Chapter 6
Gas Flow with Phase Transitions: Thermodynamics and the Navier–Stokes Equations

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6.1 Introduction

One-dimensional flows of gas or liquid are described by the following system of Navier–Stokes equations (see for example, [1]):

$$\begin{cases} \rho(u_t + uu_x) = -p_x + \eta u_{xx}, \\ \rho_t + (\rho u)_x = 0, \\ T\rho(s_t + us_x) - kT_{xx} - \eta(u_x)^2 = 0. \end{cases} \tag{6.1}$$

Here $\rho(t, x)$ is the density of the gas, $u(t, x)$ is the velocity, $p(t, x)$ is the pressure, $s(t, x)$ is the specific entropy, $T(t, x)$ is the temperature, k and η are coefficients of thermal conductivity and viscosity correspondingly, which are assumed to be constants.

The first equation of system (6.1) corresponds to the momentum conservation law of the medium, the second one is the continuity equation and the third one is the equation of heat conduction, which represents the energy conservation law. System (6.1) is incomplete. It consists of three equations for five unknown functions

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$\rho(t, x), u(t, x), p(t, x), s(t, x), T(t, x)$. To make it complete we need two additional equations describing thermodynamic properties of the gas—the equations of state.

The paper has the following structure. In Sect. 6.2, we give a geometrical description of the thermodynamic state. We consider the thermodynamic state as two-dimensional Lagrangian manifold, which can be defined by two equations with compatibility condition.

In Sect. 6.3, we study state equations and corresponding Lagrangian manifolds for van der Waals gases and get its applicable domains with a description of phase transitions.

In Sect. 6.4, we look for solutions as asymptotic expansions and analyse the zeroth and the first-order approximations.

In Sect. 6.5, we show space-time domains corresponding to different phases of the medium.

Essential computations in this paper were done in Maple with the Differential Geometry package created by I. Anderson, the corresponding files could be found in http://d-omega.org/appendices/.

6.2 Geometric Representation of Thermodynamic States

Let $\mathbb{R}^5$ be a 5-dimensional contact space equipped with the coordinates (p, ρ, e, T, s), where e represents the specific energy and the other coordinates represent thermodynamic quantities mentioned above, and the contact 1-form [2, 3]:

$$\theta = \frac{1}{T} de - ds - \frac{p}{T\rho^2} d\rho.$$

In our consideration, the thermodynamic state is a 2-dimensional Legendrian manifold $L \subset \mathbb{R}^5(p, \rho, e, T, s)$, such that

$$\theta|_L = 0.$$

The last condition means that the first law of thermodynamics holds on the manifold L.

If the specific entropy is a given function $s = s(e, \rho)$, the condition $\theta|_L = 0$ leads to the following relations, that define 2-dimensional Legendrian manifold $L \subset \mathbb{R}^5(p, \rho, e, T, s)$:

$$s = s(e, \rho), \quad p = -\rho^2 \frac{s_\rho}{s_e}, \quad T = \frac{1}{s_e}. \tag{6.2}$$

Since the equations of state usually include the specific energy and do not include the specific entropy, we shall eliminate the specific entropy s from our consideration. To this end, we consider the projection $\phi\colon \mathbb{R}^5 \to \mathbb{R}^4, \quad \phi\colon (\rho, p, e, T, s) \mapsto (\rho, p, e, T)$ and symplectic space $\mathbb{R}^4$ equipped with structure 2-form

$$\Omega = d\theta = \frac{1}{T^2}de \wedge dT - \frac{1}{T\rho^2}dp \wedge d\rho + \frac{p}{T^2\rho^2}dT \wedge d\rho.$$

The restriction of the map ϕ on the Legendrian surface L is a diffeomorphism on the image $\overline{L} = \phi(L)$. The surface $\overline{L} \subset \mathbb{R}^4$ is a Lagrangian manifold. Equally, the thermodynamic state can be considered as a 2-dimensional Lagrangian manifold $\overline{L} \subset \mathbb{R}^4(\rho, p, e, T)$, i.e. $\Omega|_{\overline{L}} = 0$.

