Chapter 9

The fractional heat equation using quaternionic techniques

The development of the spectral theory of the Nabla operator opens the way to a large class of fractional diffusion problems, and some of them will be treated in the next chapter. Indeed, the main aim of this chapter is to show how our theory, for the case of the Nabla operator, reproduces known results. Since it is very general, it allows us to manipulate a very large class of new fractional diffusion processes. The results presented in this chapter were originally proved in [53, 54]. Precisely, if $v(x,t)$ is the temperature at the point $x \in \mathbb{R}^3$ and the time $t > 0$ and κ is the thermal diffusivity of the considered material, then the heat equation

$$\partial_t v(x,t) - \kappa \Delta v(x,t) = 0, \tag{9.1}$$

where $\Delta = \sum_{\ell=1}^{3} \partial_{x_\ell}^2$ with $x = (x_1, x_2, x_3)^T$, describes the evolution of the temperature distribution in space and time. (For mathematical treatment, one usually sets $\kappa = 1$ and we will emulate this.) This model has, however, several unphysical properties, so scientists have tried to modify it. One approach has been the introduction of the fractional heat equation. In order to modify the properties of the equation, researchers replaced the negative Laplacian in (9.1) by its fractional powers of exponent α and considered the evolution equation

$$\frac{\partial}{\partial t} v(x,t) + (-\Delta)^\alpha v(x,t) = 0. \tag{9.2}$$

There are different approaches for defining the fractional Laplace operator, but each approach leads to a global integral operator, which, in contrast to the local differential operator Δ, is able to take long distance effects into account.

We want to develop a similar approach for defining fractional evolution equations with the generalized gradient. What we show here is that, if we replace the gradient in Fourier's law of conductivity by its fractional power instead of directly

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replacing the negative Laplacian by its fractional power in (9.1), we get the same equation. Indeed, this would lead to the equation

$$\frac{\partial}{\partial t}v(x,t) - \operatorname{div}(\nabla^\alpha v(x,t)) = 0,$$

with suitable interpretation of the symbol ∇^α. Our initial task is to understand the definition of the fractional powers ∇^α according to our theory developed in the previous chapters where we identified the gradient with the quaternionic Nabla operator.

Additionally in this chapter, we develop the spectral theory of the quaternionic Nabla operator on $L^2(\mathbb{R}^3,\mathbb{H})$. We find that the previously developed theory is not directly applicable because the Nabla operator does not belong to the class of sectorial operators. We therefore present a slightly modified approach and show that this allows us to reproduce the fractional heat equation (9.2) using quaternionic techniques. Finally, we give an example for a more general operator with non-constant coefficients that can be treated with our methods.

9.1 Spectral properties of the Nabla operator

The gradient of a function $v:\mathbb{R}^3\to\mathbb{R}$ is the vector-valued function

$$\nabla v(x) = \begin{pmatrix} \partial_{x_1} v(x) \\ \partial_{x_2} v(x) \\ \partial_{x_3} v(x) \end{pmatrix}, \quad \text{for } x=(x_1,x_2,x_3).$$

If we identify $\mathbb{R}$ with the set of real quaternions and $\mathbb{R}^3$ with the set of purely imaginary quaternions, this corresponds to the quaternionic Nabla operator

$$\nabla = \partial_{x_1} e_1 + \partial_{x_2} e_2 + \partial_{x_3} e_3.$$

In the following, we shall often denote the standard basis of the quaternions by $\mathsf{I} := e_1$, $\mathsf{J} := e_2$ and $\mathsf{K} := e_3 = \mathsf{IJ} = -\mathsf{JI}$. This suggests a relation with the complex theory, which we shall use excessively. With this notation, we have

$$\nabla = \partial_{x_1}\mathsf{I} + \partial_{x_2}\mathsf{J} + \partial_{x_3}\mathsf{K}.$$

We study the properties of a quaternionic Nabla operator on the space $L^2(\mathbb{R}^3,\mathbb{H})$ of all square-integrable quaternion-valued functions on $\mathbb{R}^3$, which is a quaternionic right Hilbert space when endowed with the scalar product

$$\langle w, v\rangle = \int_{\mathbb{R}^3} \overline{w(x)}v(x)\,dx.$$

On this space, the Nabla operator is closed and has dense domain. This follows immediately from its representation (9.4) in the Fourier space that we derive in the proof of Theorem 9.1.1.

Let $v \in L^2(\mathbb{R}^3, \mathbb{H})$ and write $v(x) = v_1(x) + v_2(x)\mathsf{J}$ with two $\mathbb{C}_\mathsf{I}$-valued functions v_1 and v_2. As $|v(x)|^2 = |v_1(x)|^2 + |v_2(x)|^2$, we have

$$\|v\|^2_{L^2(\mathbb{R}^3,\mathbb{H})} = \|v_1\|^2_{L^2(\mathbb{R}^3,\mathbb{C}_\mathsf{I})} + \|v_2\|^2_{L_2(\mathbb{R},^3,\mathbb{C}_\mathsf{I})}, \tag{9.3}$$

where $L^2(\mathbb{R}^3, \mathbb{H})$ denotes the complex Hilbert space over $\mathbb{C}_\mathsf{I}$ of all square-integrable $\mathbb{C}_\mathsf{I}$-valued functions on $\mathbb{R}^3$. Hence, $v \in L^2(\mathbb{R}^3, \mathbb{H})$ if and only if $v_1, v_2 \in L^2(\mathbb{H}, \mathbb{C}_\mathsf{I})$.

Theorem 9.1.1. *The S-spectrum of ∇ as an operator on $L^2(\mathbb{R}^3, \mathbb{H})$ is*

$$\sigma_S(\nabla) = \mathbb{R}.$$

Proof. Let us consider $L^2(\mathbb{R}^3, \mathbb{H})$ as a Hilbert space over $\mathbb{C}_\mathsf{I}$ by restricting the right scalar multiplication to $\mathbb{C}_\mathsf{I}$ and setting

$$\langle w, v\rangle_\mathsf{I} := \{\langle w, v\rangle_{L^2(\mathbb{R}^3,\mathbb{H})}\}_\mathsf{I}.$$

Here $\{\cdot\}_\mathsf{I}$ denotes the $\mathbb{C}_\mathsf{I}$-part of a quaternion: if $a = a_1 + a_2\mathsf{J} = a_1 + \mathsf{J}\overline{a_2}$ with $a_1, a_2 \in \mathbb{C}_\mathsf{I}$, then $\{a\}_\mathsf{I} := a_1$. If we write $v, w \in L^2(\mathbb{R}^3, \mathbb{H})$ as $v = v_1 + \mathsf{J}v_2$ and $w = w_1 + \mathsf{J}w_2$ with $v_1, v_2, w_1, w_2 \in L^2(\mathbb{R}^3, \mathbb{C}_\mathsf{I})$, then

$$\begin{aligned}
\langle w, v\rangle_{L^2(\mathbb{R}^3,\mathbb{H})} &= \int_{\mathbb{R}^3} \overline{(w_1(x) + \mathsf{J}w_2(x))}(v_1(x) + \mathsf{J}v_2(x))\,dx \\
&= \int_{\mathbb{R}^3} \overline{w_1(x)}v_1(x)\,dx + \int_{\mathbb{R}^3} \overline{w_2(x)}(-\mathsf{J})v_1(x)\,dx \\
&\quad + \int_{\mathbb{R}^3} \overline{w_1(x)}\mathsf{J}v_2(x)\,dx + \int_{\mathbb{R}^3} \overline{w_2(x)}(-\mathsf{J}^2)v_2(x)\,dx \\
&= \int_{\mathbb{R}^3} \overline{w_1(x)}v_1(x)\,dx + \int_{\mathbb{R}^3} \overline{w_2(x)}v_2(x)\,dx \\
&\quad + \mathsf{J}\left(-\int_{\mathbb{R}^3} w_2(x)v_1(x)\,dx + \int_{\mathbb{R}^3} w_1(x)v_2(x)\,dx\right).
\end{aligned}$$

