{\frac{r_0}{q}} + \alpha^\gamma_{k^m_\gamma h^v_\gamma} \le t^\gamma_{k^m_\gamma h^v_\gamma},$$

$$\big((\|\mathfrak{g}^\gamma_k\|^{\frac{1}{q}-1}_{\ell^{r_0}} t^\gamma_{kh})_{h\in F^\gamma_k}\big)_{k\in E_\gamma} \le \big((\|\mathfrak{g}^\gamma_k\|^{\frac{1}{q}-1}_{\ell^{r_0}} \|\mathfrak{g}^\gamma_k\|^{1-\frac{r_0}{q}}_{\ell^{r_0}} |g^\gamma_{kh}|^{\frac{r_0}{q}})_{h\in F^\gamma_k}\big)_{k\in E_\gamma}$$

and

$$\|\mathfrak{g}^\gamma_{k^m_\gamma}\|^{\frac{1}{q}-1}_{\ell^{r_0}}\Big(\|\mathfrak{g}^\gamma_{k^m_\gamma}\|^{1-\frac{r_0}{q}}_{\ell^{r_0}} |g^\gamma_{k^\gamma_m h^\gamma_v}|^{\frac{r_0}{q}} + \alpha^\gamma_{k^m_\gamma h^v_\gamma}\Big)\,\mathbf{e}_{k^m_\gamma h^v_\gamma} \le \|\mathfrak{g}^\gamma_{k^m_\gamma}\|^{\frac{1}{q}-1}_{\ell^{r_0}} t^\gamma_{k^m_\gamma h^v_\gamma}\,\mathbf{e}_{k^m_\gamma h^v_\gamma} \tag{13}$$

$$\le \|\mathfrak{g}^\gamma_{k^m_\gamma}\|^{\frac{1}{q}-1}_{\ell^{r_0}} \|\mathfrak{g}^\gamma_{k^m_\gamma}\|^{1-\frac{r_0}{q}}_{\ell^{r_0}} |g^\gamma_{k^m_\gamma h^v_\gamma}|^{\frac{r_0}{q}}\,\mathbf{e}_{k^m_\gamma h^v_\gamma} \le \big((\|\mathfrak{g}^\gamma_k\|^{\frac{1}{q}-1}_{\ell^{r_0}} \|\mathfrak{g}^\gamma_k\|^{1-\frac{r_0}{q}} |g^\gamma_{kh}|^{\frac{r_0}{q}})_{h\in F^\gamma_h}\big)_{k\in E_\gamma}. \tag{14}$$

By (9) we obtain $\lim_{\gamma,\mathscr{U}} \Big\| \|\mathfrak{g}^\gamma_{k^m_\gamma}\|^{\frac{1}{q}-1}_{\ell^{r_0}} \alpha^\gamma_{k^m_\gamma h^v_\gamma}\,\mathbf{e}_{k^m_\gamma h^v_\gamma} \Big\|_{\ell^1[\ell^q]} = 0$ and so, by (6), (13) and (14) we deduce

$$\mathbf{V}^q_{\mathbf{k}^m,\mathbf{h}^v} \le \big((\|\mathfrak{g}^\gamma_{k^m_\gamma}\|^{\frac{1}{q}-1}_{\ell^{r_0}} t^\gamma_{k^m_\gamma h^v_\gamma}\,\mathbf{e}_{k^m_\gamma h^v_\gamma})\big)_{\mathscr{U}} \le \big(\big((\|\mathfrak{g}^\gamma_k\|^{\frac{1}{q}-1}_{\ell^{r_0}} t^\gamma_{kh})_{h\in F^\gamma_k}\big)_{k\in E_\gamma}\big)_{\mathscr{U}} \le \mathbf{V}^q_{(E_\gamma),(\vec{F^\gamma})}. \tag{15}$$

As φ_q is an order isomorphism we have $\mathbf{V}^q_{(E_\gamma),(\vec{F^\gamma})} = \bigvee_{(m,v)\in\mathbb{M}} \mathbf{V}^q_{\mathbf{k}^m,\mathbf{h}^v}$ and from the arbitrariness of $(m, v) \in \mathbb{M}$ in (15) we obtain $\mathbf{V}^q_{(E_\gamma),(\vec{F^\gamma})} = \big(\big((\|\mathfrak{g}^\gamma_k\|^{\frac{1}{q}-1}_{\ell^{r_0}} t^\gamma_{kh})_{h\in F^\gamma_k}\big)_{k\in E_\gamma}\big)_{\mathscr{U}}$. Then $\|\mathfrak{g}^\gamma_k\|^{\frac{1-r_0}{q}}_{\ell^{r_0}} |g^\gamma_{kh}|^{\frac{r_0}{q}} = \|\mathfrak{g}^\gamma_k\|^{\frac{1}{q}-1}_{\ell^{r_0}} t^\gamma_{kh} + b^\gamma_{kh}$, $h \in F^\gamma_k$, $k \in E_\gamma$, $\gamma \in \mathscr{F}$ for some $\big((b^\gamma_{kh})\big) \in \ell^1[\ell^q]^{\mathscr{F}}$ verifying $\lim_{\gamma,\mathscr{U}} \big\|\big((b^\gamma_{kh})\big)\big\|_{\ell^1[\ell^q]} = 0$ and $b^\gamma_{kh} = 0$ if $k \notin E_\gamma$ or $k \in E_\gamma$ but $h \notin F^\gamma_k$, $\gamma \in \mathscr{F}$. As $q \ge 1$, one has

$$\lim_{\gamma,\mathscr{U}} \Big\|\big((\|\mathfrak{g}^\gamma_h\|^{1-\frac{1}{q}}_{\ell^{r_0}} b^\gamma_{kh})\big)\Big\|_{\ell^1[\ell^q]} = 0$$

and hence

$$\mathbf{T} = \Big(\Big(\Big(\big\|\mathfrak{g}_h^\gamma\big\|_{\ell^{r_0}}^{1-\frac{1}{q}} \big\|\mathfrak{g}_h^\gamma\big\|_{\ell^{r_0}}^{\frac{1}{q}-1} t_{kh}^\gamma\Big)_{h\in F_k^\gamma}\Big)_{k\in E_\gamma}\Big)_{\mathscr{U}}$$

$$= \Big(\Big(\big\|\mathfrak{g}_h^\gamma\big\|_{\ell^{r_0}}^{1-\frac{1}{q}} \Big(\big\|\mathfrak{g}_h^\gamma\big\|_{\ell^{r_0}}^{\frac{1-r_0}{q}} \big|g_{kh}^\gamma\big|^{\frac{r_0}{q}} + b_{kh}^\gamma\Big)_{h\in F_k^\gamma}\Big)_{k\in E_\gamma}\Big)_{\mathscr{U}} = \mathbf{U}^q_{(E_\gamma),(\vec{F^\gamma})}.$$

It follows that $\mathbf{S}_q = \mathbf{U}^q_{(E_\gamma),(\vec{F^\gamma})}$. As $\mathbf{S}_q$ is a weak order unit in $B(\mathbf{S}_q)$ ([1, Page 36]) and $B(\mathbf{S}_q)$ is order continuous (because X_q does), Freudenthal's spectral theorem gives the density of the linear span $\big\langle\mathfrak{C}\big(\mathbf{U}^q_{(E_\gamma),(\vec{F^\gamma})}\big)\big\rangle$ in $B\big(\mathbf{U}^q_{(E_\gamma),(\vec{F^\gamma})}\big)$. □

φ_1 being a boolean algebra isomorphism there is $\mathbf{V}^1_{(N_\gamma),(\vec{M_\gamma})} \in \mathfrak{C}(\mathbf{V}^1)$ such that

$$\varphi_1^{-1}(\Omega_0) = \mathbf{V}^1_{(N_\gamma),(\vec{M^\gamma})}. \tag{16}$$

It is clear that for every $\mathbf{X} \in (\ell^p[\ell^q])_{\mathscr{U}}$, $0 < p, q \leq \infty$ one has

$$\mathbf{X} = \mathbf{X}_{(N_\gamma),(\vec{M^\gamma})} + \mathbf{X}_{(P_\gamma),(\vec{Q^\gamma})} + \mathbf{X}_{(P_\gamma\setminus N_\gamma),(\mathbb{N}\setminus\vec{Q^\gamma})} + \mathbf{U}_{(N_\gamma\setminus P_\gamma),(\mathbb{N}\setminus\vec{M^\gamma})}$$
$$+ \mathbf{X}_{(\mathbb{N}\setminus(P_\gamma\cup N_\gamma)),(\mathbb{N})} + \mathbf{X}_{(P_\gamma\cup N_\gamma),(\mathbb{N}\setminus(\overrightarrow{Q^\gamma\cup M^\gamma}))} - \mathbf{X}_{(P_\gamma\cap N_\gamma),(\overrightarrow{Q^\gamma\cap M^\gamma})}. \tag{17}$$

where the notations $\overrightarrow{Q^\gamma \cap M^\gamma}$, $\mathbb{N}\setminus\vec{Q^\gamma}$ and $\mathbb{N}\setminus\vec{M^\gamma}$ means the families of sets $\big\{Q_m^\gamma \cap M_m^\gamma \;\big|\; m \in P_\gamma \cap N_\gamma\big\}$, $\big\{\mathbb{N}\setminus Q_m^\gamma \;\big|\; m \in P_\gamma\setminus N_\gamma\big\}$ and $\big\{\mathbb{N}\setminus M_m^\gamma \;\big|\; m \in N_\gamma\setminus P_\gamma\big\}$ respectively.

Lemma 2.4 *One has* $\mathbf{V}^1_{(P_\gamma\cap N_\gamma),(\overrightarrow{Q^\gamma\cap M^\gamma})} = 0$, $\mathbf{V}^1_{(P_\gamma\setminus N_\gamma),(\mathbb{N}\setminus\vec{Q^\gamma})} = 0$, $\mathbf{V}^1_{(N_\gamma\setminus P_\gamma),(\mathbb{N}\setminus\vec{M^\gamma})} = 0$, $\mathbf{V}^1_{(\mathbb{N}\setminus(P_\gamma\cup N_\gamma)),(\mathbb{N})} = 0$ *and* $\mathbf{V}^1_{(P_\gamma\cup N_\gamma),(\mathbb{N}\setminus(\overrightarrow{Q^\gamma\cup M^\gamma}))} = 0$.

Proof As φ_1 is an isomorphism of boolean algebras and $\varphi_1(\mathbf{V}^1_{(P_\gamma\cap N_\gamma),(\overrightarrow{Q^\gamma\cap M^\gamma})}) \subset \varphi_1(\mathbf{V}^1_{(P_\gamma),(\vec{Q^\gamma})}) \cap \varphi_1(\mathbf{V}^1_{(N_\gamma),(\vec{M^\gamma})}) = \Omega_1 \cap \Omega_0 = \emptyset$ we obtain first result. If there would be an atom in $\Omega_1 \cap \varphi_1(\mathbf{V}^1_{(P_\gamma\setminus N_\gamma),(\mathbb{N}\setminus\vec{Q^\gamma})})$ we would have

$$0 < \mathbf{V}^1_{(P_\gamma),(\vec{Q^\gamma})} \wedge \mathbf{V}^1_{(P_\gamma\setminus N_\gamma),(\mathbb{N}\setminus\vec{Q^\gamma})} = 0,$$

a contradiction. Then $\varphi_1(\mathbf{V}^1_{(P_\gamma\setminus N_\gamma),(\mathbb{N}\setminus\vec{Q^\gamma})})$ is a purely non atomic set and so $\mathbf{V}^1_{(P_\gamma\setminus N_\gamma),(\mathbb{N}\setminus\vec{Q^\gamma})} = \mathbf{V}^1_{(N_\gamma),(\vec{M^\gamma})} \wedge \mathbf{V}^1_{(P_\gamma\setminus N_\gamma),(\mathbb{N}\setminus\vec{Q^\gamma})} = 0$. The proof for $\mathbf{V}^1_{(N_\gamma\setminus P_\gamma),(\mathbb{N}\setminus\vec{M^\gamma})}$ is the same. □

Remark 2.3 We point out that the possibility $\mathbf{Z}_{(N_\gamma\cap P_\gamma)} \neq 0$ (or $\mathbf{V}^1_{(N_\gamma\cap P_\gamma),(\mathbb{N})} \neq 0$) is not excluded. Then Ξ_1 can be different from the atomic part of $\psi(\mathbf{Z})$.

Lemma 2.5 *If* $1 < q < \infty$, $\mathfrak{Q}_q^{\infty,0} := K_q \circ P_{(N_\gamma),(\vec{M^\gamma})}(X_q^\infty)$ *and* $\mathfrak{Q}_q^{\infty,1} := K_q \circ P_{(P_\gamma),(\vec{Q^\gamma})}(X_q^\infty)$, *one has* $\mathfrak{Q}_q^\infty = \mathfrak{Q}_q^{\infty,0} \oplus \mathfrak{Q}_q^{\infty,1}$.

Proof As $\mathfrak{g} = \big((\mathfrak{g}_{mk}^\gamma)\big)_{\mathscr{U}} \in X_{r_0}$ we deduce easily $\mathbf{U}^{q'} \leq \left\|\mathfrak{g}\right\|_{X_{r_0}}^{1-\frac{1}{q'}} \mathbf{V}^{q'}$ and so, by Lemma 2.4, $\mathbf{U}^{q'}_{(P_\gamma\cap N_\gamma),(\vec{Q^\gamma\cap M^\gamma})} \leq \left\|\mathfrak{g}\right\|_{X_{r_0}}^{1-\frac{1}{q'}} \mathbf{V}^{q'}_{(P_\gamma\cap N_\gamma),(\vec{Q^\gamma\cap M^\gamma})} = 0$. In the same way we obtain $\mathbf{U}^{q'}_{(P_\gamma\setminus N_\gamma),(\mathbb{N}\setminus\vec{Q^\gamma})} = 0$, $\mathbf{U}^{q'}_{(N_\gamma\setminus P_\gamma),(\mathbb{N}\setminus\vec{M^\gamma})} = 0$, $\mathbf{U}^{q'}_{(\mathbb{N}\setminus(P_\gamma\cup N_\gamma)),(\mathbb{N})} = 0$ and $\mathbf{U}^{q'}_{(P_\gamma\cup N_\gamma),(\mathbb{N}\setminus(\vec{Q^\gamma\cup M^\gamma}))} = 0$. Then, by (17) and Lemma 2.4, $\mathbf{U}^{q'} = \mathbf{U}^{q'}_{(N_\gamma),(\vec{M^\gamma})} + \mathbf{U}^{q'}_{(P_\gamma),(\vec{Q^\gamma})}$. Hence, given $\mathbf{X} \in X_q^\infty$ we have $\big\langle \mathbf{X}_{(P_\gamma\setminus N_\gamma),(\mathbb{N}\setminus\vec{Q^\gamma})}, \mathbf{U}^{q'}\big\rangle = 0$, $\big\langle \mathbf{X}_{(N_\gamma\setminus P_\gamma),(\mathbb{N}\setminus\vec{M^\gamma})}, \mathbf{U}^{q'}\big\rangle = 0$, $\big\langle \mathbf{X}_{(P_\gamma\cup N_\gamma),(\mathbb{N}\setminus(\vec{Q^\gamma\cup M^\gamma}))}, \mathbf{U}^{q'}\big\rangle = 0$ and

$$\big\langle \mathbf{X}_{(P_\gamma\cap N_\gamma),(\vec{Q^\gamma\cap M^\gamma})}, \mathbf{U}^{q'}\big\rangle = 0.$$

Once again by (17) the element $\mathbf{W} := \mathbf{X} - \mathbf{X}_{(N_\gamma),(\vec{M^\gamma})} - \mathbf{X}_{(P_\gamma),(\vec{Q^\gamma})}$ verifies $\mathbf{W} \in (\mathbf{U}^{q'})^\circ$. We deduce

$$\begin{aligned} K_q(\mathbf{X}) = \iota_q^{-1} \circ \mathfrak{q}_{q'}(\mathbf{X}) &= \iota_q^{-1} \circ \mathfrak{q}_{q'}\big(\mathbf{X}_{(N_\gamma),(\vec{M^\gamma})} + \mathbf{X}_{(P_\gamma),(\vec{Q^\gamma})}\big) \\ &= K_q(\mathbf{X}_{(N_\gamma),(\vec{M^\gamma})}) + K_q(\mathbf{X}_{(P_\gamma),(\vec{Q^\gamma})}) \end{aligned}$$

and the result follows easily. □

Using the family (N_γ) of (16) and the family $\big\{P_\gamma,\ \gamma \in \mathscr{F}\big\}$ introduced in Lemma 2.2 we define two important subsets Ξ_1 and Ξ_0 of Ξ which will be basic in the sequel:

$$\Xi_1 := \psi\big(\mathbf{Z}_{(P_\gamma)}\big), \qquad \Xi_0 := \psi\big(\mathbf{Z}_{(N_\gamma)}\big).$$

2.2.1 On the Atomic Part of $\mathfrak{R} \times \mathfrak{T}$

We consider on Ξ_1 the measure ν_1 such that $\nu_1(\bar{\mathbf{e}}_{\mathbf{k}^m}) = 1$ for every $m \in \mathbb{A}_1$. For every $0 < q < \infty$ let $\mathfrak{P}_q$ be the natural projection from $\ell^1[\Xi_1, \nu_1, \ell^q(\Omega_1, \mu)]$ onto the closure of the linear span $\mathscr{H}_q := \big\langle \chi_{\bar{\mathbf{e}}_{\mathbf{k}^m}}(t)\chi_{\bar{\mathbf{e}}_{\mathbf{k}^m},\mathbf{h}^\nu}(x)\big\rangle_{(m,\nu)\in\mathbb{P}}$. The motivation for the use of the measure ν_1 is the following result.

Lemma 2.6 ([16, Lemma 13]) *Let* $0 < q < \infty$. *The map*

$$\mathfrak{W}_q : \sum_{i=1}^{h}\sum_{\nu=1}^{\nu_i} \alpha_{i\nu}\mathbf{U}^q_{\mathbf{k}^{m_i},\mathbf{h}^{k_\nu}} \longrightarrow \sum_{i=1}^{h}\sum_{\nu=1}^{\nu_1} \left\|\mathfrak{g}_{\mathbf{k}^{m_i}}\right\|^{1-\frac{1}{q}} \alpha_{i\nu}\chi_{\bar{\mathbf{e}}_{\mathbf{k}^{m_i}}}(t)\chi_{\bar{\mathbf{e}}_{\mathbf{k}^{m_i}},\mathbf{h}^{k_\nu}}(x)$$

from $\left\langle \mathbf{U}^q_{\mathbf{k}^m,\mathbf{h}^\nu}\right\rangle_{(m,\nu)\in\mathbb{P}}$ *into* $\ell^1[\Xi_1,\nu_1,\ell^q(\Omega_1,\mu)]$ *can be extended to an isometry, again denoted by* $\mathfrak{W}_q$ *from* $\overline{X^1_q}$ *onto the complemented subspace* $\overline{\mathscr{H}}_q$ *of* $\ell^1[\Xi_1,\nu_1,\ell^q(\Omega_1,\mu)]$.

Lemma 2.7 ([16, Lemma 14]) *If* $1 < q < \infty$, *the map* Δ^1_q *sending each* $\mathbf{X} = \left((x^\gamma_{mk})\right)_{\mathscr{U}} \in X^\infty_q$ *to*

$$\Delta^1_q(\mathbf{X}) = \left(\left(\|\mathfrak{g}_{\mathbf{k}^m}\|^{\frac{r_0-1}{q}}\left(\lim_{\gamma,\mathscr{U}} x^\gamma_{k^m_\gamma h^\nu_\gamma}\left|g^\gamma_{k^m_\gamma h^\nu_\gamma}\right|^{-\frac{r_0}{q}}\right)\right)_{\bar{\mathbf{e}}_{\mathbf{k}^m,\mathbf{h}^\nu}\subset\Omega_1}\right)_{m\in\mathbb{A}_1} \in \ell^\infty[\Xi_1,\nu_1,\ell^q(\Omega_1,\mu)]$$

is continuous. Moreover, the map

$$\widetilde{\Delta}^1_q : \widetilde{\mathbf{X}} \in \mathfrak{Q}^\infty_q \longrightarrow \Delta^1_q(\mathbf{X}) \in \ell^\infty[\Xi_1,\nu_1,\ell^q(\Omega_1,\mu)]$$

is well defined and continuous.

2.2.2 On the Purely Non Atomic Part of $\mathfrak{R} \times \mathfrak{T}$

Now we work on $\Xi_0 := \psi\big(\mathbf{Z}_{(N_\gamma)}\big)$. We consider the boolean algebra $\mathfrak{C}\big(\mathbf{Z}_{(N_\gamma)}\big)$ provided with the measure ν_0 defined for every $\mathbf{Z}_{(E_\gamma)} \in \mathfrak{C}\big(\mathbf{Z}_{(N_\gamma)}\big)$ verifying $E_\gamma \subset N_\gamma$ for every γ in certain set $U \in \mathscr{U}$ as

$$\nu_0\big(\mathbf{Z}_{(E_\gamma)}\big) = \lim_{\gamma,\mathscr{U}} \sum_{m\in E_\gamma}\Big(\sum_{j\in M^\gamma_m}\left|g^\gamma_{mj}\right|^{r_0}\Big)^{\frac{1}{r_0}}.$$

Clearly there is an isometry π from the measure algebra $\big(\mathfrak{C}(\mathbf{Z}_{(N_\gamma)}),\nu_0\big)$ onto the measure sub-σ-algebra $(\mathfrak{Z},\mu)$ of $\mathfrak{C}\big(\mathbf{V}^1_{(N_\gamma),(\vec{M}^\gamma)}\big)$ defined as

$$\mathfrak{Z} := \left\{\mathbf{V}^1_{(G_\gamma),(\vec{M}^\gamma)} \mid G_\gamma \subset N_\gamma \ \ \forall\, \gamma \in \mathscr{F}\right\} \subset \mathfrak{C}\big(\mathbf{V}^1_{(N_\gamma),(\vec{M}^\gamma)}\big).$$

By Maharam's results [19, Theorem 1] and [18, Definition 10.2, Theorem 10.6 and Sect. 17] there are a component $\mathbf{V}^1_{(V_\gamma),(\vec{W}^\gamma)} \in \mathfrak{C}\big(\mathbf{V}^1_{(N_\gamma),(\vec{M}^\gamma)}\big)$, a sub-$\sigma$-algebra $\mathfrak{E} \subset \mathfrak{C}\big(\mathbf{V}^1_{(V_\gamma),(\vec{W}^\gamma)}\big) \subset \mathfrak{C}\big(\mathbf{V}^1_{(N_\gamma),(\vec{M}^\gamma)}\big)$, a measure ρ_0 on $\mathfrak{E}$ and a surjective isometry

$$\mathfrak{W} : (\mathfrak{C}\big(\mathbf{V}^1_{(N_\gamma),(\vec{M}^\gamma)}\big),\mu) \longrightarrow (\mathfrak{Z},\mu)\times(\mathfrak{E},\rho_0) \approx \big(\mathfrak{C}(\mathbf{Z}_{(N_\gamma)}),\nu_0\big)\times(\mathfrak{E},\rho_0)$$

from the measure algebra $(\mathfrak{C}\big(\mathbf{V}^1_{(N_\gamma),(\vec{M}^\gamma)}\big),\mu)$ onto the product measure algebra $\big(\mathfrak{C}(\mathbf{Z}_{(N_\gamma)}),\nu_0\big)\times(\mathfrak{E},\rho_0)$. Moreover, $\mathfrak{W}\big(\mathbf{V}^1_{(G_\gamma),(\vec{F}^\gamma)}\big) = \mathbf{V}^1_{(G_\gamma),(\vec{M}^\gamma)} \times \mathbf{V}^1_{(G_\gamma),(\vec{F}^\gamma)}$ for every $\mathbf{V}^1_{(G_\gamma),(\vec{F}^\gamma)} \in \mathfrak{E}$ because $\mathbf{V}^1_{(G_\gamma),(\vec{F}^\gamma)} = \mathbf{V}^1_{(G_\gamma),(\vec{F}^\gamma)} \wedge \mathbf{V}^1_{(G_\gamma),(\vec{M}^\gamma)}$ in $\mathfrak{C}(\mathbf{V}^1_{(N_\gamma),(\vec{M}^\gamma)})$.

If P_1 (resp. P_2) denote the canonical projections from the cartesian product $\mathfrak{Z} \times \mathfrak{E}$ onto $\mathfrak{Z}$ (resp. $\mathfrak{E}$), from now on, given $\mathbf{V}^1_{(E_\gamma),(\vec{D^\gamma})} \in \mathfrak{C}\big(\mathbf{V}^1_{(N_\gamma),(\vec{M^\gamma})}\big)$ we shall put

$$E := \psi \circ \pi^{-1} \circ P_1 \circ \mathfrak{W}\big(\mathbf{V}^1_{(E_\gamma),(\vec{D_\gamma})}\big), \qquad D := \varphi_1 \circ P_2 \circ \mathfrak{W}\big(\mathbf{V}^1_{(E_\gamma),(\vec{D^\gamma})}\big). \tag{18}$$

Note that $\mu\big(\mathbf{V}^1_{(E_\gamma),(\vec{D^\gamma})}\big) = \nu_0(E)\rho_0(D)$. We define $\overline{\Omega} := \varphi_1\big(\mathbf{V}^1_{(V_\gamma),(\vec{W^\gamma})}\big) \subset \Omega_0$ provided with the σ-algebra $\overline{\mathscr{B}} := \varphi_1\big(\mathfrak{E}\big)$ and the measure $\overline{\rho}_0(\varphi_1(A)) = \rho_0(A)$ if $A \in \mathfrak{E}$.

In general given a measure space $(\Sigma, \mathscr{K}, \upsilon)$ and a set $B \in \mathscr{K}$ we will denote by $\mathscr{K}_B$ the σ-algebra induced by $\mathscr{K}$ on B. Then we have

Lemma 2.8 ([16, Lemma 15]) *There is a linear isometry Φ from $L^1(\Omega_0, \mathscr{B}_{\Omega_0}, \mu)$ onto the Lebesgue-Bochner space $L^1[\Xi_0, \mathscr{C}_{\Xi_0}, \nu_0, L^1(\overline{\Omega}, \overline{\mathscr{B}}, \overline{\rho}_0)]$.*

We point out that the map Φ is the extension by density of the map sending every simple function $S = \sum_{h=1}^{k} \sum_{v=1}^{h_v} \alpha_{kv} \chi_{A_{hv}}(x) \in L^1(\Omega_0, \mathscr{B}_{\Omega_0}, \mu)$, where $A_{hv} = \varphi_1\big(\mathbf{V}^1_{(E^h_\gamma),(\vec{D^\gamma_{hv}})}\big)$ with pairwise disjoint components $\big\{\mathbf{V}^1_{(E^h_\gamma),(\vec{D^\gamma_{hv}})},\ 1 \leq v \leq h_v\big\}$ and $\big\{\mathbf{Z}_{(E^h_\gamma)},\ 1 \leq h \leq k\big\}$, to the element

$$\Phi(S) = \sum_{h=1}^{k} \sum_{v=1}^{h_v} \alpha_{kv} \chi_{E_h}(t) \chi_{D_{hv}}(x) \in L^1[\Xi_0, \mathscr{C}_{\Xi_0}, \nu_0, L^1(\overline{\Omega}, \overline{\mathscr{B}}, \overline{\rho}_0)].$$

Using the famous theorem of Mazur and Ulam [20] and [2, Chap. VII, §1, page 142] asserting that every isometric bijection f between Banach spaces such that $f(0) = 0$ must be a linear map, we can prove next crucial result:

Proposition 2.3 ([16, Lemma 16]) *If $1 \leq q < \infty$ there is a linear isometry Ψ_q from the band $B\big(\mathbf{U}^q_{(N_\gamma),(\vec{M^\gamma})}\big) \subset P_{(N_\gamma),(\vec{M^\gamma})}(X_q)$ onto $L^1[\Xi_0, \mathscr{C}_{\Xi_0}, \nu_0, L^q(\overline{\Omega}, \overline{\mathscr{B}}, \overline{\rho}_0)]$ such that for every $\mathbf{V}^1_{(E_\gamma),(\vec{D^\gamma})} \in \mathfrak{C}\big(\mathbf{V}^1_{(N_\gamma),(\vec{M^\gamma})}\big)$ one has*

$$\Psi_q^{-1}(\chi_E(t)\chi_D(x))$$

$$= \overline{\rho}_0(D)^{\frac{1-q}{q}} \Bigg(\bigg(\Big(\big\| \mathfrak{g}^\gamma_m \big\|^{1-r_0} \Big(\sum_{h \in D^\gamma_m} \big| g^\gamma_{mh} \big|^{r_0} \Big)^{\frac{q-1}{q}} \big| g^\gamma_{mk} \big|^{\frac{r_0}{q}} \Big)_{k \in D^\gamma_m} \bigg)_{m \in E_\gamma} \Bigg)_{\mathscr{U}}.$$

After some technical computations we find indeed that

Corollary 2.1 ([16, Corollary 18]) *Ψ_q and Ψ'_q are order isomorphisms if $1 < q < \infty$.*

Proposition 2.4 *$L^\infty[\Xi_0, \nu_0, L^q(\overline{\Omega}, \overline{\rho}_0)]$ is σ-order continuous if $1 < q < \infty$.*

Proof Let $\{f_s(t,x)\}_{s=1}^{\infty} \subset L^{\infty}[\Xi_0, \nu_0, L^q(\overline{\Omega}, \overline{\rho}_0)]$ be a decreasing sequence such that $\bigwedge_{s=1}^{\infty} f_s(t,x) = 0$. By Corollary 2.1 we obtain $\bigwedge_{s=1}^{\infty} \Psi'_{q'}\big(f_s(t,x)\big) = 0$ in $(X_{q'})'$ and by Proposition 2.2 $\big\{(\iota_q)^{-1} \circ \mathfrak{q}_{q'} \circ \Psi'_{q'}\big(f_s(t,x)\big)\big\}_{s=1}^{\infty}$ is a decreasing sequence in the order complete Banach lattice $\mathfrak{Q}_q^{\infty}$. By definition of the order in $\mathfrak{Q}_q^{\infty}$ there are sequences $\{\overline{\mathbf{X}}_s\}_{s=1}^{\infty} \subset X_q^{\infty}$ and $\{\overline{\mathbf{Y}}_s\}_{s=2}^{\infty} \subset X_q^{\infty}$ such that $K_q(\overline{\mathbf{X}}_s) = K_q(\overline{\mathbf{Y}}_s) = (\iota_q)^{-1} \circ \mathfrak{q}_{q'} \circ \Psi'_{q'}\big(f_s(t,x)\big)$ for every $s \geq 2$ and $\overline{\mathbf{X}}_s \geq \overline{\mathbf{Y}}_{s+1}$ for every $s \in \mathbb{N}$. We check that there is a decreasing sequence $\{\mathbf{X}_s\}_{s=1}^{\infty} \subset X_q^{\infty}$ such that $K_q(\mathbf{X}_s) = K_q(\overline{\mathbf{X}}_s) = (\iota_q)^{-1} \circ \mathfrak{q}_{q'} \circ \Psi'_{q'}\big(f_s(t,x)\big)$ for every $s \geq 1$. In fact, define $\mathbf{X}_1 := \overline{\mathbf{X}}_1$ and assume that $\mathbf{X}_i$, $1 \leq i \leq s$ is defined with the quoted properties. Then there is $\mathbf{Z}_s \in (\mathbf{A}^q)^{\perp}$ such that $\mathbf{X}_s = \overline{\mathbf{X}}_s + \mathbf{Z}_s \geq \overline{\mathbf{Y}}_{s+1} + \mathbf{Z}_s$. Defining $\mathbf{X}_{s+1} = \overline{\mathbf{Y}}_{s+1} + \mathbf{Z}_s$ the step $s+1$ is reached and by induction we obtain the desired sequence.

It follows that $\mathfrak{q}_{q'}(\mathbf{X}_s) = \iota_q\big(K_q(\mathbf{X}_s)\big) = \mathfrak{q}_{q'}\big(\Psi'_{q'}(f_s(t,x))\big)$. That means that $\Psi'_{q'}(f_s(t,x)) - \mathbf{X}_s \in (\mathbf{U}_{q'})^{\circ}$ and hence, $\mathbf{U}_{q'}$ being a weak order unit in $B(\mathbf{U}_{q'})$, by Proposition 2.3 and order continuity

$$(\Psi'_{q'})^{-1}\big(\Psi'_{q'}(f_s(t,x)) - \mathbf{X}_s\big) = f_s(t,x) - (\Psi'_{q'})^{-1}(\mathbf{X}_s) \in L^1[\Xi_0, \nu_0, L^{q'}(\overline{\Omega}, \overline{\rho}_0)]^{\circ} = \{0\}.$$

We obtain $f_s(t,x) = (\Psi'_{q'})^{-1}(\mathbf{X}_s),\ s \in \mathbb{N}$ and $\mathbf{X}_s = \Psi'_{q'}(f_s(t,x)),\ s \in \mathbb{N}$. By Corollary 2.1 we deduce $\bigwedge_{s=1}^{\infty}(\mathbf{X}_s) = 0$ in $(X_{q'})'$ and identifying X_q^{∞} with a sublattice of $(X_{q'})'$ we see that $\bigwedge_{s=1}^{\infty}(\mathbf{X}_s) = 0$ in X_q^{∞} too. By theorem [10, Proposition 4.7] of Henson and Moore (see alternatively [24, Lemma 0.2]) X_q^{∞} is σ-order continuous and $\lim_{s\to\infty} \mathbf{X}_s = 0$ in X_q^{∞}. By continuity of the inclusion $X_q^{\infty} \subset (X_{q'})'$ and the isomorphism $(\Psi'_{q'})^{-1}$ we obtain finally $\lim_{s\to\infty} f_s(t,x) = 0$ as desired. □

This result allows us to define the maps $\mathfrak{A}_{r'_j}$, $1 \leq j \leq n$ of the global diagram:

Proposition 2.5 ([16, Lemma 19]) *The map*

$$\mathfrak{A}_q : K_q(\mathbf{X}) \in K_q\big(P_{(N_\gamma),(\vec{M^\gamma})}(X_q^{\infty})\big) \longrightarrow (\Psi'_{q'})^{-1}(\mathbf{X}) \in L^{\infty}[\Xi_0, \nu_0, L^q(\overline{\Omega}, \overline{\rho}_0)]$$

is well defined, continuous and linear for every $q \in]1, \infty[$.

We finish this subsubsection with an important and unexpected result:

Proposition 2.6 ([16, Lemma 20]) *If* $t_{\mathbf{r}} < 1$ *we have* $\mathfrak{S}\big((\mathbf{X}^j_{(\mathbb{N}\setminus P_\gamma),(\mathbb{N})})_{j=1}^n\big) = 0$ *and* $\mathfrak{S}\big((\mathbf{X}^j_{(P_\gamma),(\mathbb{N}\setminus \vec{Q^\gamma})})_{j=1}^n\big) = 0$ *for every* $(\mathbf{X}^j)_{j=1}^n \in \prod_{j=1}^n X_{r'_j}^{\infty}$.