Any 2-dimensional Lagrangian manifold $\overline{L} \subset \mathbb{R}^4(\rho, p, e, T)$ is defined by the two equations

$$\begin{cases} f(\rho, p, e, T) = 0, \\ g(\rho, p, e, T) = 0, \end{cases} \tag{6.3}$$

and the condition that the surface $\overline{L}$ is Lagrangian can be written as:

$$[f, g] = 0 \text{ on } \overline{L}, \tag{6.4}$$

where $[f, g]$ is the Poisson bracket with respect to the symplectic form Ω, i.e.

$$[f, g]\ \Omega \wedge \Omega = df \wedge dg \wedge \Omega.$$

In coordinates (ρ, p, e, T) this bracket has the following form:

$$[f, g] = T\rho^2\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial \rho} - \frac{\partial f}{\partial \rho}\frac{\partial g}{\partial p}\right) + T^2\left(\frac{\partial f}{\partial T}\frac{\partial g}{\partial e} - \frac{\partial f}{\partial e}\frac{\partial g}{\partial T}\right) + Tp\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial e} - \frac{\partial f}{\partial e}\frac{\partial g}{\partial p}\right).$$

Condition (6.4) means the integrability of the following system of PDEs:

$$\begin{cases} f\left(\rho, -\rho^2\frac{s_\rho}{s_e}, e, \frac{1}{s_e}\right) = 0, \\ g\left(\rho, -\rho^2\frac{s_\rho}{s_e}, e, \frac{1}{s_e}\right) = 0. \end{cases}$$

Thus, in what follows, by the system of Navier–Stokes equations we shall understand system (6.1) together with two additional equations of state (6.3) satisfying relation (6.4).

6.3 Van der Waals Gases

6.3.1 *The Equations of State*

The most important class of real gases is described by the van der Waals equation:

$$f(\rho, p, e, T) = (p + a\rho^2)\left(\frac{1}{\rho} - b\right) - RT, \tag{6.5}$$

here a is a characteristic of the gas responsible for the interaction between the particles and b is particles' volume, R is the universal gas constant. To find the second equation we use the condition of compatibility, which is expressed in (6.4) and gives the following result:

Proposition 6.1 *Assuming that the specific energy is a function of density ρ and temperature T*

$$g(\rho, p, e, T) = e - \beta(\rho, T),$$

the second state equation for the van der Waals gas has to be in the form

$$\beta(\rho, T) = -a\rho + E(T),$$

where $E(T)$ is a smooth function.

Proposition 6.2 *Since the specific energy is the sum of the energy of the particles' motion and the energy of their interaction, the function $E(T)$ has to be as follows*

$$E(T) = \frac{fR}{2}T,$$

here f is a degree of freedom.

Thus, the equations of state for the van der Waals gas are

$$\begin{cases} (p + a\rho^2)\left(\frac{1}{\rho} - b\right) - RT = 0, \\ e - \frac{fRT}{2} + a\rho = 0. \end{cases} \tag{6.6}$$

To get the specific entropy s as function of the specific energy e and density ρ, we integrate system (6.6) using (6.2). We have

$$s(e, \rho) = \frac{fR}{2}\ln(e + a\rho) + R\ln\left(\frac{1}{\rho} - b\right) + s_0. \tag{6.7}$$

Thus, formulae (6.6) and (6.7) define the thermodynamic state of van der Waals gases or Legendrian manifold L.

6.3.2 Applicable Domains for the Van der Waals Gas

In this section we discuss domains where the van der Waals model is valid. We call them applicable.

Let $V = 1/\rho$ be a specific volume. First of all, we note that due to (6.5) we have restriction for volumes to consider:

$$V > b.$$

This condition is absolutely clear from the physical point of view: the volume occupied by the gas cannot be less than the particles' volume.