Therefore, we have

$$\langle w, v\rangle_\mathsf{I} := \langle w_1, v_1\rangle_{L^2(\mathbb{R}^3,\mathbb{C}_\mathsf{I})} + \langle w_2, v_2\rangle_{L^2(\mathbb{R}^3,\mathbb{C}_\mathsf{I})}$$

and hence $L^2(\mathbb{R}^3, \mathbb{H})$ considered as a $\mathbb{C}_\mathsf{I}$-complex Hilbert space with the scalar product $\langle\cdot,\cdot\rangle_\mathsf{I}$ equals $L^2(\mathbb{R}^3, \mathbb{C}_\mathsf{I}) \oplus L^2(\mathbb{R}^3, \mathbb{C}_\mathsf{I})$. Moreover, because of (9.3), the quaternionic scalar product $\langle\cdot,\cdot\rangle$ and the $\mathbb{C}_\mathsf{I}$-complex scalar product $\langle\cdot,\cdot\rangle_\mathsf{I}$ induce the same norm on $L^2(\mathbb{R}^3, \mathbb{H})$. Applying the Nabla operator to $v = v_1 + \mathsf{J}v_2$, we find

$$\begin{aligned}
\nabla v(x) &= (\mathsf{I}\partial_{x_1} + \mathsf{J}\partial_{x_2} + \mathsf{K}\partial_{x_3})(v_1(x) + \mathsf{J}v_2(x)) \\
&= \mathsf{I}\partial_{x_1}v_1(x) + \mathsf{J}\partial_{x_2}v_1(x) + \mathsf{K}\partial_{x_3}v_1(x) \\
&\quad + \mathsf{I}\partial_{x_1}\mathsf{J}v_2(x) + \mathsf{J}\partial_{x_2}\mathsf{J}v_2(x) + \mathsf{K}\partial_{x_3}\mathsf{J}v_2(x) \\
&= \mathsf{I}\partial_{x_1}v_1(x) - \partial_{x_2}v_2(x) - \mathsf{I}\partial_{x_3}v_2(x) \\
&\quad + \mathsf{J}\left(-\mathsf{I}\partial_{x_1}v_2(x) + \partial_{x_2}v_1(x) - \mathsf{I}\partial_{x_3}v_1(x)\right).
\end{aligned}$$

Writing this in terms of the components $L^2(\mathbb{R}^3,\mathbb{H}) \cong L^2(\mathbb{R}^3,\mathbb{C}_{\mathsf{I}}) \oplus L^2(\mathbb{R}^3,\mathbb{C}_{\mathsf{I}})$, we obtain

$$\nabla \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} = \begin{pmatrix} \mathsf{I}\partial_{x_1} v_1(x) - \partial_{x_2} v_2(x) - \mathsf{I}\partial_{x_3} v_2(x) \\ -\mathsf{I}\partial_{x_1} v_2(x) + \partial_{x_2} v_1(x) - \mathsf{I}\partial_{x_3} v_1(x) \end{pmatrix}.$$

If we apply the Fourier transform on $L^2(\mathbb{R}^3,\mathbb{C}_{\mathsf{I}})$ componentwise, this turns into

$$\widehat{\nabla} \begin{pmatrix} \widehat{v_1}(x) \\ \widehat{v_2}(x) \end{pmatrix} = \begin{pmatrix} -\xi_1 & -\mathsf{I}\xi_2 + \xi_3 \\ \mathsf{I}\xi_2 + \xi_3 & \xi_1 \end{pmatrix} \begin{pmatrix} \widehat{v_1}(\xi) \\ \widehat{v_2}(\xi) \end{pmatrix}. \tag{9.4}$$

Hence, in the Fourier space, the Nabla operator corresponds to the multiplication operator $M_G : \widehat{v} \mapsto G\widehat{v}$ on $\widehat{X} := L^2(\mathbb{R}^3,\mathbb{C}_{\mathsf{I}}) \oplus L^2(\mathbb{R}^3,\mathbb{C}_{\mathsf{I}})$ that is generated by the matrix valued function

$$G(\xi) := \begin{pmatrix} -\xi_1 & -\mathsf{I}\xi_2 + \xi_3 \\ \mathsf{I}\xi_2 + \xi_3 & \xi_1 \end{pmatrix}. \tag{9.5}$$

For $s \in \mathbb{C}_{\mathsf{I}}$, we find

$$s\mathcal{I}_{\widehat{X}} - G(\xi) = \begin{pmatrix} s + \xi_1 & \mathsf{I}\xi_2 - \xi_3 \\ -\mathsf{I}\xi_2 - \xi_3 & s - \xi_1 \end{pmatrix}.$$

For $s \in \mathbb{C}_{\mathsf{I}}$, the inverse of $s\mathcal{I}_{\widehat{X}} - M_G$ is hence given by the multiplication operator $M_{(s\mathcal{I}-G)^{-1}}$ determined by the matrix-valued function

$$(s\mathcal{I}_{\widehat{X}} - G(\xi))^{-1} = \frac{1}{s^2 - \xi_1^2 - \xi_2^2 - \xi_3^2} \begin{pmatrix} s - \xi_1 & -\mathsf{I}\xi_2 + \xi_3 \\ \mathsf{I}\xi_2 + \xi_3 & s + \xi_1 \end{pmatrix}.$$

This operator is bounded if and only if the function $\xi \mapsto (s\mathcal{I} - G(\xi))^{-1}$ is bounded on $\mathbb{R}^3$, that is if and only $s \notin \mathbb{R}$. Hence, $\sigma(M_G) = \mathbb{R}$.

The componentwise Fourier transform Ψ is a unitary $\mathbb{C}_{\mathsf{I}}$-linear operator from the space $L^2(\mathbb{R}^3,\mathbb{H}) \cong L^2(\mathbb{R}^3,\mathbb{C}_{\mathsf{I}}) \oplus L^2(\mathbb{R}^2,\mathbb{C}_{\mathsf{I}})$ to $\widehat{X}$ under which ∇ corresponds to M_G, that is $\nabla = \Psi^{-1} M_G \Psi$. The spectrum $\sigma_{\mathbb{C}_{\mathsf{I}}}(\nabla)$ of ∇ considered as a $\mathbb{C}_{\mathsf{I}}$-linear operator on $L^2(\mathbb{R}^3,\mathbb{H})$, therefore, equals $\sigma_{\mathbb{C}_{\mathsf{I}}}(\nabla) = \sigma(M_G) = \mathbb{R}$. By Theorem 3.1.8, we however have $\sigma_{\mathbb{C}_{\mathsf{I}}}(\nabla) = \sigma_S(\nabla) \cap \mathbb{C}_{\mathsf{I}}$ and so $\sigma_S(\nabla) = \mathbb{R}$. □

The above result shows that the gradient does not belong to the class of sectorial operators as $(-\infty, 0) \not\subset \rho_S(T)$, so the theory developed in Chapter 8 is not directly applicable. Even worse, we cannot find any other slice hyperholomorphic functional calculus that allows us to define fractional powers ∇^α of ∇ because the scalar function s^α is not slice hyperholomorphic on $(-\infty, 0]$ and hence not slice hyperholomorphic on $\sigma_S(\nabla)$.

However, we shall now show another characterization of the S-spectrum of the Nabla operator on the quaternionic right Hilbert space $L^2(\mathbb{R}^3,\mathbb{H})$ that makes use of the relation $\nabla^2 = -\Delta$ and will be fundamental later on.