2.2.3 Construction of the Spaces $\mathfrak{R}$ and $\mathfrak{T}$

Now we are ready to define our definitive measure spaces. If $t_{\mathbf{r}} < 1$ we define

$$(\mathfrak{R}, \mathfrak{H}, \nu) := (\Xi_1, \mathscr{C}_{\Xi_1}, \nu_1) \qquad (\mathfrak{T}, \mathfrak{V}, \tau) := (\Omega_1, \mathscr{B}_{\Omega_1}, \mu).$$

If $t_{\mathbf{r}} \geq 1$ the construction is more involved. Roughly speaking we take $\mathfrak{R}$ (resp. $\mathfrak{T}$) as the *disjoint union* of Ξ_0 and Ξ_1 (since the fact $\Xi_0 \cap \Xi_1 \neq \emptyset$ is not necessarily excluded) (resp. of $\overline{\Omega}$ and Ω_1) provided with the natural σ-algebra and measure. To proceed formally consider the sets

$$\Xi_0^d := \big\{(x,0) \mid x \in \Xi_0\big\},\ \Xi_1^d := \big\{(x,1) \mid x \in \Xi_1\big\},\ \overline{\Omega}^d := \big\{(x,0) \mid x \in \overline{\Omega}\big\}$$

and $\Omega_1^d := \big\{(x,1) \mid x \in \Omega_1\big\}$ provided with the σ-algebras and measures defined by

$$\mathscr{C}_{\Xi_0}^d := \Big\{A^0 := \big\{(x,0) \mid x \in A\big\} \Bigm| A \in \mathscr{C}_{\Xi_0}\Big\},\ \ \nu_0^d(A^0) = \nu_0(A)\ \ \forall A^0 \in \mathscr{C}_{\Xi_0}^d,$$

$$\mathscr{C}_{\Xi_1}^d := \Big\{A^1 := \big\{(x,1) \mid x \in A\big\} \Bigm| A \in \mathscr{C}_{\Xi_1}\Big\},\ \ \nu_1^d(A^1) = \nu_1(A)\ \ \forall A^1 \in \mathscr{C}_{\Xi_1}^d,$$

$$\overline{\mathscr{B}}^d := \Big\{A^0 := \big\{(x,0) \mid x \in A\big\} \Bigm| A \in \overline{\mathscr{B}}\Big\},\quad \overline{\rho}_0^{\,d}(A^0) = \overline{\rho}_0(A)\ \ \forall A^0 \in \overline{\mathscr{B}}^d,$$

$$\mathscr{B}_{\Omega_1}^d := \Big\{A^1 := \big\{(x,1) \mid x \in A\big\} \Bigm| A \in \mathscr{B}_{\Omega_1}\Big\},\ \ \mu_1^d(A^1) = \mu(A)\ \ \forall A^1 \in \mathscr{B}_{\Omega_1}^d$$

respectively. Finally we take

$$\mathfrak{R} = \Xi_0^d \cup \Xi_1^d,$$

$\mathfrak{H} := \{A^0 \cup A^1,\ A^0 \in \mathscr{C}_{\Xi_0}^d,\ A^1 \in \mathscr{C}_{\Xi_1}^d\}$ and $\nu(A^0 \cup A^1) = \nu_0^d(A^0) + \nu_1^d(A^1)$ for every $A^0 \cup A^1 \in \mathfrak{H}$. Analogously we define

$$\mathfrak{T} = \overline{\Omega}^d \cup \Omega_1^d,$$

$\mathfrak{V} := \big\{A^0 \cup A^1,\ A^0 \in \overline{\mathscr{B}}^d,\ A^1 \in \mathscr{B}_{\Omega_1}^d\big\}$ and $\tau(A^0 \cup A^1) = \overline{\rho}_0^d(A^0) + \mu_1^d(A^1)$ for every $A^0 \cup A^1 \in \mathfrak{V}$.

To simplify notation, for every $0 < q < \infty$ and $s = 1$ or $s = \infty$ put

$$\ell_{\Xi_1}^s[\ell_{\Omega_1}^q] := \ell^s[\Xi_1, \mathscr{C}_{\Xi_1}, \nu_1, \ell^q(\Omega_1, \mathscr{B}_{\Omega_1}, \mu)],$$

$$\ell_{\Xi_1^d}^s[\ell_{\Omega_1^d}^q] := \ell^s[\Xi_1^d, \mathscr{C}_{\Xi_1}^d, \nu_1^d, \ell^q(\Omega_1^d, \mathscr{B}_{\Omega_1}^d, \mu_1^d)]$$

and denote by $\mathrm{i}_{1,s}^q : \ell_{\Xi_1}^s[\ell_{\Omega_1}^q] \longrightarrow \ell_{\Xi_1^d}^s[\ell_{\Omega_1^d}^q]$ the natural surjective isometric isomorphism. In the same way we write $\mathrm{i}_{0,s}^q$ for the natural isometry from $L_{\Xi_0}^s[L_{\overline{\Omega}}^q] := L^s[\Xi_0, \mathscr{C}_{\Xi_0}, \nu_0, L^q(\overline{\Omega}, \overline{\mathscr{B}}, \overline{\rho}_0)]$ onto

$$L_{\Xi_0^d}^s[L_{\overline{\Omega}^d}^q] := L^s[\Xi_0^d, \mathscr{C}_{\Xi_0}^d, \nu_0^d, L^q(\overline{\Omega}^{\,d}, \overline{\mathscr{B}}^d, \overline{\rho}_0^d)].$$

The corresponding inverse mappings will be denoted by $\mathfrak{j}^q_{0,s}$ and $\mathfrak{j}^q_{1,s}$ respectively. Moreover, $J^{0,s}_q$ (resp. $J^{1,s}_q$) will be the natural embedding from $L^s_{\Xi^d_0}[L^q_{\overline{\Omega}^d}]$ (resp. $\ell^s_{\Xi^d_1}[\ell^q_{\Omega^d_1}]$) into $L^s[\mathfrak{R}, \nu, L^q(\mathfrak{T}, \tau)]$ and $\mathfrak{p}^0_q$ (resp. $\mathfrak{p}^1_q$) will denote the natural projection from $L^1[\mathfrak{R}, \nu, L^q(\mathfrak{T}, \tau)]$ onto $L^1_{\Xi^d_0}[L^q_{\overline{\Omega}^d}]$ (resp. $\ell^1_{\Xi^d_1}[\ell^q_{\Omega^d_1}]$). Finally, keeping in mind previous auxiliary notations let $J_{r_{n+1}}$ be the natural inclusion from $\ell^1_{\Xi^d_1}[\ell^{r_{n+1}}_{\Omega^d_1}] \oplus L^1_{\Xi^d_0}[L^{r_{n+1}}_{\overline{\Omega}^d}]$ into $L^1[\mathfrak{R}, \nu, L^{r_{n+1}}(\mathfrak{T}, \tau)]$ and

$$J^\infty_q := J^{1,\infty}_q \circ \mathfrak{i}^q_{1,\infty} \oplus J^{0,\infty}_q \circ \mathfrak{i}^q_{0,\infty}.$$

2.3 Completing the Operators of the Global Diagram

For every $1 \le j \le n$, let $\widetilde{\mathfrak{X}}_{r'_j}$ be the restriction to $\mathfrak{Q}^{\infty,1}_{r'_j}$ of $\widetilde{\Delta}_{r'_j}$ and let $\mathfrak{s}^0_{r'_j}$ (resp. $\mathfrak{s}^1_{r'_j}$) the natural inclusion of $L^\infty_{\Xi_0}[L^{r'_j}_{\overline{\Omega}}]$ (resp. of $\ell^\infty_{\Xi_1}[\ell^{r'_j}_{\Omega_1}]$) into $\ell^\infty_{\Xi_1}[\ell^{r'_j}_{\Omega_1}] \oplus L^\infty_{\Xi_0}[L^{r'_j}_{\overline{\Omega}}]$. Then we put $\mathfrak{Y}_{r'_j} := \widetilde{\mathfrak{X}^1_{r'_j}}$ if $t_\mathbf{r} < 1$ and $\mathfrak{Y}_{r'_j} = \mathfrak{s}^0_{r'_j} \circ \mathfrak{A}_{r'_j} \oplus \mathfrak{s}^1_{r'_j} \circ \widetilde{\mathfrak{X}}^1_{r'_j}$ if $t_\mathbf{r} \ge 1$.

Concerning to $\mathfrak{M}$ we define

$$\mathfrak{M} = \mathfrak{W}_{r_{n+1}} \circ P_{(P_\gamma),(\vec{Q}^\gamma)} \circ (I_{1,1} \otimes I_{t_\mathbf{r},r_{n+1}})_{\mathscr{U}} \circ \mathfrak{W}^{-1}_{t_\mathbf{r}} \circ \mathfrak{P}_{t_\mathbf{r}} \quad \text{if} \;\; t_\mathbf{r} < 1$$

and in the case $t_\mathbf{r} \ge 1$

$$\mathfrak{M} := J_{r_{n+1}} \circ \Big(\mathfrak{i}^{r_{n+1}}_{0,1} \circ \Psi_{r_{n+1}} \circ P_{(N_\gamma),(\vec{M}^\gamma)} \circ (I_{1,1} \otimes I_{t_\mathbf{r},r_{n+1}})_{\mathscr{U}} \circ \Psi^{-1}_{t_\mathbf{r}} \circ \mathfrak{j}^{t_\mathbf{r}}_{0,1} \circ \mathfrak{p}^0_{t_\mathbf{r}} +$$

$$+ \mathfrak{i}^{r_{n+1}}_{1,1} \circ \mathfrak{W}_{r_{n+1}} \circ P_{(P_\gamma),(\vec{Q}^\gamma)} \circ (I_{1,1} \otimes I_{t_\mathbf{r},r_{n+1}})_{\mathscr{U}} \circ \mathfrak{W}^{-1}_{t_\mathbf{r}} \circ \mathfrak{P}_{t_\mathbf{r}} \circ \mathfrak{j}^{t_\mathbf{r}}_{1,1} \circ \mathfrak{p}^1_{t_\mathbf{r}} \Big).$$

In both cases the definition of $\mathfrak{M}$ is nothing else that the composition of the maps in the down right sector of the general diagram. We notice that to simplify notation, in this diagram we have written $\mathfrak{Y} := (\mathfrak{Y}_{r'_j})^n_{j=1}$, $Q_0 := P_{(N_\gamma),(\vec{M}^\gamma)}$, $Q_1 := P_{(P_\gamma),(\vec{Q}^\gamma)}$ and that the canonical inclusions $Q_j(X_q) \subset Q_1(X_q) \oplus Q_0(X_q) \subset X_q$, $j = 0, 1$, and $\overline{\mathscr{H}}_{r_{n+1}} \subset \ell^1_{\Xi_1}[\ell^{r_{n+1}}_{\Omega_1}]$ do not have written in any special way neither in the diagram nor in the definition of $\mathfrak{M}$ (moreover, recall Lemma 2.3 and Proposition 2.3).

It can be shown ([16, Theorem 11, step 3, 2.c)] the important fact that for every $f \in L^1[\mathfrak{R}, \nu, L^{t_\mathbf{r}}(\mathfrak{T}, \tau)]$ it turns out that $Supp(\mathfrak{M}(f)) \subset Supp(f)$.

2.3.1 Definition of the Function *g*

(a) First of all we consider for every $0 < t_\mathbf{r} < \infty$ the function

$$g_1(t,x) := \sum_{m\in\mathbb{A}_1} \left\| \mathfrak{g}_{\mathbf{k}^m} \right\|^{1-\frac{1}{r_0}} \chi_{\bar{\mathbf{e}}_{\mathbf{k}^m}}(t) \sum_{v\in\mathbb{P}_m} \chi_{\bar{\mathbf{e}}_{\mathbf{k}^m,\mathbf{h}^v}}(x)$$

where we have defined $\mathbb{P}_m := \{v \in \mathbb{N} \mid (m,v) \in \mathbb{P}\}$ for every $m \in \mathbb{A}_1$. Then $\|g_1\|_{L^1[\Xi_1,\nu_1,L^{r_0}(\Omega_1,\mu)]} \le \|\mathfrak{g}\|_{X_{r_0}}$ (see [16, Theorem 11, step 4, part 2)] and hence the diagonal map $S_{g_1} \in \mathscr{L}^n\big(\prod_{j=1}^n \ell^\infty[\Xi_1,\nu_1,\ell^{r'_j}(\Omega_1,\mu)], \ell^1\big[\Xi_1,\nu_1,\ell^{t_\mathbf{r}}(\Omega_1,\mu)\big]\big)$ is well defined. It is important to check later on the commutativity of the global diagram to remark that actually

$$S_{g_1}\big(\prod_{j=1}^n \ell^\infty[\Xi_1,\nu_1,\ell^{r'_j}(\Omega_1,\mu)]\big) \subset \mathfrak{P}_{t_\mathbf{r}}\big(\ell^1[\Xi_1,\nu_1,\ell^{t_\mathbf{r}}(\Omega_1,\mu)]\big).$$

(b) Then in the case $t_\mathbf{r} < 1$ we define simply $g(t,x) := g_1(t,x)$.

(c) Let $t_\mathbf{r} \ge 1$. First of all we consider the map

$$\mathscr{Z} := \Psi_{t_\mathbf{r}} \circ P_{(N_\gamma),(\vec{M^\gamma})} \circ \widehat{(S_{g^\gamma})_\mathscr{U}} \circ \big((\iota_{r'_j}^{-1} \circ \mathfrak{q}_{r_j} \circ \Psi'_{r_j})_{j=1}^n\big)$$

from $\prod_{j=1}^n L^\infty[\Xi_0,\nu_0,L^{r'_j}(\overline{\Omega},\overline{\rho}_0)]$ into $L^1[\Xi_0,\nu_0,L^{t_\mathbf{r}}(\overline{\Omega},\overline{\rho}_0)]$. Consider the function $g_0(t,x) := \mathscr{Z}\big((\chi_{\Xi_0}(t)\chi_{\overline{\Omega}}(x)))_{j=1}^n\big)(t,x) \in L^1[\Xi_0,\nu_0,L^{t_\mathbf{r}}(\overline{\Omega},\overline{\rho}_0)]$. It follows from Proposition 2.3, Corollary 2.1 and the properties of the maps arising in the definition of $\mathscr{Z}$ that $g_0(t,x) \ge 0$.

Proposition 2.7 *For every $(h_j(t,x))_{j=1}^n \in \prod_{j=1}^n L^\infty[\Xi_0,\nu_0,L^{r'_j}(\overline{\Omega},\overline{\rho}_0)]$ the equality $\mathscr{Z}\big((h_j)_{j=1}^n\big) = g_0(t,x)\prod_{j=1}^n h_j(t,x)$ holds.*

Proof It is shown in [16, Theorem 11, step 4, part 1.b] that the proposition is true when every $h_j \in L^\infty(\Xi_0,\nu_0)\widehat{\otimes}_\varepsilon L^{r'_j}(\overline{\Omega},\overline{\rho}_0)$, $1 \le j \le n$. We proceed now to the proof of the general result. If $\widetilde{\mathscr{Z}}$ is the linearization of $\mathscr{Z}$ defined on

$$\big(\bigotimes_{j=1}^n L^\infty[\Xi_0,\nu_0,L^{r'_j}(\overline{\Omega},\overline{\rho}_0];\pi\big)$$

we only need to show that

$$g_0(t,x)\prod_{j=1}^n h_j(t,x) = \widetilde{\mathscr{Z}}\big(\otimes_{j=1}^n h_j(t,x)\big) = \mathscr{Z}\big((h_j(t,x))_{j=1}^n\big) \in L^1[\Xi_0,\nu_0,L^{t_\mathbf{r}}(\overline{\Omega},\overline{\rho}_0)] \tag{19}$$

and by linearity of $\widetilde{\mathscr{Z}}$, since every $h_j(t,x) = h_j^+(t,x) - h_j^-(t,x)$, $1 \le j \le n$, we can assume that $h_j(t,x) \ge 0$ for each $1 \le j \le n$.

Assume we have proved that (19) holds for every

$$h_j(t,x) \in L^\infty[\Xi_0,\nu_0,L^{r'_j}(\overline{\Omega},\overline{\rho}_0)],\ 1 \le j \le k-1$$

and every $h_j(t,x) \in L^\infty(\Xi_0,\nu_0)\widehat{\otimes}_\varepsilon L^{r_j'}(\overline{\Omega},\overline{\rho}_0)$, $j = k, ..., n$ for some $1 \le k < n$ (if $k = 1$ there is no initial assumption). We claim that (19) holds when $h_j(t,x) \in L^\infty[\Xi_0,\nu_0,L^{r_j'}(\overline{\Omega},\overline{\rho}_0)]$, $1 \le j \le k$ and

$$h_j(t,x) \in L^\infty(\Xi_0,\nu_0)\widehat{\otimes}_\varepsilon L^{r_j'}(\overline{\Omega},\overline{\rho}_0),\ j = k+1, ..., n.$$

By induction this fact will prove the desired proposition.

(a) First we will prove our assertion in the case that $h_k(t,x)$ be a simple function in $L^\infty[\Xi_0,\nu_0,L^{r_k'}(\overline{\Omega},\overline{\rho}_0)]$. Clearly, to do this we can assume simply that $h_k(t,x) = \chi_D(t,x)$ where D is a $(\mathscr{C}_{\Xi_0} \times \overline{\mathscr{B}})$-measurable set in $\Xi_0 \times \overline{\Omega}$. For each $v \in \mathbb{N}$ there is a sequence $\{A_{vm} \times B_{vm}\}_{m=1}^\infty$ of measurable rectangles such that setting $M_v := \bigcup_{m=1}^\infty (A_{vm} \times B_{vm})$ one has $D \subset M_v$, $(\nu_0 \times \overline{\rho}_0)\big(M_v \backslash D\big) \le \frac{1}{v}$ and $M_{v+1} \subset M_v$ for every $v \in \mathbb{N}$. Clearly

$$(\nu_0 \times \overline{\rho}_0)\left(\left(\bigcap_{v=1}^\infty M_v\right)\backslash D\right) = 0 \text{and} \bigwedge_{v=1}^\infty \chi_{M_v \backslash D}(t,x) = 0.$$

From Proposition 2.4 we deduce $\lim_{v\to\infty} \chi_{M_v\backslash D}(t,x) = 0$ in $L^\infty[\Xi_0,\nu_0,L^{r_k'}(\overline{\Omega},\overline{\rho}_0)]$ and hence $\lim_{v\to\infty} \chi_{M_v}(t,x) = \chi_D(t,x)$ in $L^\infty[\Xi_0,\nu_0,L^{r_k'}(\overline{\Omega},\overline{\rho}_0)]$. If $M_{vh} := \bigcup_{m=1}^h (A_{vm} \times B_{vm})$ one has $\chi_{M_{vh}}(t,x) \in L^\infty(\Xi_0,\nu_0) \otimes_\varepsilon L^{r_k'}(\overline{\Omega},\overline{\rho}_0)$ and $\bigwedge_{h=1}^\infty \big(\chi_{M_v}(t,x) - \chi_{M_{vh}}(t,x)\big) = 0$. Once again by Proposition 2.4 we obtain $\chi_{M_v}(t,x) = \lim_{h\to\infty} \chi_{M_{vh}}(t,x)$ in $L^\infty[\Xi_0,\nu_0,L^{r_k'}(\overline{\Omega},\overline{\rho}_0)]$ and so

$$\chi_{M_v}(t,x) \in L^\infty(\Xi_0,\nu_0)\widehat{\otimes}_\varepsilon L^{r_k'}(\overline{\Omega},\overline{\rho}_0)$$

and consequently $\chi_D(t,x) \in L^\infty(\Xi_0,\nu_0)\widehat{\otimes}_\varepsilon L^{r_k'}(\overline{\Omega},\overline{\rho}_0)$.

To simplify notation write for each function $f(t,x)$ defined on $\Xi_0 \times \overline{\Omega}$

$$\otimes_k(f(t,x)) = h_1(t,x) \otimes ... \otimes h_{k-1}(t,x) \otimes f(t,x) \otimes h_{k+1}(t,x) \otimes ... \otimes h_n(t,x)$$

and $\boxtimes_k(f(t,x)) = g_0(t,x) f(t,x) \prod_{1=j\neq k}^n h_j(t,x)$. It follows by continuity of the involved maps and (19) that

$$\widetilde{\mathscr{Z}}\big(\otimes_k(\chi_D(t,x))\big) = \lim_{v\to\infty} \widetilde{\mathscr{Z}}\big(\otimes_k(\chi_{M_v}(t,x))\big) = \lim_{v\to\infty} \boxtimes_k\big(\chi_{M_v}(t,x)\big).$$

But $\bigwedge_{v=1}^\infty \boxtimes_k\big(\chi_{M_v}(t,x)\big) = \boxtimes_k\big(\chi_D(t,x)\big)$ and since $L^1[\Xi_0,\nu_0,L^{t_{\mathbf{r}'}}(\overline{\Omega},\overline{\rho}_0)]$ is order continuous we obtain $\widetilde{\mathscr{Z}}\big(\otimes_k(\chi_D(t,x))\big) = \lim_{v\to\infty} \boxtimes_k\big(\chi_{M_v}(t,x)\big) = \boxtimes_k\big(\chi_D(t,x)\big)$ in $L^1[\Xi_0,\nu_0,L^{t_{\mathbf{r}}}(\overline{\Omega},\overline{\rho}_0)]$, that is, we have proved (19) in the case $h_k(t,x) = \chi_D(t,x)$.

(b) Now we assume that $h_k(t,x)$ is a general nonnegative measurable function. Let $G_h := \big\{(t,x) \in \Xi_0 \times \overline{\Omega} \mid g_0(t,x) \vee (\bigvee_{j=1}^n h_j(t,x)) \le h\big\}$. As $\chi_{G_h}(t,x) h_k(t,x) \ge 0$ is a $(\mathscr{C}_{\Xi_0} \times \overline{\mathscr{B}})$-measurable function, it is well known (see [11, Theorem 11.35] for instance) that there is an increasing sequence $\{S_m(t,x)\}_{m=1}^{\infty}$ of non negative simple functions uniformly convergent to $\chi_{G_h}(t,x)h_k(t,x)$. As $\nu_0(\Xi_0) < \infty$ and $\overline{\rho}_0(\overline{\Omega}) < \infty$ it is easy to check that $\chi_{G_h}(t,x)h_k(t,x) = \lim_{m\to\infty} S_m(t,x)$ in $L^{\infty}[\Xi_0, \nu_0, L^{r'_k}(\overline{\Omega}, \overline{\rho}_0)]$. By continuity of $\widetilde{\mathscr{Z}}$ and part a)

$$\widetilde{\mathscr{Z}}\big(\otimes_k\big(\chi_{G_h}(t,x)h_k(t,x)\big)\big) = \lim_{m\to\infty} \widetilde{\mathscr{Z}}\big(\otimes_k\big(\chi_{G_h}(t,x)S_m(t,x)\big)\big)$$

$$= \lim_{m\to\infty} \boxtimes_k\big(\chi_{G_h}(t,x)S_m(t,x)\big)$$

in $L^1[\Xi_0, \nu_0, L^{t_\mathbf{r}}(\overline{\Omega}, \overline{\rho}_0)]$. But by (3) and Hölder's inequality

$$\lim_{m\to\infty}\Big\|\boxtimes_k\big(\chi_{G_h}(t,x)h_k(t,x)\big) - \boxtimes_k\big(\chi_{G_h}(t,x)S_m(t,x)\big)\Big\|_{L^1[\Xi_0, L^{t_\mathbf{r}}(\overline{\Omega})]}$$

$$\leq \lim_{m\to\infty} K\,\Big\|\chi_{G_h}(t,x)h_k(t,x) - S_m(t,x)\Big\|_{L^{\infty}[\Xi_0, L^{r'_k}(\overline{\Omega})]} = 0$$

where we have used $K := h\,\big((\nu_0\times\overline{\rho}_0)(G_h)\big)^{\frac{1}{r_0}} \prod_{1=j\neq k}^{n}\big\|h_j(t,x)\big\|_{L^{\infty}[\Xi_0, L^{r'_j}(\overline{\Omega})]}$. Then

$$\widetilde{\mathscr{Z}}\big(\otimes_k(\chi_{G_h}(t,x)h_k(t,x))\big) = \lim_{m\to\infty}\boxtimes_k\big(\chi_{G_h}(t,x)S_m(t,x)\big) = \boxtimes_k\big(\chi_{G_h}(t,x)h_k(t,x)\big) \tag{20}$$

in $L^1[\Xi_0, \nu_0, L^{t_\mathbf{r}}(\overline{\Omega}, \overline{\rho}_0)]$. Let

$$D_{h_j}^{\infty} := \big\{(t,x)\in \Xi_0\times\overline{\Omega} \mid h_j(t,x) = \infty\big\},\ 1\leq j\leq n.$$

As $\Big\|\big\|\chi_{D_{h_j}^{\infty}}(t,x)h_j(t,x)\big\|_{L^{r'_j}(\overline{\Omega})}\Big\|_{L^{\infty}(\Xi_0)} < \infty$ we have $\overline{\rho}_0\big(\{x\in\overline{\Omega} \mid (t,x) \in D_{h_j}^{\infty}\}\big) = 0$ almost everywhere on Ξ_0 and by Fubini's theorem we obtain $(\nu_0 \times \overline{\rho}_0)(D_{h_j}^{\infty}) = 0,\ 1\leq j\leq n$. Recalling Lemma 1.1 this tell us that

$$(\nu_0\times\overline{\rho}_0)\Big((\Xi_0\times\overline{\Omega})\backslash\bigcup_{h=1}^{\infty} G_h\Big) = 0. \tag{21}$$

It follows that $h_k(t,x) = \bigvee_{s=1}^{\infty}\chi_{G_s}(t,x)h_k(t,x)$ and by Proposition 2.4 we have $h_k(t,x) = \lim_{s\to\infty}\chi_{G_s}(t,x)h_k(t,x)$ in $L^{\infty}[\Xi_0, \nu_0, L^{r'_k}(\overline{\Omega}, \overline{\rho}_0)]$. By continuity of $\widetilde{\mathscr{Z}}$ and (20)

$$\widetilde{\mathscr{Z}}\big(\otimes_k\big(h_k(t,x)\big)\big) = \lim_{s\to\infty}\widetilde{\mathscr{Z}}\big(\otimes_k\big(\chi_{G_s}(t,x)h_k(t,x)\big)\big) = \lim_{s\to\infty}\chi_{G_s}(t,x)g_0(t,x)\prod_{j=1}^{h} h_j(t,x)$$

in $L^1[\Xi_0, \nu_0, L^{t_\mathbf{r}}(\overline{\Omega}, \overline{\rho}_0)]$. Then for every $v \in \mathbb{N}$ we have

$$\chi_{G_v}(t,x)\widetilde{\mathscr{F}}\big(\otimes_k\big(h_k(t,x)\big)\big) = \lim_{s\to\infty}\chi_{G_v}(t,x)\chi_{G_s}(t,x)g_0(t,x)\prod_{j=1}^{h}h_j(t,x)$$

$$= \chi_{G_v}(t,x)g_0(t,x)\prod_{j=1}^{h}h_j(t,x)$$

and it turns out that $\widetilde{\mathscr{F}}\big(\otimes_k\big(h_k(t,x)\big)\big)$ coincides with $g_0(t,x)\prod_{j=1}^{h}h_j(t,x)$ on the set G_v. As v is arbitrary in $\mathbb{N}$, by (21) we deduce (19). □

It is noteworthy to remark that the key of the proof of Proposition 2.7 is Proposition 2.4 which is a reflection of the close relation of $L^\infty[\Xi_0, L^q(\overline{\Omega})]$, $1\le q<\infty$, with an ultraproduct. On the other hand, an important consequence of Proposition 2.7 is the fact that $g_0\in D\big(\prod_{j=1}^n L^\infty[\Xi_0, L^{r'_j}(\overline{\Omega})]; L^1[\Xi_0, L^{t_{\mathbf{r}}}(\overline{\Omega})]\big)$. Then by Proposition 1.9 one has $g_0(t,x)\in L^1[\Xi_0, L^{r_0}(\overline{\Omega})]$ and hence the diagonal map S_{g_0} from $\prod_{j=1}^n L^\infty[\Xi_0, L^{r'_j}(\overline{\Omega})]$ into $L^1[\Xi_0, L^{t_{\mathbf{r}}}(\overline{\Omega})]$ is well defined. We denote by $\widetilde{g_0}$ (resp. $\widetilde{g_1}$) the function defined on $\mathfrak{R}\times\mathfrak{T}$ by $\widetilde{g_0}\big((t,i),(x,j)\big)=g_0(t,x)$ if $i=j=0$ (resp. $\widetilde{g_1}\big((t,i),(x,j)\big)=g_1(t,x)$ if $i=j=1$) and $\widetilde{g_0}\big((t,i),(x,j)\big)=0$ in other case (resp. $\widetilde{g_1}\big((t,i),(x,j)\big)=0$ in other case), $0\le i,j\le 1$. Now, as final function $g\big((t,i),(x,j)\big)\in L^1[\mathfrak{R},\nu,L^{r_0}(\mathfrak{T},\tau)]$ we consider $g\big((t,i),(x,j)\big) := \widetilde{g_0}\big((t,i),(x,j)\big)+\widetilde{g_1}\big((t,i),(x,j)\big)$.

All the spaces and operators appearing in the global diagram are defined now. To reach our goal we only have to check its commutativity.

(a) Assume first $t_{\mathbf{r}}<1$. For every $(\mathbf{X}^j)_{j=1}^n\in\prod_{j=1}^n X_{r'_j}^\infty$ one has

$$\mathbf{X}^j = \mathbf{X}^j_{(P_\gamma),(\vec{Q^\gamma})} + \mathbf{X}^j_{(\mathbb{N}\setminus P_\gamma),(\vec{\mathbb{N}})} + \mathbf{X}^j_{(P_\gamma),(\overrightarrow{\mathbb{N}\setminus Q^\gamma})}, \quad 1\le j\le n$$

and hence by Proposition 2.6 $\mathfrak{S}\big((\mathbf{X}^j)_{j=1}^n\big)=\mathfrak{S}\big((\mathbf{X}^j_{(P_\gamma),(\vec{Q^\gamma})})_{j=1}^n\big)$. From now on the proof is the same that the developed one in [16, Theorem 11, step 4, a)] and it will not be repeated here.

(b) Let now $t_{\mathbf{r}}\ge 1$. As $\Omega=\Omega_0\cup\Omega_1$ and $\Omega_0\cap\Omega_1=\emptyset$ it follows that $\mathbf{V}^{t_{\mathbf{r}}} = \mathbf{V}^{t_{\mathbf{r}}}_{(N_\gamma),(\vec{M^\gamma})}+\mathbf{V}^{t_{\mathbf{r}}}_{(P_\gamma),(\vec{Q^\gamma})}$. Keeping in mind Lemma 2.4 and the definitions of the involved spaces and mappings, the proof of [16, Theorem 11, step 4, b)] can be repeated.

2.3.2 Final Chacterization of $\alpha_{\mathbf{r}}^C$-Integral Opearators

Hence we have shown the following precise form of our conjecture:

Theorem 2.4 *Let* $T\in\mathfrak{I}_{\alpha_{\mathbf{r}}^C}\big(\prod_{j=1}^n E_j; E_{n+1}\big)$. *For each* $\varepsilon>0$ *there are measure spaces* $(\mathfrak{R},\mathfrak{H},\nu)$ *and* $(\mathfrak{T},\mathfrak{V},\tau)$, *a map* $\mathfrak{M}\in\mathscr{L}\big(L^1[\mathfrak{R},L^{t_{\mathbf{r}}}(\mathfrak{T})], L^1[\mathfrak{R},L^{r_{n+1}}(\mathfrak{T})]\big)$

verifying $Supp(\mathfrak{M}(f)) \subset Supp(f)$ *for each* $f \in L^1[\mathfrak{R}, L^{t_{\mathbf{r}}}(\mathfrak{T})]$ *and a function* $g \in L^1\big[\mathfrak{R}, L^{r_0}(\mathfrak{T})\big]$ *such that* $J_{E_{n+1}} \circ T$ *factorizes as*

$$\begin{array}{ccccc} \prod_{j=1}^n E_j & \xrightarrow{\quad T \quad} & E_{n+1} & \xrightarrow{\quad J_{E_{n+1}} \quad} & E''_{n+1} \\ \Big\downarrow (A_j)_{j=1}^n & & & & \Big\uparrow C \\ \prod_{j=1}^n L^\infty[\mathfrak{R}, L^{r'_j}(\mathfrak{T})] & \xrightarrow[\quad S_g \quad]{} & L^1[\mathfrak{R}, L^{t_{\mathbf{r}}}(\mathfrak{T})] & \xrightarrow[\quad \mathfrak{M} \quad]{} & L^1[\mathfrak{R}, L^{r_{n+1}}(\mathfrak{T})]. \end{array}$$

and $\|C\| \; \|\mathfrak{M}\| \; \|g\|_{L^1[\mathfrak{R}, L^{r_0}(\mathfrak{T})]} \prod_{j=1}^n \|A_j\| \leq \mathbf{I}_{\alpha_{\mathbf{r}}^C} + \varepsilon.$

In the converse direction we have next result

Theorem 2.5 ([16, Theorem 22]) *Let* $(\Xi, \mathscr{A}, \nu)$ *and* $(\Omega, \mathscr{M}, \mu)$ *be measure spaces. If* $g \in L^1[\Xi, L^{r_0}(\Omega)]$ *and* $\mathfrak{M} \in \mathscr{L}\big(L^1[\Xi, L^{t_{\mathbf{r}}}(\Omega)], L^1[\Xi, L^{r_{n+1}}(\Omega)]\big)$ *verify* $Supp(\mathfrak{M}(f)) \subset Supp(f)$ *for every* $f \in L^1[\Xi, L^{t_{\mathbf{r}}}(\Omega)]$, *the composition map*

$$T := \mathfrak{M} \circ S_g : \prod_{j=1}^n L^\infty[\Xi, L^{r'_j}(\Omega)] \longrightarrow L^1[\Xi, L^{r_{n+1}}(\Omega)]$$

is $\alpha_{\mathbf{r}}^C$*-integral and* $\mathbf{I}_{\alpha_{\mathbf{r}}^C}(T) \leq \|\mathfrak{M}\| \; \|g\|_{L^1[\Xi, L^{r_0}(\Omega)]}$.

Hence we obtain the main result of our work, a characterization of the $\alpha_{\mathbf{r}}^C$-integral operators:

Theorem 2.6 ([16, Theorem 23, part a)]) *Let* E_j, $1 \leq j \leq n+1$, *be Banach spaces and let* $T \in \mathscr{L}^n(\prod_{j=1}^n E_j, E_{n+1})$. *The following assertions are equivalent:*

(1) T *is* $\alpha_{\mathbf{r}}^C$*−integral.*

(2) There are measurable spaces $(\Xi, \mathscr{A}, \nu)$ *and* $(\Omega, \mathscr{M}, \mu)$, *a function* $g \in L^1[\Xi, L^{r_0}(\Omega)]$ *and a map* $\mathfrak{M} \in \mathscr{L}\big(L^1[\Xi, L^{t_{\mathbf{r}}}(\Omega)], L^1[\Xi, L^{r_{n+1}}(\Omega)]\big)$ *verifying* $Supp\big(\mathfrak{M}(f)\big) \subset Supp(f)$ *for each* $f \in L^1[\Xi, L^{t_{\mathbf{r}}}(\Omega)]$ *such that* $J_{E_{n+1}} \circ T$ *can be factorized as*

$$\begin{array}{ccccc} \prod_{j=1}^n E_j & \xrightarrow{\quad T \quad} & E_{n+1} & \xrightarrow{\quad J_{E_{n+1}} \quad} & E''_{n+1} \\ \Big\downarrow (A_j)_{j=1}^n & & & & \Big\uparrow C \\ \prod_{j=1}^n L^\infty[\Xi, L^{r'_j}(\Omega)] & \xrightarrow[\quad S_g \quad]{} & L^1[\Xi, L^{t_{\mathbf{r}}}(\Omega)] & \xrightarrow[\quad \mathfrak{M} \quad]{} & L^1[\Xi, L^{r_{n+1}}(\Omega)]. \end{array}$$

(3) A factorization as in (2) but with finite measure spaces $(\Xi, \mathscr{A}, \nu)$ *and* $(\Omega, \mathscr{M}, \mu)$ *exists.*

Moreover, $\mathbf{I}_{\alpha_{\mathbf{r}}^C}(T) = \inf \|C\| \ \|\mathfrak{M}\| \ \|g\|_{L^1[\Xi, L^{r_0}(\Omega)]} \prod_{j=1}^n \|A_j\|$ *taking the* inf *over all such possible factorizations of type used in (2) or (3).*

Next particular case of previous theorem is noteworthy:

Theorem 2.7 ([16, Theorem 23, part b)]) *If* $t_{\mathbf{r}} < 1$ *and* $E_j, 1 \leq j \leq n+1$, *are Banach spaces an operator* $T \in \mathscr{L}^n(\prod_{j=1}^n E_j, E_{n+1})$ *is* $\alpha_{\mathbf{r}}^C-$*integral if and only if* T *is* $\alpha_{\mathbf{r}}^C-$*nuclear.*

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On Compacta K for Which $C(K)$ Has Some Good Renorming Properties

In Honour of Manuel López-Pellicer

Aníbal Moltó

Abstract By renorming it is usually meant obtaining equivalent norms in a Banach space with better properties, like being local uniformly rotund (LUR) or Kadets. In these notes we are concerned with C(K) spaces and pointwise lower semicontinuous Kadets or LUR renormings on them. If a C(K) space admits some of such equivalent norms then this space, endowed with the pointwise topology, has a countable cover by sets of small local norm-diameter (SLD). This property may be considered as the topological baseline for the existence of a pointwise lower semicontinuous Kadets, or even LUR renorming, since in many concrete cases it is the first step to obtain such a norm. In these notes we survey some methods, appearing in the literature, to prove that some C(K) spaces have this property.