There is another condition for thermodynamic quantities to be applicable. The Lagrangian manifold $\overline{L}$ is equipped with quadratic differential form $\kappa|_{\overline{L}}$, which has to be negative [6]:

$$\kappa|_{\overline{L}} = d(T^{-1}) \cdot de + d(pT^{-1}) \cdot dV.$$

This allows to select domains on $\overline{L}$ where the model of van der Waals gas is applicable. In case of van der Waals gases the form $\kappa|_{\overline{L}}$ is following:

$$\kappa|_{\overline{L}} = -\frac{fR}{2(e+a/V)^2} de \cdot de + \frac{fRa}{(e+a/V)^2 V^2} de \cdot dV + \left(\frac{fRa}{V^3(e+a/V)} - \frac{fRa^2}{2V^4(e+a/V)^2} - \frac{R}{(V-b)^2} \right) dV \cdot dV.$$

The form $\kappa|_{\overline{L}}$ is negative if and only if its determinant is positive, which leads to the following inequality:

$$eV^3 - a(f-1)V^2 + 2abfV - ab^2 f > 0.$$

Using $e = fRT/2 - a/V$ we get:

$$\frac{1}{2} RTV^3 - aV^2 + 2abV - ab^2 > 0. \tag{6.8}$$

Let us introduce contact transformation

$$\tilde{T} = \frac{T}{T_{\text{crit}}}, \quad \tilde{V} = \frac{V}{V_{\text{crit}}}, \quad \tilde{p} = \frac{p}{p_{\text{crit}}}, \quad \tilde{e} = \frac{e}{e_{\text{crit}}}, \quad \tilde{s} = \frac{s}{s_{\text{crit}}},$$

where $T_{\text{crit}}, V_{\text{crit}}, p_{\text{crit}}, e_{\text{crit}}, s_{\text{crit}}$ are critical parameters for van der Waals gases:

$$T_{\text{crit}} = \frac{8a}{27Rb}, \quad V_{\text{crit}} = 3b, \quad p_{\text{crit}} = \frac{a}{27b^2}, \quad e_{\text{crit}} = \frac{a}{9b}, \quad s_{\text{crit}} = \frac{3R}{8}.$$

Then inequality (6.8) can be written in dimensionless variables $\tilde{T}$ and $\tilde{V}$:

$$4\tilde{V}^3\tilde{T} - 9\tilde{V}^2 + 6\tilde{V} - 1 > 0,$$

which defines the applicable domains of specific volume and temperature for the van der Waals gas. They are shown in Fig. 6.1. The picture shows that the van der Waals model is correct at any point (V, T) over the critical one. The forbidden area corresponds to phase transitions.

Fig. 6.1 Applicable domains for the van der Waals gas. White area corresponds to forbidden volumes and temperatures

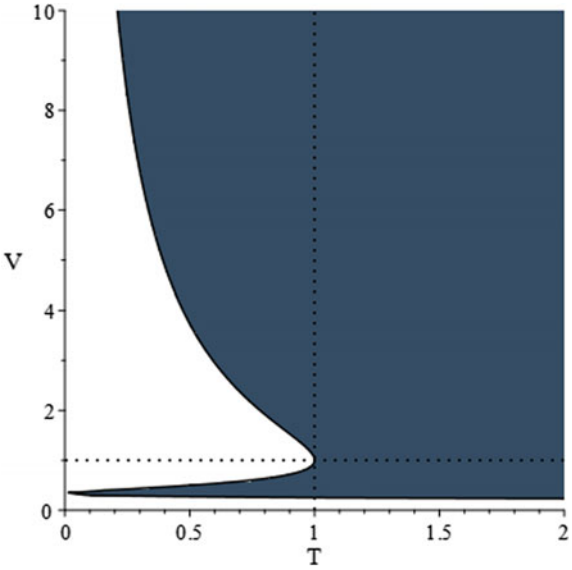
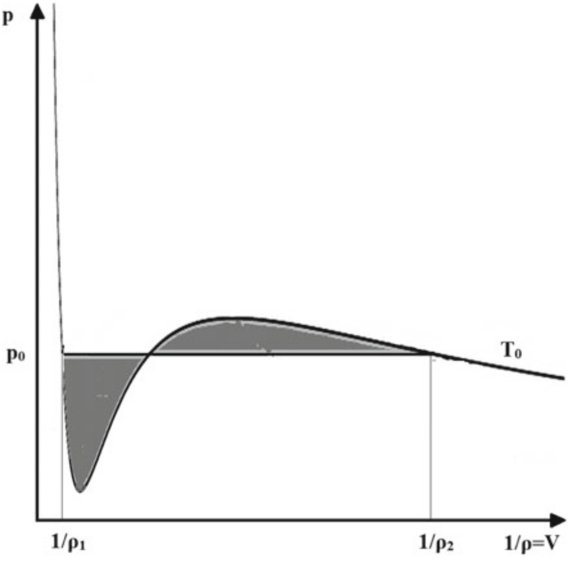


