Chapter 10
Integrability of Geodesics of Totally Geodesic Metrics

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10.1 Introduction

In [11], a class of totally geodesic metrics were given. For convenience, we outline here the main steps referring interested reader to the paper for details.

The starting point is to decompose the Weyl tensor in the base of 2-forms, which are eigenvectors of the corresponding Weyl operator. Then it results that the space-time contains totally geodesic distributions [11] of hyperbolic (H) and elliptic (E) tangent planes. This induces the solutions of the Einstein's equations with the cosmological constant Λ in the form

$$\begin{gathered} g = g^H \oplus g^E, \\ g^H = e^{\alpha(x_0,x_1)}(dx_0^2 - dx_1^2), \quad g^E = -e^{\beta(x_2,x_3)}(dx_2^2 + dx_3^2). \end{gathered} \tag{10.1}$$

The functions α and β are the solutions of the hyperbolic and elliptic Liouville equations, correspondingly, [4]

$$\begin{cases} \frac{\partial^2\alpha(x_0,x_1)}{\partial^2 x_0} - \frac{\partial^2\alpha(x_0,x_1)}{\partial^2 x_1} + 2\Lambda e^{\alpha(x_0,x_1)} = 0, \\ \frac{\partial^2\beta(x_2,x_3)}{\partial^2 x_2} + \frac{\partial^2\beta(x_2,x_3)}{\partial^2 x_3} - 2\Lambda e^{\beta(x_2,x_3)} = 0. \end{cases} \tag{10.2}$$

The solutions are as follows:

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$$\alpha(x_0, x_1) = ln(h_1(v)(v_{x_0}^2 - v_{x_1}^2)),$$
$$\beta(x_2, x_3) = ln(h_2(u)(u_{x_2}^2 + u_{x_3}^2)), \tag{10.3}$$

where u and v are the solutions of the two-dimensional hyperbolic and elliptic equations:

$$v_{x_0x_0} - v_{x_1x_1} = 0,$$
$$u_{x_2x_2} + u_{x_3x_3} = 0, \tag{10.4}$$

and where h_1 and h_2 are the solutions of a second-order ODEs. Full list of the solutions is presented [11].

In this paper, we analyse the geodesic governed by (10.1). Computations of the geodesic equations were performed using the Mathematica package CCGRG, see [14, 18, 19], and symmetries were computed using the *Differential Geometry* Maple package.

This paper is organized as follows: In the next section, it is shown that there is no true singularities of geodesics in the model of [11], i.e. the space-time is totally geodesic. Then the analysis of Liouville integrability [1] of the geodesics equations is provided. Finally, the analogous model with additional coupling to the electromagnetic field described in [12] is considered in the terms of integrability of geodesics.

The presentation starts with the analysis of the singularities of geodesics.

10.2 Singularities

The metric tensors described in [11] have obvious singularities. Generally, the singularities in the General Relativity have two origins [17]:

- singularities of the coordinates which results from the fact that in the coordinate patch ill-defined coordinate functions are used over regular points of manifold;
- true singularities which indicate geodesic incompleteness of the manifold;

True singularities are usually visible as the singularities of some invariants of curvature. The simplest second-order one is the square of the Riemann curvature (called Kretschmann scalar [3])

$$K = R_{abcd}R^{abcd}. \tag{10.5}$$

For (10.1) that are solutions of (10.2), this invariant is constant

$$K = 8\Lambda^2, \tag{10.6}$$

which suggests no singularities, i.e. completeness of the pseudo-Riemannian manifold. The answer is affirmative as it is provided by the following Lemma[1]

[1]RK would like to thank Igor Khavkine for discussion on this subject and suggestions of the outline of the proof.

Lemma 10.1 *The pseudoriemannian manifold (10.1) with (10.2) is complete.*

Proof From the metric decomposition (10.1) and the fact that

$$R^H = \sum_{i,j=0}^{1} R^{ij}_{..ij} = 2\Lambda, \quad R^E = \sum_{i,j=2}^{3} R^{ij}_{..ij} = 2\Lambda, \tag{10.7}$$

it results that the space factorizes into two-dimensional subspaces of constant curvature. These subspaces are isometric to spaces with no singularities according to the well-known Killing–Hopf theorem (see, e.g. Theorem 6.3 in [2]). □

The lemma states that any singularity of (10.1), (10.2) is an artificial singularity only and can be removed by a suitable change of coordinates.