Keywords Renorming · Small local diameter · Kadets norms · LUR norms · Dyadic compacta · Valdivia compacta · Trees · Rosenthal compacta

1 Introduction

By renorming it is usually meant a branch of Banach space theory whose original goal is obtaining equivalent norms with better properties. In these notes we are focused in two of them, Kadets and local uniformly rotundity (**LUR**). An account of their role in this theory and their relation with other important parts of this field may be found in [6]; in [25] there is an up-to-date survey of these properties for $C(K)$ spaces.

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Definition 1.1 A Banach space X (or its norm) is said to be *local uniformly rotund* (**LUR**) if for any $x \in X$ and any sequence $\{x_k\} \subset X$ we have $\lim_k \| x_k - x \| = 0$ whenever $\lim_k \| (x_k + x)/2 \| = \lim_k \| x_k \| = \| x \|$.

If τ is a topology on a Banach space X, coarser than the norm topology on X, we say that X has the τ-*Kadets* property whenever τ and the norm topology coincide in the unit sphere. The following result, due to Troyanski, shows the relation between them.

Theorem 1.1 ([6]) *Let X be a Banach space, then X admits an equivalent* **LUR** *norm if, and only if, it has an equivalent weak–Kadets norm and an equivalent rotund norm.*

Let us recall that a norm is rotund if its unit sphere contains no non trivial segment. The following property may be considered the topological baseline for the existence of a Kadets renorming.

Definition 1.2 (*Jayne, Namioka, Rogers* [15]) Given a topological space Z and ρ a lower-semicontinuous metric on it, we say that it has a *countable cover by sets of small local ρ-diameter*, whenever for a given $\varepsilon > 0$ the set Z can be expressed as a union $Z = \bigcup_{i=1}^{\infty} Z_i$, in such a way that for each i and each $x \in Z_i$ there exists an open subset W in Z such that $x \in W$ and the ρ-diameter of $W \cap Z_i$ is less than ε.

In this notes we will restrict ourselves to the study of those properties on $C(K)$ spaces. It is well known that any Banach space X is a subspace of $C(K)$, where K stands for the dual ball of X, endowed with the weak*-topology. This is one of the motivations of the following

Problem 1.1 Characterize the compacta K for which $C(K)$ admits a (pointwise lower-semicontinuous) **LUR** renorming.

Problem 1.2 Assume the Banach space X admits a **LUR** renorming and K is the dual unit ball endowed with the weak*-topology. Does $C(K)$ has a (pointwise lower-semicontinuous) **LUR** equivalent norm?

In [11], from the existence of a **LUR** renorming in $C(K)$ for a particular class of compacta K, Haydon deduces the following

Theorem 1.2 (Haydon) *Let X be a Banach space whose dual norm is* **LUR**. *Every continuous real–valued function on X may be uniformly approximated by functions of class $\mathscr{C}^1$.*

Since we will be concerned with $C(K)$ spaces, from now on we will say that $C(K)$ has the property **SLD**, or just **SLD**, when $C(K)$ endowed with the pointwise topology has a countable cover by sets of small local $\| \cdot \|$-diameter. By $C(K)\langle\mathbf{K}\rangle$ (respectively $C(K)\langle\mathbf{LUR}\rangle$) we mean that the space has an equivalent norm with the pointwise–Kadets property (respectively pointwise lower-semicontinuous **LUR**). It is known that

$$C(K)\langle\mathbf{LUR}\rangle \implies C(K)\langle\mathbf{K}\rangle \implies C(K)\,\mathbf{SLD}. \tag{1}$$

The second implication was proved in [15], Haydon [9] showed that the converse of the first is not true. Nevertheless the following problem is still open.

Problem 1.3 Does the implication $C(K)$ **SLD** $\implies C(K)\langle\mathbf{K}\rangle$ hold true for every compact space K?

This is an open problem even assuming in addition that the space $C(K)$ admits an equivalent rotund norm.

M. Raja showed in [22], roughly speaking, that the class of $C(K)$'s spaces with **SLD** is *very close* to those in which there exists a pointwise Kadets renorming, namely from [22] it follows that a Banach space X endowed with the weak topology has a countable cover by sets of small local $\|\cdot\|$-diameter if, and only if, there exists a non negative symmetric homogeneous weak-lower-semicontinuous function (that may be not convex) F on X with $\|\cdot\| \le F \le 3\|\cdot\|$ such that the norm and the weak topologies coincide on the set $\{x \in X : \; F(x) = 1\}$.

2 Some Characterizations

In [21] a linear topological characterization of Banach spaces that admit an equivalent **LUR** norm is presented, this was improved in [8] that, in turn, opened the gate to obtain a similar result for **LUR** and **SLD** in $C(K)$ spaces, [18–20]. These results enable us to prove the existence of such a renorming with a unified approach in a variety of $C(K)$'s spaces whose existence was originally deduced by methods ad hoc.

An exposition of how this method may provide a **LUR** renormig for such $C(K)$'s is beyond the scope of these notes, we restrict ourselves to survey some methods to obtain, for some compacta K, the specific decomposition required in the property **SLD** on $C(K)$ spaces, those are essentially contained in [13, 17–20]. Even with this restriction, some proofs will be only outlined and sometimes skipped. Nevertheless obtaining a concrete decomposition in $C(K)$ that satisfies the requirement of **SLD** for $C(K)$ may be useful to show the existence of an equivalent **LUR** or **K** norm in these spaces, since it may give the topological baseline to prove those properties [18, 20, 21].

If A is a bounded set of a normed space X, the *Kuratowski index of non–compactness* of A is defined by

$$\alpha(A) = \inf\{\varepsilon > 0 : A \text{ can be covered by finitely many sets of diameter less than } \varepsilon\}.$$

A subspace F of a normed space X is said to be *norming* for X whenever

$$|||x||| := \sup\left\{\left|x^*(x)\right| : \; x^*(x) \in B_{X^*} \cap F\right\}$$

is an equivalent norm on X.

Theorem 2.1 ([8, 21, 23]) *Let X be a Banach space and let F be a norming linear subspace of X^*. The following assertions are equivalent:*

1. *X admits an equivalent $\sigma(X, F)-$lower-semicontinuous **LUR** norm;*
2. *there is a countable family of subsets $\{X_n : n \in \mathbf{N}\}$ in X such that given $\varepsilon > 0$ and $x \in X$ there exists $n \in \mathbf{N}$ with $x \in X_n$ and a $\sigma(X, F)-$open half space H containing x such that diam $(H \cap X_n) < \varepsilon$;*
3. *there is a countable family of subsets $\{X_n : n \in \mathbf{N}\}$ in X such that given $\varepsilon > 0$ and $x \in X$ there exists $n \in \mathbf{N}$ with $x \in X_n$ and a $\sigma(X, F)-$open half space H containing x such that $\alpha(H \cap X_n) < \varepsilon$.*

To state some similar characterization for $C(K)$ spaces, let us remember some elementary facts of general topology. In what follows the term compact space, or just compactum, will mean a Hausdorff compact topological space. It is well–known that any compactum K is homeomorphic to a subspace of $[0, 1]^\Gamma$ for some set Γ, so we may and do assume that $K \subset [0, 1]^\Gamma$; then it makes sense to speak about the coordinates of a point of K. Moreover for such K, each $x \in C(K)$ must be uniformly continuous on K, then for any $\varepsilon > 0$ there exist $\Lambda \subset \Gamma$, Λ finite, and $\delta > 0$ such that

$$|x(s) - x(t)| < \varepsilon \text{ whenever } s,\ t \in K \text{ and } \sup_{\gamma \in \Lambda} |s(\gamma) - t(\gamma)| < \delta. \tag{2}$$

Definition 2.1 Given a compact space K we will say that the finite set Λ *ε-controls x with δ whenever* (2) *holds.*

If there is some $\delta > 0$ for which (2) holds then we will write that Λ *ε–controls* x. A subset $U \subset \Gamma$ is said to *control* $x \in C(K)$ if for every $\varepsilon > 0$ there exists a finite subset $\Lambda \subset U$ such that U ε–controls x.

Remark 2.1 Let $K \subset [0, 1]^\Gamma$ be a compact space and let U a (non necessarily finite) subset of Γ. Observe that if U does not ε–control $x \in C(K)$ there exist $s_0, t_0 \in K$ with

$$s_0(\gamma) = t_0(\gamma) \text{ for all } \gamma \in U, \text{ and } |x(s_0) - x(t_0)| \geq \varepsilon. \tag{3}$$

Indeed, given $n \in \mathbf{N}$ and a finite subset $F \subset U$, since F does not $1/n$-controls x, there must exist $s_{n,F}, t_{n,F} \in K$ such that $\left|x\left(s_{n,F}\right) - x\left(t_{n,F}\right)\right| \geq \varepsilon$ and $\left|s_{n,F}(\gamma) - t_{n,F}(\gamma)\right| < 1/n$ whenever $\gamma \in F$. By compactness there must exist a convergent subnet to a point $(s_0, t_0) \in K \times K$ that satisfies (3).

In [20], a characterization of those properties in terms of controlling coordinates is deduced from Theorem 2.1.

Theorem 2.2 ([20]) *Let K be a compact subset of $[0, 1]^\Gamma$ and let F be a norming subspace of $C(K)^*$. The following assertions are equivalent*

1. *$C(K)$ admits a $\sigma(C(K), F)$−lower-semicontinuous equivalent* **LUR** *norm;*
2. *for each $\varepsilon > 0$ we have $C(K) = \bigcup_{n=1}^{+\infty} C_{n,\varepsilon}$ in such a way that for each $n \in \mathbf{N}$ and each $x \in C_{n,\varepsilon}$ there exists a $\sigma(C(K), F)$−open half space S which contains x, a finite set $T \subseteq \Gamma$ and a $\delta > 0$ such that T ε-controls each $y \in S \cap C_{n,\varepsilon}$ with δ.*

We have a similar result for the existence of the property **SLD**.

Theorem 2.3 ([19]) *Let K be a compact subset of $[0, 1]^\Gamma$, the following assertions are equivalent*

1. *$C(K)$ has* **SLD***;*
2. *for each $\varepsilon > 0$ we have $C(K) = \bigcup_{n=1}^{+\infty} C_{n,\varepsilon}$ in such a way that for each $n \in \mathbf{N}$ and each $x \in C_{n,\varepsilon}$ there exist a pointwise–open set S which contains x, a finite set $T \subseteq \Gamma$ and a $\delta > 0$ such that T ε-controls each $y \in S \cap C_{n,\varepsilon}$ with δ.*

Problem 2.1 Characterize the compacta K for which $C(K)$ has **SLD**.

Such characterization is known within the class of scattered compacta.

Theorem 2.4 ([19]) *Let K be a scattered compact space. Then $C(K)$ has* **SLD** *if, and only if, the algebra of clopen (= open and closed) sets of K is pointwise σ–discrete.*

Examples of compacta K for which $C(K)$ has not **SLD** may be found in [6, 10].

Despite it is not known whether the converse of the second implication of (1) holds, as we pointed out above, a concrete decomposition of $C(K)$ that satisfies the requirement of **SLD** may be the first step to prove the existence of a **LUR** renorming, since sometimes a refinement of this decomposition allows to apply Theorem 2.2; for this the following result may be of some use. Given two sets A, B with $B \subset A$, and a real function $\varphi : A \to \mathbf{R}$, we write $\text{osc}_B\varphi = \sup\{\varphi(u) - \varphi(v) : u, v \in B\}$.

Theorem 2.5 ([20]) *Let φ_k, $1 \le k \le n$, either continuous linear functional on a locally convex linear topological space or non negative convex lower-semicontinuous maps on a convex set B of X. Let $A_0 \subset B$ for which we have*

$$osc_{A_0}\varphi_k \le 1, \quad 1 \le k \le n.$$

Let δ and θ such that

$$0 < 4\delta^{1/n} \le \theta \le 1.$$

Fix $x \in A_0$ and set $A_k = \{y \in A_{k-1} : \varphi_k(x) - \varphi_k(y) < \delta\}$, $1 \le k \le n$. Suppose that

$$\varphi_k(x) \ge \sup_{A_{k-1}} \varphi_k - \delta, \quad 1 \le k \le n;$$
$$\{y \in A_{k-1} : \delta \le \varphi_k(x) - \varphi_k(y) < \theta\} = \emptyset. \quad \textit{[Rigidity condition]}$$

Then there exists a continuous linear functional f on X such that

$$\{y \in A_0 : f(x - y) < 1\} \subset A_n.$$

3 The Property SLD in Some Classes of Compacta K

To deduce that a space $C(K)$ has **SLD** from Theorem 2.3, roughly speaking, we should obtain a suitable method to associate to each $x \in C(K)$ a set of controlling coordinates of x. For some compacta K, *methods* of this sort have been considered to obtain extensions of the Mibu Theorem [7]. Given a set Γ and a compact subspace $K \subset [0, 1]^{\Gamma}$ let $\Omega : C(K) \longrightarrow c_0(\Gamma)$ be the *oscillation map*

$$\Omega x(\gamma) = \sup\left\{x(t) - x(s) : t, s \in K,\ (t - s)|_{\Gamma\setminus\{\gamma\}} = 0\right\}, \quad x \in C(K). \quad (4)$$

It is clear that if Δ controls $x \in C(K)$ then supp $\Omega x = \{\gamma \in \Gamma : \Omega x(\gamma) \neq 0\} \subset \Delta$. In some compacta K we have that the set supp Ωx controls every $x \in C(K)$, this is so when $K = [0, 1]^{\Gamma}$ or $K = \{0, 1\}^{\Gamma}$ [7]. Let K be one of such compacta and fix $\varepsilon > 0$, given $x \in C(K)$ there must exist $n_x \in \mathbf{N}$ such that the finite set $\{\gamma \in \Gamma : \Omega x(\gamma) > 1/n_x\}$ ε–controls x with $\delta = 1/m_x$. Let $C_{n,m,k,\varepsilon}$ be the set of all $x \in C(K)$ such that the cardinal of $\{\gamma \in \Gamma : \Omega x(\gamma) > 1/n_x\}$ is k, $n_x = n$ and $m_x = m$. It is easy to see that given $n, m, k \in \mathbf{N}$ and $x \in C_{n,m,k,\varepsilon}$ there exists a pointwise open set W such that $\{\gamma \in \Gamma : \Omega x(\gamma) > 1/n\} \subset \{\gamma \in \Gamma : \Omega y(\gamma) > 1/n\}$ for all $y \in W$. Then the choice of k gives that the former inclusion is, in fact, an equality for $y \in W \cap C_{n,m,k,\varepsilon}$. Now from Theorem 2.3 it follows that if $K = [0, 1]^{\Gamma}$ or $K = \{0, 1\}^{\Gamma}$ then $C(K)$ has **SLD**. In both cases we have that $C(K)\,\langle\mathbf{LUR}\rangle$ [21], it is possible to show this by refining the above decomposition to apply Theorems 2.2 and 2.5.

If K is a dyadic compactum (i.e. a continuous image of $\{0, 1\}^{\Gamma}$ for some set Γ) then $C(K)$ is a linear subspace of $C\left(\{0, 1\}^{\Gamma}\right)$ so we have that $C(K)\langle\mathbf{LUR}\rangle$; in particular, this happens whenever K is a compact topological group, [26, p. 81], see also [2].

If Λ is an ordinal and then the map $\Psi(\lambda) = \left\{1\!\!1_{[0,\alpha]}(\lambda)\right\}_{\alpha\in[0,\Lambda[}$, $\lambda \in [0, \Lambda]$, is an embedding of $K = [0, \Lambda]$ into $\{0, 1\}^{[0,\Lambda[}$. In this case, let Ω be as in (4), we have supp $\Omega x = \{\alpha \in [0, \Lambda[:\ x(\alpha + 1) - x(\alpha) \neq 0\}$. It is easy to see that given $p, q \in K$ if $p|_{\text{supp } \Omega x} = q|_{\text{supp } \Omega x}$ then $x(p) = x(q)$, i.e. according to Remark 2.1 supp Ωx controls x.

Fix $\varepsilon > 0$, given $x \in C(K)$ there exists $n \in \mathbf{N}$ such that if

$$F_n(x) := \{\alpha \in [0, \Lambda[:\ |x(\alpha + 1) - x(\alpha)| > 1/n\}$$

then $F_n(x)$ ε–controls x for some $\delta = 1/m$. Once more let $C_{\varepsilon,n,m,k}$ be the set of all $y \in C(K)$ such that $F_n(y)$ ε–controls y for some $\delta = 1/m$, and the cardinal of the set $F_n(y)$ is k. The argument above shows that the decomposition made up by all $C_{\varepsilon,n,m,k}$'s satisfies the conditions of Theorem 2.3, so this $C(K)$ has **SLD**. Moreover from Theorems 2.2 and 2.5 it is possible to show that, in fact, $C(K)$ admits an equivalent **LUR** norm [1].

Since every metric compact space K may be embedded in $K \subset [0,1]^{\mathbf{N}}$, from the Borsuk–Dugundji theorem we have that $C(K)$ is a subspace of $C\left([0,1]^{\mathbf{N}}\right)$, then $C(K)\langle\mathbf{LUR}\rangle$, see also [21, Corollary 2.68]. It is known that any such K belongs to the class of the Valdivia compacta, let us remember the definition of this class. Given a set Γ let

$$\Sigma(\Gamma) := \left\{u \in [0,1]^{\Gamma} : \#\{\gamma \in \Gamma : u(\gamma) \neq 0\} \leq \aleph_0\right\}.$$

Definition 3.1 ([5]) A compact space K is a *Valdivia compact* if, for some set Γ, there exists an homeomorphic embedding $h : K \longrightarrow [0,1]^{\Gamma}$ such that $h(K) = \overline{\Sigma(\Gamma) \cap h(K)}^{[0,1]^{\Gamma}}$.

We can suppose, without loss of generality, that a Valdivia compact space K satisfies $K \subset [0,1]^{\Gamma}$ and $K = \overline{\Sigma(\Gamma) \cap K}^{[0,1]^{\Gamma}}$.

Theorem 3.1 ([6]) *If K is a Valdivia compact space then $C(K)\langle\mathbf{LUR}\rangle$.*

In the class of Valdivia compacta the following result is the topological key to deduce the existence of a **LUR** renorming.

Lemma 3.1 ([6, Lemma VI.7.5]) *Let Γ be a set and let K be a closed subset of $[0,1]^{\Gamma}$ such that $K \cap \Sigma(\Gamma)$ is dense in K. Let μ be the smallest ordinal such that $\#\mu = dens(K \cap \Sigma(\Gamma))$. Then there exists an increasing family $\{\Gamma_\alpha : \omega \leq \alpha \leq \mu\}$ of subsets of Γ such that for every α*

1. $\#\Gamma_\alpha \leq \#\alpha$;
2. $\Gamma_\alpha = \bigcup_{\beta<\alpha} \Gamma_{\beta+1}$ *and* $\Gamma_\mu = \Gamma$;
3. $R_{\Gamma_\alpha}(K) \subseteq K$ *where* $R_I : [0,1]^{\Gamma} \longrightarrow [0,1]^{\Gamma}$ *is defined for* $I \subseteq \Gamma$ *by*

$$s \longmapsto R_I(s)(\gamma) = \begin{cases} s(\gamma) & \text{if } \gamma \in I; \\ 0 & \text{if } \gamma \notin I \end{cases}$$

This lemma may be used to deduce the existence of a *projectional resolution of the identity* suitable to obtain a **LUR** renorming [6]. Nevertheless it is possible to deduce the existence of this renorming from Theorem 2.2 by transfinite induction. We only present a sketch of the proof that $C(K)$ is **SLD**.

We can assume that Γ is a limit ordinal and identify Γ with $[0,\Gamma[$. For $\Gamma = \omega$, $[0,1]^{\Gamma}$ is a metric space and we have already seen that in this case $C(K)$ has **SLD**. The idea for the induction is based on the following two observations: For each $\varepsilon > 0$ if

$$E(x,\varepsilon) := \{\alpha < \Gamma : \|x \circ R_{\alpha+1} - x \circ R_\alpha\|_\infty > \varepsilon\},$$

the uniform continuity of each $x \in C(K)$ shows that $E(x,\varepsilon)$ is finite. Moreover from the properties of the R_α's it can be shown that for every $\varepsilon > 0$ and every $x \in C(K)$ there is $n = n(\varepsilon, x) \in \mathbf{N}$ such that

$$\alpha > \max E(x, 1/n) \quad \Longrightarrow \quad \|x - x \circ R_\alpha\|_\infty \leq \varepsilon. \tag{5}$$

For a given $\varepsilon > 0$ we may consider the countable decomposition $C(K) = \bigcup_{n,m \in \mathbf{N}} C_{\varepsilon,n,m}$, where $C_{\varepsilon,n,m}$ is made up by all $x \in C(K)$ such that $n(\varepsilon, x) = n$ and the cardinal of $E(x, 1/n)$ is m. Reasoning as in the previous cases, given $n, m \in \mathbf{N}$ and $x \in C_{\varepsilon,n,m}$ it is easy to obtain a pointwise open set W such that $x \in W$ and for every $y \in W \cap C_{\varepsilon,n,m}$ we have $E(y, \varepsilon) = E(x, \varepsilon)$. Now since each each $R_\alpha(K)$ is a Valdivia compact space, from our inductive hypothesis $C\,(R_\alpha(K))$ has a countable decomposition that fulfills the requirement of Theorem 2.2; combining it with the $C_{\varepsilon,n,m}$'s we may prove that $C(K)$ has **SLD**. Refining this decomposition it is possible to deduce from Theorems 2.2 and 2.5 that, in fact, $C(K)$ admits an equivalent **LUR** norm.

In what follows $(\Upsilon, \preceq)$, or namely Υ, stands for a tree, i.e. a partially ordered set such that for every $t \in \Upsilon$ the set $]\leftarrow, t] = \{s \in \Upsilon : s \preceq t\}$ is well–ordered by $\preceq$. Then for each $t \in \Upsilon$ there exists an ordinal $r(t)$ such that $]\leftarrow, t]$ has the same order type as $r(t)$. From now on all trees will be *Hausdorff*, i.e. for every $s, t \in \Upsilon$, if $r(t)$ is a limit ordinal and $]\leftarrow, s[=]\leftarrow, t[$ then $s = t$, these trees can be endowed with the coarsest topology for which the sets $]\leftarrow, s]$ are clopen for any $s \in \Upsilon$, and we consider every tree Υ endowed with this topology.

If $t \in \Upsilon$ we will denote by t^+ the set of immediate successors of t, that is the set of all $s \in \Upsilon$, $t \prec s$ such that $]t, s[= \emptyset$. Following [10] given an increasing function on a tree $\rho : \Upsilon \to \mathbf{R}$ a point $t \in \Upsilon$ is called in [10] a *good point* for ρ if there is a finite subset $F \subset t^+$ such that $\rho(t) < \inf_{u \in t^+ \setminus F} \rho(u)$; if $t \in \Upsilon$ is not a good point we say that is *bad point* for ρ.

In [10] there is a characterization of those trees Υ for which $C\,(\Upsilon^*)\,\langle\mathbf{K}\rangle$ and $\langle\mathbf{LUR}\rangle$ where Υ^* stands for its Alexandroff compactification $\Upsilon^* = \Upsilon \cup \{\infty\}$. There Haydon shows that the converse of the first implication in (1) does not hold. Here we restrict ourselves to discuss the case $C\,(\Upsilon^*)\,\langle\mathbf{LUR}\rangle$, as above we just sketch the proof of the property **SLD**. Let us recall that a subset S of a tree Υ is ever-branching if for every $t \in S$ the set $\{u \in S : u \succeq t\}$ is not totally ordered [10].

Theorem 3.2 (Haydon [10]) *For any tree Υ we have that $C\,(\Upsilon^*)\,\langle\mathbf{LUR}\rangle$ if, and only if, there exists an increasing function $\rho : \Upsilon \to \mathbf{R}$ which is constant on no ever-branching subset of Υ and which has no bad points.*

We may embed Υ^* into $\{0,1\}^\Upsilon$ by the map $t \to \left\{\mathbb{1}_{]\leftarrow,s]}(t)\right\}_{s\in\Upsilon}$. Write $\tilde{x} := x - x(\infty)$ for $x \in C\,(\Upsilon^*)$. In [10] it is shown that for the trees of Theorem 3.2, there exists a bounded linear operator $T : C_0(\Upsilon) \longrightarrow c_0(\Upsilon)$ such that for every $x \in C\,(\Upsilon^*)$ we have $\{t \in \Upsilon : \tilde{x}(t) \neq 0\} \subset \bigcup_{s \in S(x)}]\leftarrow, s]$ where $S(x) := \bigcup \{u \in \Upsilon \colon T\tilde{x}(u) \neq 0\}$.

Fix $\varepsilon > 0$. Set $S_n(x) := \bigcup \{\gamma \in \Upsilon : |T\tilde{x}(\gamma)| > 1/n\}, n \in \mathbf{N}$. Since T is $c_0(\Upsilon)$–valued we have that the set $S_n(x)$ is finite. Given $x \in C\,(\Upsilon^*)$ a compactness argument shows that there must exist $n_x \in \mathbf{N}$ such that the finite set $S_{n_x}(x)$ satisfies

$$\{t \in \Upsilon : |\tilde{x}(t)| \geq \varepsilon/2\} \subset \bigcup_{s \in S_{n_x}(x)}]\leftarrow, s]. \tag{6}$$

It is clear that if a finite set $F_x \subset \bigcup_{s\in S_{n_x}(x)}]\leftarrow, s]$ ε–controls $x|_{\bigcup_{s\in S_{n_x}(x)}]\leftarrow,s]}$ for some δ_x then $F_x \cup S_{n_x}(x)$ ε–controls x. Set $C(\Upsilon^*) = \bigcup_{n=1}^{+\infty} C_{n,m,\ell,\varepsilon}$, where $C_{n,m,\ell,\varepsilon}$ is the set of all $x \in C(\Upsilon^*)$ for which (6) holds when $n_x = n$, the cardinal $\#S_n(x) = m$ and $\|x\|_\infty \leq \ell$. Fix $n, m, \ell \in \mathbf{N}$ and $x \in C_{n,m,\ell,\varepsilon}$. We have that

$$G := \left\{y \in C_{n,m,\ell,\varepsilon} : |T\tilde{y}(\gamma)| > 1/n \text{ for all } \gamma \in S_n(x)\right\}$$

is relatively pointwise open in $C_{n,m,\ell,\varepsilon}$. For any $y \in G$ we get $S_n(x) \subset S_n(y)$, once more the choice of m implies that $S_n(x) = S_n(y)$. Roughly speaking, now we have only to *describe* the finite set F_x above; for this observe that in the finite union $\bigcup_{s\in S_n(x)}]\leftarrow, s]$, each $]\leftarrow, s]$ is an ordinal and we have already seen such *description* in those spaces. Then combining both techniques it is possible to prove the property **SLD** from Theorem 2.3. As above, this decomposition may be refined to show the existence of a **LUR** renorming, for the trees considered, from Theorems 2.2 and 2.5, [18].

In some of the previous classes of compacta, the oscillation map Ω has played an essential role in associating a controlling set of coordinates to each continuous function, nevertheless there are compacta where Ω seems to be useless for this purpose. For instance let us consider the Helly compact space H, i.e. the space of all increasing functions from the real interval $[0, 1]$ in itself, $H \subset [0, 1]^{[0,1]}$, and the map $x : H \longrightarrow \mathbf{R}$, $x(s) := \int_{[0,1]} s$, when $x \in C(H)$. It is clear that for any $\gamma \in [0, 1]$ and any $s, t \in C(H)$ such that $(t - s)|_{\Gamma\setminus\{\gamma\}} = 0$ we have that $\int_{[0,1]} s = \int_{[0,1]} t$, i.e. $x(s) = x(t)$ so supp $\Omega x = \emptyset$. Since x is not constant we have that supp Ωx does not control x. Nonetheless there exists an equivalent pointwise lower-semicontinuous **LUR** norm in $C(H)$ [21]. In fact since each $s \in H$ has at most countably many discontinuities, the existence of this norm may be deduced form the following

Theorem 3.3 ([13]) *Let Γ be a Polish space and let K be a separable and pointwise compact set of functions on Γ. Assume further that each function in K has only countably many discontinuities. Then $\mathscr{C}(K)$ admits a pointwise-lower-semicontinuous and local uniformly rotund norm, equivalent to the supremum norm.*

The compact space K above belongs to the class of Rosenthal compacta, i.e. a space made up by of Baire–1 functions over a Polish space that is pointwise compact. Todorcevič showed that there exists a Rosenthal compact space K for which $C(K)$ has no equivalent **LUR** norm [27]. For separable Rosenthal compact spaces it has been conjectured a positive answer to the following question [13].

Problem 3.1 If K is a separable Rosenthal compact then does $C(K)$ admit a locally uniformly rotund renorming?

In [20] Theorem 3.3 is deduced from Theorem 2.2, we will only point out some ideas to associate to each $x \in C(K)$ a controlling sets of coordinates, that can be regarded as the first step of a proof by this method. Let us recall some notation.

For a topological space X the derived space X' is the set of all points of X that are not isolated, by transfinite induction for an ordinal β we define the βth–derived set $X^{(\beta)}$ by

$$X^{(0)} = X, \quad X^{(\gamma+1)} = \left(X^{(\gamma)}\right)', \quad X^{(\alpha)} = \bigcap_{\gamma<\alpha} X^{(\gamma)},$$

for arbitrary ordinals γ and limit ordinals α. A topological space X is said to be *scattered* if no nonempty subset of X is dense in itself, equivalently if there exists an ordinal γ such that $X^{(\gamma)} = \emptyset$.

Fix $\varepsilon > 0$ and Q a countable dense subset of Γ. Let $C_{F,1/n}$ be set of all the functions $x \in C(K)$ that are ε–controlled by a finite subset $F \subset Q$ and $\delta = 1/n$. Observe that we cannot assert that the countable family $C_{F,1/n}$ covers $C(K)$, because there may be functions $y \in C(K)$ that are not ε–controlled by Q; from now on we will consider only those functions y's. Associate to each one of those y finite subsets $F_y \subset \Gamma \setminus Q$, $R_y \subset Q$ and $n_y \in \mathbf{N}$ such that $|y(u) - y(v)| < \varepsilon$ whenever $u, v \in K$ and $\sup_{\gamma\in F_y\cup R_y} |u(\gamma) - v(\gamma)| \leq 1/n_y$. Observe that $F_y \neq \emptyset$ for such a function y.

By compactness there must exist an $\ell_y \in \mathbf{N}$ such that

$$|y(u) - y(v)| \leq \varepsilon - 1/\ell_y \text{ whenever } u,\ v \in K \text{ and } \sup_{\gamma\in F_y\cup R_y} |u(\gamma) - v(\gamma)| \leq 1/n_y. \tag{7}$$

On the other hand, according to Remark 2.1 there must exist $s, t \in K$ such that

$$s|_Q = t|_Q \text{ and } |y\,(s) - y\,(t)| \geq \varepsilon. \tag{8}$$

In particular

$$s|_Q = t|_Q \text{ and } |y\,(s) - y\,(t)| > \varepsilon - 1/\ell_y. \tag{9}$$

For $u \in K$ and $\delta > 0$ write

$$J(u, \delta) := \{\gamma \in \Gamma : \ \operatorname{osc}(u, G) \geq \delta \text{ for all } G \text{ open with } \gamma \in G\}$$

where $\operatorname{osc}(u, G)$ stands for the oscillation of u in G, and $J(u, v; \delta) := J(u, \delta) \cup J(v, \delta)$. From (7) it follows that for any couple $s, t \in K$ verifying (9) there exists $\gamma \in F_y$ such that $|s\,(\gamma) - t\,(\gamma)| > 1/n_y$, moreover such γ must satisfy $\gamma \in F_y \cap J\left(s, t;\, 1/(2n_y)\right)$; in particular

$$F_y \cap J\left(s, t;\, 1/(2n_y)\right) \neq \emptyset. \tag{10}$$

On the other hand, for any $u \in K$ and $\delta > 0$ the set $J(u, \delta)$ is closed and, according to our hypothesis, countable, so it does not contain any perfect subset, therefore there exists a countable ordinal α_u such that $J(u,\delta)^{(\alpha_u)} = \emptyset$. For any couple s, $t \in K$ satisfying (9) let $\alpha_{s,t,y}$ the maximum of all the ordinals α such that $F_y \cap \left(J\left(s,t;1/(2n_y)\right)\right)^{(\alpha)} \neq \emptyset$, and $\alpha_{0,y}$ the minimum for all such $\alpha_{s,t,y}$. Let $k_{0,y}$ the minimum of the cardinals of $F_y \cap \left(J\left(s,t;1/(2n_y)\right)\right)^{(\alpha_{0,y})}$ for $s,t \in K$ that fulfill (9) and

$$F_y \cap \left(J\left(s,t;1/(2n_y)\right)\right)^{(\alpha_{0,y}+1)} = \emptyset. \tag{11}$$

For such $y \in C(K)$ choose s_y, $t_y \in K$ that verify (9), and (11) and the cardinal of $F_y \cap \left(J\left(s_y,t_y;1/(2n_y)\right)\right)^{(\alpha_{0,y})}$ is $k_{0,y}$.

Let $\mathscr{B}$ be a countable base of open sets in Γ, since $J\left(s,t;1/(2n_y)\right)$ is a totally inconnected compact space, for $y \in C(K)$, not controlled by Q, we may fix $V_j^{(y)} \in \mathscr{B}$, $1 \leq j \leq j_{0,y}$, pairwise disjoint, in such a way that $F_y \subset \bigcup_{j=1}^{j_{0,y}} V_j^{(y)}$, the cardinal of each $F_y \cap V_j^{(y)}$ is one, and there exist $i_{0,y} \in \mathbf{N}$, $i_{0,y} \leq j_{0,y}$, and ordinals $\{\beta_j\}_{j=1}^{i_{0,y}}$ such that if $F_y \cap V_j^{(y)} \cap J\left(s_y,t_y;1/(2n_y)\right) = \emptyset$ then $V_j^{(y)} \cap J\left(s_y,t_y;1/(2n_y)\right) = \emptyset$ and

$$\begin{aligned} &\text{if } F_y \cap V_j^{(y)} \cap \left(J\left(s_y,t_y,1/(2n_y)\right)\right) \neq \emptyset \text{ then } \#\left(V_j^{(y)} \cap J\left(s_y,t_y;1/(2n_y)\right)^{\left(\beta_j^{(y)}\right)}\right) = 1 \\ &\text{and} \quad V_j^{(y)} \cap J\left(s_y,t_y;1/(2n_y)\right)^{\left(\beta_j^{(y)}+1\right)} = \emptyset. \end{aligned} \tag{12}$$

From the choice of $\alpha_{0,y}$ and $k_{0,y}$ we have $\beta_j^{(y)} \leq \alpha_{0,y}$, $V_j^{(y)} \cap J\left(s_y,t_y;1/(2n_y)\right)^{(\alpha_{0,y})} = \emptyset$ if $\beta_j^{(y)} < \alpha_{0,y}$ and the set $J_{0,y} := \left\{j : \beta_j^{(y)} = \alpha_{0,y}\right\}$ has cardinal $k_{0,y}$. So from (12) we get

$$V_j^{(y)} \cap J\left(s_y,t_y;1/(2n_y)\right)^{(\alpha_{0,y})} = \emptyset \text{ whenever } j \notin J_{0,y}. \tag{13}$$

Now we have several parameters and sets associated to each y, namely n_y, ℓ_y, R_y, $i_{0,y}$, $j_{0,y}$, $k_{0,y}$, $V_j^{(y)}$'s, $\alpha_{0,y}$'s, $\beta_j^{(y)}$'s, and $J_{0,y}$, we have seen that all of them but the $\alpha_{0,y}$'s and $\beta_j^{(y)}$'s belong to a countable set. In [13] it is deduced from the Rank Theorem [3, 16], that there exists a countable ordinal Ω such that $J(u,\delta)^{(\Omega)} = \emptyset$ for all $u \in K$ and all $\delta > 0$, therefore the $\alpha_{0,y}$'s and $\beta_j^{(y)}$'s belong to the countable set $[0,\Omega]$.