Fig. 6.2 Isotherm for the van der Waals gas



6.3.3 Phase Transitions

Phase transitions for the van der Waals gas can be described by means of Fig. 6.2. Grey domains in this picture correspond to intermediate state. At points $1/\rho_2$ and $1/\rho_1$, phase transition starts and finishes correspondingly. To find these points, we use the following condition of thermodynamic equilibrium, which claims that the chemical potential of different phases of our system is the same:

$$\mu(T_0, p_0, \rho_1) = \mu(T_0, p_0, \rho_2) = \mu_0,$$

where T_0 and p_0 are the temperature and the pressure of phase transition. The expression for the chemical potential of gases is

$$\mu = e - Ts + \frac{p}{\rho},$$

and for van der Waals gases it can be expressed in terms of pressure p, temperature T and density ρ:

$$\mu = \frac{fRT}{2} - \rho a - T\left(\frac{fR}{2}\ln\left(\frac{fR}{2}T\right) + R\ln\left(\frac{1}{\rho} - b\right)\right) + \frac{p}{\rho}.$$

Moreover, the equation of state of the gas must be satisfied at the points (T_0, p_0, ρ_1) and (T_0, p_0, ρ_2). As a result we obtain the following system of equations for ρ_1 and ρ_2:

$$\mu_0 = \frac{fRT_0}{2} - \rho_1 a - T_0\left(\frac{fR}{2}\ln\left(\frac{fR}{2}T_0\right) + R\ln\left(\frac{1}{\rho_1} - b\right)\right) + \frac{p_0}{\rho_1}, \tag{6.9}$$

$$\mu_0 = \frac{fRT_0}{2} - \rho_2 a - T_0\left(\frac{fR}{2}\ln\left(\frac{fR}{2}T_0\right) + R\ln\left(\frac{1}{\rho_2} - b\right)\right) + \frac{p_0}{\rho_2}, \tag{6.10}$$

$$p_0 - p_0\rho_1 b + a\rho_1^2 - ab\rho_1^3 - \rho_1 RT_0 = 0, \tag{6.11}$$

$$p_0 - p_0\rho_2 b + a\rho_2^2 - ab\rho_2^3 - \rho_2 RT_0 = 0. \tag{6.12}$$

Eliminating μ_0 and p_0 from (6.9)–(6.12) we get the following equations:

$$(\rho_1 - \rho_2)(RT_0 - a(\rho_1 + \rho_2)(b\rho_1 - 1)(b\rho_2 - 1)) = 0,$$

$$\rho_1 RT_0(b\rho_2 - 1)\ln\left(\frac{\rho_1(1 - b\rho_2)}{\rho_2(1 - b\rho_1)}\right) + (\rho_1 - \rho_2)(a\rho_1(1 - b\rho_2) + ab\rho_2^2 + RT_0 - a\rho_2) = 0.$$

There is the trivial solution $\rho_1 = \rho_2$, which is out of interest, because the temperature is assumed to be under the critical value. In general case the solution is given by Fig. 6.3. We can see that the straight line and the points C and D correspond to

