Chapter 7

The H^∞-Functional Calculus

The H^∞-functional calculus was originally introduced in [170] by Alan McIntosh. His approach was generalized to quaternionic sectorial operators that are injective and have dense range in [30]. Moreover, under the above assumptions, in [30], it is also treated the case of n-tuples of noncommuting operators. The H^∞-functional calculus stands out among all holomorphic (resp. slice hyperholomorphic) functional calculi because it allows to define functions f of an operator T such that $f(T)$ is unbounded.

This chapter is based on our paper [54], where we defined the H^∞-functional calculus for arbitrary sectorial operators following the strategy of [165]. This provides also the techniques to introduce fractional powers of quaternionic linear operators. The approach in [54] requires neither the injectivity of T nor that T has dense range. Several proofs do not need much additional work and the strategies of the complex setting can be applied in a quite straightforward way. We shall therefore, in particular, focus on the proof of the chain rule and of the the spectral mapping theorem, since more severe technical difficulties arise in these proofs.

7.1 The S-functional calculus for sectorial operators

In order to define the notion of a sectorial operator, we introduce the sector Σ_φ for $\varphi \in (0, \pi]$ as

$$\Sigma_\varphi := \{s \in \mathbb{H} : \quad \arg(s) < \varphi\}.$$

Definition 7.1.1 (Sectorial operator). Let $\omega \in [0, \pi)$. An operator $T \in \mathcal{K}(X)$ is called sectorial of angle ω if

(i) we have $\sigma_S(T) \subset \overline{\Sigma_\omega}$ and

(ii) for every $\varphi \in (\omega, \pi)$ there exists a constant $C > 0$ such that for $s \notin \overline{\Sigma_\varphi}$

$$\left\|S_L^{-1}(s,T)\right\| \leq \frac{C}{|s|} \quad \text{and} \quad \left\|S_R^{-1}(s,T)\right\| \leq \frac{C}{|s|}. \tag{7.1}$$

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We denote the infimum of all these constants by C_φ and additionally by $C_{\varphi,T}$ if we also want to stress its dependence on T.

Next we introduce the following notations:

(a) We denote the set of all operators in $\mathcal{K}(X)$ that are sectorial of angle ω by $\mathrm{Sect}(\omega)$. Furthermore, if T is a sectorial operator, we call

$$\omega_T = \min\{\omega : \quad T \in \mathrm{Sect}(\omega)\}$$

the spectral angle of T.

(b) A family of operators $\{T_\ell\}_{\ell\in\Lambda}$ is called uniformly sectorial of angle ω if $T_\ell \in \mathrm{Sect}(\omega)$ for all $\ell \in \Lambda$ and $\sup_{\ell\in\Lambda} C_{\varphi,T_\ell} < +\infty$ for all $\varphi \in (\omega, \pi)$.

The class of slice hyperholomorphic functions that will be considered in order to define the H^∞-functional calculus is specified in the next definitions.

Definition 7.1.2. Let f be a slice hyperholomorphic function.

(i) We say that f has polynomial limit $c \in \mathbb{H}$ in Σ_φ at 0 if there exists $\alpha > 0$ such that $f(p) - c = O(|p|^\alpha)$ as $p \to 0$ in Σ_φ and that it has polynomial limit ∞ in Σ_φ at 0 if f^{-*_L} (resp. f^{-*_R}) has polynomial limit 0 at 0 in Σ_φ. (By (2.26) this is equivalent to $1/|f(p)| \in O(|p|^\alpha)$ for some $\alpha > 0$ as $p \to 0$ in Σ_φ.)

(ii) Similarly, we say that f has polynomial limit $c \in \mathbb{H}_\infty$ at ∞ in Σ_φ if $p \mapsto f(p^{-1})$ has polynomial limit c at 0.

(iii) If a function has polynomial limit 0 at 0 or ∞, we say that it decays regularly at 0 (resp. ∞).

Observe that the mapping $p \mapsto p^{-1}$ leaves Σ_φ invariant such that the above relation between polynomial limits at 0 and ∞ makes sense.

Definition 7.1.3. Let $\varphi \in (0, \pi]$.

(i) We define $\mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$ as the set of all bounded functions in $\mathcal{SH}_L(\Sigma_\varphi)$ that decay regularly at 0 and ∞.

(ii) Similarly, we define $\mathcal{SH}^\infty_{R,0}(\Sigma_\varphi)$ and $\mathcal{SH}^\infty_0(\Sigma_\varphi)$ as the set of all bounded functions in $\mathcal{SH}_R(\Sigma_\varphi)$ (resp. $\mathcal{N}(\Sigma_\varphi)$) that decay regularly at 0 and ∞.

The following Lemma is an immediate consequence of Theorem 2.1.3.