Then we can split up the set of all functions y, that are not controlled by Q, into a countable number of subsets C_p, $p \in \mathbf{N}$, in such a way that the parameters and sets above are the same for all functions $y \in C_p$; fix C_p and writte those common parameters and sets as n_0, ℓ_0, R_0, j_0, i_0, k_0, V_j's, α_0's, β_j's and J_0. Now take $x \in C_p$. Then

$$W := \{y \in C_k : |y(s_x) - y(t_x)| > \varepsilon - 1/\ell_0\}$$

is a pointwise open set in C_k than contains x. We claim that for $y \in W$ we have

$$F_y \cap J\,(s_x, t_x; 1/(2n_0))^{(\alpha_0)} = F_x \cap J\,(s_x, t_x; 1/(2n_0))^{(\alpha_0)} \neq \emptyset. \qquad (14)$$

Indeed from the choice of W and (12) we have

$$F_y \cap (J\,(s_x, t_x; 1/(2n_0)))^{(\alpha_0+1)} \subset \left(\bigcup_{j=1}^{j_0} V_j\right) \cap J\,(s_x, t_x; 1/(2n_0))^{(\alpha_0+1)} = \emptyset;$$

i.e. $F_y \cap (J\,(s_x, t_x; 1/(2n_0)))^{(\alpha_0+1)} = \emptyset$. So, the cardinal of $F_y \cap (J\,(s_x, t_x; 1/(2n_y)))^{(\alpha_0)}$ is greater or equal than k_0. Since $y \in C_k$, from (13) and the choice of $\left\{V_j\right\}_{j=0}^{j_0}$ and α_0 we have

$$F_y \cap J\,(s_x, t_x; 1/(2n_0))^{(\alpha_0)} \subset J\,(s_x, t_x; 1/(2n_0))^{(\alpha_0)} \cap \left(\bigcup_{j \in J_0} V_j\right).$$

So the cardinal of the last set above is k_0 therefore both sets are equal and (14) is proved.

Iterating this process we get a countable decomposition of $C(K)$ in such a way that the hypotheses of Theorem 2.3 hold, so $C(K)$ has **SLD**. In [20] Theorem 3.3 is deduced from Theorems 2.2 and 2.5 refining this decomposition.

In all previous classes of compacta K, the presence the property **SLD** has been deduced, roughly speaking, from *a good method to associate finite sets of controlling coordinates* to each $x \in C(K)$, this notion comes from the elementary topological property in (2). It is elementary too that for any $x \in C(K)$ and any $\varepsilon > 0$, there exists a finite covering of K such that the oscillation of x, in each element of the covering, is not bigger than ε. The next result states that in Theorems 2.2 and 2.3 it is possible to replace the first property by the second, namely

Theorem 3.4 *Let K be a compact space. The Banach space $C(K)$ has the property* **SLD** *(respectively admits an equivalent (pointwise lower-semicontinuous)* **LUR** *norm) if, and only if, there is a countable family of subsets $\{C_n : n \in \mathbf{N}\}$ in $C(K)$ such that, for every $x \in C(K)$ and every $\varepsilon > 0$, there are $q \in \mathbf{N}$, a pointwise open set (respectively pointwise open half space) H with $x \in H \cap C_q$ together with a finite covering $\mathscr{L}$ of K such that*

$$|y(s) - y(t)| < \varepsilon \ \textit{ whenever } s, t \in L,\ y \in H \cap C_q \textit{ and } L \in \mathscr{L}.$$

From this result it is possible to deduce the existence of a **LUR** renorming in $C(K)$ when K is a Namioka–Phelps compact, a class introduced by M. Raja [24]. Let us recall that a family $\mathscr{H} = \{H_i : i \in I\}$ of subsets of a topological space $(X, \mathscr{T})$ is said to be $\mathscr{T}$*-isolated* whenever $H_i \cap \overline{\bigcup\{H_j : j \in I, j \neq i\}}^{\mathscr{T}} = \emptyset$, for every $i \in I$.

We will say that $\mathscr{H}$ is a $\mathscr{T}$-*σ-isolated* family if $\mathscr{H}$ is a countable union of $\mathscr{T}$-isolated families.

A collection $\mathscr{N}$ of subsets of a topological space $(X, \mathscr{T})$ is said to be a *network* for the topology $\mathscr{T}$ if for every $U \in \mathscr{T}$ and every $x \in U$ there exists $N \in \mathscr{N}$ such that $x \in N \subset U$.

Definition 3.2 ([24]) A compact space $(K, \mathscr{T})$ is said to be a *Namioka–Phelps compact* if there is a $\mathscr{T}$-lower-semicontinuous metric ρ on K such that the metric topology induced by ρ has a network which is $\mathscr{T}$-σ-isolated.

Theorem 3.5 (Haydon [11]) *Let K be a Namioka–Phelps compact space. Then $C(K)$ admits an equivalent pointwise lower-semicontinuous* **LUR** *norm.*

In [20] this result is deduced from Theorem 3.4, the proof is technically involved and it is not included here.

If X is a space satisfying the hypotheses of Theorem 1.2, then its dual unit ball endowed with the weak* topology is a Namioka–Phelps compact [24]; using this fact in [11] Theorem 1.2 is deduced from Theorem 3.5 above.

Since every σ-discrete compact space is Namioka–Phelps compact it follows the following

Corollary 3.1 (Haydon) *Let K be a σ-discrete compact space. Then $C(K)$ admits an equivalent pointwise lower semicontinuous* **LUR** *norm.*

If ω_1 stands for the first uncountable ordinal and K is a compact space such that $K^{(\omega_1)} = \emptyset$, a compactness argument gives a countable ordinal $\Lambda < \omega_1$ such that $K^{(\Lambda)}$ is finite. Since $K = \bigcup_{\gamma \leq \Lambda} \left(K^{(\gamma)} \setminus K^{(\gamma+1)}\right)$ and each $K^{(\gamma)} \setminus K^{(\gamma+1)}$ is discrete, from Corollary 3.1 we have

Theorem 3.6 (Haydon, Rogers [14]) *Let K be a nonempty compact space such that $K^{(\omega_1)} = \emptyset$ then $C(K)$ admits an equivalent pointwise lower-semicontinuous* **LUR** *norm.*

From Theorem 3.4 it follows that $C(K)$ has **SLD** for a class of compacta K whose topology is linked with its order. In [4] it is proved that if K is a product of totally ordered spaces, that are compact for the order topology, then $C(K)$ has a pointwise lower-semicontinuous Kadets norm. We are going to see how to deduce from Theorem 3.4 the property **SLD** for a subclass of such compacta, namely for totally ordered compact spaces K (see [12]).

Thus let K be a totally ordered space, that is compact for its order topology; 0, 1 denote the minimum and the maximum of K. In what follows given $a, b \in K$, the symbol $< a, b >$ stands for the pair made up by these two points, and by (a, b) the open interval whose extreme points are a and b. We say that $x \in C(K)$ ε*–leaps* on the couple $< a, b >$, $a < b$, $a, b \in K$, whenever $|x(b) - x(a)| > \varepsilon$. Given $a, b \in K$, the couple $< a, b >$ is called a *gate* in K whenever $a < b$ and $(a, b) = \emptyset$. If x ε–leaps on a gate we will say that ε*–jumps* on it. It is easy to check that

Remark 3.1 Let $x \in C(K)$, $a, b \in I$, $a < b$, $M := \sup_{[a,b]} x$, $m := \inf_{[a,b]} x$, if for any $u, v \in [a, b]$, $u < v$, x does not ε–jump in $< u, v >$ then $[m, M] \setminus x([a, b])$ contains no interval of length bigger or equal than ε.

Given $x : K \to \mathbf{R}$ and $\varepsilon > 0$ we will say that x *can* ε*–leap* n *times* on $\{\langle a_i, b_i \rangle\}_{i=1}^n$, $a_i \le b_i, a_i, , b_i \in K$, $1 \le i \le n$, whenever

(i) either $b_i \le a_j$ or $b_j \le a_i$, $i \ne j$, $1 \le i, j \le n$;
(ii) if x ε–jumps on $< a, b >$ then there exists i, $1 \le i \le n$, such that $a = a_i$ and $b = b_i$;
(iii) x ε–leaps on each $\langle a_i, b_i \rangle$, $1 \le i \le n$.

Given $\langle a_i, b_i \rangle$, $a_i, b_i \in K$, $1 \le i \le n$, set

$$S := \{0, 1\} \cup \{a_i : 1 \le i \le n\} \cup \{b_i : 1 \le i \le n\}.$$

It is clear that there exists one, and only one, covering $\mathscr{I}$ of K such that each $I \in \mathscr{I}$ is a closed interval whose extreme points belong to S, and

(I) if $I = [a, b]$ with $a \ne b$ then $(a, b) \ne \emptyset$ and $(a, b) \cap S = \emptyset$;
(II) $I \not\subset J$ for $I, J \in \mathscr{I}$ with $I \ne J$.

We say that $\mathscr{I}$ is the *covering of* K *associated to* $\{\langle a_i, b_i \rangle\}_{i=1}^n$.

It is clear that given $x \in C(K)$ and $\varepsilon > 0$ there exists $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_m = 1$ such that the oscillation of x in each $\left[\alpha_{i-1}, \alpha_i\right]$, $1 \le i \le m$, is strictly less than ε. Then x cannot ε–leap more than m times, therefore it makes sense to consider the maximum $n(x, \varepsilon)$ of those n for which x can leap n times. Now it is easy to prove that

Remark 3.2 Given $x \in C(K)$ and $\varepsilon > 0$ if x can ε–leap $n(x, \varepsilon)$ times on $\{\langle a_i, b_i \rangle\}_{i=1}^n$, $a_i, b_i \in K$, $1 \le i \le n(x, \varepsilon)$, and if $\mathscr{I}$ is the covering of K associated to them, then $\operatorname{osc}(x, I) \le 3\varepsilon$ for every $I \in \mathscr{I}$.

For $n \in \mathbf{N}$ let $C_{n,\varepsilon}$ be the set of all $x \in C(K)$ for which $n(x, \varepsilon) = n$. We have that $C(K) = \bigcup_{n=1}^{+\infty} C_{n,\varepsilon}$. Given $n \in \mathbf{N}$, $\varepsilon > 0$ and $x_0 \in C_{n,\varepsilon}$, set $\{\langle a_i, b_i \rangle\}_{i=1}^n$ satisfying (i)–(iii) above for $n(x, \varepsilon) = n$, and $r > 0$ such that $r < |x_0(b_i) - x_0(a_i)| - \varepsilon$, $1 \le i \le n$. If W is the pointwise open neighbourhood of x_0 defined by

$$W := \left\{ y \in C(K) : \max_{1 \le i \le n} \left(|x_0(a_i) - y(a_i)|, |x_0(b_i) - y(b_i)|\right) < r/2 \right\},$$

from Remark 3.2 it follows that there is a finite partition of K made up by sets I such that $\operatorname{osc}(y, I) < 3\varepsilon$ for all $y \in W \cap C_{n,\varepsilon}$. Then the hypotheses of Theorem 3.4 for **SLD** are fulfilled.

If K is a compact space endowed with an order for which is a distributive lattice such that the supremum, $\wedge$, and the infimum, $\vee$, are continuous, then we will say that K is a *compact distributive lattice.* (A lattice $(K, \wedge, \vee)$ is *distributive* if it satisfies $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for every $a, b, c \in K$.) If K is a closed sublattice of a finite product of linearly ordered compacta it is called a *finite dimensional compact lattice*. In [17], using an extension of the notion of *gate*, from Theorem 3.4 it is deduced the following

Theorem 3.7 *Let K be a finite-dimensional compact lattice then $C(K)$ has the property* **SLD**.

We will finish with the following problem that arises in a natural way after the previous result.

Problem 3.2 If K is a finite-dimensional compact lattice has $C(K)$ an equivalent Kadets norm?

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A Fixed Point Theory Linked to the Zeros of the Partial Sums of the Riemann Zeta Function

In Honour of Manuel López-Pellicer

Gaspar Mora

Abstract For each $n > 2$ we consider the corresponding nth-partial sum of the Riemann zeta function $\zeta_n(z) := \sum_{j=1}^{n} j^{-z}$ and we introduce two real functions $f_n(c)$, $g_n(c)$, $c \in \mathbb{R}$, associated with the end-points of the interval of variation of the variable x of the analytic variety $|\zeta_n^*(z)| = p_{k_n}^{-c}$, where $\zeta_n^*(z) := \zeta_n(z) - p_{k_n}^{-z}$ and p_{k_n} is the last prime not exceeding n. The analysis of fixed point properties of f_n, g_n and the behavior of such functions allow us to explain the distribution of the real parts of the zeros of $\zeta_n(z)$. Furthermore, the fixed points of f_n, g_n characterize the set $\mathscr{P}^*$ of prime numbers greater than 2 and the set $\mathscr{C}^*$ of composite numbers greater than 2, proving in this way how close those functions from Arithmetic are. Finally, from the study of the graphs of f_n, g_n we deduce important properties about the set $R_{\zeta_n(z)} := \overline{\{\Re z : \zeta_n(z) = 0\}}$ and the bounds $a_{\zeta_n(z)} := \inf\{\Re z : \zeta_n(z) = 0\}$, $b_{\zeta_n(z)} := \sup\{\Re z : \zeta_n(z) = 0\}$ that define the critical strip $[a_{\zeta_n(z)}, b_{\zeta_n(z)}] \times \mathbb{R}$ where are located all the zeros of $\zeta_n(z)$.

Keywords Exponential polynomials · Zeros of the partial sums of the Riemann zeta function · Diophantine approximation

1 Introduction

Since the non-trivial zeros of the **Riemann zeta function** $\zeta(z)$, until now found, lie on the line $\Re z = 1/2$ (the assertion that all them are situated on that line is the **Riemann Hypothesis**) and the trivial ones are on the real axis (they are the negative even numbers [9, p. 8]), it seems that the zeros of $\zeta(z)$ are situated on those two perpendicular lines. However that is not so for the zeros of the partial sums $\zeta_n(z) := \sum_{j=1}^{n} j^{-z}$ of the series $\sum_{j=1}^{\infty} j^{-z}$ that defines the Riemann zeta function $\zeta(z)$ on the half-plane $\Re z > 1$. Indeed, except for $\zeta_2(z)$ whose zeros all are imaginary

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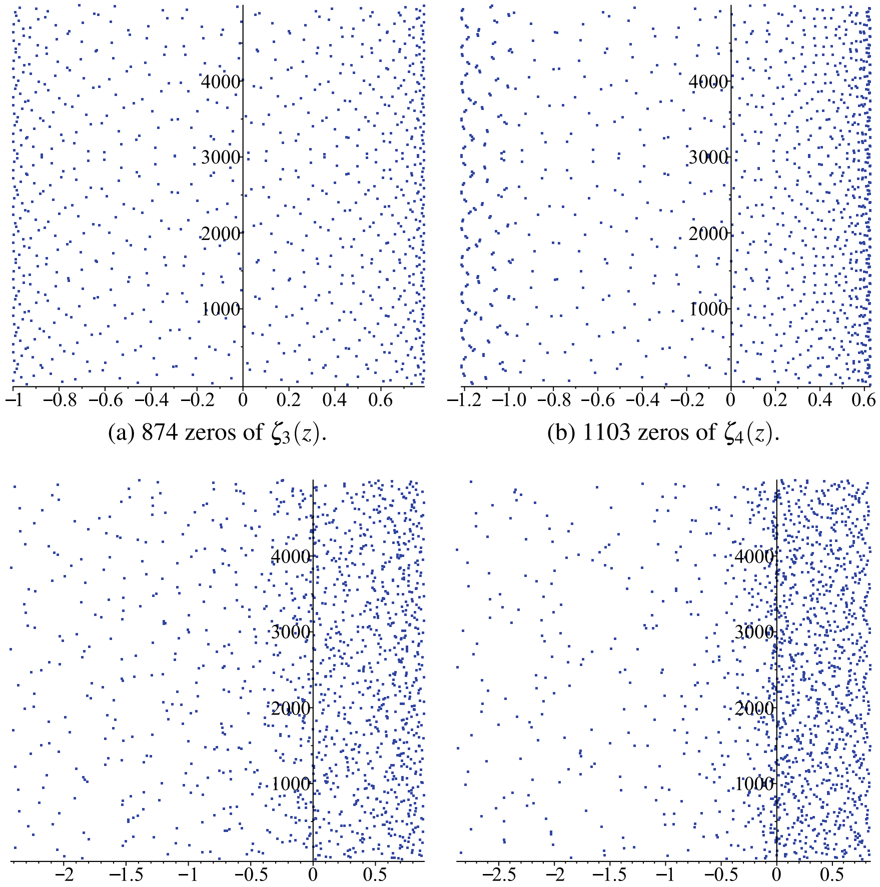


(a) 874 zeros of $\zeta_3(z)$.

(b) 1103 zeros of $\zeta_4(z)$.

(c) 1281 zeros of $\zeta_5(z)$.

(d) 1426 zeros of $\zeta_6(z)$.

Fig. 1 Graphs of the zeros of $\zeta_n(z)$ for some values of n, with $\Re z \in [-3, 1]$ and $\Im z \in [0, 5000]$

(it is immediate to check that the zeros of $\zeta_2(z)$ are $z_{2,j} = \frac{(2j+1)\pi i}{\log 2}$, $j \in \mathbb{Z}$), so aligned, the zeros of each $\zeta_n(z)$ for any $n > 2$ are dispersed in a vertical strip forming a sort of cloud, more or less uniform, that extends up, down and left as n increases, whereas at the right the cloud of zeros is upper bounded (essentially) by the line $\Re z = 1$ (see Fig. 1).

An explanation *grosso modo* why the zeros of the $\zeta_n(z)$'s are distributed of such a form is supported by the following facts:

(a) Any exponential polynomial (EP for short) of the form

$$P(z) := 1 + \sum_{j=1}^{N} a_j e^{-z\lambda_j}, \quad z \in \mathbb{C}, \quad a_j \in \mathbb{C} \setminus \{0\}, \quad 0 < \lambda_1 < \ldots < \lambda_N, \quad N \geq 1, \tag{1}$$

has zeros as a consequence of Hadamard's Factorization Theorem or from Pólya's Theorem [13, p. 71]. For $N = 1$, it is immediate that an EP of the form (1) has its zeros aligned. For $N > 1$, noticing that for any y,

$$\lim_{x \to +\infty} P(z) = \lim_{x \to -\infty} Q(z) = 1,$$

where $Q(z) := a_N^{-1} e^{z\lambda_N} P(z)$ (observe that $P(z)$ and $Q(z)$ have exactly the same zeros), it follows that the zeros of $P(z)$ are situated in a vertical strip. Therefore, for every EP $P(z)$ of the form (1), there exist two real numbers

$$a_{P(z)} := \inf\{\Re z : P(z) = 0\}, \quad b_{P(z)} := \sup\{\Re z : P(z) = 0\}, \tag{2}$$

that define an interval $[a_{P(z)}, b_{P(z)}]$, called *critical interval* associated with $P(z)$. Therefore the set $[a_{P(z)}, b_{P(z)}] \times \mathbb{R}$, called *critical strip* associated with $P(z)$, is the minimal vertical strip that contains all the zeros of $P(z)$.

It is immediate that any partial sum $\zeta_n(z) := \sum_{j=1}^{n} j^{-z}, n \geq 2$, is an EP of the form (1). Therefore the zeros of each $\zeta_n(z)$ are situated on its critical strip $[a_{\zeta_n(z)}, b_{\zeta_n(z)}] \times \mathbb{R}$ (a detailed proof on the existence of the zeros of $\zeta_n(z)$ and their distribution with respect to the imaginary axis can be found in [14, Prop. 1, 2, 3]). Regarding the bounds $a_{\zeta_n(z)}, b_{\zeta_n(z)}$, taking into account that all the zeros of $\zeta_2(z)$ lie on the imaginary axis, we get the property

$$a_{\zeta_2(z)} = b_{\zeta_2(z)} = 0; \quad a_{\zeta_n(z)} < 0 < b_{\zeta_n(z)}, \quad n > 2, \tag{3}$$

that will be proved below in Lemma 2.3, Part (ii). A much more precise estimation of such bounds is given by the formulas:

$$b_{\zeta_n(z)} = 1 + \left(\frac{4}{\pi} - 1 + o(1) \right) \frac{\log \log n}{\log n}, \quad n \to \infty, \tag{4}$$

obtained by Montgomery and Vaughan [12] in 2001, by completing a previous work of Montgomery [11] of 1983, and

$$a_{\zeta_n(z)} = -\frac{\log 2}{\log(\frac{n-1}{n-2})} + \Delta_n, \quad \limsup_{n \to \infty} |\Delta_n| \leq \log 2, \tag{5}$$

found by Mora [17] in 2015. Consequently, from (5) and (4), we have

$$\lim_{n \to \infty} a_{\zeta_n(z)} = -\infty, \quad \lim_{n \to \infty} b_{\zeta_n(z)} = 1,$$

what justifies the fact of the cloud of zeros of $\zeta_n(z)$ moves to the left as n increases but not to the right, where the cloud is upper bounded (essentially) by the line $\Re z = 1$ (it does not mean that some $\zeta_n(z)$ can have zeros with real part greater than 1; in fact, many works prove the existence of such zeros [10, 22, 23, 25], among others).

(b) Since the zeros of an analytic function are isolated, and all the $\zeta_n(z)$'s are entire functions, by taking into account the real parts of the zeros of each $\zeta_n(z)$ are bounded (the real parts are contained in the critical interval $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$ for every fixed n), their imaginary parts cannot be. Furthermore, as the coefficients of every $\zeta_n(z)$ are real, its zeros are conjugate. Consequently the zeros of the $\zeta_n(z)$'s are located up and down, symmetrically with respect to the real axis.

(c) From (3) we deduce that, for any $n > 2$, $\zeta_n(z)$ has zeros with positive and negative real parts.

With the aim to understand what law controls the distribution of the real projections of the zeros of $\zeta_n(z)$, we introduce a Fixed Point Theory focused on two real functions, f_n and g_n, for every $n > 2$. Firstly, such functions, by virtue of a recent result [19, Theorem 3], allow us to have an easy characterization of the sets

$$R_{\zeta_n(z)} := \overline{\{\Re z : \zeta_n(z) = 0\}}. \tag{6}$$

Secondly, among others relevant results deduced from the fixed point properties of f_n and g_n, we stress those that characterize some notable *arithmetic sets* such as $\mathscr{P}^*$ and $\mathscr{C}^*$, the set of primes greater than 2 and the set of composite numbers greater than 2, respectively. In this way, we show how close the arithmetic sets $\mathscr{P}^*$ and $\mathscr{C}^*$ from the law of the distribution of the zeros of the partial sums of the Riemann zeta function are. Furthermore, our point fixed theory proves the existence of a *minimal density interval* for each $\zeta_n(z)$, that is, a closed interval $[A_n, b_{\zeta_n(z)}]$, with $a_{\zeta_n(z)} \leq A_n < b_{\zeta_n(z)}$ contained in the set $R_{\zeta_n(z)}$, for any integer $n > 2$, which means that there is no vertical sub-trip contained in $[A_n, b_{\zeta_n(z)}] \times \mathbb{R}$ zero-free for $\zeta_n(z)$. Then, since it is always true that $R_{\zeta_n(z)} \subset [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$, when the bound A_n coincides with $a_{\zeta_n(z)}$ it follows that $R_{\zeta_n(z)} = [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$. In this case we will say that $\zeta_n(z)$ has a *maximum density interval*, and it is exactly the critical interval $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$. Finally, we will give a sufficient condition in terms of the quantity of fixed points of f_n for $\zeta_n(z)$ have a maximum density interval.

2 The Functions f_n and g_n

The functions f_n and g_n that we are going to introduce below, are directly linked to the interval of variation of the variable x of the Cartesian equation of an analytic variety associated with the nth-partial sum $\zeta_n(z) := \sum_{j=1}^{n} j^{-z}$, $n > 2$. First we consider the EP

$$\zeta_n^*(z) := \zeta_n(z) - p_{k_n}^{-z}, \quad n > 2, \tag{7}$$

where p_{k_n} is the last prime not exceeding n. The bounds $a_{\zeta_n^*(z)}, b_{\zeta_n^*(z)}$ defined in (2) corresponding to $\zeta_n^*(z)$ satisfy the following crucial property (for details see [16, Theorem 15]) :

$$a_{\zeta_n^*(z)} = b_{\zeta_n^*(z)} = 0, \quad \text{for } n = 3, 4; \quad a_{\zeta_n^*(z)} < 0 < b_{\zeta_n^*(z)}, \quad \text{for all } n > 4. \tag{8}$$

Now our objective is to analyse the behavior of the end-points of the interval of variation of the variable x of the analytic variety, or *level curve* [24, p. 121], of equation

$$|\zeta_n^*(z)| = p_{k_n}^{-c}, \quad n > 2, \quad c \in \mathbb{R}. \tag{9}$$

To do it, we square (9) and by using elementary formulas of trigonometry we obtain the Cartesian equation of $|\zeta_n^*(z)| = p_{k_n}^{-c}$, namely

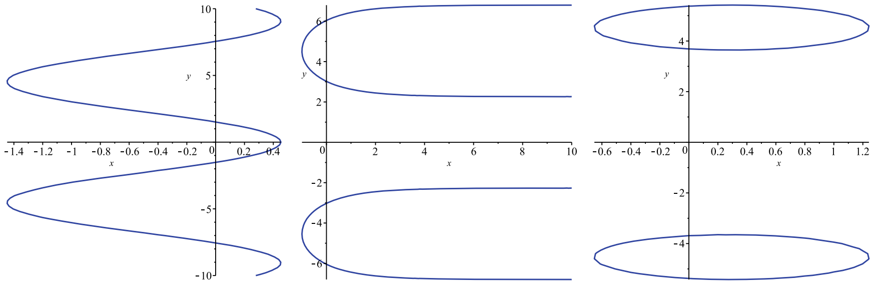
$$\begin{aligned} &\sum_{j=1, j\neq p_{k_n}}^{n} j^{-2x} + 2\cdot 1^{-x} \sum_{j=2, j\neq p_{k_n}}^{n} j^{-x}\cos(y\log(\frac{j}{1})) + \\ &2\cdot 2^{-x} \sum_{j=3, j\neq p_{k_n}}^{n} j^{-x}\cos(y\log(\frac{j}{2})) + \cdots + \\ &2(n-1)^{-x} \sum_{j=n, j\neq p_{k_n}}^{n} j^{-x}\cos(y\log(\frac{j}{n-1})) = p_{k_n}^{-2c}. \end{aligned} \tag{10}$$

It is immediate to see that for any value of y, the left-hand side of (10) tends to $+\infty$ as $x \to -\infty$. Then, as the right-hand side of (10) is a constant, the variation of x is always lower bounded by a number denoted by $a_{n,c}$. On the other hand, the limit of the left-hand side of (10) is 1 when $x \to +\infty$. Then, if $c \neq 0$, the variation of x is upper bounded by a number denoted by $b_{n,c}$. Therefore, fixed an integer $n > 2$, we have:

If $c \neq 0$, the variable x in the Eq. (10) varies on an open interval $(a_{n,c}, b_{n,c})$ satisfying the properties: (a) Given $x \in (a_{n,c}, b_{n,c})$, there is at least a point of the level curve $|\zeta_n^*(z)| = p_{k_n}^{-c}$ with abscissa x. Exceptionally $|\zeta_n^*(z)| = p_{k_n}^{-c}$ could have points of abscissas $a_{n,c}$, $b_{n,c}$. In this case we will say that $a_{n,c}$, $b_{n,c}$ are *accessible*. Otherwise the lines $x = a_{n,c}$, $x = b_{n,c}$ are asymptotes of the variety. (b) For $x < a_{n,c}$ or $x > b_{n,c}$ there is no point of the variety $|\zeta_n^*(z)| = p_{k_n}^{-c}$.

If $c = 0$, x varies on $(a_{n,0}, +\infty)$, so $b_{n,0}$ can be defined as $+\infty$, satisfying: (c) Given $x \in (a_{n,0}, +\infty)$, there is at least a point of the variety $|\zeta_n^*(z)| = 1$ with abscissa x. If there is a point of $|\zeta_n^*(z)| = 1$ with abscissa $a_{n,0}$, we will say that $a_{n,0}$ is accessible. Otherwise the line $x = a_{n,0}$ is an asymptote of the variety. (d) For $x < a_{n,0}$ there is no point of $|\zeta_n^*(z)| = 1$.

We show in Fig. 2 the varieties $|\zeta_n^*(z)| = p_{k_n}^{-c}$ for $n = 3$ and some values of c.



(a) Graph of $|\zeta_3^*(z)| = 3^{1/2}$. (b) Graph of $|\zeta_3^*(z)| = 1$. (c) Graph of $|\zeta_3^*(z)| = 3^{-1/2}$.

Fig. 2 Graphs of the varieties $|\zeta_3^*(z)| = 3^{-c}$ for some values of c

The end-points $a_{3,c}, b_{3,c}$ corresponding to the variety $|\zeta_3^*(z)| = p_{k_3}^{-c}$ can be easily determined by a completely similar way to those of the variety $|\zeta_3^*(-z)| = p_{k_3}^{c}$ (see [8, p. 49]). Each bound $a_{3,c}, b_{3,c}$ as a function of c is given by the formulas

$$a_{3,c} = -\frac{\log(1+3^{-c})}{\log 2}, \quad c \in \mathbb{R}; \quad b_{3,c} = \begin{cases} -\frac{\log(3^{-c}-1)}{\log 2}, & \text{if } c < 0 \\ -\frac{\log(1-3^{-c})}{\log 2}, & \text{if } c > 0 \end{cases}. \tag{11}$$

By virtue of above considerations (a), (b), (c), (d), and by using an elementary geometric reasoning, similar to that it was used to find the graphs of $|\zeta_n^*(-z)| = p_{k_n}^{c}$ (see [16, Proposition 8]), the graphs of the varieties $|\zeta_n^*(z)| = p_{k_n}^{-c}$ are described in the next result.

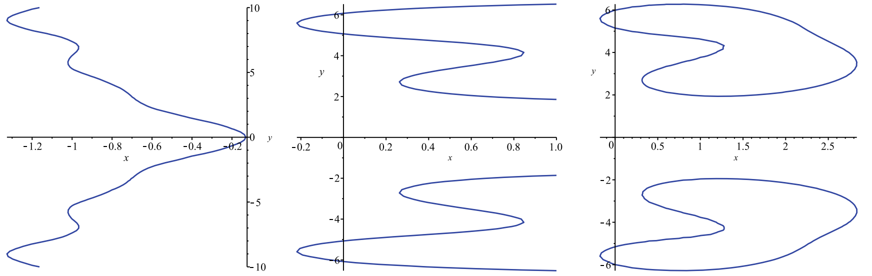
Proposition 2.1 *Fixed an integer $n > 2$, we have:*

(i) If $c > 0$, $|\zeta_n^*(z)| = p_{k_n}^{-c}$ has infinitely many arc-connected components which are closed curves and x varies on a finite interval $(a_{n,c}, b_{n,c})$, where $a_{n,c}, b_{n,c}$ could be accessible.

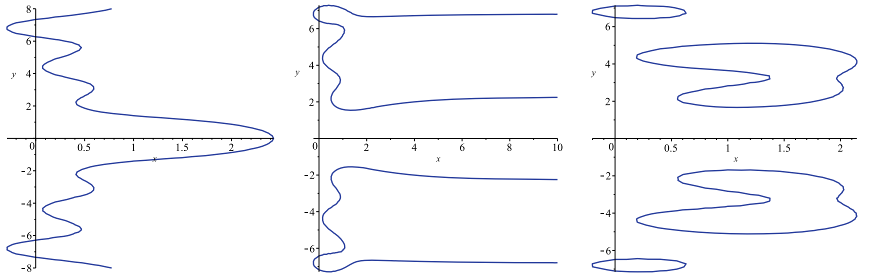
(ii) If $c = 0$, $|\zeta_n^*(z)| = 1$ has infinitely many arc-connected components which are open curves with horizontal asymptotes of equations $y = (2j+1)\frac{\pi}{2\log 2}$, $j \in \mathbb{Z}$, and x varies on the infinite interval $(a_{n,0}, +\infty)$, where $a_{n,0}$ could be accessible.

(iii) If $c < 0$, $|\zeta_n^*(z)| = p_{k_n}^{-c}$ has only one arc-connected component which is an open curve; x varies on a finite interval $(a_{n,c}, b_{n,c})$, where $a_{n,c}, b_{n,c}$ could be accessible. The variable y takes all real values. Furthermore, $|\zeta_n^*(z)| = p_{k_n}^{-c}$ intersects the real axis at a unique point of abscissa $b_{n,c}$, so $b_{n,c}$ is always accessible when $c < 0$.

In Fig. 3 we show the graph of $|\zeta_n^*(z)| = p_{k_n}^{-c}$ for some values of $n > 3$ and c. From Proposition 2.1, a simple consequence is deduced:



(a) Graph of $|\zeta_7^*(z)| = 7$. (b) Graph of $|\zeta_7^*(z)| = 1$. (c) Graph of $|\zeta_7^*(z)| = 7^{-1/15}$.



(d) Graph of $|\zeta_{12}^*(z)| = 11^{1/8}$. (e) Graph of $|\zeta_{12}^*(z)| = 1$. (f) Graph of $|\zeta_{12}^*(z)| = 11^{-1/12}$.

Fig. 3 Graphs of the varieties $|\zeta_7^*(z)| = 7^{-c}$ and $|\zeta_{12}^*(z)| = 11^{-c}$ for some values of c

Corollary 2.1 *Fixed an integer $n > 2$, if $u \in \mathbb{C}$ satisfies $|\zeta_n^*(u)| < p_{k_n}^{-c}$ (in this case we will say that u is an interior point of the variety $|\zeta_n^*(z)| = p_{k_n}^{-c}$), then there exists a point w of $|\zeta_n^*(z)| = p_{k_n}^{-c}$, so $a_{n,c} \leq \Re w \leq b_{n,c}$, such that $\Re w < \Re u$.*

Definition 2.1 Given an integer $n > 2$, we define the real functions

$$f_n(c) := a_{n,c}, \quad c \in \mathbb{R}; \qquad g_n(c) := b_{n,c}, \quad c \in \mathbb{R} \setminus \{0\}, \tag{12}$$

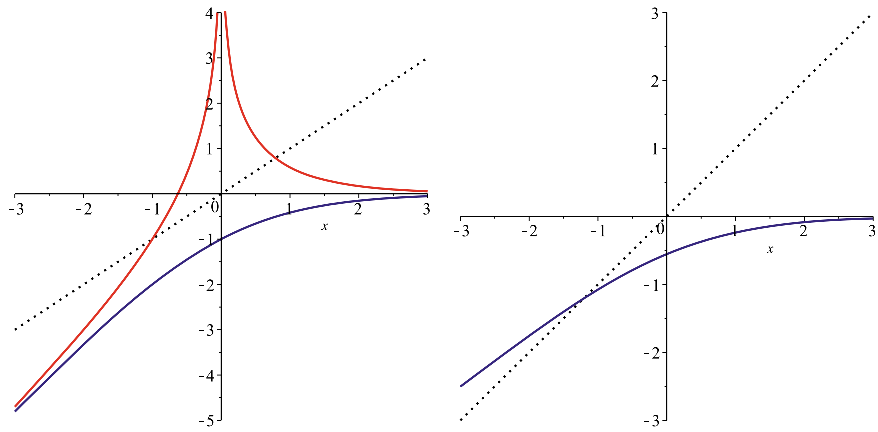
where $a_{n,c}, b_{n,c}$ are the end-points of the interval of variation of the variable x in the Eq. (10).

We show in Fig. 4 the graph of the functions $f_3(c)$ and $g_3(c)$, defined by the Eq. (11), and the function $f_4(c)$.

Since $|\zeta_n^*(z)| = p_{k_n}^{-d}$ tends to $|\zeta_n^*(z)| = p_{k_n}^{-c}$ as d tends to c, it is immediate that f_n, g_n are both continuous on $\mathbb{R} \setminus \{0\}$, and f_n is continuous on whole of $\mathbb{R}$. For $c = 0$, by Part (ii) of Proposition 1 we can agree $b_{n,0} = +\infty$, and then we should define $g_n(0) := +\infty$.