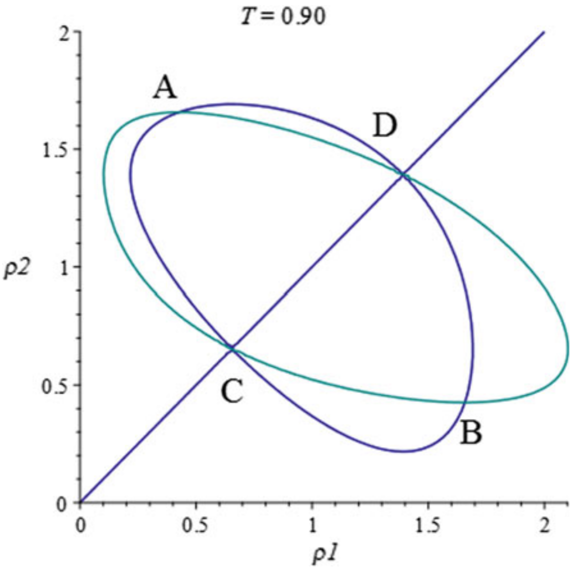


Fig. 6.3 Solution

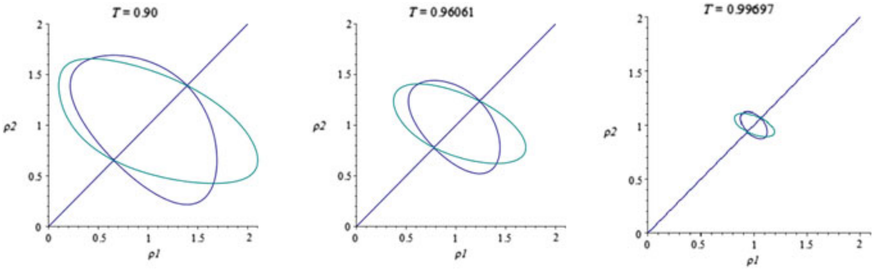


Fig. 6.4 Dynamics of the solution

trivial solution $\rho_1 = \rho_2$. The two other points A and B of intersection of the curves correspond to solution for ρ_1 and ρ_2. Since we have not specified which density is greater, the points A and B define the same solution. If we change the temperature of phase transition T_0, we can see that values ρ_1 and ρ_2 become closer and there is only one solution $\rho_1 = \rho_2$ when $T_0 = T_{\text{crit}}$ (Fig. 6.4).

6.4 Asymptotic Expansions for Solution

6.4.1 *Zeroth-Order Approximation*

Recall that we consider the system of equations:

$$\begin{cases} \rho(u_t + uu_x) = -p_x + \eta u_{xx}, \\ \rho_t + (\rho u)_x = 0, \\ T\rho(s_t + us_x) - kT_{xx} - \eta(u_x)^2 = 0, \end{cases} \tag{6.13}$$

extended by the equations of state (Legendrian manifold L):

$$\begin{cases} s(e,\rho) = \frac{fR}{2}\ln(e + a\rho) + R\ln\left(\frac{1}{\rho} - b\right) + s_0, \\ (p + a\rho^2)\left(\frac{1}{\rho} - b\right) - RT = 0, \\ e - \frac{fRT}{2} + a\rho = 0. \end{cases} \tag{6.14}$$

We are looking for asymptotical solution of system (6.13)-(6.14) with respect to van der Waals parameters a and b:

$$u(t,x) = u_0(t,x) + au_1(t,x) + bu_2(t,x) + \cdots,$$

$$\rho(t,x) = \rho_0(t,x) + a\rho_1(t,x) + b\rho_2(t,x) + \cdots,$$

$$e(t,x) = e_0(t,x) + ae_1(t,x) + be_2(t,x) + \cdots.$$

For simplicity, we shall continue to use $u(t,x)$, $\rho(t,x)$, $e(t,x)$ instead of $u_0(t,x)$, $\rho_0(t,x)$, $e_0(t,x)$ and get the following equations which describe the zeroth-order approximation:

$$\rho(u_t + uu_x) + \frac{2}{f}(\rho e)_x - \eta u_{xx} = 0, \tag{6.15}$$

$$\rho_t + \rho_x u + \rho u_x = 0, \tag{6.16}$$

$$\rho(e_t + ue_x) - \frac{2}{f}e(\rho_t + u\rho_x) - \eta(u_x)^2 - \frac{2k}{Rf}e_{xx} = 0. \tag{6.17}$$