Lemma 7.1.4. *Let $\varphi \in (0, \pi]$.*

(i) *If $f, g \in \mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$ and $a \in \mathbb{H}$, then $fa + g \in \mathcal{SH}^\infty_{L,0}(\Sigma_\varphi)$. If in addition $f \in \mathcal{SH}^\infty_0(\Sigma_\varphi)$, then $fg \in \mathcal{SH}^\infty_0(\Sigma_\varphi)$.*

(ii) *If $f, g \in \mathcal{SH}^\infty_{R,0}(\Sigma_\varphi)$ and $a \in \mathbb{H}$, then $af + g \in \mathcal{SH}^\infty_{R,0}(\Sigma_\varphi)$. If in addition $g \in \mathcal{SH}^\infty_0(\Sigma_\varphi)$, then $fg \in \mathcal{SH}^\infty_0(\Sigma_\varphi)$.*

(iii) *The space* $\mathcal{SH}_0^\infty(\Sigma_\varphi)$ *is a real algebra.*

Definition 7.1.5 (S-functional calculus for sectorial operators)**.** Let $T \in \mathrm{Sect}(\omega)$. For $f \in \mathcal{SH}_{L,0}^\infty(\Sigma_\varphi)$ with $\omega < \varphi < \pi$, we choose φ' with $\omega < \varphi' < \varphi$ and $j \in \mathbb{S}$ and define

$$f(T) := \frac{1}{2\pi} \int_{\partial(\Sigma_{\varphi'} \cap \mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s). \tag{7.2}$$

Similarly, for $f \in \mathcal{SH}_{R,0}^\infty(\Sigma_\varphi)$ with $\omega < \varphi < \pi$, we choose φ' with $\omega < \varphi' < \varphi$ and $j \in \mathbb{S}$ and define

$$f(T) := \frac{1}{2\pi} \int_{\partial(\Sigma_{\varphi'} \cap \mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,T). \tag{7.3}$$

Remark 7.1.1. Since T is sectorial of angle ω, the estimates in (7.1) assure the convergence of the above integrals. A standard argument using the slice hyperholomorphic version of Cauchy's integral theorem shows that the integrals are independent of the choice of the angle φ', and standard slice hyperholomorphic techniques, based on the representation formula, show that they are independent of the choice of the imaginary unit $j \in \mathbb{S}$. Finally, computations as in the proof of Theorem 3.4.6 show that (7.2) and (7.3) yield the same operator if f is intrinsic.

If $T \in \mathrm{Sect}(\omega)$, then $f(T)$ in Definition 7.1.5 can be defined for any function that belongs to $\mathcal{SH}_{L,0}^\infty(\Sigma_\varphi)$ for some $\varphi \in (\omega, \pi]$. We thus introduce a notation for the space of all such functions.

Definition 7.1.6. Let $\omega \in (0, \pi)$. We define

$$\mathcal{SH}_{L,0}^\infty[\Sigma_\omega] = \bigcup_{\omega<\varphi\leq\pi} \mathcal{SH}_{L,0}^\infty(\Sigma_\varphi),$$

$$\mathcal{SH}_{R,0}^\infty[\Sigma_\omega] = \bigcup_{\omega<\varphi\leq\pi} \mathcal{SH}_{R,0}^\infty(\Sigma_\varphi),$$

$$\mathcal{SH}_{0}^\infty[\Sigma_\omega] = \bigcup_{\omega<\varphi\leq\pi} \mathcal{SH}_{0}^\infty(\Sigma_\varphi).$$

The following properties of the S-functional calculus for sectorial operators can be proved by standard slice-hyperholomorphic techniques, see Theorem 3.5.1 or see also [30, Theorem 4.12].

Lemma 7.1.7. *If* $T \in \mathrm{Sect}(\omega)$*, then the following statements hold true.*

(i) *If* $f \in \mathcal{SH}_{L,0}^\infty[\Sigma_\omega]$ *or* $f \in \mathcal{SH}_{R,0}^\infty[\Sigma_\omega]$*, then the operator* $f(T)$ *is bounded.*

(ii) *If* $f, g \in \mathcal{SH}_{L,0}^\infty[\Sigma_\omega]$ *and* $a \in \mathbb{H}$*, then* $(fa + g)(T) = f(T)a + g(T)$*. Similarly, if* $f, g \in \mathcal{SH}_{R,0}^\infty[\Sigma_\omega]$ *and* $a \in \mathbb{H}$*, then* $(af + g)(T) = af(T) + g(T)$*.*

(iii) *If* $f \in \mathcal{SH}_0^\infty[\Sigma_\omega]$ *and* $g \in \mathcal{SH}_{L,0}^\infty[\Sigma_\omega]$*, then* $(fg)(T) = f(T)g(T)$*. Similarly, if* $f \in \mathcal{SH}_{R,0}^\infty[\Sigma_\omega]$ *and* $g \in \mathcal{SH}_0^\infty[\Sigma_\omega]$*, then also* $(fg)(T) = f(T)g(T)$*.*

We recall that a closed operator $A \in \mathcal{K}(X)$ is said to commute with $B \in \mathcal{B}(X)$, if $BA \subset AB$.