If $j, i \in \mathbb{S}$ with $i \perp j$, then any $v \in L^2(\mathbb{R}^3,\mathbb{H})$ can be written as $v = v_1 + v_2 i$ with components v_1, v_2 in $L^2(\mathbb{R}^3,\mathbb{C}_j)$, i.e., $L^2(\mathbb{R}^3,\mathbb{H}) = L^2(\mathbb{R}^3,\mathbb{C}_j) \oplus L^2(\mathbb{R}^3,\mathbb{C}_j)i$.

Contrary to the decomposition $v = v_1 + iv_1$, which we used in the proof of Theorem 9.1.1 with $j = \mathsf{I}$ and $i = \mathsf{J}$, this decomposition is not compatible with the $\mathbb{C}_j$-right vector space structure of $L^2(\mathbb{R}^3, \mathbb{H})$ as $va = v_1 a + v_2 \overline{a} i$ for any $a \in \mathbb{C}_j$. However, this identification has a different advantage: any closed $\mathbb{C}_j$-linear operator $A : \mathcal{D}(A) \subset L^2(\mathbb{R}^3, \mathbb{C}_{\mathsf{I}}) \to L^2(\mathbb{R}^3, \mathbb{C}_{\mathsf{I}})$ extends to a closed $\mathbb{H}$-linear operator on $L^2(\mathbb{R}^3, \mathbb{H})$ with domain $\mathcal{D}(A) \oplus \mathcal{D}(A)i$, namely to the operator $A(v_1 + v_2 i) := A(v_1) + A(v_2)i$. Moreover, if A is bounded, then its extension to $L^2(\mathbb{R}^3, \mathbb{H})$ has the same norm as A. We shall denote an operator on $L^2(\mathbb{R}^3, \mathbb{C}_j)$ and its extension to $L^2(\mathbb{R}^3, \mathbb{H}) = L^2(\mathbb{R}^3, \mathbb{C}_j) \oplus L^2(\mathbb{R}^3, \mathbb{C}_j)i$ via componentwise application by the same symbol. This will not cause any confusion as it will be clear from the context to which we refer.

Theorem 9.1.2. *Let Δ be the Laplace operator on $L^2(\mathbb{H}, \mathbb{C}_j)$ and let $R_z(-\Delta)$ be the resolvent of $-\Delta$ at $z \in \mathbb{C}_j$. We have*

$$\sigma_S(\nabla)^2 = \left\{ s^2 \in \mathbb{H} : \quad s \in \sigma_S(T) \right\} = \sigma(-\Delta) \tag{9.6}$$

and

$$\mathcal{Q}_{c,s}(\nabla)^{-1} = R_{s^2}(-\Delta), \quad \forall s \in \mathbb{C}_j \setminus \mathbb{R}. \tag{9.7}$$

Proof. Since the components of ∇ commute and $e_\kappa e_\ell = -e_\ell e_\kappa$ for $1 \leq \kappa, \ell \leq 3$ with $\kappa \neq \ell$, we have

$$\begin{aligned}
\nabla^2 &= \sum_{\ell,\kappa=1}^{3} \partial_{x_\ell} \partial_{x_\kappa} e_\ell e_\kappa \\
&= \sum_{\ell=1}^{3} -\partial_{x_\ell}^2 + \sum_{1 \leq \ell < \kappa \leq 3} \left(\partial_{x_\ell} \partial_{x_\kappa} - \partial_{x_\kappa} \partial_{x_\ell} \right) e_\ell e_\kappa \\
&= \sum_{\ell=1}^{3} -\partial_{x_\ell}^2 = -\Delta.
\end{aligned}$$

As $\nabla_0 = 0$, we have $\overline{\nabla} = -\nabla$ and in turn

$$\mathcal{Q}_{c,s}(\nabla) = s^2 \mathcal{I} - 2s\nabla_0 + \nabla \overline{\nabla} = s^2 \mathcal{I} - \nabla^2 = s^2 \mathcal{I} - (-\Delta)$$

Hence, $\mathcal{Q}_{c,s}(\nabla)$ is invertible if and only if $s^2 \mathcal{I} - (-\Delta)$ is invertible. In this case

$$\mathcal{Q}_{c,s}(\nabla) = \left(s^2 \mathcal{I} - (-\Delta)\right)^{-1} = R_{s^2}(-\Delta). \qquad \square$$

9.2 A relation with the fractional heat equation

As one can easily verify, the Nabla operator is self-adjoint on $L^2(\mathbb{R}^3, \mathbb{H})$. From the spectral theorem for unbounded normal quaternionic linear operators (see the paper [14] or the book [57]), we hence deduce the existence of a unique spectral measure E on $\sigma_S(\nabla) = \mathbb{R}$, the values of which are orthogonal quaternionic linear projections on $L^2(\mathbb{R}^3, \mathbb{H})$, such that

$$\nabla = \int_{\mathbb{R}} s \, dE(s).$$

Using the measurable functional calculus for intrinsic slice functions (see [14] or the book [57]), it is now possible to define $P_\alpha(s) = s^\alpha \chi_{[0,+\infty)}(s)$ of T as

$$P_\alpha(\nabla) = \int_{\mathbb{R}} s^\alpha \chi_{[0,+\infty)}(s) \, dE(s),$$

where $\chi_{[0,+\infty)}$ denotes the characteristic function of the set $[0, +\infty)$. This corresponds to defining ∇^α, at least on the subspace associated with the spectral values $[0, +\infty)$, on which s^α is defined. (We stress that, even with the measurable functional calculus, the operator ∇^α cannot be defined because s^α is not defined on $(-\infty, 0)$.)

We shall now give an integral representation for this operator via an approach similar to the one of the slice hyperholomorphic H^∞-functional calculus. Surprisingly, this yields a possibility to obtain the fractional heat equation via quaternionic operator techniques applied to the Nabla operator. For $\alpha \in (0, 1)$, we define

$$P_\alpha(\nabla)v := \frac{1}{2\pi} \int_{-j\mathbb{R}} S_L^{-1}(s, \nabla) \, ds_j \, s^{\alpha-1} \nabla v, \quad \forall v \in \mathcal{D}(\nabla). \tag{9.8}$$

Intuitively, this corresponds to Balakrishnan's formula for ∇^α, where only spectral values on the positive real axis, i.e., points where s^α is actually defined, are taken into account, because the path of integration surrounds only the positive real axis.

Theorem 9.2.1. *The integral* (9.8) *converges for any* $v \in \mathcal{D}(\nabla)$ *and hence defines a quaternionic linear operator on* $L^2(\mathbb{R}^3, \mathbb{H})$.

Proof. If we write the integral (9.8) explicitly, we have

$$\begin{aligned} P_\alpha(\nabla)v &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_L^{-1}(-jt, \nabla) \, (-j)^2 \, (-jt)^{\alpha-1} \nabla v \\ &= -\frac{1}{2\pi} \int_0^{+\infty} S_L^{-1}(-jt, \nabla)(-jt)^{\alpha-1} \nabla v \, dt \\ &\quad - \frac{1}{2\pi} \int_0^{+\infty} S_L^{-1}(jt, \nabla)(jt)^{\alpha-1} \nabla v \, dt \\ &= -\frac{1}{2\pi} \int_0^{+\infty} S_L^{-1}(-jt, \nabla) t^{\alpha-1} e^{-j\frac{(\alpha \quad 1)\pi}{2}} \nabla v \, dt \\ &\quad - \frac{1}{2\pi} \int_0^{+\infty} S_L^{-1}(jt, \nabla) t^{\alpha-1} e^{j\frac{(\alpha \quad 1)\pi}{2}} \nabla v \, dt, \end{aligned} \tag{9.9}$$

where $P_\alpha(\nabla)v$ is defined if and only if the last two integrals converge in $L^2(\mathbb{R}^3, \mathbb{H})$.