This system corresponds to equations (6.1) for the ideal gas. It defines a smooth submanifold $\mathcal{E} \subset J^2(\pi)$, here π is a 3-dimensional bundle [4, 5]:

$$\pi : \mathbb{R}^5 \to \mathbb{R}^2, \quad \pi : (t,x,u,\rho,e) \mapsto (t,x).$$

Proposition 6.3 *The symmetry algebra* $\mathfrak{g}$ *of the system* $\mathcal{E}$ *is solvable and generated by the following vector fields on the space* $J^0(\pi)$*:*

Table 6.1 The Lie algebra structure

Field	X_1	X_2	X_3	X_4	X_5
X_1	0	0	X_2	X_1	0
X_2	0	0	0	0	X_2
X_3	$-X_2$	0	0	$-X_3$	X_3
X_4	$-X_1$	0	X_3	0	0
X_5	0	$-X_2$	$-X_3$	0	0

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = t\partial_x + \partial_u,$$

$$X_4 = t\partial_t + \rho\partial_\rho - u\partial_u - 2e\partial_e,$$

$$X_5 = x\partial_x - 2\rho\partial_\rho + u\partial_u + 2e\partial_e,$$

The Lie algebra structure is represented in Table 6.1. The table shows that the Lie algebra $\mathfrak{g}$ is solvable:

$$\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \langle X_1, X_2, X_3 \rangle$$

$$\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = \langle X_2 \rangle, \quad \mathfrak{g}^{(3)} = [\mathfrak{g}^{(2)}, \mathfrak{g}^{(2)}] = 0.$$

We are going to find solutions of system (6.15)–(6.17) invariant with respect to a one-dimensional subalgebra of $\mathfrak{g}$ and also we want to get the reduced ordinary system having as many symmetries as possible. Since the symmetries in the normalizer of one-dimensional subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ are the symmetries of the reduced equations, we compute the normalizers of all admissible one-dimensional subalgebras in $\mathfrak{g}$:

$$N_X = \{\, Y \in \mathfrak{g} \mid [X, Y] = \lambda X \,\}, \quad \text{where } \lambda \text{ is a parameter.}$$

One may show that in our case one-dimensional subalgebra

$$\mathfrak{h} = \langle \alpha_2 X_2 + \alpha_3 X_3 + \alpha_5 X_5 \rangle,$$

where α_j are constants, has the biggest normalizer

$$N_{\mathfrak{h}} = \left\langle \frac{\alpha_2}{\alpha_3} X_1 + \frac{\alpha_3}{\alpha_5} X_3 + X_5, -\frac{\alpha_3}{\alpha_5} X_3 + X_4, X_1 - \frac{\alpha_3}{\alpha_5} X_2 \right\rangle.$$

The $\mathfrak{h}$-invariant solution of system (6.15)–(6.17) has the following form:

$$e(t,x) = (\alpha_3 t + \alpha_5 x + \alpha_2)^2 F_1(t), \quad \rho(t,x) = \frac{F_2(t)}{(\alpha_3 t + \alpha_5 x + \alpha_2)^2},$$

$$u(t,x) = (\alpha_3 t + \alpha_5 x + \alpha_2) F_3(t) - \frac{\alpha_3}{\alpha_5},$$

and the reduced ordinary equations are

$$F_3' + \alpha_5 F_3^2 = 0, \quad -F_2' + \alpha_5 F_2 F_3 = 0, \tag{6.18}$$

$$F_1' F_2 R f - 2 R F_1 F_2' - \alpha_5 \left(F_1 (4k\alpha_5 - 2R(f+2) F_2 F_3) + R f \eta \alpha_5 F_3^2 \right) = 0. \tag{6.19}$$

After the integration of these equations we get the following:

$$\rho(t,x) = \frac{C_2(\alpha_5 t + C_1)}{(\alpha_3 t + \alpha_5 x + \alpha_2)^2}, \quad u(t,x) = \frac{\alpha_3 t + \alpha_5 x + \alpha_2}{\alpha_5 t + C_1} - \frac{\alpha_3}{\alpha_5}, \tag{6.20}$$

$$e(t,x) = (\alpha_3 t + \alpha_5 x + \alpha_2)^2 \left(\frac{\alpha_5 R f \eta}{2(RC_2 - 2k\alpha_5)(\alpha_5 t + C_1)^2} + C_3 (\alpha_5 t + C_1)^{-2 - \frac{2}{f} + \frac{4k\alpha_5}{C_2 R f}} \right), \tag{6.21}$$

where C_1, C_2 and C_3 are constants.