10.3 Geodesics

In this section, the analysis of the geodesic equations will be provided. In the first part, the canonical form of the geodesic equations and their symmetries will be presented. Then the (Liouville) integrable cases will be singled out.

10.3.1 Geodesic Equations

Since the tangent space decomposes into two-dimensional subspaces, therefore, the geodesic equations consist of two pairs of two coupled ODEs for $\gamma(s) = (x_0(s), x_1(s), x_2(s), x_3(s))$, namely,

$$\begin{cases} x_0'' + x_0' x_1' \alpha_{x_1} + \frac{1}{2}\left(x_0'\right)^2 \alpha_{x_0} + \frac{1}{2}\left(x_1'\right)^2 \alpha_{x_0} = 0 \\ x_1'' + \frac{1}{2}\left(x_0'\right)^2 \alpha_{x_1} + \frac{1}{2}\left(x_1'\right)^2 \alpha_{x_1} + x_0' x_1' \alpha_{x_0} = 0, \end{cases} \tag{10.8}$$

$$\begin{cases} x_2'' + x_2' x_3' \beta_{x_3} + \frac{1}{2}\left(x_2'\right)^2 \beta_{x_2} - \frac{1}{2}\left(x_3'\right)^2 \beta_{x_2} = 0 \\ x_3'' - \frac{1}{2}\left(x_2'\right)^2 \beta_{x_3} + \frac{1}{2}\left(x_3'\right)^2 \beta_{x_3} + x_2' x_3' \beta_{x_2} = 0, \end{cases} \tag{10.9}$$

where $' = \frac{d}{ds}$, $\alpha_{x_i} = \frac{\partial \alpha}{\partial x_i}$ and $\beta_{x_i} = \frac{\partial \beta}{\partial x_i}$.

These equations can be significantly simplified. Adding and subtracting Eq. (10.8) and then introducing the light-cone variables (characteristics of the wave equation): $x_0 = \frac{z_0+z_1}{2}$ and $x_1 = \frac{z_0-z_1}{2}$ one gets

$$\Delta_1(z_0, z_1) : \begin{cases} z_0'' + \frac{\partial \alpha(z_0,z_1)}{\partial z_0}\left(z_0'\right)^2 = 0 \\ z_1'' + \frac{\partial \alpha(z_0,z_1)}{\partial z_1}\left(z_1'\right)^2 = 0. \end{cases} \tag{10.10}$$

Symmetries of (10.10) can be found by assuming that the generator of a symmetry is of the form: $X = f(s, z_0, z_1)\partial_s + g(s, z_0, z_1)\partial_{z_0} + h(s, z_0, z_1)\partial_{z_1}$, and solving the following system of PDEs:

$$£_{X^{(2)}}\Delta_1(z_0, z_1)|_{\Delta_1(z_0,z_1)} = 0, \tag{10.11}$$

where £ is the Lie derivative along $X^{(2)}$—the second prolongation of X to the jet space [5, 10, 15, 16]. The result is

$$X_1 = (As + B)\partial_s, \tag{10.12}$$

where A and B are constants. This gives a scaling and a translation symmetry of s variable, and it results from the fact that (10.8) does not depends explicitly on s. The symmetry reflects the fact that the geodesics should not depend on re-parametrization in s and is also connected with the fact that the geodesic equations are variational and should posses such symmetries.

The same procedure can be applied to the second system of (10.9). In this case we have positively defined ('elliptic') metric, which suggests complex characteristics. It is, therefore, more appropriate to use complex-valued characteristics of an elliptic equation, i.e. the substitution $x_2 = \frac{z_2+z_3}{2i}$ and $x_3 = \frac{z_2-z_3}{2}$, where $i = \sqrt{-1}$. Then adding and subtracting from the first equation of (10.9) multiplied by the imaginary unity the second one one gets the system which resembles (10.10), namely,

$$\Delta_1(z_2, z_3) : \begin{cases} z_2'' + \frac{\partial\beta(z_2,z_3)}{\partial z_2}\left(z_2'\right)^2 = 0 \\ z_3'' + \frac{\partial\beta(z_2,z_3)}{\partial z_3}\left(z_3'\right)^2 = 0. \end{cases} \tag{10.13}$$

Since the equations are the same as in the previous case, symmetry analysis indicates, as above, the following generator:

$$X_2 = (Cs + D)\partial_s, \tag{10.14}$$

where C and D are some constants.

In the next section, integrability of geodesics equations will be investigated.