Let us consider $L^2(\mathbb{R}^3, \mathbb{H})$ as a Hilbert space over $\mathbb{C}_j$ as in the proof of Theorem 9.1.1. If we write $v \in L^2(\mathbb{R}^3, \mathbb{H})$ as $v = v_1 + iv_2$ with $v_1, v_2 \in L^2(\mathbb{R}, \mathbb{C}_j)$ and apply the Fourier transform componentwise, we obtain an isometric $\mathbb{C}_j$-linear isomorphism $\Psi : v \mapsto (\widehat{v}_1, \widehat{v}_2)^T$ between $L^2(\mathbb{R}^3, \mathbb{H})$ and

$$\widehat{X} := L^2(\mathbb{R}^3, \mathbb{C}_j) \oplus L^2(\mathbb{R}^3, \mathbb{C}_j).$$

For any quaternionic linear operator T on $L^2(\mathbb{R}^3, \mathbb{H})$, the composition $\Psi T \Psi^{-1}$ is a $\mathbb{C}_j$-linear operator on $\widehat{X}$ with $\mathcal{D}(\Psi T \Psi^{-1}) = \Psi\, \mathcal{D}(T)$.

Applying ∇ to $v \in \mathcal{D}(\nabla) \subset L^2(\mathbb{R}^3, \mathbb{H})$, corresponds to applying the multiplication operator M_G associated with the matrix-valued function $G(\xi)$ defined in (9.5) to $\widehat{v}(\xi) = (\widehat{v_1}(\xi), \widehat{v_2}(\xi))^T$. Hence, $\nabla = \Psi^{-1} M_G \Psi$ and

$$\begin{aligned} \Psi\, \mathcal{D}(\nabla) = \mathcal{D}(M_G) &= \left\{ \widehat{v} \in \widehat{X} : G(\xi)\widehat{v}(\xi) \in \widehat{X} \right\} \\ &= \left\{ \widehat{v} \in \widehat{X} : |\xi|\widehat{v}(\xi) \in \widehat{X} \right\}. \end{aligned} \tag{9.10}$$

The last identity holds, for $\widehat{v}(\xi) = (\widehat{v_1}(\xi), \widehat{v_2}(\xi))^T \in \widehat{X}$, as straightforward computations show that

$$\begin{aligned} |G(\xi)\widehat{v}(\xi)|^2 &= \left| \begin{pmatrix} -\xi_1 \widehat{v_1}(\xi) + (-j\xi_2 + \xi_3)\widehat{v_2}(\xi) \\ (j\xi_2 + \xi_3)\widehat{v_1}(\xi) + \xi_1 \widehat{v_2}(\xi) \end{pmatrix} \right|^2 \\ &= (\xi_1^2 + \xi_2^2 + \xi_3^2)(|\widehat{v_1}(\xi)|^2 + |\widehat{v_2}(\xi)|^2) = |\xi|^2 |\widehat{v}(\xi)|^2. \end{aligned} \tag{9.11}$$

Because of (9.9), we have

$$\begin{aligned} P_\alpha(\nabla)v = &-\Psi \frac{1}{2\pi} \int_0^{+\infty} \left(\Psi^{-1} S_L^{-1}(-jt, \nabla) t^{\alpha-1} e^{-j\frac{(\alpha-1)\pi}{2}} \nabla \Psi^{-1} \right) \Psi v \, dt \\ &- \Psi \frac{1}{2\pi} \int_0^{+\infty} \left(\Psi^{-1} S_L^{-1}(jt, \nabla) t^{\alpha-1} e^{j\frac{(\alpha-1)\pi}{2}} \nabla \Psi^{-1} \right) \Psi v \, dt. \end{aligned} \tag{9.12}$$

Since $jv = j(v_1 + iv_2) = v_1 j - i(v_2 j)$ and Ψ is $\mathbb{C}_j$-linear, we find

$$\Psi j \Psi^{-1} (\widehat{v_1}, \widehat{v_2})^T = (\widehat{v_1} j, \widehat{v_2}(-j))^T,$$

i.e., multiplication with j on $L^2(\mathbb{R}^3, \mathbb{H})$ from the left corresponds to the multiplication with the matrix $E := \operatorname{diag}(j, -j)$ on $\widehat{X}$. As

$$\mathcal{Q}_{-jt}(\nabla)^{-1} = (\nabla^2 + t^2)^{-1} = (-\Delta + t^2)^{-1}$$

is a scalar operator and hence commutes with any quaternion, we have

$$S_L^{-1}(-jt, \nabla) = \mathcal{Q}_{-jt}(\nabla)^{-1} jt - \nabla \mathcal{Q}_{-jt}(\nabla)^{-1} = (jt - \nabla)\mathcal{Q}_{-jt}(\nabla)^{-1},$$

and in turn

$$\begin{aligned}
&\Psi^{-1} S_L^{-1}(-jt,\nabla) t^{\alpha-1} e^{-j\frac{(\alpha\ \ 1)\pi}{2}} \nabla \Psi^{-1} \\
&= \Psi^{-1}\left(jt\mathcal{Q}_{-jt}(\nabla)^{-1} - \nabla \mathcal{Q}_{-jt}(\nabla)^{-1}\right) t^{\alpha-1} e^{-j\frac{(\alpha\ \ 1)\pi}{2}} \nabla \Psi^{-1} \\
&= \left(tM_E \mathcal{Q}_{-jt}(M_G)^{-1} - M_G \mathcal{Q}_{-jt}(M_G)^{-1}\right) t^{\alpha-1} M_{\exp\left(-\frac{(\alpha\ \ 1)\pi}{2}E\right)} M_G.
\end{aligned}$$

The operator $\mathcal{Q}_{jt}(M_G)^{-1}$ is

$$\mathcal{Q}_{jt}(M_G)^{-1} = (M_G^2 + t^2\mathcal{I})^{-1} = M_{(G^2+t^2\mathcal{I})\ \ 1} = M_{(t^2+|\xi|^2)\ \ 1\mathcal{I}}$$

with $|\xi|^2 = \xi_1^2+\xi_2^2+\xi_3^2$ and the operator in the first integral of (9.12) therefore equals

$$\begin{aligned}
&\Psi^{-1} S_L^{-1}(-jt,\nabla) t^{\alpha-1} e^{-j\frac{(\alpha\ \ 1)\pi}{2}} \nabla \Psi^{-1} \\
&= M_{tE(t^2+|\xi|^2)\ \ 1 - G(t^2+|\xi|^2)\ \ 1} t^{\alpha-1} M_{\exp\left(-\frac{(\alpha\ \ 1)\pi}{2}E\right)} M_G.
\end{aligned}$$

It is hence the multiplication operator $M_{A_1(t,\xi)}$ determined by the matrix-valued function

$$\begin{aligned}
&A_1(t,\xi) = \frac{t^{\alpha\ \ 1}}{t^2+|\xi|^2}\,(tE - G(\xi)) \exp\left(-\frac{(\alpha-1)\pi}{2}E\right) G(\xi) \\
&= \frac{t^{\alpha\ \ 1}}{t^2+\xi_1^2+\xi_2^2+\xi_3^2} \\
&\cdot \begin{pmatrix} e^{\ \ j\frac{\alpha\pi}{2}}\xi_1(t-j\xi_1) + je^{j\frac{\alpha\pi}{2}}\left(\xi_2^2+\xi_3^2\right) & \left(e^{j\frac{\alpha\pi}{2}}\xi_1 + e^{\ \ j\frac{\alpha\pi}{2}}(\xi_1+jt)\right)(\xi_2+j\xi_3) \\ \left(je^{\ \ j\frac{\alpha\pi}{2}}\xi_1 + e^{j\frac{\alpha\pi}{2}}(-t+j\xi_1)\right)(j\xi_2+\xi_3) & e^{j\frac{\alpha\pi}{2}}(-t+j\xi_1)\xi_1 - je^{\ \ j\frac{\alpha\pi}{2}}\left(\xi_2^2+\xi_3^2\right) \end{pmatrix}.
\end{aligned}$$