This solution represents the zeroth-order approximation of the solution for the van der Waals gas. Since the flows of vector fields X_2 and X_3 are the shift and Galilean transformation correspondingly, their influence on the solution is not crucial: Galilean transformation makes the frame of reference move with constant velocity and the shift along the x-axis just changes the location of the origin. Assuming that our frame of reference does not move and the point $x = 0$ corresponds to the origin we shall take $\alpha_3 = \alpha_2 = 0$, $\alpha_5 = 1$ in (6.20)–(6.21).

6.4.2 First-Order Approximation

The equations for the first-order corrections $u_1(t,x)$, $\rho_1(t,x)$ and $e_1(t,x)$ can be written in the following form:

$$\begin{pmatrix} u_1 \\ \rho_1 \\ e_1 \end{pmatrix}_t = A(t,x) \begin{pmatrix} u_1 \\ \rho_1 \\ e_1 \end{pmatrix}_{xx} + B(t,x) \begin{pmatrix} u_1 \\ \rho_1 \\ e_1 \end{pmatrix}_x + C(t,x) \begin{pmatrix} u_1 \\ \rho_1 \\ e_1 \end{pmatrix} + D(t,x), \tag{6.22}$$

here matrixes A, B, C and D depend on the functions $u(t,x)$, $\rho(t,x)$ and $e(t,x)$ of the zeroth-order approximation found in Sect. 6.4.1:

$$A(t,x) = \frac{1}{\rho}\begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{2k}{fR} \end{pmatrix}, \quad B(t,x) = -\begin{pmatrix} u & \frac{2e}{f\rho} & \frac{2}{f} \\ \rho & u & 0 \\ \frac{2e}{f} - \frac{2\eta u_x}{\rho} & 0 & u \end{pmatrix}$$

$$C(t,x) = -\begin{pmatrix} u_x & \frac{2Re_x + fR(u_t(t,x) + uu_x)}{\rho f R} & \frac{\rho_x}{\rho f} \\ \rho_x & u_x & 0 \\ e_x & \frac{2Reu_x + fR(e_t + ue_x)}{\rho f R} & -\frac{2(\rho_t + u\rho_x)}{\rho f} \end{pmatrix}, \quad D(t,x) = \begin{pmatrix} 2\rho_x\left(1 - \frac{2}{f}\right) \\ 0 \\ \frac{2k\rho_{xx} + \rho R(\rho_t + u\rho_x)(2-f)}{\rho f R} \end{pmatrix}$$

System (6.22) is linear non-homogeneous system of partial differential equations and its general solution can be represented as the sum of general solution of the corresponding homogeneous system ($D = 0$) and particular solution of non-homogeneous system.

Let us consider homogeneous system:

$$\begin{pmatrix} u_1 \\ \rho_1 \\ e_1 \end{pmatrix}_t = A(t,x)\begin{pmatrix} u_1 \\ \rho_1 \\ e_1 \end{pmatrix}_{xx} + B(t,x)\begin{pmatrix} u_1 \\ \rho_1 \\ e_1 \end{pmatrix}_x + C(t,x)\begin{pmatrix} u_1 \\ \rho_1 \\ e_1 \end{pmatrix},$$