10.3.2 Integrability of Geodesic Equations

First, let us consider the hyperbolic part of the metric, namely define the Hamiltonian

$$H_{0,\alpha} = e^{\alpha(x_0,x_1)}(p_0^2 - p_1^2), \tag{10.15}$$

which surfaces of constant value determine the movement of the particles (positive-massive particles, zero-massless particles). Since the submanifold dimension is 2, therefore in order to find its foliation, according the the Liouville theorem [1], one additional function that the Poisson brackets with $H_{0,\alpha}$ vanishes, is needed. It is assumed in the polynomial form in p_0 and p_1, namely,

$$H_{1,\alpha} = \sum_{k=0}^{n} f_i(x_0, x_1) p_0^k p_1^{n-k}, \tag{10.16}$$

where n is natural number that is fixed degree. Complete integrability is equivalent to the existence of a solution of

$$\{H_{0,\alpha}, H_{1,\alpha}\}_{PB} = 0, \tag{10.17}$$

where $\{.,.\}_{PB}$ is the standard Poisson bracket. Equation (10.17) gives the set of PDEs.[2] In order to check closeness of this system the Kruglikov–Lychagin multi-bracket [6–9, 13] is used. When applied on the system (10.17), it gives compatibility condition in terms of PDEs for $\alpha(x, y)$, which solutions up to $n = 5$ are

1. $n = 1, 2$:

$$\begin{aligned} \alpha(x_0, x_1) = F \tanh(B(y-x)+A)^3 + E \tanh(B(y-x)+A)^2 \\ + D \tanh(B(y-x)+A) + C; \end{aligned} \tag{10.18}$$

   where A, B, C, D, E, F are the constants of integration and parametrize α.
2. $n = 3, 4$:

$$\alpha(x_0, x_1) = Ax + By + C; \tag{10.19}$$

   where A, B, C are the constants of integration and parametrize α.

Surprisingly, these solutions fulfil the first equation of (10.2) only when the cosmological constant $\Lambda = 0$. This is a very prominent example of the role of the cosmological constant in integrability of geodesic equations.

For the case (10.18), integration can be easily performed using (10.10) and gives

$$\begin{cases} z_0(s) = As + J, \\ \int_0^{z_1(s)} \exp(F \tanh(Ba+A)^3 + E \tanh(Ba+A)^2 + D \tanh(Ba+A) + C) da + Gs + H = 0, \end{cases} \tag{10.20}$$

where the second solution is expressed in the implicit form, $A, B, \ldots, F$ are as in (10.18) and G, H, J are the constants dependent on initial data.

The second case (10.19) can be explicitly expressed in terms of elementary functions, namely,

$$\begin{aligned} z_0(s) &= -2\frac{\ln\left(\frac{2}{(Ds+E)(A+B)}\right)}{A+B}, \\ z_1(s) &= -2\frac{ln\left(\frac{2}{(Fs+G)(A+B)}\right)}{A+B}, \end{aligned} \tag{10.21}$$

[2] All calculations for this section are available as Maple files on: https://github.com/rkycia/GeodesicsIntegrability.

where D, E, F, G are the constants depending on initial data.

Similar analysis performed for the elliptic part of the metric by taking

$$H_{0,\beta} = e^{\beta(x_2,x_3)}(p_2^2 + p_3^2), \tag{10.22}$$

and

$$H_{1,\beta} = \sum_{k=0}^{n} f_i(x_2, x_3) p_2^k p_3^{n-k}, \tag{10.23}$$

and checking when

$$\{H_{0,\beta}, H_{1,\beta}\}_{PB} = 0. \tag{10.24}$$

The two solutions for β are obtained up to the degree $n = 5$, namely:

1. $n = 1, 2$:

$$\begin{aligned}\beta(x_0, x_1) = {} & F \tanh(B(y - xi) + A)^3 + E \tanh(B(y - xi) + A)^2 \\ & + D \tanh(B(y - xi) + A) + C;\end{aligned} \tag{10.25}$$

   where A, B, C, D, E, F are the constants of integration and parametrize β, and i is the imaginary unit.
2. $n = 3, 4$:

$$\beta(x_0, x_1) = Ax + By + C; \tag{10.26}$$

   where A, B, C are the constants of integration and parametrize β.

As in the previous case, these βs solve (10.2) only when the cosmological constant $\Lambda = 0$.

For (10.25) the solution of (10.13) is

$$\begin{cases} z_2(s) = As + J, \\ \int_0^{z_3(s)} \exp(F \tanh(Ba + A)^3 + E \tanh(Ba + A)^2 + D \tanh(Ba + A) + C) da + Gs + H = 0, \end{cases} \tag{10.27}$$

where, as before, G, H, J are the integration constants depending on initial data.