Now we are ready to give a characterization of the set $R_{\zeta_n(z)}$, defined in (6), by using the functions f_n and g_n.

Theorem 2.1 *Let $n > 2$ be a fixed integer. A real number $c \in R_{\zeta_n(z)}$ if and only if*



Graph of $f_3(c)$ and $g_3(c)$. Graph of $f_4(c)$.

Fig. 4 Left: Graph of the functions $f_3(c)$ (blue), $g_3(c)$ (red) and $y = x$ (plotted). Right: Graph of the function $f_4(c)$ (blue) and $y = x$ (plotted)

$$f_n(c) \le c \le g_n(c). \tag{13}$$

Proof If $c \in R_{\zeta_n(z)}$, there exists a sequence $(z_m)_{m=1,2,\ldots}$ of zeros of $\zeta_n(z)$ such that $\lim_{m\to\infty} \Re z_m = c$. From (7), $\zeta_n^*(z_m) = -p_{k_n}^{-z_m}$ for each $m = 1, 2, \ldots$. By taking the modulus, we have $|\zeta_n^*(z_m)| = p_{k_n}^{-x_m}$, where $x_m := \Re z_m$. This means that each z_m is a point of the variety $|\zeta_n^*(z)| = p_{k_n}^{-x_m}$, so $x_m \in [a_{n,x_m}, b_{n,x_m}]$ and then we get

$$f_n(x_m) = a_{n,x_m} \le x_m \le b_{n,x_m} = g_n(x_m), \quad \text{for all } m.$$

Now by taking the limit when $m \to \infty$, noticing that $\lim_{m\to\infty} x_m = c$, because of the continuity of f_n and g_n, the inequalities (13) follow. Conversely, if $f_n(c) < c < g_n(c)$, by taking into account the definitions of f_n, g_n, the value c is in the interval of variation of x of the variety $|\zeta_n^*(z)| = p_{k_n}^{-c}$ and then the line $x = c$ intersects the variety. Therefore, by applying [16, Theorem 3], $c \in R_{\zeta_n(z)}$. If $f_n(c) = c$ or $g_n(c) = c$, the line $x = c$ intersects the variety $|\zeta_n^*(z)| = p_{k_n}^{-c}$ provided that $a_{n,c}$ or $b_{n,c}$ be accessible. Otherwise the line $x = c$ is an asymptote of $|\zeta_n^*(z)| = p_{k_n}^{-c}$. Therefore, in both cases, again by [19, Theorem 3], the point $c \in R_{\zeta_n(z)}$. □

As we can easily check, the function $f_3(c) := a_{3,c}$, with $a_{3,c}$ given in (11), is strictly increasing; this property is true for all the functions $f_n(c)$, $n > 2$, defined in (12), as we prove below.

Lemma 2.1 *For every integer $n > 2$, f_n is a strictly increasing function on $\mathbb{R}$.*

Proof Firstly, for each fixed $c \in \mathbb{R}$, we claim that f_n satisfies

$$\inf\{|\zeta_n^*(f_n(c) + iy)| : y \in \mathbb{R}\} = p_{k_n}^{-c}. \tag{14}$$

Indeed, we put $\lambda_{n,c} := \inf\{|\zeta_n^*(f_n(c) + iy)| : y \in \mathbb{R}\}$. By assuming $\lambda_{n,c} < p_{k_n}^{-c}$, there exists a point $z_c := f_n(c) + iy_c$ such that

$$\lambda_{n,c} \leq |\zeta_n^*(z_c)| < p_{k_n}^{-c},$$

and then it means that z_c is an interior point of $|\zeta_n^*(z)| = p_{k_n}^{-c}$. By Corollary 2.1 there exists w belonging to the variety $|\zeta_n^*(z)| = p_{k_n}^{-c}$, so $a_{n,c} \leq \Re w \leq b_{n,c}$, such that $\Re w < \Re z_c = f_n(c) = a_{n,c}$. But this is a contradiction, and then necessarily

$$\lambda_{n,c} \geq p_{k_n}^{-c}. \tag{15}$$

For $\varepsilon > 0$ sufficiently small, we consider the strip

$$S_\varepsilon := \{z \in \mathbb{C} : a_{n,c} \leq \Re z < a_{n,c} + \varepsilon\},$$

and put

$$\lambda_{n,c,\varepsilon} := \inf\{|\zeta_n^*(z)| : z \in S_\varepsilon\}.$$

From the definition of $a_{n,c}$, the set S_ε contains infinitely many points of the variety $|\zeta_n^*(z)| = p_{k_n}^{-c}$. Then $\lambda_{n,c,\varepsilon} \leq p_{k_n}^{-c}$ for all $\varepsilon > 0$, so $\lambda_{n,c} \leq p_{k_n}^{-c}$. Therefore, according to (15), $\lambda_{n,c} = p_{k_n}^{-c}$ and then (14) follows. Let d be a real number such that $d < c$, so $p_{k_n}^{-d} > p_{k_n}^{-c}$. Let η be such that $0 < \eta < p_{k_n}^{-d} - p_{k_n}^{-c}$. From (14), there exists some point $z_\eta := f_n(c) + iy_\eta$ such that

$$p_{k_n}^{-c} \leq |\zeta_n^*(z_\eta)| < p_{k_n}^{-c} + \eta < p_{k_n}^{-d},$$

so z_η is interior of $|\zeta_n^*(z)| = p_{k_n}^{-d}$. By Corollary 2.1, there exists a point w_η of $|\zeta_n^*(z)| = p_{k_n}^{-d}$, so $a_{n,d} \leq \Re w_\eta \leq b_{n,d}$, such that $\Re w_\eta < \Re z_\eta$. Then

$$f_n(d) = a_{n,d} \leq \Re w_\eta < \Re z_\eta = f_n(c),$$

which definitely proves the lemma. □

In the next result we prove that f_n is upper bounded by the number $a_{\zeta_n^*(z)}$ defined in (2) corresponding to the EP $\zeta_n^*(z)$, defined in (7).

Lemma 2.2 *For every $n > 2$, the function f_n satisfies*

$$f_n(c) < a_{\zeta_n^*(z)} \textit{ for any } c \in \mathbb{R}.$$

Proof Let c be an arbitrary real number. By taking into account the definition of $a_{\zeta_n^*(z)}$, there exists a sequence $(z_m)_{m=1,2,\ldots}$ of zeros of $\zeta_n^*(z)$, with $\Re z_m \geq a_{\zeta_n^*(z)}$, such that

$$\lim_{m\to\infty} \Re z_m = a_{\zeta_n^*(z)}. \tag{16}$$

Since $\zeta_n^*(z_m)=0$, we get $|\zeta_n^*(z_m)|<p_{k_n}^{-c}$, for all m. Then, from Corollary 2.1, there exists a sequence $(w_m)_{m=1,2,\ldots}$ of points of $|\zeta_n^*(z)|=p_{k_n}^{-c}$, so $a_{n,c}\leq \Re w_m\leq b_{n,c}$, such that $\Re w_m<\Re z_m$, for all m. Therefore, since $f_n(c)=a_{n,c}$, we have

$$f_n(c)\leq \Re w_m<\Re z_m,\quad \text{for all } m.$$

Now, by taking the limit in the above inequality when $m\to\infty$, by (16), we get

$$f_n(c)\leq a_{\zeta_n^*(z)}\quad \text{for any } c\in\mathbb{R},$$

implying, noticing that by Lemma 2.1 f_n is strictly increasing, that $f_n(c)<a_{\zeta_n^*(z)}$ for any $c\in\mathbb{R}$. □

For every $n>2$, let $a_{\zeta_n(z)}$, $b_{\zeta_n(z)}$ be the bounds, defined in (2), corresponding to the EP $\zeta_n(z)$. The function g_n, defined in (12), has the following properties.

Lemma 2.3 *For every $n>2$, the function g_n satisfies:*

(i) *g_n is strictly increasing on $(-\infty,0)$ and decreasing on $(0,+\infty)$.*
(ii) *If n is composite, then $c\leq g_n(c)$ for any $c\in(-\infty,b_{\zeta_n(z)}]\setminus\{0\}$ and the inequality is strict for all $c\in(-\infty,b_{\zeta_n(z)})\setminus\{0\}$; if $c\in(b_{\zeta_n(z)},+\infty)$, then $c>g_n(c)$.*
(iii) *If n is prime, then $c\leq g_n(c)$ for any $c\in[a_{\zeta_n(z)},b_{\zeta_n(z)}]\setminus\{0\}$ and the inequality is strict for all $c\in(a_{\zeta_n(z)},b_{\zeta_n(z)})\setminus\{0\}$; if $c\in(-\infty,a_{\zeta_n(z)})\cup(b_{\zeta_n(z)},+\infty)$, then $c>g_n(c)$.*

Proof Part (i). Let c,d be real numbers such that $c<d<0$. From Proposition 2.1, $b_{n,c}$ and $b_{n,d}$ are the unique points of $|\zeta_n^*(z)|=p_{k_n}^{-c}$ and $|\zeta_n^*(z)|=p_{k_n}^{-d}$ that intersect the real axis, respectively. Therefore $b_{n,c}$ and $b_{n,d}$ satisfy the equations

$$\sum_{\substack{m=1\\ m\neq p_{k_n}}}^{n} m^{-x}=p_{k_n}^{-c},\qquad \sum_{\substack{m=1\\ m\neq p_{k_n}}}^{n} m^{-x}=p_{k_n}^{-d},\tag{17}$$

respectively. Each equation of (17) has only one real solution by virtue of [20, p. 46] and then, since $p_{k_n}^{-c}>p_{k_n}^{-d}$, the real solution of the first equation is obviously greater than the second one. Therefore $-b_{n,c}>-b_{n,d}$, equivalently, $b_{n,c}<b_{n,d}$. Consequently, $g_n(c)<g_n(d)$ and then g_n is strictly increasing in $(-\infty,0)$. Let c,d be such that $c>d>0$. From Proposition 2.1, $|\zeta_n^*(z)|=p_{k_n}^{-c}$ and $|\zeta_n^*(z)|=p_{k_n}^{-d}$ have infinitely many arc-connected components which are closed curves. Since $p_{k_n}^{-c}<p_{k_n}^{-d}$, any point of $|\zeta_n^*(z)|=p_{k_n}^{-c}$ is interior of $|\zeta_n^*(z)|=p_{k_n}^{-d}$, so $b_{n,c}\leq b_{n,d}$. That is, $g_n(c)\leq g_n(d)$, which means that g_n is decreasing on $(0,+\infty)$.

Part (ii). We firstly demonstrate that the bounds $a_{\zeta_n(z)}$, $b_{\zeta_n(z)}$ defined in (2) corresponding to $\zeta_n(z)$ satisfy the second inequality of (3), that is

$$a_{\zeta_n(z)}<0<b_{\zeta_n(z)}\quad \text{for all } n>2.\tag{18}$$

Indeed, we introduce the EP

$$G_n(z) := \zeta_n(-z). \tag{19}$$

In [7, Chap. 3, Theorem 3.20] was shown that

$$b_{G_n(z)} := \sup\{\Re z : G_n(z) = 0\} > 0 \quad \text{for all } n > 2,$$

now we claim that

$$a_{G_n(z)} := \inf\{\Re z : G_n(z) = 0\} < 0 \quad \text{for all } n > 2.$$

Otherwise, if all the zeros of $G_n(z)$, say $(z_{n,k})_{k=1,2,\ldots}$, satisfy $\Re z_{n,k} \geq 0$, since $b_{G_n(z)} > 0$, there is at least a zero z_{n,k_0} with $\Re z_{n,k_0} > 0$. Then, as $G_n(z)$ is almost-periodic (see for instance [4, 5] and [10, Chap. VI]), $G_n(z)$ has infinitely many zeros in the strip

$$S_\varepsilon := \{z : \Re z_{n,k_0} - \varepsilon < \Re z < \Re z_{n,k_0} + \varepsilon\}, \quad 0 < \varepsilon < \Re z_{n,k_0},$$

and consequently

$$\sum_{k=1}^{\infty} \Re z_{n,k} = +\infty. \tag{20}$$

However, as all the coefficients of $G_n(z)$ are equal to 1, [21, formula (9)] applies and then we get $\sum_{k=1}^{\infty} \Re z_{n,k} = O(1)$, contradicting (20). Therefore the claim follows, that is, $a_{G_n(s)} < 0$ for all $n > 2$. By (19) we have $a_{\zeta_n(z)} = -b_{G_n(z)}$ and $b_{\zeta_n(z)} = -a_{G_n(z)}$, so (18) follows.

We now consider the point $b_{\zeta_n(z)}$. It is immediate that $b_{\zeta_n(z)}$ belongs to the set $R_{\zeta_n(z)}$ defined in (6). Then from Theorem 2.1 we have $b_{\zeta_n(z)} \leq g_n(b_{\zeta_n(z)})$, so the property $c \leq g_n(c)$ is true for $c = b_{\zeta_n(z)}$. From (18) and by using that g_n is decreasing on $(0, \infty)$ by virtue of Part (i), for any $c \in (0, b_{\zeta_n(z)})$ we have

$$0 < c < b_{\zeta_n(z)} \leq g_n(b_{\zeta_n(z)}) \leq g_n(c). \tag{21}$$

Consequently, Part (ii) follows for $c \in (0, b_{\zeta_n(z)}]$. We now assume $c < 0$ and n composite, so $p_{k_n} < n$. If $b_{n,c} \geq 0$, then $c < b_{n,c} = g_n(c)$ and again Part (ii) is true. Finally, we suppose $b_{n,c} < 0$. Since $c < 0$, $b_{n,c}$ satisfies the first equation of (17) and then $p_{k_n}^{-c} > n^{-b_{n,c}}$. Consequently $-c > -b_{n,c}$, so $c < b_{n,c}$ and then Part (ii) follows for $c \in (-\infty, b_{\zeta_n(z)}] \setminus \{0\}$. Finally, we claim that $c > g_n(c)$ when $c > b_{\zeta_n(z)}$. Indeed, because of Lemma 2.2 and (8), we have $f_n(c) < a_{\zeta_n^*(z)} \leq 0$ for any c. Therefore, since $c > b_{\zeta_n(z)}$, by (18) c is positive and then $f_n(c) < c$. Assume $c > g_n(c)$ is not true. Then we would have $f_n(c) < c \leq g_n(c)$ and by Theorem 2.1, $c \in R_{\zeta_n(z)} \subset [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$ which means that $c \leq b_{\zeta_n(z)}$. This is a contradiction because $c > b_{\zeta_n(z)}$, so the claim follows. This definitely proves Part (ii).

Part (iii). We first note that, since n is prime, $p_{k_n} = n$. Therefore the first equation in (17) becomes $\sum_{m=1}^{n-1} m^{-x} = n^{-c}$. By assuming $c < 0$, $b_{n,c}$ satisfies the above equation and then we have

$$\sum_{m=1}^{n-1} m^{-b_{n,c}} = n^{-c}. \tag{22}$$

For every $n \geq 2$, we consider the number $\beta_{G_n(z)}$, defined as the unique real solution of the equation $\sum_{m=1}^{n-1} m^x = n^x$ (see [20, p. 46]). By [6, Proposition 5], $\beta_{G_n(z)} \geq b_{G_n(z)}$ and the equality is attained for n prime. Therefore the set $\mathbb{R}$ of real numbers is partitioned in two sets:

$$(-\infty, \beta_{G_n(z)}] = \{x \in \mathbb{R} : \sum_{m=1}^{n-1} m^x \geq n^x\}, \tag{23}$$

and

$$(\beta_{G_n(z)}, \infty) = \{x \in \mathbb{R} : \sum_{m=1}^{n-1} m^x < n^x\}. \tag{24}$$

Now we claim that $c \leq g_n(c)$ when $a_{\zeta_n(z)} \leq c < 0$. Indeed, by (19), $b_{G_n(z)} = -a_{\zeta_n(z)}$, so c is such that $0 < -c \leq b_{G_n(z)} = \beta_{G_n(z)}$. Then, according to (23), we have

$$\sum_{m=1}^{n-1} m^{-c} \geq n^{-c}. \tag{25}$$

Therefore, if we assume $c > g_n(c) = b_{n,c}$, by applying (25) and taking into account (22), we get

$$n^{-c} \leq \sum_{m=1}^{n-1} m^{-c} < \sum_{m=1}^{n-1} m^{-b_{n,c}} = n^{-c},$$

which is a contradiction. Therefore $c \leq g_n(c)$ is true for c such that $a_{\zeta_n(z)} \leq c < 0$. Consequently, taking into account (21), it follows

$$c \leq g_n(c), \text{ for any } c \in [a_{\zeta_n(z)}, b_{\zeta_n(z)}] \setminus \{0\},$$

where the inequality is strict for all c of $(a_{\zeta_n(z)}, b_{\zeta_n(z)}) \setminus \{0\}$. Now suppose $c \in (-\infty, a_{\zeta_n(z)})$. Then, since $-c > -a_{\zeta_n(z)} = b_{G_n(z)} = \beta_{G_n(z)}$, by applying (24) we have

$$\sum_{m=1}^{n-1} m^{-c} < n^{-c}. \tag{26}$$

It implies that $c > g_n(c)$. Indeed, by supposing $c \leq g_n(c) = b_{n,c}$, from (22) and (26) we are led to the following contradiction:

$$n^{-c} = \sum_{m=1}^{n-1} m^{-b_{n,c}} \leq \sum_{m=1}^{n-1} m^{-c} < n^{-c}.$$

Therefore $c > g_n(c)$ if $c \in (-\infty, a_{\zeta_n(z)})$. Finally, if $c \in (b_{\zeta_n(z)}, +\infty)$, the reasoning used to demonstrate the end of Part (ii) of the lemma proves that $c > g_n(c)$. □

As a consequence of Lemma 2.3 we find the fixed points of the function g_n.

Corollary 2.2 *For every composite number $n > 2$, $b_{\zeta_n(z)}$ is the fixed point of the function g_n. If $n > 2$ is prime, $a_{\zeta_n(z)}$, $b_{\zeta_n(z)}$ are the fixed points of g_n.*

Proof Fixed an integer $n > 2$, by (18) $a_{\zeta_n(z)}, b_{\zeta_n(z)} \neq 0$, so g_n is well defined at $a_{\zeta_n(z)}$ and $b_{\zeta_n(z)}$. By applying Part (ii) of Lemma 2.3 for $n > 2$ composite, it is immediate, by the continuity of g_n, that the unique fixed point of g_n is $b_{\zeta_n(z)}$. If $n > 2$ is prime, by Part (iii) of Lemma 2.3, we get $g_n(a_{\zeta_n(z)}) = a_{\zeta_n(z)}$ and $g_n(b_{\zeta_n(z)}) = b_{\zeta_n(z)}$. Furthermore, Part (iii) of Lemma 2.3 also proves that $a_{\zeta_n(z)}, b_{\zeta_n(z)}$ are the unique fixed points of g_n. □

In the next result we obtain a characterization of $\mathscr{P}^*$, the set of prime numbers greater than 2.

Theorem 2.2 *An integer $n > 2$ belongs to $\mathscr{P}^*$ if and only if $a_{\zeta_n(z)}$ is a fixed point of the function g_n.*

Proof Assume $n > 2$ is prime, from Corollary 2.2, $a_{\zeta_n(z)}$ is a fixed point of g_n. Conversely, if

$$g_n(a_{\zeta_n(z)}) = a_{\zeta_n(z)}, \tag{27}$$

by supposing n composite, from Part (ii) of Lemma 2.3, we have $c < g_n(c)$ for all $c \in (-\infty, b_{\zeta_n(z)}) \setminus \{0\}$. From (18), $a_{\zeta_n(z)} \in (-\infty, b_{\zeta_n(z)}) \setminus \{0\}$. Then, $a_{\zeta_n(z)} < g_n(a_{\zeta_n(z)})$. This contradicts (27). Consequently n is a prime number and then the theorem follows. □

3 The Fixed Points of f_n and the Sets $R_{\zeta_n(z)}$

For every integer $n > 2$, the function f_n defined in (12) allows us to give a sufficient condition to have points of the set $R_{\zeta_n(z)}$, defined in (6).

Theorem 3.1 *For every integer $n > 2$, if a point $c \in [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$ satisfies $f_n(c) \leq c$, then $c \in R_{\zeta_n(z)}$.*

Proof We first claim that

$$a_{\zeta_n(z)}, 0, b_{\zeta_n(z)} \in R_{\zeta_n(z)} \quad \text{for every } n \geq 2. \tag{28}$$

Indeed, for $n = 2$, the claim trivially follows because as we have seen in Introduction all the zeros of $\zeta_2(z)$ are imaginary, so $a_{\zeta_2(z)} = b_{\zeta_2(z)} = 0$ and then $R_{\zeta_2(z)} = \{0\}$. Therefore we assume $n > 2$. By taking into account the definitions of $a_{\zeta_n(z)}, b_{\zeta_n(z)}$, both numbers obviously belong to $R_{\zeta_n(z)}$. Regarding the fact that $0 \in R_{\zeta_n(z)}$for all $n > 2$, it was proved in [18, (3.7)]. Then (28) is true. Hence it only remains to prove the theorem for $c \in (a_{G_n(z)}, b_{G_n(z)}) \setminus \{0\}$. But in this case,since by hypothesis $f_n(c) \leq c$, by using Parts (ii) and (iii) of Lemma 2.3 we are lead to $f_n(c) \leq c < g_n(c)$ and then, by Theorem 2.1, $c \in R_{\zeta_n(z)}$. □

An important conclusion is deduced from the above theorem.

Theorem 3.2 *For every integer $n > 2$, if c belongs to $R_{\zeta_n(z)}$ then*

$$[f_n(c), c] \cap [a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}. \tag{29}$$

If $n > 2$ is composite and c belongs to $R_{\zeta_n(z)}$, then

$$[f_n(c), c] \subset R_{\zeta_n(z)}. \tag{30}$$

Proof Assume $c \in R_{\zeta_n(z)}$. Then, by Theorem 2.1, $f_n(c) \leq c \leq g_n(c)$. Therefore the interval $[f_n(c), c]$ is well defined. If $f_n(c) = c$ the theorem trivially follows. Suppose $f_n(c) < c$. Let t be a point of $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$ such that $f_n(c) < t < c$. By Lemma 2.1, $f_n(t) < f_n(c)$. Therefore we have

$$f_n(t) < f_n(c) < t < c,$$

and then, by applying Theorem 3.1, $t \in R_{\zeta_n(z)}$. Consequently

$$(f_n(c), c) \cap [a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)},$$

and from the closedness of $R_{\zeta_n(z)}$, (29) follows.

Assume $n > 2$ is composite. Since $c \in R_{\zeta_n(z)}$ and

$$R_{\zeta_n(z)} \subset [a_{\zeta_n(z)}, b_{\zeta_n(z)}],$$

we have $c \leq b_{\zeta_n(z)}$. Furthermore, from Theorem 2.1, $f_n(c) \leq c \leq g_n(c)$. Then, if $f_n(c) = c$, (30) is obviously true. Suppose $f_n(c) < c$. Consider a number t such that $f_n(c) \leq t < c$. Then, we get

$$f_n(c) \leq t < c \leq b_{\zeta_n(z)}. \tag{31}$$

If $t = 0$, by virtue of (28), $t \in R_{\zeta_n(z)}$. If $t \neq 0$, from (31), $t \in (-\infty, b_{\zeta_n(z)}) \setminus \{0\}$. Then, as n is composite, by Part (ii) of Lemma 2.3, $t < g_n(t)$. On the other hand, since $t < c$, from Lemma 2.1, $f_n(t) < f_n(c)$ and then, again by (31), we have

$$f_n(t) < f_n(c) \leq t < g_n(t).$$

Now, by applying Theorem 2.1, $t \in R_{\zeta_n(z)}$. Consequently $[f_n(c), c) \subset R_{\zeta_n(z)}$ and then, since by hypothesis $c \in R_{\zeta_n(z)}$, we get $[f_n(c), c] \subset R_{\zeta_n(z)}$. The proof is now completed. □

As a consequence of the two preceding results we characterize the set $\mathscr{C}^*$ of composite numbers $n > 2$.

Corollary 3.1 *For every $n \in \mathscr{C}^*$, $a_{\zeta_n(z)}$ is a fixed point of the function f_n.*

Proof Assume $n \in \mathscr{C}^*$. From (28), $a_{\zeta_n(z)} \in R_{\zeta_n(z)}$. Since n is composite and greater than 2, by (30) we have $[f_n(a_{\zeta_n(z)}), a_{\zeta_n(z)}] \subset R_{\zeta_n(z)}$. Noticing $R_{\zeta_n(z)} \subset [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$, necessarily $f_n(a_{\zeta_n(z)}) = a_{\zeta_n(z)}$. □

In the next result we prove that $a_{\zeta_n(z)}$ is not a fixed point of f_n for any $n \in \mathscr{P}^*$.

Corollary 3.2 *For every $n \in \mathscr{P}^*$, $f_n(a_{\zeta_n(z)}) < a_{\zeta_n(z)}$.*

Proof For every $n > 2$, the variety $|\zeta_n^*(z)| = p_{k_n}^{-c}$, for arbitrary $c \in \mathbb{R}$, by virtue of equation (10) is not contained in a vertical line, so the interval of the variation of the variable x in the variety $|\zeta_n^*(z)| = p_{k_n}^{-c}$ is not degenerate. Therefore, taking into account (12), we have

$$f_n(c) < g_n(c) \text{ for every integer } n > 2, \quad \text{for all } c \in \mathbb{R}. \tag{32}$$

Assume $n > 2$ prime. By Corollary 2.2, $g_n(a_{\zeta_n(z)}) = a_{\zeta_n(z)}$. Then, by taking $c = a_{\zeta_n(z)}$ in (32), the corollary follows. □

As a simple consequence from Corollary 3.2 we obtain a characterization of $\mathscr{C}^*$.

Theorem 3.3 *An integer $n > 2$ belongs to $\mathscr{C}^*$ if and only if $a_{\zeta_n(z)}$ is a fixed point of the function f_n.*

Proof From Corollary 3.1, if $n > 2$ is composite, $a_{\zeta_n(z)}$ is a fixed point of f_n. Reciprocally, if $a_{\zeta_n(z)}$ is a fixed point of f_n, by assuming $n > 2$ is not composite, by applying Corollary 3.2 we are led to a contradiction. Therefore, the theorem follows. □

The bounds $a_{\zeta_n(z)}, a_{\zeta_n^*(z)}$ satisfy the following inequality.

Proposition 3.1 *For every integer $n > 2$, $a_{\zeta_n(z)} < a_{\zeta_n^*(z)}$.*

Proof By taking $c = a_{\zeta_n^*(z)}$ in Lemma 2.2 we have

$$f_n(a^*_{\zeta_n(z)}) < a^*_{\zeta_n(z)} \text{ for all } n > 2. \tag{33}$$

Again from Lemma 2.2, for $c = a_{\zeta_n(z)}$, we get $f_n(a_{\zeta_n(z)}) < a^*_{\zeta_n(z)}$. If n is composite, by Corollary 3.1 $f_n(a_{\zeta_n(z)}) = a_{\zeta_n(z)}$ and from (33) we then deduce that $a_{\zeta_n(z)} < a^*_{\zeta_n(z)}$. This proves the proposition for n composite.

Assume n is prime. Then $p_{k_n} = n$ and, from (7), $\zeta_n^*(z) = \zeta_{n-1}(z)$, so $a_{\zeta_n^*(z)} = a_{\zeta_{n-1}(z)}$. Now we consider the function $G_n(z)$ defined in (19). As we have seen in the proof of Lemma 2.3, because of [6, Proposition 5] we have $b_{G_n(z)} \leq \beta_{G_n(z)}$ for all $n \geq 2$ and the equality is attained for n prime. Noticing [17, Lemma 1], $\beta_{G_{n-1}(z)} < \beta_{G_n(z)}$ for all $n > 2$. Then we get

$$b_{G_{n-1}(z)} \leq \beta_{G_{n-1}(z)} < \beta_{G_n(z)} = b_{G_n(z)}, \quad \text{for all prime } n > 2, \tag{34}$$

or equivalently

$$-b_{G_{n-1}(z)} \geq -\beta_{G_{n-1}(z)} > -\beta_{G_n(z)} = -b_{G_n(z)}, \quad \text{for all prime } n > 2.$$

Now, since from (19) $a_{\zeta_n(z)} = -b_{G_n(z)}$ for all $n \geq 2$, from the above chain of inequalities we deduce

$$a_{\zeta_n^*(z)} = a_{\zeta_{n-1}(z)} = -b_{G_{n-1}(z)} > -b_{G_n(z)} = a_{\zeta_n(z)}, \quad \text{for all prime } n > 2.$$

The proof is now completed. □

Corollary 3.3 *For every integer $n > 2$, $a^*_{\zeta_n(z)} \in R_{\zeta_n(z)}$.*

Proof For $n = 3, 4$, because of (8) we have $a^*_{\zeta_n(z)} = 0$. Therefore, from (28), $a^*_{\zeta_n(z)} \in R_{\zeta_n(z)}$ for $n = 3, 4$. Assume $n > 4$. By Proposition 3.1, $a_{\zeta_n(z)} < a_{\zeta_n^*(z)}$ for all $n > 2$. Then, from (8) and (18), $a^*_{\zeta_n(z)} \in [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$ for all $n > 4$. Therefore, by using (33) and applying Theorem 3.1, $a^*_{\zeta_n(z)} \in R_{\zeta_n(z)}$ for all $n > 4$. This proves the corollary. □

In the next result we prove the existence of a minimal density interval for every $\zeta_n(z), n > 2$.

Theorem 3.4 *For every integer $n > 2$ there exists a number $A_n \in [a_{\zeta_n(z)}, a_{\zeta_n^*(z)})$ such that $[A_n, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}$.*

Proof Firstly we note that, by Proposition 3.1, the interval $[a_{\zeta_n(z)}, a_{\zeta_n^*(z)})$ is well defined. On the other hand, by (18) $b_{\zeta_n(z)} > 0$ and, by (8) $a_{\zeta_n^*(z)} \leq 0$ for all $n > 2$, so by Proposition 3.1 we have

$$a_{\zeta_n(z)} < a_{\zeta_n^*(z)} \leq 0 < b_{\zeta_n(z)}, \quad \text{for all } n > 2. \tag{35}$$

This means that $[a_{\zeta_n^*(z)}, b_{\zeta_n(z)}]$ is a non-degenerate sub-interval of $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$ for any $n > 2$. By Lemma 2.2, we have $f_n(b_{\zeta_n(z)}) < a_{\zeta_n^*(z)}$. Then, according to (35), we get

$$f_n(b_{\zeta_n(z)}) \leq a_{\zeta_n^*(z)} < b_{\zeta_n(z)},$$

so

$$[a_{\zeta_n^*(z)}, b_{\zeta_n(z)}] \subset [f_n(b_{\zeta_n(z)}), b_{\zeta_n(z)}].$$

Now, since $b_{\zeta_n(z)} \in R_{\zeta_n(z)}$, because of Theorem 3.2 we obtain

$$[a_{\zeta_n^*(z)}, b_{\zeta_n(z)}] \subset [f_n(b_{\zeta_n(z)}), b_{\zeta_n(z)}] \cap [a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}. \tag{36}$$

This implies that $a_{\zeta_n^*(z)} \in R_{\zeta_n(z)}$ (observe that from Corollary 3.3 we already knew that $a_{\zeta_n^*(z)} \in R_{\zeta_n(z)}$)) so, again by Theorem 3.2, we have

$$[f_n(a_{\zeta_n^*(z)}), a_{\zeta_n^*(z)} \cap [a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}. \tag{37}$$

If $f_n(a_{\zeta_n^*(z)}) \leq a_{\zeta_n(z)}$, from (37) we deduce that $[a_{\zeta_n(z)}, a_{\zeta_n^*(z)}] \subset R_{\zeta_n(z)}$ and then, by (36) we get $[a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}$. In this case by taking $A_n = a_{\zeta_n(z)}$, the theorem follows. Moreover, $\zeta_n(z)$ has a maximum density interval and it coincides with its critical interval $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$.

If $f_n(a_{\zeta_n^*(z)}) > a_{\zeta_n(z)}$, from (37) we deduce

$$[f_n(a_{\zeta_n^*(z)}), a_{\zeta_n^*(z)}] \subset R_{\zeta_n(z)}. \tag{38}$$

Therefore $f_n(a_{\zeta_n^*(z)}) \in R_{\zeta_n(z)}$ and, again by Theorem 3.2, we have

$$[f_n^{(2)}(a_{\zeta_n^*(z)}), f_n(a_{\zeta_n^*(z)})] \cap [a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}, \tag{39}$$

where $f_n^{(2)}$ denotes f_n composed with itself. Then, if $f_n^{(2)}(a_{\zeta_n^*(z)}) \leq a_{\zeta_n(z)}$, from (39), we have $[a_{\zeta_n(z)}, f_n(a_{\zeta_n^*(z)})] \subset R_{\zeta_n(z)}$ and by (38), we get $[a_{\zeta_n(z)}, a_{\zeta_n^*(z)}] \subset R_{\zeta_n(z)}$. Therefore taking into account (36) we obtain $[a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}$. Consequently, by taking $A_n = a_{\zeta_n(z)}$, the theorem follows and $\zeta_n(z)$ has a maximum density interval that coincides with its critical interval $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$. If $f_n^{(2)}(a_{\zeta_n^*(z)}) > a_{\zeta_n(z)}$, from (39), we get

$$[f_n^{(2)}(a_{\zeta_n^*(z)}), f_n(a_{\zeta_n^*(z)})] \subset R_{\zeta_n(z)}.$$

Therefore $f_n^{(2)}(a_{\zeta_n^*(z)}) \in R_{\zeta_n(z)}$ and, again by Theorem 3.2, we have

$$[f_n^{(3)}(a_{\zeta_n^*(z)}), f_n^{(2)}(a_{\zeta_n^*(z)})] \cap [a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)},$$

and so on. Therefore, by denoting $f_n^{(k)} = f_n^{(k-1)} \circ f_n$ for $k \geq 2$ and repeating the process above, we are led to one of the two cases:

(i) There is some $k \geq 1$ such that $f_n^{(k)}(a_{\zeta_n^*(z)}) \leq a_{\zeta_n(z)}$. In this case, as we have seen $A_n = a_{\zeta_n(z)}$ and then $\zeta_n(z)$ has a maximum density interval that coincides with its critical interval $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$.

(ii) For all k, $f_n^{(k)}(a_{\zeta_n^*(z)}) > a_{\zeta_n(z)}$ and then, by virtue of Lemma 2.1 and (33), we have

$$a_{\zeta_n(z)} < \cdots < f_n^{(k)}(a_{\zeta_n^*(z)}) < \cdots < f_n^{(2)}(a_{\zeta_n^*(z)}) < f_n(a_{\zeta_n^*(z)}) < a_{\zeta_n^*(z)}.$$

Consequently there exists $\lim_{k\to\infty} f_n^{(k)}(a_{\zeta_n^*(z)})$ and then, by defining

$$A_n := \lim_{k\to\infty} f_n^{(k)}(a_{\zeta_n^*(z)}),$$

we have $a_{\zeta_n(z)} \leq A_n < a_{\zeta_n^*(z)}$. On the other hand, by reiterating Theorem 3.2, we get

$$[f_n^{(k)}(a_{\zeta_n^*(z)}), f_n^{(k-1)}(a_{\zeta_n^*(z)})] \subset R_{\zeta_n(z)}, \text{ for all } k \geq 2. \tag{40}$$

Then taking into account (36) and (38), by (40) we deduce that $[A_n, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}$. This definitely proves the theorem. □

Remark 3.5 Observe that if the case (ii) of above theorem holds, A_n will be a fixed point of f_n by virtue of the continuity of f_n. Then if $n \in \mathscr{C}^*$, by Theorem 14, the point A_n could be $a_{\zeta_n(z)}$. But if $n \in \mathscr{P}^*$, from Corollary 3.2, A_n can not be equal to $a_{\zeta_n(z)}$.

In the next result we prove that the number of fixed points of f_n influences on the existence of a maximum density interval of $\zeta_n(z)$.