Similarly, the operator in the second integral of (9.12) is

$$\begin{aligned}
&\Psi^{-1} S_L^{-1}(jt,\nabla) t^{\alpha-1} e^{j\frac{(\alpha\ \ 1)\pi}{2}} \nabla \Psi^{-1} \\
&= M_{-tE(t^2+|\xi|^2)\ \ 1 - G(t^2+|\xi|^2)\ \ 1} t^{\alpha-1} M_{\exp\left(\frac{(\alpha\ \ 1)\pi}{2}E\right)} M_G.
\end{aligned}$$

It is hence the multiplication operator $M_{A_2(t,\xi)}$ determined by the matrix-valued function

$$\begin{aligned}
&A_2(t,\xi) = \frac{t^{\alpha}}{t^2+|\xi|^2}\,(-tE - G(\xi)) \exp\left(\frac{(\alpha-1)\pi}{2}E\right) G(\xi) \\
&= \frac{t^{\alpha\ \ 1}}{t^2+\xi_1^2+\xi_2^2+\xi_3^2} \\
&\cdot \begin{pmatrix} e^{j\frac{\alpha\pi}{2}}\xi_1(t+j\xi_1) - je^{\ \ j\frac{\alpha\pi}{2}}\left(\xi_2^2+\xi_3^2\right) & -\left(e^{j\frac{\alpha\pi}{2}}(-jt+\xi_1) + e^{\ \ j\frac{\alpha\pi}{2}}\xi_1\right)(\xi_2+j\xi_3) \\ \left(e^{j\frac{\alpha\pi}{2}}\xi_1 + e^{\ \ j\frac{\alpha\pi}{2}}(-jt+\xi_1)\right)(\xi_2-j\xi_3) & -e^{\ \ j\frac{\alpha\pi}{2}}(t+j\xi_1)\xi_1 + je^{j\frac{\alpha\pi}{2}}\left(\xi_2^2+\xi_3^2\right) \end{pmatrix}.
\end{aligned}$$

Hence, we have

$$P_\alpha(\nabla)v = \Psi^{-1}P_\alpha(M_G)\Psi v$$

with

$$P_\alpha(M_G)\widehat{v} := -\frac{1}{2\pi}\int_0^{+\infty} M_{A_1(t,\xi)}\widehat{v}\,dt - \frac{1}{2\pi}\int_0^{+\infty} M_{A_2(t,\xi)}\widehat{v}\,dt, \tag{9.13}$$

for $\widehat{v} = \Psi v \in \Psi\,\mathcal{D}(\nabla)$. We show now that these integrals converge for any $\widehat{v} \in \Psi\,\mathcal{D}(\nabla)$. As Ψ is isometric, this is equivalent to (9.8) converging for any $v \in \mathcal{D}(\nabla)$. Since all norms on a finite-dimensional vector space are equivalent, there exists a constant $C > 0$ such that

$$\|M\| \le C \max_{\ell,\kappa\in\{1,2\}} |m_{\ell,\kappa}|, \quad \forall M = \begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2}\end{pmatrix} \in \mathbb{C}_j^{2\times 2}. \tag{9.14}$$

The modulus of the $(1,1)$-entry of $A_1(t,\xi)$ with $t \ge 0$ is

$$\begin{aligned}
&\frac{t^{\alpha-1}}{t^2+\xi_1^2+\xi_2^2+\xi_3^2}\left|e^{-j\frac{\alpha\pi}{2}}\xi_1(t-j\xi_1) + je^{j\frac{\alpha\pi}{2}}\left(\xi_2^2+\xi_3^2\right)\right| \\
&= \frac{t^{\alpha-1}}{t^2+\xi_1^2+\xi_2^2+\xi_3^2}\left(|\xi_1 t| + |\xi|^2\right) \le \frac{t^{\alpha-1}}{t^2+|\xi|^2}\left(|\xi|t+|\xi|^2\right).
\end{aligned}$$

Similarly, one sees that the $(2,2)$-entry of $A_1(t,\xi)$ satisfies this estimate. For the $(1,2)$-entry we have on the other hand

$$\begin{aligned}
&\frac{t^{\alpha-1}}{t^2+\xi_1^2+\xi_2^2+\xi_3^2}\left|\left(je^{-j\frac{\alpha\pi}{2}}\xi_1 + e^{j\frac{\alpha\pi}{2}}(-t+j\xi_1)\right)(j\xi_2+\xi_3)\right| \\
&\le \frac{t^{\alpha-1}}{t^2+\xi_1^2+\xi_2^2+\xi_3^2}\left(2|\xi_1||\xi_2+j\xi_3| + t|\xi_2+j\xi_3|\right) \\
&\le \frac{2t^{\alpha-1}}{t^2+|\xi|^2}\left(|\xi|^2+t|\xi|\right).
\end{aligned}$$

Similar computations show that the $(2,1)$-entry also satisfies this estimate and hence we deduce from (9.14) that

$$\|A_1(t,\xi)\| \le 2C\frac{t^{\alpha-1}}{t^2+|\xi|^2}\left(|\xi|t+|\xi|^2\right).$$

Analogous arguments show that this estimate is also satisfied by $\|A_2(t,\xi)\|$. For the integrals in (9.13) we hence obtain

$$\begin{aligned}
&\int_0^{+\infty}\|M_{A_1(t,\xi)}\widehat{v}\|_{\widehat{X}}\,dt + \int_0^{+\infty}\|M_{A_2(t,\xi)}\widehat{v}\|_{\widehat{X}}\,dt \\
&\le 2\int_0^{+\infty} 2C\left\|\frac{t^{\alpha-1}}{t^2+|\xi|^2}\left(|\xi|t+|\xi|^2\right)|\widehat{v}(\xi)|\right\|_{L^2(\mathbb{R}^3)} dt \\
&\le 4C\int_0^1 t^{\alpha-1}\left\|\frac{|\xi|t}{t^2+|\xi|^2}|\widehat{v}(\xi)| + \frac{|\xi|^2}{t^2+|\xi|^2}|\widehat{v}(\xi)|\right\|_{L^2(\mathbb{R}^3)} dt \\
&\quad + 4C\int_1^{+\infty} t^{\alpha-2}\left\|\frac{t^2}{t^2+|\xi|^2}|\xi\widehat{v}(\xi)| + \frac{t|\xi|}{t^2+|\xi|^2}|\xi\widehat{v}(\xi)|\right\|_{L^2(\mathbb{R}^3)} dt.
\end{aligned}$$

Now observe that

$$\frac{t^2}{t^2+|\xi|^2} \leq 1, \quad \frac{|\xi|^2}{t^2+|\xi|^2} \leq 1, \quad \frac{t|\xi|}{t^2+|\xi|^2} \leq \frac{1}{2} < 1.$$