For (10.26), the solution of (10.13) is

$$\begin{aligned} z_2(s) &= -2 \frac{\ln\left(\frac{2}{(A-iB)(Ds+E)}\right)}{A - iB}, \\ z_3(s) &= -2 \frac{\ln\left(\frac{2}{(A+iB)(Fs+G)}\right)}{A + iB}, \end{aligned} \tag{10.28}$$

where D, E, F, G are again the constants depending on initial data. These solutions are complex-valued, however, since x_2 and x_3 fulfil real equations for geodesic, therefore, transforming to the original variables one gets real solutions.

In general, the geodesic solutions can be constructed by selecting the solution (10.20) or (10.21) for the hyperbolic part of the subspace, and (10.27) or (10.28) for the elliptic subspace. Therefore, in total $4 = 2 \times 2$ integrable solutions were obtained.

10.4 Einstein–Maxwell Solutions

The results from the previous section can be used for analysis of the geodesics of the solutions for coupled the Einstein and Maxwell equations described in [12]. In this model the totally geodesic solutions, the same as the solution of (10.1) for the metric, were obtained. However, now α and β are solutions of

$$\begin{cases} \frac{\partial^2\alpha(x_0,x_1)}{\partial^2 x_0} - \frac{\partial^2\alpha(x_0,x_1)}{\partial^2 x_1} + k_1 e^{\alpha(x_0,x_1)} = 0, \\ \frac{\partial^2\beta(x_2,x_3)}{\partial^2 x_2} + \frac{\partial^2\beta(x_2,x_3)}{\partial^2 x_3} + k_2 e^{\beta(x_2,x_3)} = 0, \end{cases} \tag{10.29}$$

where

$$k_1 = 2\left(\frac{kJ}{c^4} + \Lambda\right), \quad k_2 = \left(\frac{kJ}{c^4} - \Lambda\right), \tag{10.30}$$

where Λ is the cosmological constant, k is the gravitational constant, and the new parameter J is connected with the solution for the Faraday tensor of electromagnetic field

$$F = -2le^{\alpha(x_0,x_1)}dx_0 \wedge dx_1 + 2me^{\beta(x_2,x_3)}dx_2 \wedge dx_3, \tag{10.31}$$

where

$$l^2 = \frac{J - I_1}{2}, \quad m^2 = \frac{J + I_1}{2}, \tag{10.32}$$

and where I_1 is the invariant of the characteristic polynomial of the skew symmetric operator $\hat{F}$ (associated to F by $g(\hat{F}X, Y) = F(X, Y)$), namely, its determinant. The parameters $\pm l$ and $\pm im$, where $l, m \in \mathbb{R}$, are the eigenvalues of the hyperbolic and the elliptic parts of the operator $\hat{F}$.

The straightforward result from (10.32) is that

$$J = l^2 + m^2, \quad I_1 = m^2 - l^2. \tag{10.33}$$

From our previous considerations, the geodesic equations are (Liouville) integrable when $k_1 = 0 = k_2$, i.e., when $J = 0$ and $\Lambda = 0$. And therefore, since $l, m \in \mathbb{R}$, from the first equation of (10.33) it results that $l = 0$ and $m = 0$, and therefore, the Faraday tensor vanishes. This shows that the integrable solutions for geodesics exist when no electromagnetic field and no cosmological constant is present in this model. The solutions for geodesics are exactly the same as in the previous section for the Einstein equations only, since the electromagnetic field vanishes.

10.5 Discussion

The semi-Riemmanian metric of [11] describes anisotropic space-time, which distinguished the space direction x_1, and therefore cannot describe the observed space-time where assumption on spherical symmetry is imposed. The presence of this distinguished space direction resembles the phenomena from the phase transitions in solid state physics, and therefore it suggests that the model can be applied in some phenomena that occur when the universe undertake some kind of phase transition, e.g., in the early state of the universe. A similar description applies also to the coupled Einstein–Maxwell system.

Intriguing correspondence between vanishing of the cosmological constant and integrability of the geodesic equations was noted. In the case of electromagnetism for integrability also electromagnetic field must vanish.

10.6 Conclusions

In this paper, analysis of the geodesic of the solution of the Einstein vacuum equations resulting from the Weyl tensor bivector structure was provided. In particular, integrable geodesic equations of special solutions of the Einstein vacuum equation were found and described. A similar analysis was also performed for the Einstein–Maxwell system.

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