Theorem 3.6 *For every integer $n > 2$, if f_n has at most a fixed point in the interval $(a_{\zeta_n(z)}, a_{\zeta_n^*(z)})$ then $\zeta_n(z)$ has a maximum density interval that coincides with the critical interval $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$ associated with $\zeta_n(z)$.*

Proof We first assume f_n has no fixed point in $(a_{\zeta_n(z)}, a_{\zeta_n^*(z)})$. Then we claim that $f_n(c) < c$ for all $c \in (a_{\zeta_n(z)}, a_{\zeta_n^*(z)}]$. Indeed, we define the function $h_n(c) := f_n(c) - c$. Then h_n is continuous on $\mathbb{R}$, and by virtue of Lemma 2.2 and (33), h_n is negative on $[a_{\zeta_n^*(z)}, \infty)$. Then, since f_n by hypothesis has no fixed point on $(a_{\zeta_n(z)}, a_{\zeta_n^*(z)})$, $h_n(c)$ has no zero on $(a_{\zeta_n(z)}, \infty)$. Consequently, $h_n(c) < 0$ for any $c \in (a_{\zeta_n(z)}, \infty)$ and in particular we have

$$f_n(c) < c \text{ for all } c \in (a_{\zeta_n(z)}, a_{\zeta_n^*(z)}]. \tag{41}$$

Hence the claim follows. On the other hand, by Corollary 3.3 $a_{\zeta_n^*(z)} \in R_{\zeta_n(z)} \subset [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$, so

$$(a_{\zeta_n(z)}, a_{\zeta_n^*(z)}] \subset [a_{\zeta_n(z)}, b_{\zeta_n(z)}].$$

Consequently, by taking into account (41) and by applying Theorem 3.1 we have

$$(a_{\zeta_n(z)}, a_{\zeta_n^*(z)}] \subset R_{\zeta_n(z)}.$$

Therefore, since from (28) $a_{\zeta_n(z)} \in R_{\zeta_n(z)}$, we get $[a_{\zeta_n(z)}, a_{\zeta_n^*(z)}] \subset R_{\zeta_n(z)}$ and then by (36) it follows that $[a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}$. As always is true that $R_{\zeta_n(z)} \subset [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$ we deduce that $R_{\zeta_n(z)} = [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$, i.e. $\zeta_n(z)$ has a maximum density interval. Then the theorem follows in this case.

We now suppose f_n has only one fixed point, say c_1, in $(a_{\zeta_n(z)}, a_{\zeta_n^*(z)})$. Then the function $h_n(c) := f_n(c) - c$, continuous on $\mathbb{R}$, is non-positive on $[c_1, +\infty)$ by virtue of Lemma 2.2. Therefore, in particular, $f_n(c) \leq c$ for all $c \in [c_1, a_{\zeta_n^*(z)}]$. Since $[c_1, a_{\zeta_n^*(z)}] \subset [a_{\zeta_n(z)}, b_{\zeta_n(z)}]$, by applying the Theorem 3.1 at any $c \in [c_1, a_{\zeta_n^*(z)}]$ we have

$$[c_1, a_{\zeta_n^*(z)}] \subset R_{\zeta_n(z)}. \tag{42}$$

Now we claim that h_n is negative on $(a_{\zeta_n(z)}, c_1)$. Indeed, if we assume that h_n is non-negative on $(a_{\zeta_n(z)}, c_1)$, since c_1 is the unique fixed point of f_n in $(a_{\zeta_n(z)}, a_{\zeta_n^*(z)})$, then $f_n(c) > c$ for all $c \in (a_{\zeta_n(z)}, c_1)$. Then, by Theorem 2.1, $c \notin R_{\zeta_n(z)}$ for all $c \in (a_{\zeta_n(z)}, c_1)$. This means that $\zeta_n(z)$ has no zero on the strip $(a_{\zeta_n(z)}, c_1) \times \mathbb{R}$. But, taking into account that $a_{\zeta_n(z)} \in R_{\zeta_n(z)}$, $a_{\zeta_n(z)}$ would be an isolated point of $R_{\zeta_n(z)}$ and it contradicts [2, Corollary 3.2]. Therefore the claim follows. Consequently, $f_n(c) < c$ for all $c \in (a_{\zeta_n(z)}, c_1)$ and then, by Theorem 3.1, $(a_{\zeta_n(z)}, c_1) \subset R_{\zeta_n(z)}$. From the closedness of $R_{\zeta_n(z)}$, we have

$$[a_{\zeta_n(z)}, c_1] \subset R_{\zeta_n(z)}. \tag{43}$$

Then, from (43), (42) and (36) we deduce that $[a_{\zeta_n(z)}, b_{\zeta_n(z)}] \subset R_{\zeta_n(z)}$. Consequently, $\zeta_n(z)$ has a maximum density interval and it coincides with its critical interval $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$. □

As a first application of the usefulness of Theorem 3.6 we prove a result on $\zeta_3(z)$ (the same result can be also deduced from others methods as we can see in [13, 15]).

Corollary 3.4 *$\zeta_3(z)$ has a maximum density interval and it coincides with its critical interval $[a_{\zeta_3(z)}, b_{\zeta_3(z)}]$.*

Proof The function $f_3(c) := a_{3,c}$ is explicitly given by the formula (11). Then it is immediate to check that $f_3(c) < c$ for all $c \in \mathbb{R}$. Therefore $f_3(c)$ has no fixed point and then, from Theorem 3.6, $\zeta_3(z)$ has a maximum density interval and it coincides with $[a_{\zeta_3(z)}, b_{\zeta_3(z)}]$. □

4 The Fixed Point Theory and the Maximum Density Interval for $\zeta_n(z)$

In this section our aim is to give a very useful result (see below Lemma 4.1) based on Kronecker Theorem [8, Theorem 444] that allows us to apply our fixed point theory to prove the existence of a maximum density interval.

Let $\mathscr{P} := \{p_j : j = 1, 2, 3, \ldots\}$ be the set of prime numbers and $U := \{1, -1\}$. For every map $\delta : \mathscr{P} \to U$, we define the function $\omega_\delta : \mathbb{N} \to U$ as

$$\omega_\delta(1) := 1, \quad \omega_\delta(m) := (\delta(p_{k_1}))^{\alpha_1} \ldots (\delta(p_{k_{l(m)}}))^{\alpha_{l(m)}}, \quad m > 1, \tag{44}$$

where $(p_{k_1})^{\alpha_1} \ldots (p_{k_{l(m)}})^{\alpha_{l(m)}}$, with $\alpha_1, \ldots, \alpha_{l(m)} \in \mathbb{N}$, is the decomposition of m in prime factors. Let Ω be the set of all the ω_δ's defined in (44). Observe that all functions of Ω are *completely multiplicative* (see for instance [1, p. 138]).

Lemma 4.1 *Let $n > 2$ a fixed integer, p_{k_n} the last prime not exceeding n and f_n defined in (12). Given an arbitrary $\omega_\delta \in \Omega$, the inequality*

$$p_{k_n}^{-c} \leq \Big|\sum_{\substack{m=1\\ m\neq p_{k_n}}}^{n} \omega_\delta(m) m^{-f_n(c)}\Big|, \tag{45}$$

holds for all $c \in \mathbb{R}$.

Proof Because of (7), $\zeta_n^*(z) := \sum_{m=1, m\neq p_{k_n}}^{n} m^{-z}$. Therefore, given $c \in \mathbb{R}$ we have

$$\zeta_n^*(f_n(c)+iy) = \sum_{\substack{m=1\\ m\neq p_{k_n}}}^{n} m^{-f_n(c)}(\cos(y\log m) - i\sin(y\log m)).$$

Then taking into account (14),

$$p_{k_n}^{-c} \leq |\sum_{\substack{m=1\\ m\neq p_{k_n}}}^{n} m^{-f_n(c)}(\cos(y\log m) - i\sin(y\log m))|, \text{ for all } y \in \mathbb{R}. \tag{46}$$

Given $n > 2$, we define $J_n := \{1, 2, 3, \ldots \pi(n)\}$, where $\pi(n)$ denotes the number of prime numbers not exceeding n. As the set $\{\log p_j : j \in J_n\}$ is rationally independent, the set $\{\frac{\log p_j}{2\pi} : j \in J_n\}$ is also rationally independent. Then by Kronecker Theorem [8, Theorem 444] fixed an arbitrary set of real numbers $\{\gamma_j : j \in J_n\}$ and given an integer $N \geq 1$, there exists a real number $y_N > N$ and integers $m_{j,N}$, such that

$$|y_N \frac{\log p_j}{2\pi} - m_{j,N} - \gamma_j| < \frac{1}{N}, \quad \text{for all } j \in J_n. \tag{47}$$

For each $n > 2$, we define the set $\mathscr{P}_n := \{p_j \in \mathscr{P} : p_j \leq n\}$. Then, given a mapping $\delta := \mathscr{P}_n \to U$, we consider the set $\{\gamma_j : j \in J_n\}$ where $\gamma_j = 1$ for those j such that $\delta(p_j) = 1$ and $\gamma_j = 1/2$ for those j such that $\delta(p_j) = -1$. Then by applying the aforementioned Kronecker Theorem for $N = 1, 2 \ldots$, we can determine a sequence $(y_N)_N$ satisfying, by virtue of (47), that

$$\cos(y_N \log p_j) \to 1, \quad \sin(y_N \log p_j) \to 0 \text{ as } N \to \infty, \text{ for } p_j \text{ with } \delta(p_j) = 1,$$

and

$$\cos(y_N \log p_j) \to -1, \quad \sin(y_N \log p_j) \to 0 \text{ as } N \to \infty, \text{ for } p_j \text{ with } \delta(p_j) = -1.$$

Therefore for each m such that $1 \leq m \leq n$ we get

$$\cos(y_N \log m) \to \omega_\delta(m), \quad \sin(y_N \log m) \to 0 \quad \text{as } N \to \infty. \tag{48}$$

Now, we substitute y by y_N in (46) and we take the limit as $N \to \infty$. Then, according to (48), the inequality (45) follows. □

Theorem 4.1 *For all prime numbers* $n > 2$ *except at most for a finite quantity,* f_n *has no fixed point in the interval* $(a_{\zeta_n(z)}, a_{\zeta_n^*(z)})$.

Proof Corollary 3.4 proves the theorem for $n = 3$. Assume $n > 3$ prime. The numbers $n - 2$ and $n - 1$ are relatively primes and both cannot be perfect squares, so there exists $\omega_\delta \in \Omega$ such that $\omega_\delta(n-2)\omega_\delta(n-1) = -1$. Since n is prime, $a_{\zeta_n^*(z)} = a_{\zeta_{n-1}(z)}$ and $p_{k_n} = n$. By supposing the existence of a fixed point $c_n \in (a_{\zeta_n(z)}, a_{\zeta_{n-1}(z)})$ for the function f_n for infinitely many prime $n > 3$, we are led to the following contradiction:

By (45) we have

$$n^{-c_n} \leq |\pm((n-1)^{-c_n} - (n-2)^{-c_n}) + \sum_{m \in P_{n-3,\omega_\delta}} m^{-c_n} - \sum_{m \notin P_{n-3,\omega_\delta}} m^{-c_n}|, \tag{49}$$

where, for a fixed integer $n > 2$ and $\omega_\delta \in \Omega$, the set P_{n,ω_δ} is defined as

$$P_{n,\omega_\delta} := \{m : 1 \leq m \leq n \text{ such that } \omega_\delta(m) = 1\}.$$

On the other hand, $\lim_{n\to\infty} \frac{a_{\zeta_n(z)}}{n} = -\log 2$ (see [3, Theorem 1] and [17, Theorem 2]). Then noticing that $a_{\zeta_n(z)} < c_n < a_{\zeta_{n-1}(z)}$, we get

$$\lim_{\substack{n \text{ prime} \\ n\to\infty}} \frac{c_n}{n-1} = -\log 2.$$

Therefore, for each fixed $j \geq 0$, it follows

$$\lim_{\substack{n \text{ prime} \\ n\to\infty}} \left(\frac{n-j}{n-1}\right)^{-c_n} = 2^{-j+1}. \tag{50}$$

Now, dividing by $(n-1)^{-c_n}$ the inequality (49), we have

$$\begin{aligned} \left(\frac{n}{n-1}\right)^{-c_n} &\leq \left| \pm\left(1 - \left(\frac{n-2}{n-1}\right)^{-c_n}\right) \right. \\ &\quad \left. + \sum_{m \in P_{n-3,\omega_\delta}} \left(\frac{m}{n-1}\right)^{-c_n} - \sum_{m \notin P_{n-3,\omega_\delta}} \left(\frac{m}{n-1}\right)^{-c_n} \right| \\ &\leq \left| \pm\left(1 - \left(\frac{n-2}{n-1}\right)^{-c_n}\right)\right| \\ &\quad + \left| \sum_{m \in P_{n-3,\omega_\delta}} \left(\frac{m}{n-1}\right)^{-c_n} - \sum_{m \notin P_{n-3,\omega_\delta}} \left(\frac{m}{n-1}\right)^{-c_n} \right| \\ &\leq \left(1 - \left(\frac{n-2}{n-1}\right)^{-c_n}\right) + \sum_{j=3}^{n-1} \left(\frac{n-j}{n-1}\right)^{-c_n}. \end{aligned} \tag{51}$$

According to (50), by taking the limit in (51) for n prime, $n \to \infty$, it follows that the limit of the left-hand side of (51) is 2 whereas the limit of the right-hand side one is $1/2 + \sum_{j=3}^{\infty} 2^{-j+1} = 1$. This is the contradiction desired. Hence the theorem follows. □

As a consequence from Theorem 4.1, an important property of the partial sums of order n prime can be deduced.

Theorem 4.2 *For all prime numbers $n > 2$ except at most for a finite quantity, $\zeta_n(z)$ has a maximum density interval and it coincides with its critical interval $[a_{\zeta_n(z)}, b_{\zeta_n(z)}]$.*

Proof It is enough to apply Theorems 3.6 and 4.1. □

5 Numerical Experiences

Simple numerical experiences carried out for some values of n in inequality (45) joint with the application of Theorem 3.6 and Lemma 4.1, allows us to prove the existence of a maximum density interval of $\zeta_n(z)$ for all $2 \leq n \leq 8$. Indeed: For $n = 2$, we have already seen in the Introduction section that the zeros of $\zeta_2(z)$ are all imaginary, so the set $R_{\zeta_2(z)} = \{0\}$ and then $a_{\zeta_2(z)} = b_{G_2(z)} = 0$ which means that we trivially have

$$R_{\zeta_2(z)} = [a_{\zeta_2(z)}, b_{\zeta_2(z)}].$$

Therefore $\zeta_2(z)$ has a maximum density interval (in this case degenerate).

For $n = 3$, Corollary 3.4 proves that

$$R_{\zeta_3(z)} = [a_{\zeta_3(z)}, b_{\zeta_3(z)}]$$

and then $\zeta_3(z)$ has a maximum density interval. In this case the end-points $a_{\zeta_3(z)}$, $b_{\zeta_3(z)}$ can be easily computed, being $a_{\zeta_3(z)} = -1$ and $b_{\zeta_3(z)} \approx 0.79$. Thus, $R_{\zeta_3(z)} \approx [-1, 0.79]$.

For $n = 4$, we firstly claim that f_4 has no fixed point in the interval $(a_{\zeta_4(z)}, a_{\zeta_4^*(z)})$. Indeed, by (8), $a_{\zeta_4^*(z)} = 0$ and from (18), $a_{\zeta_4(z)} < 0$. Therefore we only study the behavior of $f_4(c)$ for $c < 0$. We recall that from (12) $f_4(c) = a_{4,c}$, where $a_{4,c}$ is the left end-point of the interval of variation of the variable x in the Cartesian equation of the variety $|\zeta_4^*(z)| = p_{k_4}^{-c}$. By taking into account formula (10) for $n = 4$, the equation of that variety is

$$1 + 2^{-2x} + 4^{-2x} + 2 \cdot 2^{-x}(1 + 4^{-x}) \cos(y \log 2) + 2 \cdot 4^{-x} \cos(y \log 4)) = 3^{-2c}. \tag{52}$$

By putting $\cos(y \log 4)) = 2\cos^2(y \log 2) - 1$ in (52) and solving it for $\cos(y \log 2)$ we have

$$\cos(y \log 2) = \frac{-(1 + 4^{-x}) \pm \sqrt{(2 \cdot 3^{-c})^2 - (\sqrt{3}(4^{-x} - 1))^2}}{4 \cdot 2^{-x}}.$$

Then the variable x must satisfy the inequality $(\sqrt{3}(4^{-x}-1))^2 \leq (2\cdot 3^{-c})^2$ which is equivalent to say that

$$4^{-x} \in [1-2\cdot 3^{-c-\frac{1}{2}}, 1+2\cdot 3^{-c-\frac{1}{2}}]. \tag{53}$$

Since $1-2\cdot 3^{-c-\frac{1}{2}} < 0$ for all $c<0$, by noting that $4^{-x}>0$ for any x, (53) is in turn equivalent to

$$-\frac{\log(1+2\cdot 3^{-c-\frac{1}{2}})}{\log 4} \leq x.$$

Hence the minimum value for x is $-\frac{\log(1+2\cdot 3^{-c-\frac{1}{2}})}{\log 4}$, so $a_{4,c} = -\frac{\log(1+2\cdot 3^{-c-\frac{1}{2}})}{\log 4}$ and consequently for $c<0$ the function $f_4(c)$ is given by the formula

$$f_4(c) = -\frac{\log(1+2\cdot 3^{-c-\frac{1}{2}})}{\log 4}.$$

Then the fixed points of $f_4(c)$ are the solutions of the equation $f_4(c)=c$, that is

$$1+2\cdot 3^{-c-1/2} = 4^{-c}. \tag{54}$$

According to [20, p. 46] Eq. (54) has a unique real solution, say c_0, whose approached value is -1.21. On the other hand, since $n=4$ belongs to $\mathscr{C}^*$, by Theorem 3.3 $a_{\zeta_4(z)}$ is a fixed point of the function f_4. Since c_0 is the unique solution of $f_4(c)=c$, necessarily $a_{\zeta_4(z)} = c_0 \approx -1.21$ and then f_4 has no fixed point in $(a_{\zeta_4(z)}, a_{\zeta_4^*(z)})$. Hence the claim follows. Then, by applying Theorem 3.6, $\zeta_4(z)$ has a maximum density interval and consequently

$$R_{\zeta_4(z)} = [a_{\zeta_4(z)}, b_{\zeta_4(z)}].$$

For $n=5$ we take a mapping $\delta: \mathscr{P} \to U$ satisfying $\delta(2)=\delta(3)=-1$ and consider its corresponding $\omega_\delta: \mathbb{N} \to U$ defined in (44). Assume f_5 has some fixed point, say c_0, in the interval $(a_{\zeta_5(z)}, a_{\zeta_5^*(z)})$. By (8) $a_{\zeta_5^*(z)} < 0$ and then $(a_{\zeta_5(z)}, a_{\zeta_5^*(z)})$ contains only negative numbers, so $c_0<0$. By applying (45) for $n=5$, f_5 and the above defined ω_δ, under the assumption $f_5(c_0)=c_0$, we have

$$5^{-c_0} \leq |1-2^{-c_0}-3^{-c_0}+4^{-c_0}|.$$

But this inequality is clearly impossible for any $c_0<0$. Hence f_5 has no fixed point in $(a_{\zeta_5(z)}, a_{\zeta_5^*(z)})$. Then, by applying Theorem 3.6, $\zeta_5(z)$ has a maximum density interval and consequently

$$R_{\zeta_5(z)} = [a_{\zeta_5(z)}, b_{\zeta_5(z)}].$$

For $n=6$, we take a mapping $\delta: \mathscr{P} \to U$ satisfying $\delta(2)=-1$, $\delta(3)=1$ and consider its corresponding $\omega_\delta: \mathbb{N} \to U$ defined in (44). Assume f_6 has some fixed point, say c_0, in the interval $(a_{\zeta_6(z)}, a_{\zeta_6^*(z)})$. By (8) $a_{\zeta_6^*(z)} < 0$ and then $(a_{\zeta_6(z)}, a_{\zeta_6^*(z)})$ contains only negative numbers, so $c_0 < 0$. By applying (45) for $n = 6$, f_6 and the above defined ω_δ, under the assumption $f_6(c_0) = c_0$, we have

$$5^{-c_0} \leq |1 - 2^{-c_0} + 3^{-c_0} + 4^{-c_0} - 6^{-c_0}|. \tag{55}$$

Regarding inequality (55) we consider the two possible cases: (a) $1 - 2^{-c_0} + 3^{-c_0} + 4^{-c_0} - 6^{-c_0} \geq 0$, (b) $1 - 2^{-c_0} + 3^{-c_0} + 4^{-c_0} - 6^{-c_0} < 0$. In (a), according to (55), we have the inequality

$$1 + 3^{-c_0} + 4^{-c_0} \geq 2^{-c_0} + 5^{-c_0} + 6^{-c_0},$$

that as we easily can check is not possible for any $c_0 < 0$. In (b), because of (55), we get

$$1 + 3^{-c_0} + 4^{-c_0} + 5^{-c_0} \leq 2^{-c_0} + 6^{-c_0}. \tag{56}$$

By a direct computation we see that (56) is only true for $c_0 \leq a_{\zeta_6(z)} \approx -2.8$ (observe that for $c_0 \approx -2.8$, inequality (56) becomes an equality and since $n = 6$ belongs to C^*, by Theorem 3.3, $a_{\zeta_6(z)}$ is a fixed point of the function f_6). Therefore for $c_0 > a_{\zeta_6(z)}$, (56) is not possible. Hence f_6 has no fixed point in $(a_{\zeta_6(z)}, a_{\zeta_6^*(z)})$. Then, by applying Theorem 3.6, $\zeta_6(z)$ has a maximum density interval and consequently

$$R_{\zeta_6(z)} = [a_{\zeta_6(z)}, b_{\zeta_6(z)}].$$

For $n = 7$, we take a mapping $\delta : \mathscr{P} \to U$ satisfying $\delta(2) = \delta(3) = \delta(5) = -1$ and consider its corresponding $\omega_\delta : \mathbb{N} \to U$ defined in (44). Assume f_7 has some fixed point, say c_0, in the interval $(a_{\zeta_7(z)}, a_{\zeta_7^*(z)})$. By (8) $a_{\zeta_7^*(z)} < 0$ and then $(a_{\zeta_7(z)}, a_{\zeta_7^*(z)})$ contains only negative numbers, so $c_0 < 0$. By applying (45) for $n = 7$, f_7 and the above defined ω_δ, under the assumption $f_7(c_0) = c_0$, we have

$$7^{-c_0} \leq |1 - 2^{-c_0} - 3^{-c_0} + 4^{-c_0} - 5^{-c_0} + 6^{-c_0}|. \tag{57}$$

We consider the two possible cases: (a) $1 - 2^{-c_0} - 3^{-c_0} + 4^{-c_0} - 5^{-c_0} + 6^{-c_0} \geq 0$, (b) $1 - 2^{-c_0} - 3^{-c_0} + 4^{-c_0} - 5^{-c_0} + 6^{-c_0} < 0$. In (a), according to (57), we have the inequality

$$1 + 4^{-c_0} + 6^{-c_0} \geq 2^{-c_0} + 3^{-c_0} + 5^{-c_0} + 7^{-c_0},$$

that is clearly impossible for any $c_0 < 0$. In (b), because of (57), we get

$$1 + 4^{-c_0} + 6^{-c_0} + 7^{-c_0} \leq 2^{-c_0} + 3^{-c_0} + 5^{-c_0}. \tag{58}$$

It is immediate to check that inequality (58) is false for any $c_0 < 0$. Hence f_7 has no fixed point in $(a_{\zeta_7(z)}, a_{\zeta_7^*(z)})$. Then, by applying Theorem 3.6, $\zeta_7(z)$ has a maximum density interval and consequently

$$R_{\zeta_7(z)} = [a_{\zeta_7(z)}, b_{\zeta_7(z)}].$$

For $n = 8$, we take a mapping $\delta : \mathscr{P} \to U$ satisfying $\delta(2) = 1$, $\delta(3) = \delta(5) = -1$ and consider its corresponding $\omega_\delta := \mathbb{N} \to U$ defined in (44). Assume f_8 has some fixed point, say c_0, in the interval $(a_{\zeta_8(z)}, a_{\zeta_8^*(z)})$. By (8) $a_{\zeta_8^*(z)} < 0$ and then $(a_{\zeta_8(z)}, a_{\zeta_8^*(z)})$ contains only negative numbers, so $c_0 < 0$. By applying (45) for $n = 8$, f_8 and the above defined ω_δ, under the assumption $f_8(c_0) = c_0$, we have

$$7^{-c_0} \leq |1 + 2^{-c_0} - 3^{-c_0} + 4^{-c_0} - 5^{-c_0} - 6^{-c_0} + 8^{-c_0}|. \tag{59}$$

Regarding inequality (59) we consider the two possible cases: (a) $1 + 2^{-c_0} - 3^{-c_0} + 4^{-c_0} - 5^{-c_0} - 6^{-c_0} + 8^{-c_0} < 0$, (b) $1 + 2^{-c_0} - 3^{-c_0} + 4^{-c_0} - 5^{-c_0} - 6^{-c_0} + 8^{-c_0} \geq 0$. In case (a), according to (59), we have the inequality

$$3^{-c_0} + 5^{-c_0} + 6^{-c_0} \geq 1 + 2^{-c_0} + 4^{-c_0} + 7^{-c_0} + 8^{-c_0},$$

which is clearly impossible for any $c_0 < 0$. In case (b), because of (59), we get

$$1 + 2^{-c_0} + 4^{-c_0} + 8^{-c_0} \geq 3^{-c_0} + 5^{-c_0} + 6^{-c_0} + 7^{-c_0}. \tag{60}$$

By an elementary analysis we can see that (60) is only true for $c_0 \leq a_{\zeta_8(z)} \approx -4.1$ (observe that for $c_0 \approx -4.1$ inequality (60) becomes an equality and since $n = 8$ belongs to C^*, by Theorem 3.3, $a_{\zeta_8(z)} \approx -4.1$ is a fixed point of the function f_8). Therefore for $c_0 \in (a_{\zeta_8(z)}, 0)$, (60) is not possible. Then, since by (8) $a_{\zeta_8^*(z)} < 0$, in particular (60) is not possible in $(a_{\zeta_8(z)}, a_{\zeta_8^*(z)})$. Hence f_8 has no fixed point in the interval $(a_{\zeta_8(z)}, a_{\zeta_8^*(z)})$. Then, by applying Theorem 3.6, $\zeta_8(z)$ has a maximum density interval and consequently

$$R_{\zeta_8(z)} = [a_{\zeta_8(z)}, b_{\zeta_8(z)}].$$

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On Lindelöf Σ-Spaces

In Honour of Manuel López-Pellicer

María Muñoz-Guillermo

Abstract We revisit the notion of Lindelöf Σ-space giving a general overview about this question. For that, we deal with the Lindelöf property to introduce Lindelöf Σ-spaces in order to make a description of the "goodness" of such a type of spaces, making special emphasis in the duality between X and $C_p(X)$ respect to some topological properties, more specifically, topological properties in which different cardinal functions are involved. Classical results are linked with more recent results.

Keywords Lindelöf number · Cardinal inequalities · Topological properties

1 Notation and Terminology

The set-theoretic notation which will be used follows [19, 20]. Cardinal numbers κ and $\mathfrak{m}$ are the initial ordinals that will denote always *infinite* cardinals, ω is the smallest infinite cardinal number. The cardinal number assigned to the set of all real numbers is denoted by $\mathfrak{c}$. κ^+ is the smallest cardinal number after κ. The cardinality of a set E is denote by $|E|$, $P(E)$ is the power set of E and $[E]^n = \{A : A \subset E, |A| = n\}$. Respect to the notation refereed to topology the basic references used are [14, 22].

Let $(X, \mathscr{T})$ be a topological space, where X is a set and $\mathscr{T}$ is a topology. A family of sets in $\mathscr{N}$ it is called a *network* for X if for every point $x \in X$ and any neighborhood U of x there exists $N \in \mathscr{N}$ such that $x \in N \subset U$. The *network weight* of a space X, $nw(X)$, is defined as the smallest cardinal number of a network in X. A family of open sets in $\mathscr{B}$ it is called a *basis* if for every non-empty open subset $U \in \mathscr{T}$ of X can be represented as the union of a subfamily of $\mathscr{B}$. This definition is equivalent to the property that for each open set $U \in \mathscr{T}$ such that $x \in U$ there exists $B \in \mathscr{B}$ such that $x \in B \subset U$. It is clear that a basis is a network such that the

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elements of the family are open. The *weight of a topological space* $(X, \mathscr{T})$, denoted by $w(X, \mathscr{T})$, is the smallest cardinal number of the cardinality of a basis.

Let $x \in X$ be a fixed point of a topological space $(X, \mathscr{T})$ a family, $B(x) \subset \mathscr{T}$ of open subsets is called a *basis of neighborhoods at* x if for every open set $U \in \mathscr{T}$ such that $x \in U$, there exists $V \in B(x)$ such that $x \in V \subset U$. The *character of a point* x, denoted by $\chi(X, x)$ is the smallest cardinal number of the cardinality of a basis of neighborhoods at x. The *character of a topological space* $(X, \mathscr{T})$ is the supremum of all cardinal numbers $\chi(x, X)$ for $x \in X$, and it will be denoted by $\chi(X)$. We will write X a topological space instead of (X, τ) for short.

Definition 1.1 (*p. 12* [14]) A topological space X is said to be

1. *first-countable* or satisfies the *first axiom of countability* if $\chi(X) \leq \omega$, this means that each point has a countable basis of neighborhoods.
2. *second-countable* or satisfies the *second axiom of countability* if $w(X) \leq \omega$, that is, X has a countable basis.

The following definitions are standard and can be found in [14].

Definition 1.2 (*pp. 37–40* [14]) A topological space X is called a

1. T_1-*space* if for every pair of different points $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$.
2. T_2-*space*, or a *Hausdorff space*, if for every pair of different points $x, y \in X$ there exist open sets $U_1, U_2 \subset X$ such that $x \in U_1$, $y \in U_2$ and $U_1 \cap U_2 = \emptyset$.
3. T_3-*space*, or a *regular space*, if X is a T_1-space and for every $x \in X$ and every closed set $F \subset X$ such that $x \notin F$ there exist open sets U_1, U_2 such that $x \in U_1$, $F \subset U_2$ and $U_1 \cap U_2 = \emptyset$.
4. $T_{3\frac{1}{2}}$-*space*, or a *Tychonoff* space, or a completely regular space, if X is a T_1-space and for every $x \in X$ and every closed set $F \subset X$ such that $x \notin F$ there exists a continuous function $f : X \to I$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in F$. Every Tychonoff space is a regular space.
5. T_4-*space*, or a *normal space*, if X is a T_1-space an for every pair of disjoint closed subsets $A, B \subset X$ there exist open sets $U_1, U_2 \subset X$ such that $A \subset U_1$, $B \subset U_2$ and $U_1 \cap U_2 = \emptyset$.
6. T_5-*space*, or a *completely normal space*, if X is a T_1-space and for every pair of subsets A and B of X such that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ there exists open sets $U_1, U_2 \subset X$ such that $A \subset U_1$, $B \subset U_2$ and $U_1 \cap U_2 = \emptyset$.

All topological spaces in this chapter are supposed to be Hausdorff.

2 Lindelöf Spaces

It is well-known that a regular topological space X is a *Lindelöf space*, or has *the Lindelöf property*, if every open cover of X has a countable subcover. In particular, every compact space is a Lindelöf space. This is a descriptive well-known property that can be found along the bibliography thoroughly. According to R. Engelking [14], the notion of a Lindelöf space was introduced by Alexandroff and Urysohn in [1], although the property was named after Lindelöf [23] who proved in 1903 that any open covering of a subset F of $\mathbb{R}^n$ contains a countable subcovering.

The existence of a countable number of open sets is a level immediately close to the notion of compact space, in which a finite number of open sets are enough to cover it, and as we will see the number of open sets to cover a topological space is enough to establish bounds about other cardinal functions.

Basic properties related with axioms of separability follow. Thus,

Proposition 2.1 (Theorem 3.8.1 [14]) *Every regular second countable space is a Lindelöf space.*

The converse does not hold in general.

Example 2.1 Sorgenfrey line is a Lindelöf space which is not second countable.

Proof On the set of the real numbers X it is considered the right half-open interval topology, it means that τ is the family of all sets of the form $[a, b)$, where $a, b \in X$. The Sorgenfrey line $\mathscr{S} := (X, \tau)$ is a Lindelöf completely normal space which is not second countable since if $S = \{[x_i, y_i) : i \in \mathbb{Z}^+\}$ is a countable set of open sets, then there exists $a \in X$ such that $a \neq x_i$ for each $i \in \mathbb{Z}^+$, thus, for any $b > a$, we have that $[a, b)$ is an open set such that is not a union of elements of S [36, Counterexample 84, pp. 103–105]. Observe that Sorgenfrey line is not σ-compact since each compact set is countable and the real numbers is not countable. (X, τ) is Lindelöf. Let $\{U_\alpha\}$ be an open covering of X. Let $\{\text{int}(U_\alpha)\}$ be the family obtained considering the interior of U_α in the usual topology of the real numbers. Then $P = \cup_\alpha \text{int}(U_\alpha)$ is Lindelöf and there exists a countable subfamily such that $P = \cup_{n\in\mathbb{N}} \text{int}(U_{\alpha_n}) = \cup_\alpha \text{int}(U_\alpha)$. Let $A := X \setminus P$, then A is a countable set which can be covered by a countable subfamily of $\{U_\alpha\}$ and a countable subcovering can be obtained from the original one.

On the contrary, Lindelöf property implies T_4-space as it is stated in the following proposition. The proof can be found in [14].

Proposition 2.2 (Theorem 3.8.2 [14]) *Every Lindelöf space is normal.*

In the case of regular spaces to be Lindelöf is close to have the *countable intersection property*, namely,

Proposition 2.3 (Theorem 3.8.3 [14]) *A regular space X is Lindelöf if and only if every family of closed subsets of X which has the countable intersection, that is, each family $\mathscr{F}$ of closed sets such that for each countable subfamily $\mathscr{F}' \subset \mathscr{F}$ holds that $\cap_{F\in\mathscr{F}'} F \neq \emptyset$, has non-empty intersection.*

In the frame of locally compact space the Lindelöf property is characterized in the following proposition, see [14, Exercise 3.8.C, p. 195].

Proposition 2.4 *Let X be a locally compact space, that is, for every $x \in X$ there exists a neighbourhood U of the point x such that $\overline{U}$ is a compact of subspace of X. Then the following sentences are equivalent:*

1. *The space X has the Lindelöf property.*
2. *The space X is σ-compact.*
3. *There exists a sequence $A_1, A_2, \dots,$ of compact subspaces of the space X such that $A_i \subset int(A_{i+1})$ and $X = \bigcup_{i=1}^{\infty} A_i$.*

Unless otherwise was indicated, we assume that all topological spaces are non-empty, completely regular and Hausdorff.

Respect to the stability properties, the Lindelöf property has a good behaviour respect to some operations but not all.

We have that every closed subset subspace of a Lindelöf space is a Lindelöf space. Every regular space which can be represented as a countable union of Lindelöf subspaces is Lindelöf. The continuous image of a Lindelöf space X onto a regular space Y is a Lindelöf space. Inverse images of a Lindelöf space under perfect mappings are also Lindelöf. In fact, inverse images of closed mappings with Lindelöf fibers are again Lindelöf. More about the "goodness" of the Lindelöf property comes from realcompactness. A topological space is *realcompact* if and only if it is homeomorphic to a closed subspace of a power $\mathbb{R}^{\mathfrak{m}}$ of the real line, for a cardinal number $\mathfrak{m}$, and it is known, that every Lindelöf space is realcompact [14, Theorem 3.11.12]. The *realcompactification* of a topological space X is denoted by νX, whereas the Stone-Čech compactification of X, is denoted by βX [14, Sect. 3.6]. As a good property we have that every open cover of a Lindelöf space has a *locally finite open refinement* [14, Theorem 3.8.11].

On the other hand, the Cartesian product of two Lindelöf spaces is not in general a Lindelöf space, considering again the Sorgenfrey line $\mathscr{S}$, then $\mathscr{S} \times \mathscr{S}$ is not Lindelöf although $\mathscr{S}$ is it. In [35, p. 632] it is proved that $\mathscr{S} \times \mathscr{S}$ is not normal and hence it is not Lindelöf.