Because of (9.10), the relation $\widehat{v} \in \Psi\,\mathcal{D}(\nabla)$ implies that $|\widehat{v}(\xi)|$ and $||\xi|\widehat{v}(\xi)|$ both belong to $L^2(\mathbb{R}^3)$ and hence we finally find

$$\begin{aligned}&\int_0^{+\infty} \|M_{A_1(t,\xi)}\widehat{v}\|_{\widehat{X}}\,dt + \int_0^{+\infty} \|M_{A_2(t,\xi)}\widehat{v}\|_{\widehat{X}}\,dt\\ &\leq 8C\|v(\xi)\|_{L^2(\mathbb{R}^3)} \int_0^1 t^{\alpha-1}\,dt + 8C\|\xi\widehat{v}(\xi)\|_{L^2(\mathbb{R}^3)} \int_1^{+\infty} t^{\alpha-2}\,dt,\end{aligned}$$

which is finite as $\alpha \in (0,1)$. Hence, (9.13) converges for any $\widehat{v} \in \Psi\,\mathcal{D}(\nabla)$ and (9.8) converges in turn for any $v \in \mathcal{D}(\nabla)$. □

Theorem 9.2.2. *The operator $P_\alpha(\nabla)$ can be extended to a closed operator on $L^2(\mathbb{R}^3,\mathbb{H})$. For $v \in \mathcal{D}(\nabla^2) = \mathcal{D}(-\Delta)$, it is moreover given by*

$$P_\alpha(\nabla)v = (-\Delta)^{\frac{\alpha}{2}-1}\left[\frac{1}{2}(-\Delta)^{\frac{1}{2}} + \frac{1}{2}\nabla\right]\nabla v. \tag{9.15}$$

Proof. Let $v \in \mathcal{D}(\nabla^2) = \mathcal{D}(-\Delta)$. Because of (3.29), we have that

$$\begin{aligned}P_\alpha(\nabla)v &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} (-jt\mathcal{I}+\nabla)\mathcal{Q}_{c,-jt}(\nabla)^{-1}(-j)^2(-tj)^{\alpha-1}\nabla v\,dt\\ &= -\frac{1}{2\pi}\int_0^{+\infty} (-jt\mathcal{I}+\nabla)\mathcal{Q}_{c,-jt}(\nabla)^{-1}t^{\alpha-1}e^{-j(\alpha-1)\frac{\pi}{2}}\nabla v\,dt\\ &\quad -\frac{1}{2\pi}\int_0^{+\infty} (jt\mathcal{I}+\nabla)\mathcal{Q}_{c,jt}(\nabla)^{-1}t^{\alpha-1}e^{j(\alpha-1)\frac{\pi}{2}}\nabla v\,dt.\end{aligned} \tag{9.16}$$

Due to (9.7), we have moreover

$$\mathcal{Q}_{c,jt}(\nabla)^{-1} = (-t^2+\Delta)^{-1} = \mathcal{Q}_{c,-jt}(\nabla)^{-1}$$

and hence

$$\begin{aligned}P_\alpha(\nabla)v &= -\frac{1}{2\pi}\int_0^{+\infty} t^\alpha\mathcal{Q}_{c,jt}(\nabla)^{-1}j\left(e^{j(\alpha-1)\frac{\pi}{2}} - e^{-j(\alpha-1)\frac{\pi}{2}}\right)\nabla v\,dt\\ &\quad -\frac{1}{2\pi}\int_0^{+\infty} \nabla\mathcal{Q}_{c,jt}(\nabla)^{-1}t^{\alpha-1}\left(e^{j(\alpha-1)\frac{\pi}{2}} + e^{-j(\alpha-1)\frac{\pi}{2}}\right)\nabla v\,dt\\ &= \frac{\sin\left((\alpha-1)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty} t^\alpha\mathcal{Q}_{c,jt}(\nabla)^{-1}\nabla v\,dt\\ &\quad -\frac{\cos\left((\alpha-1)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty} \nabla\mathcal{Q}_{c,jt}(\nabla)^{-1}t^{\alpha-1}\nabla v\,dt.\end{aligned} \tag{9.17}$$

For the first integral, we obtain

$$\begin{aligned}
&\frac{\sin\left((\alpha-1)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty} t^{\alpha}\mathcal{Q}_{c,jt}(\nabla)^{-1}\nabla v\,dt\\
&=\frac{\sin\left((\alpha-1)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty} t^{\alpha}(-t^2+\Delta)^{-1}\nabla v\,dt\\
&=\frac{\sin\left((\alpha-1)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty} \tau^{\frac{\alpha-1}{2}}(-\tau+\Delta)^{-1}\nabla v\,d\tau\\
&=\frac{1}{2}(-\Delta)^{\frac{\alpha-1}{2}}\nabla v.
\end{aligned}\tag{9.18}$$

The last identity follows from the integral representation of the fractional power A^{β} with $\mathrm{Re}(\beta)\in(0,1)$ of a complex linear sectorial operator A given in Corollary 3.1.4 of [165], namely

$$A^{\beta}v=\frac{\sin(\pi\beta)}{\pi}\int_0^{+\infty}\tau^{\beta}\left(\tau\mathcal{I}+A^{-1}\right)^{-1}v\,d\tau,\quad v\in\mathcal{D}(A).\tag{9.19}$$

As $-\Delta$ is an injective sectorial operator on $L^2(\mathbb{R}^3,\mathbb{C}_j)$, its closed inverse $(-\Delta)^{-1}$ is also a sectorial operator. Its fractional power $\left((-\Delta)^{-1})\right)^{\frac{1-\alpha}{2}}$ is, because of (9.19), given by the last integral in (9.18). Since

$$(-\Delta)^{\frac{\alpha-1}{2}}=\left((-\Delta)^{-1}\right)^{\frac{1-\alpha}{2}},$$

we obtain the last equality. Observe that the expression $\frac{1}{2}(-\Delta)^{\frac{\alpha-1}{2}}\nabla v$ is meaningful as we chose $v\in\mathcal{D}(\nabla^2)$. Indeed, if we consider the operators in the Fourier space $\widehat{X}$ as in the proof of Theorem 9.2.1, then $-\Delta$ corresponds to the multiplication operator $M_{|\xi|^2}$ generated by the scalar function $|\xi|^2$. The operator $(-\Delta)^{\frac{\alpha-1}{2}}$ is then the multiplication operator $M_{|\xi|^{\alpha-1}}$ generated by the function $(|\xi|^2)^{\frac{\alpha-1}{2}}=|\xi|^{\alpha-1}$. Hence,

$$\begin{aligned}
\mathcal{D}(-\Delta)^{-\frac{\alpha-1}{2}}&=\left\{v\in L^2(\mathbb{R}^3,\mathbb{H}):\quad \widehat{v}\in\mathcal{D}(M_{|\xi|^{\alpha-1}})\right\}\\
&=\left\{v\in L^2(\mathbb{R}^3,\mathbb{H}):\quad |\xi|^{\alpha-1}\widehat{v}(\xi)\in\widehat{X}\right\}.
\end{aligned}$$

If $G(\xi)$ is as in (9.5), then

$$\widehat{\nabla v}(\xi)=M_G\widehat{v}(\xi)=G(\xi)\widehat{v}(\xi)\in\hat{X}$$

and because of (9.11) we have $|G(\xi)\widehat{v}(\xi)|=|\xi||\widehat{v}(\xi)|\in L^2(\mathbb{R})$. As $\alpha\in(0,1)$, we therefore find that

$$|\xi|^{\alpha-1}|M_G\widehat{v}(\xi)|=|\xi|^{\alpha}|\widehat{v}(\xi)|$$

belongs to $L^2(\mathbb{R}^3)$ and so we have $\widehat{\nabla v}\in\mathcal{D}(M_{|\xi|^{\alpha-1}})$. This is equivalent to $\nabla v\in\mathcal{D}\left((-\Delta)^{\frac{\alpha-1}{2}}\right)$.