Since its coefficients depend on the zeroth-order solution, which are invariant with respect to X_5, we are looking for the solution in the form representing the eigenfunctions of differential operator $x\partial_x = \pi_*(X_5)$:

$$\rho_1(t,x) = R(t)x^l, \quad e_1(t,x) = E(t)x^m, \quad u_1(t,x) = U(t)x^n.$$

The numbers l, m and n satisfy the linear non-homogeneous system:

$$\begin{cases} m - l = 4, \\ n - l = 3, \\ m - n = 1. \end{cases}$$

Its general solution is

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} l \\ 4 + l \\ 3 + l \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} + l\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Time-dependent part of the first-order corrections $U(t)$, $R(t)$ and $E(t)$ satisfies the following ODE system:

$$\begin{pmatrix} \dot{U} \\ \dot{R} \\ \dot{E} \end{pmatrix} = \Phi(t)\begin{pmatrix} U \\ R \\ E \end{pmatrix}.$$

Matrix $\Phi(t)$ has the following form:

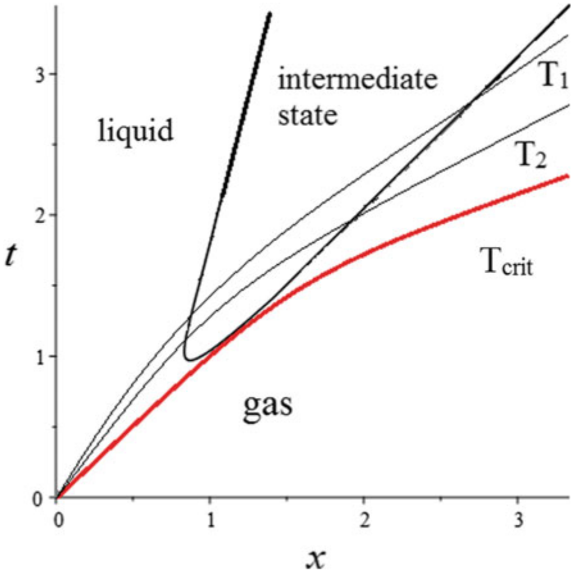


Fig. 6.5 Phase picture. The thick curve at the plane (t, x) separates domains with different phases. This allows to define which phase corresponds to the medium at given point x and at given time moment t

$$\Phi(t) = \begin{pmatrix} \frac{\eta(3+l)(2+l)-\dot{F}_2(4+l)}{F_2} & \frac{-2(2+l)F_1}{fF_2} & -\frac{2(2+l)}{f} \\ -(l+1)F_2 & -(l+1)F_3 & 0 \\ \frac{2(F_3\eta(3+l)f-F_1F_2(l+3+f))}{fF_2} & -\frac{f\dot{F}_1+2F_1F_3(f+1)}{fF_2} & \frac{2k(4+l)(3+l)-R\dot{F}_2(2+f(4+l))}{fRF_2} \end{pmatrix},$$

where functions $F_1(t)$, $F_2(t)$ and $F_3(t)$ are the solutions of reduced system (6.18)–(6.19).

If either $C_3 = 0$ or we consider gases with zero viscosity $\eta = 0$, this system can be integrated in the same way as it was done for the spatial part, because in this case components of the matrix $\Phi(t)$ are the homogeneous functions in t.

6.5 Phase Transitions Along the Gas Flow

In this section, we describe space-time domains corresponding to different phases of the medium. Since we have a solution of system (6.13)–(6.14), we can compute the corresponding set of points (t, x) of the same temperature T_0. In Sect. 6.3 we have developed a method that allows (for a given value of temperature T_0) to define the densities $\rho_1^{(0)}$ and $\rho_2^{(0)}$ of liquid and gas phases respectively, between which the phase transition occurs. The corresponding points can be found on the plane (t, x) as well. Changing the temperature, we get a set of points $(\rho_1^{(i)}, \rho_2^{(i)})$, which form a curve on the plane (t, x). This curve separates different phases of the medium. The result of this procedure for the solution obtained in Sect. 6.4 is shown in Fig. 6.5.

The curves labelled by T_1 and T_2 are the isotherms. The red curve corresponds to the critical isotherm, under which we have no phase transitions. For a given spatial coordinate x, our medium passes through three states while the time is running: gas, intermediate state and liquid.

This picture is an approximation for the real one. It can be refined by computation of further series of asymptotics.

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