2.1 The Lindelöf Number

Until now we have summarized some properties of Lindelöf spaces respect to separability axioms, countably axioms or stability properties, in a general frame. The following lines will be occupied on the relationship between Lindelöf property and other cardinal functions.

We have considered in the beginning some cardinal functions as the weight and the network weight but a more formal definition is needed about what a cardinal function is. Recall that a cardinal function is a function that assigns to every topological space an infinite cardinal number which is invariant by homeomorphisms, it means that if X and Y are homeomorphic, the cardinal function of X is equal to the cardinal function of Y. In topology the descriptive properties of the spaces are mostly determined by different cardinal functions. The generalization of the notion of Lindelöf space

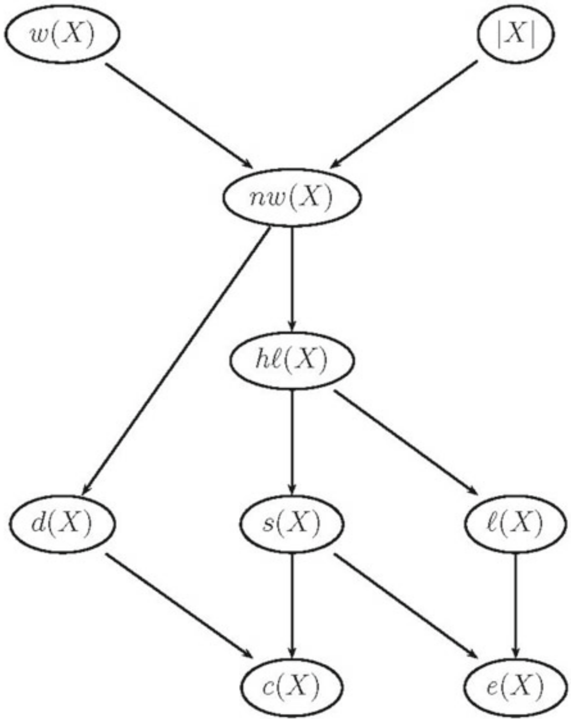

Fig. 1 General relationship between general cardinal functions. The arrow "→" means greater than or equal to. See [14] and [19, Sect. 3] for more details

gives us a new cardinal function defined for each topological space X. The *Lindelöf number*, denoted by $\ell(X)$, is the smallest cardinal number κ such that for every open cover there exists a subcovering of cardinality $\leq \kappa$.

Other cardinal functions are the following. The *density* of X, $d(X)$ is the smallest cardinal number of a set $S \subset X$, such that $\overline{S} = X$. The *Souslin number*, or *cellularity* of a topological space X, $c(X)$ is defined as the smallest cardinal number $\mathfrak{m}$ such that the cardinality of a family of pairwise disjoint non-empty open subsets of X is not greater than $\mathfrak{m}$. The *spread* of X, $s(X)$, is the smallest cardinal number $\mathfrak{m}$ such that the cardinality of every discrete subspace is not greater than $\mathfrak{m}$. While the *extent* of X, $e(X)$, is the smallest cardinal number $\mathfrak{m}$ such that the cardinality of a closed and discrete subset of X is not greater than $\mathfrak{m}$ [19, Sect. 3]. It is clear that $e(X) \leq \ell(X)$ and $e(X) \leq s(X)$. In other sense, as we have previously mentioned each closed subspace of a Lindelöf space is Lindelöf but the same does not occur for open subspaces, hence the *hereditarily Lindelöf number of* X, is defined as $h\ell(X) = \sup\{\ell(Y) : Y \subset X\}$.

Observe in Fig. 1 that the density character and the Lindelöf number are not closely related. The *Niemytzki plane* is an example of a separable space which is not a Lindelöf space [36, Example 82, p. 100]. Let $L = \{(x, y) \in \mathbb{R}^2 : \ y \geq 0\}$. Let $L_1 = \{(x, y) : \ y = 0\}$ the real axis and $L_2 = L \setminus L_1$. In L_2, the topology τ is the Euclidean topology and τ^* is the topology in L generated by τ and all the sets of the form $\{(x, 0)\} \cup D$, where D is an open disc in L_2 which is tangent to L_1 at the point $(x, 0)$. The space (L, τ^2) is called the Niemytzki plane.

The rest of the results included in Fig. 1 are classical and can be found in [19, Sect. 3].

3 Lindelöf Σ-Spaces

A subclass of Lindelöf spaces is the class of Lindelöf Σ-spaces. Σ-spaces were introduced by K. Nagami in [25]. This class of spaces has been widely used not only in Topology, see [41] but also in Functional Analysis in which it is called the class of countably K-determined spaces [11, 12, 17, 37].

3.1 Definition and First Properties

The categorical definition of Lindelöf Σ-spaces can be found in [2, p. 6].

Definition 3.1 The class of Lindelöf Σ-spaces is the smallest class of spaces containing all compacta, all spaces with a countable basis and closed under the following operations: finite products, closed subspaces and continuous images.

This definition gives us the first difference respect to the Lindelöf property, namely, the finite product of Lindelöf Σ-spaces is again a Lindelöf Σ-space, although more can be done, since the countable product of Lindelöf Σ-spaces is a Lindelöf Σ-space, see [41, Proposition 3]. Nevertheless, the categorical definition is not operative to work with it.

Following M. Talagrand [37], we use the notion of upper semicontinuous map.

Definition 3.2 Let X and Y be topological spaces. A multivalued map $\phi : X \to 2^Y$ is said to be *upper semicontinuous* in $x_0 \in X$ if $\phi(x_0)$ is not empty and for each open set V in Y with $\phi(x_0) \subset V$ there exists an open set U of x_0 such that $\phi(U) \subset V$. A multivalued map ϕ is said to be *upper semicontinuous* if it is upper semicontinuous for each point in X. We will say that a multivalued map $\phi : X \to 2^Y$ is *usco* if ϕ is upper semicontinuous and the set $\phi(x)$ is compact for each $x \in X$.

The reader can find more information about usco maps in [9] and references therein.

The number of equivalent definitions for Lindelöf Σ-space has increased because the different situations in which it appears. In [41, Theorem 1], some equivalent definitions of Lindelöf Σ-space have been summarized.

Proposition 3.1 (Theorem 1 [41]) *The following conditions are equivalent for a topological space X:*

1. *X is a Lindelöf Σ-space;*

2. *there exist spaces K compact and M second countable such that X is a continuous image of a closed subspace of $K \times M$;*
3. *there exists an usco map $\phi : M \to 2^X$, where M is a second countable space and $\cup\{\phi(x) : x \in M\} = X$;*
4. *there exits a compact cover $\mathscr{C}$ of the space X such that some countable family $\mathscr{N}$ of subsets of X is a network mod$(\mathscr{C})$ in the sense that, for any $C \in \mathscr{C}$ and any $U \in \tau(X)$ with $C \subset U$ there is $N \in \mathscr{N}$ such that $C \subset N \subset U$;*
5. *there exists a compact cover $\mathscr{C}$ of the space X such that some countable family $\mathscr{Q}$ of closed subsets of X is a network mod$(\mathscr{C})$;*
6. *there exists a countable family $\mathscr{F}$ of compact subsets of βX such that $\mathscr{F}$ separates X from $\beta X \setminus X$ in the sense that, for any $x \in X$ and $y \in \beta X \setminus X$ there exists $F \in \mathscr{F}$ for which $x \in F$ and $y \notin F$;*
7. *there exists a compactification bX of the space X and a countable family $\mathscr{K}$ of compact subsets of bX which separates X from $bX \setminus X$;*
8. *there exists a space Y such that $X \subset Y$ and, for some countable family $\mathscr{K}$ of compact subsets of Y, we have $X \subset \cup\mathscr{K}$ and $\mathscr{K}$ separates X from $Y \setminus X$.*

3.2 Generalizing Lindelöf Σ-Spaces

After characterization of Lindelöf Σ-space using usco maps, the cardinal functions *number of K-determination*, $\ell\Sigma(X)$ and *Nagami number*, $Nag(X)$, make sense.

Definition 3.3 (*Definition 2* [8]) Let X be a topological space.

(i) The number of *K-determination* of X, $\ell\Sigma(X)$, is defined as the smallest cardinal number $\mathfrak{m}$ for which there are a metric space (M, d) of weight $\mathfrak{m}$ and an usco map $\phi : M \to 2^X$ such that $X = \bigcup\{\phi(x) : x \in M\}$.
(ii) The number of *Nagami* of X, $Nag(X)$, is defined as the smallest cardinal number $\mathfrak{m}$ for which there are a topological space Y of weight $\mathfrak{m}$ and an usco map $\phi : Y \to 2^X$ such that $X = \bigcup\{\phi(y) : y \in Y\}$.

The following characterizations hold.

Proposition 3.2 (Proposition 6 [8]) *Let X be topological space and $\mathfrak{m}$ a cardinal number. The following statements are equivalent:*

1. *$Nag(X) \le \mathfrak{m}$ (resp. $\ell\Sigma(X) \le \mathfrak{m}$);*
2. *there is a family of closed sets $\{A_i : i \in \mathfrak{m}\}$ in βX, such that for every $x \in X$ there is a set $J \subset \mathfrak{m}$ (resp. with $|J| \le \omega$) such that $x \in \bigcap_{i \in J} A_i \subset X$.*
3. *there exists a topological (metric) space Y such that $w(Y) \le \mathfrak{m}$ and $\phi : Y \to 2^X$ an usco map such that $X = \cup\{\phi(y) : y \in Y\}$.*

Observe that $Nag(X) \le \ell\Sigma(X)$ and $\ell\Sigma(X) \le \omega$ implies that X is a Lindelöf Σ-space. Both notions are different, in [8, Example 9] is given an example of a space $\mathbb{Y}$ such that $Nag(\mathbb{Y}) \le w(\mathbb{Y}) < \ell\Sigma(\mathbb{Y})$, [8, Proposition 10]. Figure 2 adds to Fig. 1

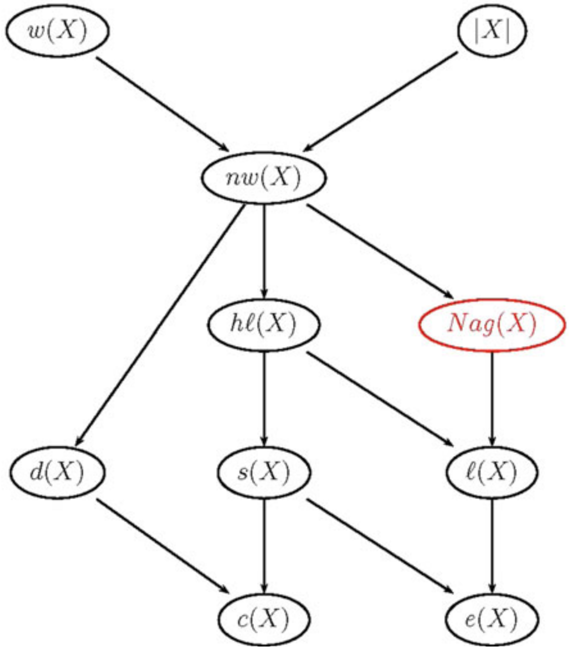

Fig. 2 Relationships between general cardinal functions for a completely regular topological space X including Nagami number. The arrow "$\rightarrow$" means $\geq$

the cardinal function $Nag(X)$ and its relationships with other cardinal functions. Thus, it is known that $Nag(X) \leq nw(X)$ for X a completely regular space, see [8, Corollary 27]. In the class of infinite metric spaces we have that

$$w(X) = \ell(X) = d(X) = \ell\Sigma(X) = Nag(X).$$

When we consider $\aleph$-spaces the relationships between cardinal functions also allow us to have more information. The class of $\aleph$-spaces was introduced by P. O'Meara in [28]. A topological space X is called an $\aleph$-*space* if X is regular and has a σ-locally finite k-network. A family $\mathscr{F}$ of subsets of X is called a k-*network* in X, if whenever $K \subset U$ with K compact and U open in X, then $K \subset \cup\mathscr{F}' \subset U$ for some finite family $\mathscr{F}' \subset \mathscr{F}$. Because the regularity of the space, the collection of subsets which is a σ-locally finite k-network can be chosen to consist of closed sets. In the class of $\aleph$-spaces, $\ell(X) = Nag(X) = \ell\Sigma(X)$.

3.3 Lindelöf Σ-Spaces in C_p-Theory

The attempt to collect all the properties even in the particular case of Lindelöf Σ-spaces is not an easy task, because the large quantity of results related, see [41]. Thus, we will show up only some results that give us a general knowledge about the behavior of Lindelöf Σ-spaces. Let X be a topological space and $C_p(X)$ stands for the space of real-valued continuous functions endowed with the pointwise convergence topology. In this section we focus on how topological properties of both spaces are related in the framework of Lindelöf Σ-spaces, since there is a special relationship between properties of X and $C_p(X)$ when Lindelöf Σ-property is involved.

Additional notation and definitions are needed and they can be found in [2]. The *tightness* of a point x in a topological space X, $t(x, X)$, is the smallest infinite cardinal number $\mathfrak{m}$ such that for any $x \in \overline{A}$, there exists $B \subset A$ such that $|B| \leq \mathfrak{m}$ and $x \in \overline{B}$. The *tightness of a topological space* X, $t(X)$, is the supremum of all $t(x, X)$ for $x \in X$.

The following definition can be found in [2, Sect. 0.2, p. 5].

Definition 3.4 Let X be a topological space, we called $iw(X)$ the smallest cardinal $\mathfrak{m}$ for which there exist a topological space Y with $w(Y) \leq \mathfrak{m}$ and a one-to-one continuous map onto $f : X \to Y$.

A space X is said to be $\mathfrak{m}$-*stable* if for every continuous image Y of X if $iw(Y) \leq \mathfrak{m}$ then $nw(Y) \leq \mathfrak{m}$. The space X is *stable* if it is $\mathfrak{m}$-stable for any infinite cardinal number $\mathfrak{m}$. The following results get us a first example of what we mean about the relationship between X and $C_p(X)$ in the frame of Lindelöf Σ-spaces.

Theorem 3.1 (Theorem II.6.21 [2]) *Every Lindelöf Σ-space is stable.*

As a consequence of previous theorem the product of an arbitrary family of Lindelöf Σ-spaces is stable [2, Corollary II.6.27]. Moreover, a space is stable if and only if $C_p(C_p(X))$ is stable, [2, Corollary II.6.11]. Following [39], let $C_{p,0}(X) = X$ and $C_{p,n+1}(X) = C_p(C_{p,n}(X))$ for each natural number $n \in \mathbb{N}$. The following proposition holds.

Proposition 3.3 (Corollary II.6.32 [2]) *Let X be a Lindelöf Σ-space then $C_{p,2n}(X)$ is stable for any $n \in \mathbb{N}$.*

3.3.1 The Iterative Process

In [41] V. Tkachuck gave a description of all possible distribution of the Lindelöf Σ-property in the iterated spaces $C_{p,n}(X)$. Only the following cases can occur:

Proposition 3.4 (Corollary 2.10 [41]) *Only the following distributions of the Lindelöf Σ-property in iterated function spaces are possible:*

1. *$C_{p,n+1}(X)$ is a Lindelöf Σ-space for every $n \in \mathbb{N}$;*
2. *$C_{p,n+1}(X)$ is a Lindelöf Σ-space only for odd $n \in \mathbb{N}$;*
3. *$C_{p,n+1}(X)$ is a Lindelöf Σ-space only for even $n \in \mathbb{N}$;*
4. *for any $n \in \mathbb{N}$ the space $C_{p,n+1}(X)$ is not a Lindelöf Σ-space.*

An example of a non-Lindelöf space X such that $C_{p,2n+1}(X)$ is Lindelöf Σ-space for every $n \in \mathbb{N}$ but $C_{p,2n}(X)$ is not Lindelöf and a space Y such that $C_{p,2n}(Y)$ is a Lindelöf Σ-space for every $n \in \mathbb{N}$ and $C_{p,2n+1}(Y)$ is not Lindelöf are shown up in [41, Examples 2.9].

Previously, particular results in the framework of compact spaces had been obtained. The following definitions are well-known. A compact subset of a Banach space in the weak topology is called *Eberlein compact*. A compact space X is called *Gul'ko compact* if $C_p(X)$ is a Lindelöf Σ-space. Finally, a compact space is said to be a *Corson compact* if it can be embedded in the subspace of the product $\mathbb{R}^{\mathfrak{m}}$ of the real line consisting of functions vanishing at all but countably many points for an infinite cardinal number $\mathfrak{m}$ [2, p. 134].

The behaviour of iterated spaces for Lindelöf Σ-spaces have been widely studied. Gul'ko proved in [18] that for any Eberlein compact X the iterated function spaces $C_{p,n}(X)$, $n \in \mathbb{N}$, are Lindelöf. Sipachova [31] proved that $C_{p,n}(X)$ is Lindelöf Σ-space for any $n \in \mathbb{N}$ whenever X is an Eberlein compact space. On the other hand, Okunev [27] proved that if X and $C_p(X)$ are Lindelöf Σ-spaces then $C_{p,n}(X)$ is a Lindelöf Σ-space for each $n \in \mathbb{N}$. In general, when X is a Lindelöf Σ-space such that $X \subset C_p(Y)$, the following result is known.

Proposition 3.5 (Theorem 2.12 [27]) *Let X and Y Lindelöf Σ-spaces such that $X \subset C_p(Y)$, then $C_{p,n}(X)$ is a Lindelöf Σ-space for any $n \in \mathbb{N}$.*

More in this sense,

Proposition 3.6 (Theorem 4.3 [27]) *Let X be a Gul'ko compact space and K be a compact subspace of $C_{p,n}(X)$ for some $n \in \mathbb{N}$, then K is a Gul'ko compact space.*

A generalization of the previous result is the following one.

Theorem 3.2 (Theorem 4.4 [27]) *Let K be a compact subspace of $C_p(X)$ such that there exists a Lindelöf Σ-space Z such that $C_p(X) \subset Z$ then K is a Gul'ko compact space.*

The following result is a characterization of Gul'ko compact spaces.

Theorem 3.3 (Theorem 4.7 [27]) *Let X be a compact space. Then the following conditions are equivalent:*

1. *X is a Gul'ko compact space;*
2. *$C_{p,n}(X)$ is a Lindelöf Σ-space for some $n \in \mathbb{N}$;*
3. *$C_{p,n}(X)$ is a Lindelöf Σ-space for any $n \in \mathbb{N}$.*

This Theorem links to the following one proved by Sokolov [33, Corollary 2].

Proposition 3.7 *If X is a Corson compact space, then $C_{p,n}(X)$ is Lindelöf for each $n \in \mathbb{N}$.*

Similarly, Gul'ko [44, Problem 27, p. 610] conjectured that the Lindelöf property of all iterated continuous spaces characterizes Corson compact, neverthe-less, Sokolov [34, Theorem 2.1] gave an example of a compact space X whose iterated continuous function spaces $C_{p,n}(X)$ for $n \in \mathbb{N}$ are Lindelöf but X is not a Corson compact space.

More similarities follow. If X is a Corson (Gul'ko) compact space and $C_{p,n}(X)$ is homeomorphic to $C_{p,n}(Y)$, for some $n \in \mathbb{N}$, then Y is Corson (Gul'ko). In 2018, see [6, Sect. 3] for details, it has been proved the same result for Eberlein compact space.

Recently (2017), Ferrando, Kąkol and López-Pellicer have characterized in [16] Gul'ko compact spaces considering the topology in $C(X)$ for σ_Y, where Y is a subset that separates the functions of $C(X)$ and σ_Y is the weak topology $\sigma(C(X), span(Y))$.

Theorem 3.4 (Theorem 4.1 [16]) *Let X be a compact space and Y be a G_δ-dense subspace. Then X is a Gul'ko compact space if and only if $(C(X), \sigma_Y)$ is a Lindelöf Σ-space.*

3.3.2 Σ_s-Products

The concept of Σ_s-product was used by Sokolov [32, Theorem 8] in order to give a different characterization of Gul'ko compact spaces. The following definitions are needed, see [30, Definition 3.1].

Definition 3.5 Let a be a point in the product space $X = \prod_{t \in T} X_t$.

1. The *support of* x, denoted by $supp(x)$, is the set $\{t \in T : x(t) \neq a(t)\}$.
2. The Σ-product of the family $\{X_t\}_{t \in T}$ centered at the point a, is the subspace of X given by
$$\Sigma(X, a) = \{x \in X : |supp(x)| \leq \omega\}.$$
3. The σ-product of the family $\{X_t\}_{t \in T}$ centered at the point a, is the subspace of X given by
$$\sigma(X, a) = \{x \in X : |supp(x)| < \omega\}.$$
4. Let s be a countable family of subsets of T and $s_x = \{E \in s : |supp(x) \cap E| < \omega\} \subseteq s$ for $x \in X$, then the Σ_s-product of the family $\{X_t\}_{t \in T}$ centered at the point a with respect to the set s is the subspace of X given by
$$\Sigma_s(X, a) = \{x \in X : T = \cup s_x\}.$$

If the point a in consideration is not relevant we will write $\Sigma(X)$, $\sigma(x)$ and $\Sigma_s(X)$.

Now the following characterization can be introduced.

Proposition 3.8 (Theorem 8 [32]) *A compact space X is Gul'ko if and only if X embeds into a Σ_s-product of real lines.*

An extension of the previous result is debt to Casarrubias-Segura et al. [6] in which it is proved that if X is a Lindelöf Σ-space contained in a Σ_s-product of real lines then $C_p(X)$ is a Lindelöf Σ-space.

More results related to Lindelöf Σ-spaces are known. Thus,

Proposition 3.9 (Theorem 3.2 [42]) *Every Σ_s-product of compact spaces is a Lindelöf Σ-space.*

Recently, in 2018,

Proposition 3.10 (Theorem 4.1 [6]) *If $X = \prod_{t \in T} X_t$ is a product, and every σ-product in X is a Lindelöf Σ-space, then each Σ_s-product in X is a Lindelöf Σ-space.*

Proposition 3.11 (Theorem 4.5 [6]) *Every Σ_s-product of K-analytic spaces is a Lindelöf Σ-space.*

In [6, Corollary 4.2] it has been proved that Proposition 3.9 holds for every Σ_s-product of σ-compact spaces.

Different questions remain open in this framework, thus, in [6, Questions 5.6 and 5.7] the following questions are posed.

1. Let X be a Lindelöf Σ-space which admits a condensation in a Σ_s-product of real lines. Must $C_p(X)$ be a Lindelöf Σ-space?
2. Let X be a Lindelöf subspace of a Σ-product (or Σ_s-product) of real lines. Must $C_p(X)$ be Lindelöf?

Remind that a map $f : X \to Y$ is a *condensation* if it is a continuous bijection; in this case we say that X condenses *onto* Y. If X condenses onto a subspace of Y, we say that X condenses *into* Y (Sect. 2 in [6]).

3.3.3 Cardinal Inequalities

In this section we focus our interest on the relationship between X and $C_p(X)$ involving *different* cardinal functions. In [8] can be found some of them in which the number of K-determination and the Nagami number appear. Thus,

Proposition 3.12 (Proposition 16 [8]) *Let X be a topological space, then $t(C_p(X)) \leq \ell\Sigma(X)$. In particular, if X is a Lindelöf Σ-space, then $t(C_p(X))$ is countable.*

Involving the network of the space the following results give us information in the particular case of the Lindelöf Σ-spaces. Classical results of Arkhangel'skii follow.

Proposition 3.13 (Theorem 10 [3]) *Let X be a topological space such that $C_p(X)$ is a Lindelöf Σ-space and the spread of $C_p(X)$ is countable then $nw(X)$ is countable (X is cosmic).*

Proposition 3.14 (Proposition 12 [3]) *Let X be a topological space such that the spread of X is countable and $C_p(X)$ is a Lindelöf Σ-space then X is a Lindelöf Σ-space.*

Proposition 3.15 (Theorem 13 [3]) *Let X be a topological space such that the spread of $X \times X$ is countable and $C_p(X)$ is a Lindelöf Σ-space then X has a countable network (X is cosmic).*

More conditions to obtain X cosmic were obtained by Tkachuck.

Proposition 3.16 (Theorem 3.6 [40]) *Let X be a topological space such that $C_p(X)$ is a Lindelöf Σ-space and $s(X)$ is countable then $nw(X)$ is countable (X is cosmic).*

Proposition 3.17 (Theorem 3.30 [43]) *Let X be a topological space such that $C_p(C_p(X))$ is a Lindelöf Σ-space and $s(X)$ is countable then $nw(X)$ is countable (X is cosmic).*

When we consider subspaces as our goal, then the following definitions are needed. Let $\mathfrak{m}$ be an infinite cardinal number, a topological space X is said to be $\mathfrak{m}$-*monolithic* if for each $A \subset X$ such that $|A| \leq \mathfrak{m}$ then $nw(\overline{A}) \leq \mathfrak{m}$. A space X is called *strongly* $\mathfrak{m}$-*monolithic* if for every $Y \subset X$ with $|Y| \leq \mathfrak{m}$, the weight of the space $\overline{Y}$ does not exceed $\mathfrak{m}$. A topological space X is said to be *monolithic* if X is $\mathfrak{m}$-monolithic for each infinite cardinal number $\mathfrak{m}$. Thus, if X is a monolithic space, then for each subspace $Y \subset X$, we have that $d(Y) = nw(Y)$, [2, p. 76].

The following result can be found in [8].

Proposition 3.18 (Proposition 17 [8]) *Let X be a topological space and $H \subset C(X)$ τ_p-compact, then H is strongly $\ell\Sigma(X)$-monolithic.*

The corollary which follows from the previous proposition is also an immediate consequence of [2, Theorem II.6.8].

Corollary 3.1 *Let X be a Lindelöf Σ-space and $H \subset C_p(X)$ then*

$$nw(H) = d(H).$$

In particular, if H is τ_p-compact subspace then H is metrizable, see [10, Corollary 1.2].

Recent work has established accurate boundedness of the weight in Lindelöf Σ-spaces. Tkachenko [38] has proved the following result.

Theorem 3.5 (Theorem 2.1 [38]) *Let X be a completely regular space then $w(X) \leq |C(X)| \leq nw(X)^{Nag(X)}$ holds.*

In [6, Theorem 8.2] this result has been proved with different arguments proving that the inequality $w(X) \leq nw(X)^{Nag(X)}$ holds for regular spaces. In particular for a Lindelöf Σ-space X such that $nw(X) \leq \mathfrak{c}$, then $|C(X)| \leq \mathfrak{c}$ and $w(X) \leq \mathfrak{c}$. Even more it is established,

Theorem 3.6 (Theorem 2.3 [38]) *Let Y be a dense subspace of a completely regular space X, then* $|C(X)| \leq nw(Y)^{Nag(Y)}$ *and* $w(X) \leq nw(Y)^{Nag(Y)}$.

Respect to the hereditarily numbers the following properties hold.

Theorem 3.7 (Theorem 5.2 [24]) *Let X be a topological space then:*

1. $h\ell(X) \leq \max\{Nag(C_p(X)), s(X)\}$.
2. $h\ell(C_p(X)) \leq \max\{Nag(X), s(C_p(X))\}$.

The corollary that follows can be found in [3].

Corollary 3.2 (Proposition 9 [3]) *If X is a Lindelöf Σ-space and the spread of* $C_p(X)$ *is countable, then* $C_p(X)$ *is hereditarily Lindelöf, and* $X \times X$ *is hereditarily separable.*

Finally, the classical theorem of Baturov [4] states that

Theorem 3.8 (Theorem III.6.1 [2]) *Let X be a Lindelöf Σ-space and* $Y \subset C_p(X)$ *a subspace, then* $\ell(Y) = e(Y)$.

If X is a countably compact space Baturov's theorem fails. Buzyakova [5, Example 3.6] showed an example of a countably compact space X such that $e(C_p(X)) < \ell(C_p(X))$.

4 Characterizing When νX Is a Lindelöf Σ-Space

Realcompactification of a space is related with the space $C_p(X)$ and as an intermediate step in order to get any of the previous results. In fact when dual spaces are considered, *envelopes* play an important role.

Theorem 4.1 (Theorem 3.5 [27]) *Let X be a topological space. Then* νX *is a Lindelöf Σ-space if and only if there exists a Lindelöf Σ-space Z such that* $C_p(X) \subset Z \subset \mathbb{R}^X$.

As a consequence of the previous result we have that

Proposition 4.1 (Corollary 3.6 [27]) *Let* $\nu C_p(X)$ *be a Lindelöf Σ-space, then* νX *is a Lindelöf Σ-space.*

Theorem 4.2 (Theorem 3.5 [26]) *Let* $C_p(X)$ *be a Lindelöf Σ-space, then* νX *is a Lindelöf Σ-space.*

Theorem 4.3 (Theorem 2.3 [39]) *Let* $C_p(X)$ *be a Lindelöf Σ-space, then* $C_p(\nu X)$ *is a Lindelöf Σ-space.*

Now, it is clear that

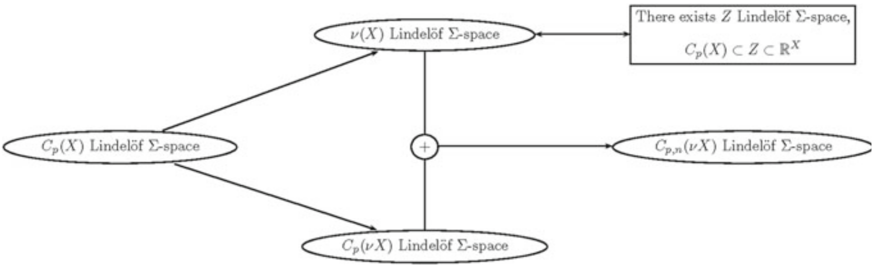

Fig. 3 Relationships between some notions that summarizes some properties involving realcompactification of a space. $A \to B$ means A implies B

Corollary 4.1 (Corollary 2.4 [39]) *If $C_p(X)$ is a Lindelöf Σ-space, then $C_{p,n}(\nu X)$ is a Lindelöf Σ-space for each $n \in \mathbb{N}$.*

Figure 3 summarizes the previous results.

Completely regular spaces X whose realcompactification νX is a Lindelöf Σ-space were studied by Ferrando in [15]. The characterization of topological spaces whose realcompactification is a Lindelöf Σ-space was also considered in [21] where the notions of *strongly web-bounded space* and *web-bounding space* are involved.

Definition 4.1 (*Definition 3* [7]) A locally convex space X is *web-bounded* if there is a family $\{A_\alpha : \alpha \in \Omega\}$ of sets covering X for some nonempty $\Omega \subset \mathbb{N}^{\mathbb{N}}$ such that if $\alpha = (n_k)_k \in \Omega$ and $x_k \in C_{n_1,n_2,\ldots,n_k} := \bigcup\{A_\beta : \beta = (m_k)_k \in \Omega,\ m_j = n_j,\ j = 1,\ldots,k\}$ then $(x_k)_k$ is bounded.

Definition 4.2 (*p. 150* [29]) A space X is *strongly web-bounding* if there is a family $\{A_\alpha : \alpha \in \Omega\}$ of sets covering X for some nonempty $\Omega \subset \mathbb{N}^{\mathbb{N}}$ such that if $\alpha = (n_k)_k \in \Omega$ and $x_k \in C_{n_1,n_2,\ldots,n_k} := \bigcup\{A_\beta : \beta = (m_k)_k \in \Omega,\ m_j = n_j,\ j = 1,\ldots,k\}$ then $(x_k)_k$ is functionally bounded, that is, $f((x_k)_k) \subset \mathbb{R}$ is bounded for each continuous function $f : X \to \mathbb{R}$.

Characterization of the realcompactification of a space X which is also a Lindelöf Σ-space was given by Kąkol and López-Pellicer in [21] giving a description of a web-bounded structure in the original space.

Theorem 4.4 (Theorem 1.2 and Corollary 2.6 [21]) *Let X be a completely regular space then the following sentences are equivalent.*

1. *νX is a Lindelöf Σ-space;*
2. *X is strongly web-bounding;*
3. *$C_p(X)$ is web-bounded;*
4. *there exits a Lindelöf Σ-space Z such that $C_p(X) \subset Z \subset \mathbb{R}^X$.*

Regarding the question if this property is in some sense "hereditary" when realcompactification is involved we have the following result.

Proposition 4.2 (Theorem 8 [13]) *Let X and Y be spaces and $h : C_p(X) \to C_p(Y)$ a surjective map that takes bounded sequences to bounded sequences. If νX is a Lindelöf Σ-space, then νY is a Lindelöf Σ-space.*

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Some Extensions of the Class of $\mathscr{L}^p$-Spaces

In Honour of Manuel López-Pellicer

María José Rivera

Abstract The ℓ_p spaces play a central role in the classical theory of operators and finitely generated tensor norms. In this setting Lindenstrauss and Pelczyǹski (Stud Math 29: 275–326, 1968) introduced the class of the $\mathscr{L}^p$-spaces in 1968. By construction the class $\mathscr{L}^p$ places us squarely in the so called **Local Theory**, that deals with the study of Banach spaces through or in terms of their finite-dimensional subspaces, see Pietsch (Stud Math 135: 273–298, 1999). Although the definition of the spaces of the class $\mathscr{L}^p$ from the ℓ_p spaces is finite-dimensional, it has many implications in the global structure of the $\mathscr{L}^p$-spaces. And it is clear that, in general, the nice behavior of the $\mathscr{L}^p$ spaces is a consequence of some good properties of the ℓ_p spaces, which make them a basic pillar of the Banach spaces theory. Anyway, if we replace ℓ_p for another sequence space λ, one expects that some important new problems will appear, since ℓ_p and hence $\mathscr{L}^p$ seem to be more or less irreplaceable. Our purpose is to present some conclusions we reached in this way in relation to the study of certain concrete tensor norms defined through different sequence spaces, making a small chronological journey, step by step, attempt after attempt. This study has an earlier version entitled “On the classes of $\mathscr{L}^\lambda$, quasi-$\mathscr{L}^\lambda$ and $\mathscr{L}^{\lambda,g}$ spaces”, published in the Proceedings of the American Mathematical Society (Rivera in Proc Amer Math Soc 133: 2035–2044, 2005).

Keywords $\mathscr{L}^p$-Spaces · Tensor products · Local theory · Finite representability · Uniform projection property · Local unconditional structure

1 Preliminaries

In the paper *Résumé de la théorie métrique des produits tensoriells topologiques*, Grothendieck [8] demostrated the relevance of the tensor products in the Banach space theory, being the first to realize the scope of the finite dimensional behavior of

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the tensor norms. And anticipating the notion of finite representability, other basic tool of the **Local Theory**, he established a partial order in the class of normed spaces, saying that a normed space E has "lower linear type" to another normed space F if there is a constant $c > 0$ such that for every finite-dimensional subspace M of E there is a finite-dimensional subspace N of F such that the Banach-Mazur distance $d(M, N) < c$. A decisive step was the introduction of Dacunha-Castelle and Krivine [3] of the ultraproducts technique in Banach spaces theory. The tensor norms of Saphar and Lapresté, defined through the ℓ_p spaces, completed what nowadays is known as the classical theory of topological tensor products of Banach spaces.

Different authors have contributed to precise what is now understood by the **Local Theory** in Banach spaces. For example, the word "localization" which appears in the title of he paper of Pelczyǹski and Rosenthal [23] refers to obtain quantitative finite-dimensional formulations of infinite-dimensional results. For Tomczak-Jaegermann [30] a property of Banach spaces (or operators acting between them) is called local if it can be defined by a quantitative statement or inequality concerning a finite number of finite-dimensional subspaces. On the other hand Lindenstrauss and Milman [17] emphasize in the relation of the structure of an infinite-dimensional space and its finite-dimensional subspaces. But in all these formulations lies the same idea: the theory of Banach spaces covers local aspects and global aspects, and certain properties of infinite-dimensional Banach spaces can be studied through their finite-dimensional spaces.