As $v \in \mathcal{D}(\nabla^2) = \mathcal{D}(-\Delta)$, we obtain similarly that the second integral in (9.17) equals

$$\begin{aligned} &-\frac{\cos\left((\alpha-1)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty}\nabla\mathcal{Q}_{c,jt}(\nabla)^{-1}t^{\alpha-1}\nabla v\,dt\\ &=\frac{\sin\left((\alpha-2)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty}\nabla(-t^2\mathcal{I}+\Delta)^{-1}t^{\alpha-1}\nabla v\,dt\\ &=\frac{\sin\left((\alpha-2)\frac{\pi}{2}\right)}{2\pi}\int_0^{+\infty}(-\tau\mathcal{I}+\Delta)^{-1}\tau^{\frac{\alpha\ 2}{2}}\nabla^2 v\,d\tau\\ &=\frac{1}{2}(-\Delta)^{\frac{\alpha}{2}-1}\nabla^2 v. \end{aligned} \tag{9.20}$$

Again this expression is meaningful as we assumed $v \in \mathcal{D}(\nabla^2)$. This is equivalent to $|\xi|^2\widehat{v}(\xi) \in \widehat{X}$ because $\widehat{\nabla^2 v}(\xi) = |\xi|^2\widehat{v}(\xi)$. Since $\alpha \in (0,1)$ and $\widehat{v} \in \mathcal{D}(M_{|\xi|^2})$, the function $|\xi|^2\widehat{v}(\xi)$ belongs to the domain of the multiplication operator $M_{|\xi|^{\alpha\ 2}}$ because

$$M_{|\xi|^{\alpha\ 2}}|\xi|^2\widehat{v}(\xi) = |\xi|^\alpha\widehat{v}(\xi) \in \widehat{X}.$$

Since $(-\Delta)^{\frac{\alpha}{2}-1}$ corresponds to $M_{|\xi|^{\alpha\ 2}}$ on the Fourier space $\widehat{X}$, we find $\nabla^2 v$ in $\mathcal{D}\left((-\Delta)^{\frac{\alpha}{2}-1}\right)$. Altogether, we find

$$P_\alpha(\nabla)v = (-\Delta)^{\frac{\alpha}{2}-1}\left[\frac{1}{2}(-\Delta)^{\frac{1}{2}}+\frac{1}{2}\nabla\right]\nabla v, \quad \forall v \in \mathcal{D}(\nabla^2). \tag{9.21}$$

Finally, we show that $P_\alpha(\nabla)$ can be extended to a closed operator. We need to show that for any sequence $v_n \in \mathcal{D}(P_\alpha(\nabla)) = \mathcal{D}(\nabla)$ that converges to 0 and for which also the sequence $P_\alpha(\nabla)v_n$ converges, we have $z := \lim_{n\to+\infty} P_\alpha(\nabla)v_n = 0$. In order to do this, we write as in (9.17)

$$\begin{aligned} P_\alpha(\nabla)v = &\frac{\sin\left((\alpha-1)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty}t^\alpha(t^2\mathcal{I}+\Delta)^{-1}\nabla v\,dt\\ &-\frac{\cos\left((\alpha-1)\frac{\pi}{2}\right)}{\pi}\int_0^{+\infty}\nabla(t^2\mathcal{I}+\Delta)^{-1}t^{\alpha-1}\nabla v\,dt. \end{aligned}$$

If we choose an arbitrary, but fixed $r > 0$, then the operator $(r\mathcal{I}+\Delta)^{-1}$ commutes with $(t^2\mathcal{I}+\Delta)^{-1}$ and ∇ and we deduce from the above integral representation that

$$(r\mathcal{I}+\Delta)^{-1}P_\alpha(\nabla)v = P_\alpha(\nabla)(r\mathcal{I}+\Delta)^{-1}v, \quad \forall v \in \mathcal{D}(\nabla).$$

We show now that the mapping $v \mapsto P_\alpha(\nabla)(r\mathcal{I}+\Delta)^{-1}v$ is a bounded linear operator on $L^2(\mathbb{R}^3,\mathbb{H})$. Since $(r\mathcal{I}+\Delta)^{-1}$ maps $L^2(\mathbb{R}^3,\mathbb{H})$ to $\mathcal{D}(\Delta) = \mathcal{D}(\nabla^2)$, the composition $\nabla^2(r\mathcal{I}+\Delta)^{-1}$ of the bounded operator $(r\mathcal{I}+\Delta)^{-1}$ and the closed operator ∇^2 is bounded itself. As we have seen above, ∇^2 and also the bounded operator $\nabla^2(r\mathcal{I}+\Delta)^{-1}$ map $L^2(\mathbb{R}^3,\mathbb{H})$ into the domain of the closed operator $(-\Delta)^{\frac{\alpha}{2}-1}$. Hence, their composition $(-\Delta)^{-\frac{\alpha}{2}-1}\nabla^2(r\mathcal{I}+\Delta)^{-1}$ is therefore bounded. Similarly, $\nabla(r+\Delta)^{-1}$ is a bounded operator that maps $L^2(\mathbb{R}^3,\mathbb{H})$ to $\mathcal{D}\big((-\Delta)^{\frac{\alpha}{2}\frac{1}{2}}\big)$ as we have seen above, and so the composition $(-\Delta)^{\frac{\alpha}{2}\frac{1}{2}}\nabla(r\mathcal{I}+\Delta)^{-1}$ is also bounded. Because of (9.21), the operator

$$P_\alpha(\nabla)(r\mathcal{I}+\Delta)^{-1} = \frac{1}{2}(-\Delta)^{\frac{\alpha}{2}\frac{1}{2}}\nabla(r\mathcal{I}+\Delta)^{-1} + \frac{1}{2}(-\Delta)^{\frac{\alpha}{2}-1}\nabla^2(r\mathcal{I}+\Delta)^{-1}$$

is the linear combination of bounded operators and hence bounded itself.

If a sequence $v_n \in \mathcal{D}(P_\alpha(\nabla))$ converges to 0 and $z = \lim_{n\to+\infty} P_\alpha(\nabla)v_n$ exists in $L^2(\mathbb{R}^3,\mathbb{H})$, then

$$(r+\Delta)^{-1}z = \lim_{n\to+\infty}(r+\Delta)^{-1}P_\alpha(\nabla)v_n = \lim_{n\to+\infty} P_\alpha(\nabla)(r+\Delta)^{-1}v_n = 0.$$

But as $(r+\Delta)^{-1}$ is the inverse of a closed operator, its kernel is trivial and so $z = \lim_{n\to+\infty} P_\alpha(\nabla)v_n = 0$. Hence, $P_\alpha(\nabla)$ can be extended to a closed operator. □

Remark 9.2.1. The identity (9.15) might seem surprising at first glance, but it is actually rather intuitive. By the spectral theorem, there exist two spectral measures $E_{(-\Delta)}$ and E_∇ on $[0,+\infty)$ (resp. $\mathbb{R}$) such that $-\Delta = \int_{[0,+\infty)} t\,dE_{-\Delta}(t)$ and $\nabla = \int_{\mathbb{R}} r\,dE_\nabla(r)$. As $\nabla^2 = -\Delta$, the spectral measure $E_{(-\Delta)}$ is furthermore the push-forward measure of E_∇ under the mapping $t\mapsto t^2$ such that

$$\int_{[0,+\infty)} f(t)\,dE_{(-\Delta)}(t) = \int_{\mathbb{R}} f\left(t^2\right)\,dE_\nabla(t)$$

for any measurable function f. Hence, we have for $v\in\mathcal{D}(\nabla^2)$ that

$$\begin{aligned}
P_\alpha(\nabla) &= \int_{\mathbb{R}} t^\alpha \chi_{[0,+\infty)}(t)\,dE_\nabla(t)v\\
&= \int_{\mathbb{R}} t^{\alpha-2}\frac{1}{2}(|t|+t)t\,dE_\nabla(t)v\\
&= \int_{\mathbb{R}} t^{\alpha-2}\,dE_\nabla(t)\frac{1}{2}\left(\int_{\mathbb{R}}|t|\,dE_\nabla(t)+\int_{\mathbb{R}} t\,dE_\nabla(t)\right)\int_{\mathbb{R}} t\,dE_\nabla(t)v\\
&= \int_{[0,+\infty)} t^{\frac{\alpha}{2}-1}\,dE_{(-\Delta)}(t)\frac{1}{2}\\
&\quad\cdot\left(\int_{[0,+\infty)}|t|^{\frac{1}{2}}\,dE_{(-\Delta)}(t)+\int_{\mathbb{R}} t\,dE_\nabla(t)\right)\int_{\mathbb{R}} t\,dE_\nabla(t)v\\
&= (-\Delta)^{-\frac{\alpha}{2}-1}\left[\frac{1}{2}(-\Delta)^{\frac{1}{2}}+\frac{1}{2}\nabla\right]\nabla v.
\end{aligned}$$