With the instruments of the **Local Theory**, Grothendieck developed a very useful theory of duality in tensor products of Banach spaces, involving in it the wonderful theory of operator ideals. In this context appeared in 1968 the $\mathscr{L}^p$ spaces of Lindenstrauss and Pelczyǹski [18] that we try to extend, study which would be enriched a year later with the paper of Lindenstrauss and Rosenthal [19]. Also in 1968 Pietsch and his school carry out a systematic investigation of operator ideals on the class of Banach spaces ignoring tensor products, without this lack affecting the quality of the results. But many of the ideas of the book od Pietsch "Operator ideals", that culminated this huge task and posterior reflections, see [25] for example, clearly came from the work of Grothendieck, in particular the use of techniques of the **Local Theory** that Pietsch leads to the theory of operators. We must highlight the great use that Pietsch makes in his book of the ultraproducts of operators. And in the same line Beauzamy [1] introduced the notion of finite representability for operators.

To facilitate the reading, we begin by briefly explaining some concepts, to fix through them our starting position, the difficulties we face, the mathematical instruments we have and the specific objectives of the paper. The notation is standard, but eventually we will have to clarify some abbreviations. In this paper we almost always work with Banach spaces. But it is necessary to highlight that it is possible to extend the concepts of the **Local Theory** of Banach spaces we are going to handle to the **Local Theory** of Banach lattices, which is the right place where locate the study of tensor norms and operator ideals associated to a sequence space λ, see [4, 9, 18–20, 22–24, 28].

1.1 Banach Sequence Spaces

Let ω be the vector space of all real sequences and φ the subspace of ω of sequences with finitely many non zero coordinates.

Definition 1.1 A real Banach space $(\lambda, \|.\|_\lambda)$ is said to be a **Banach sequence space** if it satisfies the following conditions:
(I) $\varphi \subset \lambda \subset \omega$.
(II) If $|x| \leq |y|$ with $x \in \omega$ and $y \in \lambda$, then $x \in \lambda$ and $\|x\|_\lambda \leq \|y\|_\lambda$.

We recall that every Banach sequence space λ, endowed with the pointwise order, by (II) can be considered as a **solid Banach lattice**. $S_k(\lambda)$ represents the k-sectional subspace of λ. Then we say that a Banach sequence space λ is **regular** if every element of λ is limit of its sections. If λ is not regular, their subspace $\overline{\varphi}^\lambda$ is denoted by λ_r. $\lambda^\times$ represents the Köthe dual of λ, that is the space of the scalar sequences (b_n) such that $\sum_{n=1}^{\infty} |a_n b_n| < \infty$ for every $(a_n) \in \lambda$. $\lambda^\times$ is a closed subspace of λ'.

1.2 Tensor Products and Tensor Norms

The reference text of this subsection is [4].

For every pair E and F of real linear spaces, if $B(E, F)$ is the space of bilinear forms defined in $E \times F$, every $(x, y) \in E \times F$ defines canonical linear form in $B(E, F)$ denoted by $x \otimes y$ such that $\forall w \in B(E, F), < x \otimes y, w >= w(x, y)$.

Definition 1.2 The subspace of the algebraic dual of $B(E, F)$ generated by $\{x \otimes y, x \in E, y \in F\}$ is called the **tensor product** of E and F, and it is denoted by $E \otimes F$.

Definition 1.3 A tensor norm α in the class of normed spaces assigns to every pair of normed spaces E, F a norm in $E \otimes F$ such that, if we denote by $E \otimes_\alpha F$ the corresponding normed space, the following conditions are satisfied.

- $\alpha(x \otimes y) = \|x\|\|y\|, \forall x \in E, y \in F$
- $\|x' \otimes y'\|_{(E \otimes_\alpha F)'} = \|x'\|\|y'\|, \forall x' \in E', y' \in F'$.
- If $A : E_1 \to E_2$ and $B : F_1 \to F_2$ are operators, then $A \otimes B : E_1 \otimes_\alpha F_1 \to E_2 \otimes_\alpha F_2$ is a operator with $\|A \otimes B\| \leq \|A\|\|B\|$

Given a Banach sequence space λ, there is an standard way to construct a tensor norm in the class of Banach spaces.

Definition 1.4 We say that a sequence $(x_n)_{n=1}^{\infty}$ in a Banach space E is **strongly λ-summable** if

$$\pi_\lambda((x_n)) := \|(\|x_n\|)\|_\lambda < \infty.$$

Denote $\lambda\{E\} := \{(x_n) : \pi_\lambda((x_n)) < \infty\}$. Then $(\lambda\{E\}, \pi_\lambda)$ is a Banach space

Definition 1.5 We say that a sequence $(x_n)_{n=1}^{\infty}$ in a Banach space E is **weakly** λ-**summable** if

$$\varepsilon_\lambda((x_n)) := \sup_{\|x'\|\leq 1} \|(< x_n, x' >)\|_\lambda < \infty.$$

Denote $\lambda(E) := \{(y_n) : \varepsilon_\lambda((y_n)) < \infty\}$. Then $(\lambda(E), \varepsilon_\lambda)$ is a Banach space.

If E and F are Banach spaces, for every $z \in E \otimes F$, we define

$$g_\lambda(z) := \inf\{\pi_\lambda((x_n))\ \varepsilon_{\lambda^\times}((y_n)) : z = \sum_{n=1}^{m} x_n \otimes y_n\}.$$

If $\lambda = \ell_p$, g_λ is the tensor-norm g_p of Saphar. But in more general cases g_λ is only a quasi-norm, because it does not satisfy the triangular property. Then, to get a tensor-norm in $E \otimes F$, it is necessary to take the absolutely convex hull $\Gamma(B)$ of $B = \{z \in E \otimes F : g_\lambda(z) \leq 1\}$ and then the Minkowski functional g_λ^c of $\overline{\Gamma(B)}$. Then g_λ^c is a tensor-norm equivalent to the quasi-norm g_λ. which satisfies

$$g_\lambda^c(z) = \inf\{\sum_{i=1}^{n} \pi_\lambda((x_{ij})_j)\ \varepsilon_{\lambda^\times}((y_{ij})_j) : z = \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij} \otimes y_{ij}\}.$$

The normed space $(E \otimes F, g_\lambda^c)$ is denoted by $E \otimes_{g_\lambda^c} F$, and $E \hat{\otimes}_{g_\lambda^c} F$ represent its completion. Then if If $z \in E \hat{\otimes}_{g_\lambda^c} F$, it has a representation (called here of **"good type"**)

$$z = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_{ij} \otimes y_{ij}$$

such that

$$\{(x_{ij})_{j=1}^{\infty},\ i \in \mathbb{N}\} \subset \lambda_r\{E\},$$

$$\{(y_{ij})_{j=1}^{\infty},\ i \in \mathbb{N}\} \subset \lambda^\times(F)),$$

$$\sum_{i=1}^{\infty} \pi_\lambda((x_{ij})_j)\ \varepsilon_{\lambda^\times}((y_{ij})_j) < \infty.$$

Then

$$g_\lambda^c(z) = \inf\{\sum_{i=1}^{\infty} \pi_\lambda((x_{ij})_j)\ \varepsilon_{\lambda^\times}((y_{ij})_j)\},$$

where the infimum is in the set of representations of z of **"good type"**.

Remark that g_λ^c is the tensor norm associated to $\ell_1(\lambda_r)$, but having to use $\ell_1(\lambda_r)$ instead of λ adds very little difficulties, because the real obstacle is λ itself. And to save those obstacles we need two important tools of the Local Theory of Banach spaces: **ultraproducts and finite representability**.

1.3 Ultraproducts

The ultraproducts are essential theoretical instruments in the study of models, and have been fundamental in branches of mathematics such as algebra or set theory. But since the Robinson's work, "Non-standard analysis", [27], the ultraproducts also found numerous applications in Analysis. The basic material for this subsection is in [3, 10, 29].

We recall that a filter of subsets of a set D is a non empty family $\mathscr{F}$ of subsets of D such that (1) $\emptyset \notin \mathscr{F}$, (2) $A,\ B \in \mathscr{F}$ implies $A \cap B \in \mathscr{F}$ and (3) $A \in \mathscr{F}$ and $A \subset B$ implies $B \in \mathscr{F}$. A maximal filter in D with respect to the inclusion order is said to be an ultrafilter.

Definition 1.6 Let H be a topological space, let $(x_d)_{d\in D}$ be a family of elements of H of index D and let $\mathscr{U}$ be an ultrafilter of subsets of D. We say that $\lim_{\mathscr{U}} x_d = x$, $x \in H$, if for every neighborhood V of x,

$$\{d \in D : x_d \in V\} \in \mathscr{U}.$$

From a purely set point of view, the construction of ultraproducts comes from a family of sets $\{A_d, d \in D\}$, an ultrafilter $\mathscr{U}$ of subsets of D, a binary equivalence relation $\mathscr{R}_{\mathscr{U}}$ defined in the cartesian product $\Pi_{d\in D} A_d$ defined:

$$(a_d)\mathscr{R}_{\mathscr{U}}(b_d) \leftrightarrow \{d \in D : a_d = b_d\} \in \mathscr{U}.$$

The quotient set $\Pi_{d\in D} A_d/\mathscr{R}_{\mathscr{U}}$ is called the ultraproduct of the family $\{A_d, d \in D\}$ with respect to the ultrafilter $\mathscr{U}$, and it is denoted by $(A_d)_{\mathscr{U}}$. If $A_d = A, \forall d \in D$, the corresponding $(A)_{\mathscr{U}}$ is said to be an ultrapower.

But the set definition of ultraproduct has serious difficulties if we want to incorporate it into the Banach space theory when A_d are Banach spaces. For that reason Dacunha-Castelle and Krivine [3] replaced the Cartesian product $\Pi_{d\in D} A_d$ by the space

$$\ell_\infty(A_d, d \in D) = \{(a_d) \in \Pi_{d\in D} A_d : \|(a_d)\|_\infty = \sup_{d\in D} \|a_d\| < \infty\}$$

which is a Banach space with respect to the norm $\|.\|_\infty$.

Then let $N_{\mathscr{U}}$ be the closed subspace of $\ell_\infty(A_d, d \in D)$ such that:

$$N_{\mathscr{U}} = \{(a_d) \in \Pi_{d\in D} A_d : \lim_{\mathscr{U}} \|a_d\| = 0.\}$$

The ultraproduct $(A_d)_{\mathscr{U}}$ is for definition the quotient space $\ell_\infty(A_d, d \in D)/N_{\mathscr{U}}$ which is a Banach endowed with the quotient norm which is

$$\|(a_d)_{\mathscr{U}}\| := \lim_{\mathscr{U}} \|a_d\|.$$

If $\{A_d, d \in D\}$ and $\{B_d, d \in D\}$ are two families of Banach spaces, and $\{T_d : A_d \to B_d, d \in D\}$ is a family of operators such that, $\sup_{d\in D} \|T_d\| < \infty$, then $\{T_d : A_d \to B_d, d \in D\}$ defines in a natural way an operator $T : (A_d)_{\mathscr{U}} \to (B_d)_{\mathscr{U}}$, such that $\|T\| = \lim_{\mathscr{U}} \|T_d\|$.

1.4 Finite Representability

Definition 1.7 A Banach space E is said to be finitely representable in a Banach space F if each finite-dimensional subspace of E fits almost isometrically in F. With more precision, if for every $\varepsilon > 0$ and for every finite-dimensional subspace M of E there is a finite-dimensional subspace N of F such that the Banach-Mazur distance $d(M, N) \leq 1 + \varepsilon$.

If $d(M, N) \leq b$ for a fix $b > 0$ independent of M, we say that X is b-finitely representable in F.

Refer to [12] for details and for more information.

The relationship between ultrapowers and finite representability is quite close: a Banach space E is finitely representable (b-finitely representable) in a Banach space F if and only if E is isometric (b-isomorphic) to a subspace of some ultrapower of F.

1.5 The $\mathscr{L}^p$-Spaces of Lindenstrauss and Pelczyǹski

An strong version of the notion of finite representability of a Banach space X in ℓ_p characterizes the intensively studied class of the $\mathscr{L}^p$ spaces of Lindenstrauss and Pelczyǹski [18], see also [19, 23].

Definition 1.8 We say that a Banach space X is a $\mathscr{L}^p$-space, $1 \leq p \leq \infty$, if there is $c \geq 1$ such that for each finite-dimensional subspace M of X there is a finite-dimensional subspace N containing M such that $d(N, S_{\dim(N)}(\ell_p)) \leq c$.

It is clear that, in general, the nice behavior of the $\mathscr{L}^p$-spaces, $1 \leq p \leq \infty$, is a consequence of some good properties of the ℓ_p spaces, which make them basic constructions in the Banach spaces theory. And in the setting of the classical theory of tensor norm s and operator ideals, this "nice behavior" are based on the possibility of representing every ultrapower $(\ell_p)_{\mathscr{U}}$ as an $L_p(\mu)$, for some measure space (Ω, Σ, μ), and the fact that every $L_p(\mu)$-space is an $\mathscr{L}^p$-space.

Let's see what happens when we translate the definition of the $\mathscr{L}^p$-spaces substituting ℓ_p for another Banach sequence space λ.

2 The Class of $\mathscr{L}^\lambda$-Spaces

Definition 2.1 Given a Banach sequence space λ, a Banach space X is said to be an $\mathscr{L}^\lambda$-space if there exists a real constant $c \geq 1$ such that for every finite-dimensional subspace M of X, there is a finite-dimensional subspace N of X containing M such that $d(N, S_{\dim(N)}(\lambda)) \leq c$.

The following proposition shows that all the ultrapowers of λ are $\mathscr{L}^\lambda$-spaces if and only if λ satisfies a certain property.

Proposition 2.1 *Given a Banach sequence space λ, the following conditions are equivalent:*

(1) Every ultrapower $(\lambda)_{\mathscr{U}}$ is a $\mathscr{L}^\lambda$ space.

*(2) λ satisfies the following property, denoted property (**P**): There is a constant $c \geq 1$ such that given a positive integer m, there exists some positive integer $n = n(m)$ only depending of m such that any m-dimensional subspace M of λ is contained in an n-dimensional subspace N of λ with $d(N, S_n(\lambda)) < c$.*

Proof $(1) \rightarrow (2)$: Let $\mathscr{U}$ be an ultrafilter on an index set D, and let $\{N_d, d \in D\}$ be an arbitrary family of n-dimensional subspaces of λ. If for every $d \in D$, $\{x_d^i, i = 1, \ldots, n\}$ is a basis of norm one vectors in N_d, $\{x^i = (x_d^i)_{\mathscr{U}}, i = 1, \ldots, n\}$ is an basis in $N = (N_d)_{\mathscr{U}}$, hence N is an n-dimensional subspace of $(\lambda)_{\mathscr{U}}$. From hypothesis $(\lambda)_{\mathscr{U}}$ is a $\mathscr{L}^\lambda$-space then, there are $c > 0$ and an m-dimensional subspace M of $(\lambda)_{\mathscr{U}}$ containing N such that $d(M, S_m(\lambda)) < c$. Let $\{(x_d^i)_{\mathscr{U}}, i = n+1, \ldots, m\}$ be a basis of an algebraic complement of N in M. Then $M = (M_d)_{\mathscr{U}}$ where $M_d = span(x_d^i, i = 1, \ldots, m)$. But as $\{(x_d^i)_{\mathscr{U}}, i = 1, \ldots, m\}$ is a basis in M, there is a $I_1 \in \mathscr{U}$ such that if $d \in I_1$, $\{x_d^i, i = 1, \ldots, m\}$ are linearly independent hence $dim(M_d) = m$ for every $d \in I_1$. But also there is a $I_0 \in \mathscr{U}$ such that $d(M, M_d) \leq 1 + \varepsilon$ for every $d \in I_0$, hence for every $d \in I_0 \cap I_1$, $N_d \subset M_d$ with $d(M_d, S_m(\lambda)) < c(1+\varepsilon)$. The result follows because D, $\mathscr{U}$ and $\{N_d, d \in D\}$ are arbitrary.

$(2) \rightarrow (1)$: If N is an n-dimensional subspace of $(\lambda)_{\mathscr{U}}$ and $\varepsilon > 0$, from [10] proposition 6.1, $N = (N_d)_{\mathscr{U}}$ with $dim(N_d) = n$. From hypothesis λ satisfies (**P**), hence there are $c \geq 1$ and $m = m(n)$ such that for every $d \in D$ there is a m-dimensional subspace M_d of λ containing N_d with $d(M_d, S_m(\lambda)) < c$. Then fixed a basis in every M_d, $M = (M_d)_{\mathscr{U}}$ is an m-dimensional subspace of $(\lambda)_{\mathscr{U}}$ containing N. Moreover there is $I_0 \in \mathscr{U}$ such that $d(M, M_d) < 1 + \varepsilon$ for every $d \in I_0$. Then $d(M, S_m(\lambda)) \leq c(1+\varepsilon)$ □

We emphasize that (**P**) is a very strong property that ℓ_p spaces satisfy [23], and Yves Raynaud pointed out to us that $\ell_p(\ell_q)$ also satisfies the property (**P**). But we don't know any Banach sequence space λ out of the very close ℓ_p setting with this property.

The limitation of the class $\mathscr{L}^{\lambda}$, if λ is not an ℓ_p space, with respect to the stability under ultrapowers of λ is the main motivation we had for considering that the resolution of the problem must be two stages:

I Investigate what **new property** of λ, less restrictive that the property **P**, could be desirable as a substitute for property **P**.

II Replace the class $\mathscr{L}^{p}$ for a **new class** so that if a Banach sequence space λ satisfies the **new property**, then the **new class** has a "nice behavior" in the sense that it is useful in the theory of tensor norms and operator ideals defined by λ, with basically means that the **new class** is stable under ultrapowers, biduals and complemented subspaces.

From the first moment we thought that a good candidate to **new property** could be the called **uniform projection property** defined by Pelczyński and Rosenthal in 1975 [19], because it has the same spirit as the property **P**, but is less restrictive.

Definition 2.2 A Banach space E has the **uniform projection property** if there is a positive real number $h > 0$ such that for each natural number m there is a natural number $n = n(m)$ only depending of m such that for every m-dimensional subspace $M \subset E$ there exists a k-dimensional and h-complemented subspace N of E containing M with $k \leq n$.

1) The class is quite large. Examples: reflexive Orlicz and modular spaces [21], Bochner spaces $L_p(\mu, E)$ if E does (hence $\ell_1(\lambda)$ if λ does) see [10], the Hardy space H^1 see [14].
2) The uniform projection property is stable under ultrapowers and duals, see [10, 11].

And to choose another candidate to **new class** we thought it would be wise to stay in the environment of Pelczyński, Lindenstrauss and Rosenthal.

3 DPR-Local Unconditional Structure: The Class of the Quasi-$\mathscr{L}^{\lambda}$-Spaces

The definition of the **new class** part of the observation of Dubinski, Pelczyǹski and Rosenthal [5] that, as the ℓ_p-spaces, certain classical Banach spaces admit a family of finite-dimensional subspaces having unconditional basis with respect to some constant, and that this family is dense in the family of all its finite-dimensional subspaces. With this idea, they define a notion of local unconditional structure in Banach spaces.

Definition 3.1 A Banach space X has **DPR-local unconditional structure** if there is $c \geq 1$ such that for each finite-dimensional subspace M of X there are a finite-dimensional subspace N of X containing M and a finite-dimensional Banach lattice L such that $d(N, L) \leq c$.

Every Banach lattice has *DPR*-local unconditional structure for all $c > 1$. And also every finite dimensional Banach space has *DPR*-local unconditional structure since it is isomorphic to a Banach lattice, but, as we can read in [2], it is the uniform bound on the isomorphisms of all finite-dimensional subspaces of an infinite dimensional Banach space X which restricts the class of the Banach spaces having local unconditional structure. Moreover from [13], a Banach space with *DPR*-local unconditional structure is either super-reflexive or it contains uniformly isomorphic copies of $S_n(\ell_1)$ or $S_n(\ell_\infty)$, for all $n \in \mathbb{N}$.

Inspired in the DPR-local unconditional structure we propose the following definition of the Anew class. If h is a positive real number, in short we say that a Banach space F is h-complemented in a Banach space E if the norm of the projection map $P : E \to F$ is less or equal than h.

Definition 3.2 Let λ be a Banach sequence space and let X be Banach space. We say that X is a quasi-$\mathscr{L}^\lambda$-space if there exist constants $c \geq 1$ and $h > 0$ such that for every finite-dimensional subspace M of X there are a finite-dimensional subspace N of X containing M and a h-complemented and finite-dimensional Banach subspace G of λ such that $d(N, G) \leq c$.

It is obvious that if λ satisfies the **uniform projection property** then λ is a quasi-$\mathscr{L}^\lambda$-space.

Next we have to test if the quasi-$\mathscr{L}^\lambda$ class is stable under ultrapowers, biduals and complemented subspaces.

To abbreviate, from now on for every Banach space E, *FIN*(E) represents the set of finite-dimensional subspaces of E.

Theorem 3.1 *Let λ be a Banach sequence space satisfying the **uniform projection property.** Then every ultrapower of λ (in particular λ itself) is a quasi-$\mathscr{L}^\lambda$-space.*

Proof Let $M = M = (M_d)_{\mathscr{U}}$ be a m dimensional subspace of $(\lambda)_{\mathscr{U}}$. Without loss of generality, we can assume that $\forall d \in D$ the dimension of M_d is m. As λ satisfies the uniform projection property, $\exists b > 0$, exists a natural number $n = n(m)$ only depending of m and $\exists \{N_d \in FIN(\lambda), d \in D\}$ such that $\forall d \in D$,

- $M_d \subset N_d$
- $\dim(N_d) \leq n(m)$
- N_d b-complemented in λ.

If $N = (N_d)_{\mathscr{U}}$, then $\dim(N) \leq n$ and it is b-complemented in $(\lambda)_{\mathscr{U}}$. Denote $N = (N_d)_{\mathscr{U}}$. Let $P_d : \lambda \to N_d, d \in D$ be the corresponding projections with $\|P_d\| \leq b$. Denote $P := (P_d)_{\mathscr{U}} : (\lambda)_{\mathscr{U}} \to N$ is a projection with $\|P\| \leq b$. Given $0 < \varepsilon < 1$, it s known that there exists a $A_0 \in \mathscr{U}$ such that for every $d_0 \in A_0$ and $\forall x = (x_d)_{\mathscr{U}} \in N$,

$$(1-\varepsilon)\|x\| \leq \|x_{d_0}\| \leq (1+\varepsilon)\|x\|$$

Then $d(N, N_0) \leq \frac{1+\varepsilon}{1-\varepsilon}$. □

To study the stability of the class of quasi-$\mathscr{L}^{\lambda}$-spaces under biduals we need the **Principle of Local Reflexivity** of Linndenstrauss and Rosenthal [19]:

Theorem 3.2 *Let X be a Banach space, and let $G \subset X''$ and $F \subset X'$ be finite-dimensional subspaces. Given $\varepsilon > 0$ there exists a ε-isometry $A : G \to X$ (that is if $y \in G$, $(1-\varepsilon)\|y\| \leq \|A(y)\| \leq (1+\varepsilon)\|y\|$) such that*

- $A|_{G\cap X} = id|_{G\cap X}$
- $< f, A(g) >=< g, f >$, *for every $g \in G$ and $f \in F$.*

A well known consequence of the **Principle of Local Reflexivity** says that X'' is a 1-complemented subspace of some ultrapower of X. In this direction we have the following property:

Theorem 3.3 *Let X be a quasi-$\mathscr{L}^{\lambda}$-space. Then $id_{X''}$ factors through an ultrapower of λ such that if $S_1 : X'' \to (\lambda)_{\mathscr{U}}$ and $S_2 : (\lambda)_{\mathscr{U}} \to X''$ then there is complemented subspace G of $(\lambda)_{\mathscr{U}}$ and $T : X'' \to G$ such that if $I_G : G \to (\lambda)_{\mathscr{U}}$ is the inclusion map then $S_1 = I_G T$.*

Proof The proof is based in the following fact: If $(x_d)_{\mathscr{U}} \in (X)_{\mathscr{U}}$, then $\{x_d, d \in D\}$ is bounded in X, and then it is relatively-$\sigma(X'', X')$-compact in X'', hence $\lim_{\mathscr{U}} x_d$ exists in the $\sigma(X'', X')$-topology of X''.

First of all consider the set $D = \{(M, N, \varepsilon), M \in FIN(X''), N \in FIN(X'), \varepsilon > 0\}$ endowed the order $(M_1, N_1, \varepsilon_1) \leq (M_2, N_2, \varepsilon_2)$ if and only if $M_1 \subset M_2, N_1 \subset N_2, \varepsilon_1 \geq \varepsilon_2$ We denote $R(d_0) = \{d \in D : d_0 \leq d\}$. Then $\mathscr{R} = \{R(d), d \in D\}$ is a filter basis of D. Let $\mathscr{U}$ be an ultrafilter containing $\mathscr{R}$.

For the Principle of Local Reflexivity $\forall d = (M_d, N_d, \varepsilon_d)$ such that $\lim_U \varepsilon_d = 0$, $\exists F_d \in FIN(X)$ and $T_d : M_d \to F_d$ with $\|T_d\|\ \|T_d^{-1}\| \leq 1 + \varepsilon_d$ such that

- $(T_d)_{|M_d \cap F_d} = I_{|M_d \cap F_d}$.
- $\forall x'' \in M_d, \forall x' \in N_d, < x', T_d(x'') =< x'', x' >$.

As X is a a quasi-$\mathscr{L}^{\lambda}$-space with constants $c \geq 1$ and $b > 0$, $\forall d \in D$,

- $\exists Y_d \in FIN(X)$ such that $F_d \subset Y_d \in FIN(X)$ $\exists G_d \in FIN(\lambda)$ complemented in λ with projection $P_d : \lambda \to G_d$, $\|P_d\| \leq b$
- $\exists S_d : Y_d \to G_d$, $\|S_d\|\ \|S_d^{-1}\| \leq c + \varepsilon_d$.

Then $(Y_d)_{\mathscr{U}}$ and $G := (G_d)_{\mathscr{U}}$ are isomorphic, and G is b-complemented in $(\lambda)_{\mathscr{U}}$.

We construct the map $T : X'' \to (Y_d)_{\mathscr{U}}$ such that if $T(x'') = (x_d)_{\mathscr{U}}$, then $x_d = T_d(x'')$ if $x'' \in M_d$, and $x_d = 0$ if $x'' \notin M_d$.

Given $x'' \in X''$, for all $x' \in X'$, there is $d_0 = (M_{d_0}, N_{d_0}, \varepsilon_{d_0})$ such that $x'' \in M_{d_0}$ and $x' \in N_{d_0}$. Then $\lim_{\mathscr{U}} < T_d(x''), x' >=< x'', x' >$. Hence $\lim_{\mathscr{U}} T_d(x'') = x''$ in the $\sigma(X'', X')$ topology of X''. And also $\|x''\| = \lim_{\mathscr{U}} \|(T_d(x''))_{\mathscr{U}}\|$. Then if we identify x'' with $T(x'')$, we can say that T is an isometric embedding.

Define $Q = (Q_d)_{\mathscr{U}} : (\lambda)_{\mathscr{U}} \to (Y_d)_{\mathscr{U}}$ such that $Q_d = S_d^{-1} P_d, \forall d \in D$. Identifying $(S_d^{-1} P_d(w_d))_{\mathscr{U}}$ with $\lim_{\mathscr{U}} S_d^{-1} P_d(w_d)$, which is some element of X'', we can consider that $Q : (\lambda)_{\mathscr{U}} \to X''$.

Denote I_d the inclusion map of G_d in λ. Then $\lim_{\mathscr{U}} < Q_d I_d S_d T(x''), x' >=< x'', x' >, \forall x' \in X'$ and $\forall x'' \in X''$. Hence $\lim_{\mathscr{U}} Q_d I_d S_d T(x'') = x''$. □

From Theorem 3.3 it is clear that for every finite-dimensional subspace M of the bidual of a quasi-$\mathscr{L}^\lambda$-space X where λ satisfies the **uniform projection property**, the identity map id_M factors through a complemented finite-dimensional subspace G of λ. But this no proves that M is contained in a finite-dimensional subspace N of X'' such that $d(N, G) \leq c$, for some constant c. That is, 3.3 does not prove that X'' is a quasi-$\mathscr{L}^\lambda$-space.

In the same way it Y is a complemented subspace of a quasi-$\mathscr{L}^\lambda$-space X where λ satisfies the **uniform projection property**, then there exist constants $c \geq 1$ and $h > 0$ such that if $M \in FIN(Y)$ there are a $N \in FIN(X)$ and a h-complemented and finite dimensional Banach subspace G of λ) such that $d(N, G) \leq c$, but it does not prove that Y is a quasi-$\mathscr{L}^\lambda$-space because we cannot prove that $N \in FIN(Y)$.

Then the class of quasi-$\mathscr{L}^\lambda$-spaces does not have the "nice behavior" we need in the theory of tensor norms and operator ideals associated to the Banach sequence space λ, hence we have to reject it. But the information obtained through these two failures illuminates a new possibility.

4 (GL)-Local Unconditional Structure: The Class of $\mathscr{L}^{\lambda,g}$-Spaces

In view of the difficulties we have had with the class of quasi-$\mathscr{L}^\lambda$-spaces, whose definition is inspired by the notion of (DPR)-local unconditional structure, we propose the definition of a new class based on another way of understanding the concept of local unconditional structure given by Gordon and Lewis in 1974 [7], which is more flexible and more manageable than the (DPR)-local unconditional structure, but nonetheless has the same spirit and plays a very similar role.

Definition 4.1 We say that a Banach space X has **(GL)-local unconditional structure** if there exists a real constants $b > 0$ such that for every finite-dimensional subspace M of X, there are a finite-dimensional lattice L and linear operators $u : F \to L$ and $v : L \to X$ such that $\|u\| \, \|v\| \leq b$ and $v\, u = I_{M,X}$.

We recall that the **local unconditional structure of Gordon and Lewis** has been a basic tool in the study of many important topics in the theory of Banach spaces, Banach lattices, Banach algebras and operators, for example, in the study of unconditional basis (it is significant that the James space J does not have **(GL)-local unconditional structure**) and other geometrical aspects of Banach spaces, Banach lattices and operator ideals. The following characterization of Figiel, Johnson and

Tzafriri, [6], see also [15, 16], of the Banach spaces having **(GL)-local unconditional structure** is very significant.

Theorem 4.1 *A Banach space X gas* ***(GL)-local unconditional structure*** *if and only if X'' is isomorphic to a complemented subspace of a Banach lattice.*

Inspired by the notion of (GL)-local unconditional structure, we will propose the definition of another **new class** of Banach spaces, denoted the class of the $\mathscr{L}^{\lambda,g}$-spaces, as a candidate to replace the class $\mathscr{L}^p$ when we substitute ℓ_p by a more general Banach sequence space λ.

Definition 4.2 Given a Banach sequence space λ, a Banach space X is said to be a $\mathscr{L}^{\lambda,g}$-space if exists real constants $h, b > 0$ such that for every finite-dimensional subspace M of X, there exist a finite-dimensional and h-complemented subspace G of λ and linear operators $u : M \to G$ and $v : G \to X$ such that $\|u\| \, \|v\| \leq b$ and $v\, u = I_{M,X}$.

Obviously quasi-$\mathscr{L}^\lambda \subset \mathscr{L}^{\lambda,g}$. Hence if λ satisfies the **uniform projection property** then $\lambda \in \mathscr{L}^{\lambda,g}$ and $(\lambda)_{\mathscr{U}} \in \mathscr{L}^{\lambda,g}$, for every ultrafilter $\mathscr{U}$.

Theorem 4.2 *Let λ be a Banach sequence space and let X be a Banach space. Consider the following statements:*

(i) $X \in \mathscr{L}^{\lambda,g}$.
(ii) $I_{X,X''}$ *factors through an ultrapower of* λ.
(iii) $id_{X''}$ *factors through an ultrapower of* λ.

Then,
(a) (i) $\to$ (ii) $\leftrightarrow$ (iii).
(b) If λ satisfies the ***uniform projection property****, all are equivalent.*

Proof (a) Consider the ultrafilter of the proposition 3.3. By hypothesis, there are $h, b > 0$ such that for every $d \in D$, there are an h-complemented finite-dimensional subspace S_d of λ and maps $u_d : M_d \to S_d$ and $v_d : S_d \to X$ such that $v_d u_d = I_{M_d,X}$ and $\|u_d\| \, \|v_d\| \leq b$. Then,

(i)$\to$ (ii) because X is isometric to a subspace of $(M_d)_{\mathscr{U}}$, by local reflexivity X'' is isometric to a 1-complemented subspace of $(X)_{\mathscr{U}}$ and $(S_d)_{\mathscr{U}}$ is a complemented subspace of $(\lambda)_{\mathscr{U}}$ with projection norm less or equal to h.

(ii) $\to$ (iii): Since $I_{X,X''} = S_2 S_1$ with $S_1 : X \to (\lambda_r)_{\mathscr{U}}$ and $S_2 : (\lambda_r)_{\mathscr{U}} \to X''$, then by bidualization we have $(I_{X,X''})'' = S_2'' S_1''$, which provides a factorization of the biconjugate of the inclusion map through $((\lambda_r)_{\mathscr{U}})''$. Note that although $(I_{X,X''})''$ is an into isometry $X'' \to X''''$ which differs generally from the natural inclusion map $I_{X'',X''''}$, it has nevertheless the same left inverse $P = (I_{X',X'''})'$. Hence $I_{X''} = P S_2'' S_1''$ provides a factorization of the identity of X'' through $((\lambda)_{\mathscr{U}})''$. Now the usual local reflexivity argument gives a factorization of $((\lambda)_{\mathscr{U}})''$ through an iterated ultrapower $((\lambda_r)_{\mathscr{U}})_{\mathscr{V}}$, which is isometric with $(\lambda_r)_{\mathscr{U}\times\mathscr{V}}$.

(b) it is enough to see that $(iii) \to (i)$. As λ satisfies the uniform projection property, $(\lambda)_{\mathscr{U}}$ is a quasi-$\mathscr{L}^\lambda$-space. If $id_{X''} = S_2 S_1$, $S_1 : X'' \to (\lambda)_{\mathscr{U}}$ and

$S_2 : (\lambda)_{\mathscr{U}} \to X''$, given a finite-dimensional subspace M of X, $S_1(M)$ is a finite dimensional subspace of $(\lambda)_{\mathscr{U}}$, then there are a $N \in FIN(\lambda)_{\mathscr{U}})$ containing $S_1(M)$, an h-complemented subspace G of λ and an isomorphism $C : N \to G$ such that $\|C\| \, \|C^{-1}\| \le c$. The result follows defining $u := C I_{S_1(M),N} S_1$ and $v := S_2 C^{-1}$, taking into account that $S_2 S_1(X) = X$. □

Theorem 4.3 *Let λ be a Banach sequence space satisfying the* ***uniform projection property****. Then the class $\mathscr{L}^{\lambda,g}$ is stable*

(1) Under complemented subspaces.
(2) Under ultrapowers.
(3) Under biduals.

Proof (1) Let Y be a complemented subspace of a $\mathscr{L}^{\lambda,g}$-space X with projection P. If M is a finite-dimensional subspace of Y it is also a finite-dimensional subspace of X. Then there are real constants $h, b > 0$ a finite-dimensional and h-complemented subspace G of λ and linear operators $u : M \to G$ and $v : G \to X$ such that $\|u\| \, \|v\| \le b$ and $v\, u = I_{M,X}$. The result follows taking $U = u$ and $V = Pv$.

(2) If X is a $\mathscr{L}^{\lambda,g}$, then $I_{X,X''}$ factors through some $(\lambda)_{\mathscr{U}}$. Hence the inclusion map $(I_{X,X''})_{\mathscr{V}} : (X)_{\mathscr{V}} \to (X'')_{\mathscr{V}}$ factors through $((\lambda)_{\mathscr{U}})_{\mathscr{V}} = (\lambda)_{\mathscr{U}\times\mathscr{V}}$. The result follows because $(X'')_{\mathscr{V}}$ is a subspace of $((X)_{\mathscr{V}})''$, and then $I_{(X)_{\mathscr{V}},((X)_{\mathscr{V}})''}$ factors through $(\lambda)_{\mathscr{U}\times\mathscr{V}}$.

(3) The result follows from (1) and (2), because the bidual of a Banach space X is a 1-complemented subspace of some ultrapower of X, and for (2) if X is a $\mathscr{L}^{\lambda,g}$-space Every ultrapower of X is also a $\mathscr{L}^{\lambda,g}$-space, hence the bidual of X is a complemented subspace of a $\mathscr{L}^{\lambda,g}$-space, and for (1) it is also a $\mathscr{L}^{\lambda,g}$-space. □

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