The vector part of $P_\alpha(\nabla)$ is, because of (9.15), given by

$$\operatorname{Vec} P_\alpha(\nabla)v = \frac{1}{2}(-\Delta)^{\frac{\alpha}{2}\frac{1}{2}}\nabla v.$$

If we apply the divergence to this equation with sufficiently regular v, we find

$$\operatorname{div}(\operatorname{Vec} P_\alpha(\nabla)v) = \frac{1}{2}(-\Delta)^{\frac{\alpha-1}{2}}\Delta v = -\frac{1}{2}(-\Delta)^{\frac{\alpha+1}{2}}.$$

We can thus reformulate the fractional heat equation (9.2) with $\alpha \in (1/2, 1)$ as

$$\frac{\partial}{\partial t}v - 2\operatorname{div}(\operatorname{Vec} f_\beta(\nabla)v) = 0, \quad \beta = 2\alpha - 1.$$

9.3 An example with non-constant coefficients

As pointed out before, the advantage of the above procedure is that is does not only apply to the gradient to reproduce the fractional Laplacian. Rather it applies to a large class of vector operators, in particular generalized gradients with non-constant coefficients. As a first example, we consider the operator

$$T := \xi_1 \frac{\partial}{\partial \xi_1} e_1 + \xi_2 \frac{\partial}{\partial \xi_2} e_2 + \xi_3 \frac{\partial}{\partial \xi_3} e_3$$

on the space $L^2(\mathbb{R}^3_+, \mathbb{H}, d\mu)$ of $\mathbb{H}$-valued functions on

$$\mathbb{R}^3_+ = \{\xi = (\xi_1, \xi_2, \xi_3)^T \in \mathbb{R}^3 : \quad \xi_\ell > 0\}$$

that are square integrable with respect to

$$d\mu(\xi) = \frac{1}{\xi_1\xi_2\xi_3} d\lambda(\xi),$$

where λ denotes the Lebesgue measure on $\mathbb{R}^3$. In order to determine $\mathcal{Q}_s(T)^{-1}$ we observe that the operator given by the change of variables $J : f \mapsto f \circ \iota$ with $\iota(x) = (e^{x_1}, e^{x_2}, e^{x_3})^T$ is an isometric isomorphism between $L^2(\mathbb{R}^3, \mathbb{H}, d\lambda(x))$ and $L^2(\mathbb{R}^3_+, \mathbb{H}, d\mu(\xi))$. Moreover, $T = J^{-1}\nabla J$ such that

$$Q_s(T) = (s^2\mathcal{I} + T\overline{T}) = J^{-1}(s^2\mathcal{I} + \Delta)J$$

and in turn

$$Q_s(T)^{-1} := (s^2\mathcal{I} - T\overline{T})^{-1} = J^{-1}(s^2\mathcal{I} + \Delta)^{-1}J.$$

We therefore have for sufficiently regular v with calculations analogue to those in (9.16) and (9.17) that

$$P_\alpha(T)v = \frac{\sin((\alpha-1)\pi)}{\pi}\int_0^{+\infty} t^\alpha(-t^2\mathcal{I} + T\overline{T})^{-1}T\,dt + \frac{\cos((\alpha-1)\pi)}{\pi}\int_0^{+\infty} t^{\alpha-1}T(-t^2\mathcal{I} + T\overline{T})^{-1}Tv\,dt.$$

Clearly, the vector part of this operator is again given by the first integral such that

$$\begin{aligned}
\operatorname{Vec} P_\alpha(T)v &= \frac{\sin((\alpha-1)\pi)}{\pi}\int_0^{+\infty} t^\alpha(-t^2\mathcal{I}+T\overline{T})^{-1}Tv\,dt \\
&= \frac{\sin((\alpha-1)\pi)}{\pi}\int_0^{+\infty} t^\alpha J^{-1}(-t^2\mathcal{I}+\Delta)^{-1}JTv\,dt \\
&= J^{-1}\frac{\sin((\alpha-1)\pi)}{\pi}\int_0^{+\infty} t^\alpha(-t^2\mathcal{I}+\Delta)^{-1}\,dt\,JTv \\
&= \frac{1}{2}J^{-1}(-\Delta)^{\frac{\alpha}{2}-\frac{1}{2}}JTv,
\end{aligned}$$

where the last equation follows from computations as in (9.21). Choosing $\beta = 2\alpha+1$, we thus find for sufficiently regular v that

$$\begin{aligned}
&\operatorname{Vec} f_\beta(T)v(\xi) \\
&= \frac{1}{2}J^{-1}(-\Delta)^\alpha JTv(\xi_1,\xi_2,\xi_3) \\
&= \frac{1}{2}J^{-1}(-\Delta)^\alpha \begin{pmatrix} e^{x_1}v_{\xi_1}(e^{x_1},e^{x_2},e^{x_3}) \\ e^{x_2}v_{\xi_2}(e^{x_1},e^{x_2},e^{x_3}) \\ e^{x_3}v_{\xi_3}(e^{x_1},e^{x_2},e^{x_3}) \end{pmatrix} \\
&= \frac{1}{2}J^{-1}\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} -|y|^{2\alpha}e^{iz\cdot y}e^{-x\cdot y}\begin{pmatrix} e^{x_1}v_{\xi_1}(e^{x_1},e^{x_2},e^{x_3}) \\ e^{x_2}v_{\xi_2}(e^{x_1},e^{x_2},e^{x_3}) \\ e^{x_3}v_{\xi_3}(e^{x_1},e^{x_2},e^{x_3}) \end{pmatrix}dx\,dy \\
&= \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} -|y|^{2\alpha}e^{i\sum_{k=1}^3 \xi_k y_k}e^{-ix\cdot y}\begin{pmatrix} e^{x_1}v_{\xi_1}(e^{x_1},e^{x_2},e^{x_3}) \\ e^{x_2}v_{\xi_2}(e^{x_1},e^{x_2},e^{x_3}) \\ e^{x_3}v_{\xi_3}(e^{x_1},e^{x_2},e^{x_3}) \end{pmatrix}dx\,dy.
\end{aligned}$$

The above computations are elementary and illustrate that more complicated operators than the Nabla operator can be considered with the introduced techniques. In particular, one can define and study new types of fractional evolution equations derived from generalized gradient operators with non-constant coefficients of the form

$$T = a_1(x)\frac{\partial}{\partial x_1}e_1 + a_2(x)\frac{\partial}{\partial x_2}e_2 + a_3(x)\frac{\partial}{\partial x_3}e_3. \tag{9.22}$$

The version of the S-functional calculus for operators with commuting components, which we applied in order to study the Nabla operator, simplifies the computations considerably. In the next chapter, we will investigate a more involved example that shows the power of our theory.