Lemma 7.1.8. *Let $T \in \operatorname{Sect}(\omega)$ and $A \in \mathcal{K}(X)$ commute with $\mathcal{Q}_s(T)^{-1}$ and $T\mathcal{Q}_s(T)^{-1}$ for any $s \in \rho_S(T)$. Then A commutes with $f(T)$ for any $f \in \mathcal{SH}_0^\infty[\Sigma_\omega]$. In particular, $f(T)$ commutes with T for any $f \in \mathcal{SH}_0^\infty[\Sigma_\omega]$.*

Proof. If $f \in \mathcal{SH}_0^\infty[\Sigma_\omega]$, then for suitable $\varphi \in (\omega, \pi)$ and $j \in \mathbb{S}$, we have

$$\begin{aligned} f(T) &= \frac{1}{2\pi} \int_{\partial(\Sigma_\varphi \cap \mathbb{C}_j)} f(s)\, ds_j S_R^{-1}(s,T) \\ &= \frac{1}{2\pi} \int_{-\infty}^{0} f\left(-te^{j\varphi}\right)\left(-e^{j\varphi}\right)(-j)\left(-te^{-j\varphi} - T\right) \mathcal{Q}_{-te^{j\varphi}}(T)^{-1}\, dt \\ &\quad + \frac{1}{2\pi} \int_{0}^{+\infty} f\left(te^{-j\varphi}\right)\left(e^{-j\varphi}\right)(-j)\left(te^{j\varphi} - T\right) \mathcal{Q}_{te^{-j\varphi}}(T)^{-1}\, dt. \end{aligned}$$

After changing $t \mapsto -t$ in the first integral, we find

$$\begin{aligned} f(T) &= \frac{1}{2\pi} \int_{0}^{+\infty} f\left(te^{j\varphi}\right)\left(e^{j\varphi} j\right)\left(te^{-j\varphi} - T\right) \mathcal{Q}_{te^{j\varphi}}(T)^{-1}\, dt \\ &\quad + \frac{1}{2\pi} \int_{0}^{+\infty} f\left(te^{-j\varphi}\right)\left(-e^{-j\varphi} j\right)\left(te^{j\varphi} - T\right) \mathcal{Q}_{te^{-j\varphi}}(T)^{-1}\, dt \\ &= \frac{1}{2\pi} \int_{0}^{+\infty} 2\mathrm{Re}\left[f\left(te^{j\varphi}\right) jt\right] \mathcal{Q}_{te^{j\varphi}}(T)^{-1}\, dt \\ &\quad - \frac{1}{2\pi} \int_{0}^{+\infty} 2\mathrm{Re}\left[f\left(te^{j\varphi}\right) je^{j\varphi}\right] T\mathcal{Q}_{te^{j\varphi}}(T)^{-1}\, dt, \end{aligned}$$

where the last identity holds because $f(\overline{s}) = \overline{f(s)}$ as f is intrinsic and

$$\mathcal{Q}_{te^{j\varphi}}(T)^{-1} = \mathcal{Q}_{te^{-j\varphi}}(T)^{-1}.$$

If now $y \in \mathcal{D}(A)$, then the fact that A commutes with $\mathcal{Q}_s(T)^{-1}$ and $T\mathcal{Q}_s(T)^{-1}$ and any real scalar implies

$$\begin{aligned} f(T)Ay &= \frac{1}{2\pi} \int_{0}^{+\infty} 2\mathrm{Re}\left[f\left(te^{j\varphi}\right) jt\right] \mathcal{Q}_{te^{j\varphi}}(T)^{-1} Ay\, dt \\ &\quad - \frac{1}{2\pi} \int_{0}^{+\infty} 2\mathrm{Re}\left[f\left(te^{j\varphi}\right) je^{j\varphi}\right] T\mathcal{Q}_{te^{j\varphi}}(T)^{-1} Ay\, dt \\ &= A\frac{1}{2\pi} \int_{0}^{+\infty} 2\mathrm{Re}\left[f\left(te^{j\varphi}\right) jt\right] \mathcal{Q}_{te^{j\varphi}}(T)^{-1} y\, dt \\ &\quad - A\frac{1}{2\pi} \int_{0}^{+\infty} 2\mathrm{Re}\left[f\left(te^{j\varphi}\right) je^{j\varphi}\right] T\mathcal{Q}_{te^{j\varphi}}(T)^{-1} y\, dt = Af(T)y. \end{aligned}$$

We thus find $y \in \mathcal{D}(Af(T))$ with $f(T)Ay = Af(T)y$ and in turn $f(T)A \subset Af(T)$. □

Lemma 7.1.9. *Let $T \in \operatorname{Sect}(\omega)$. If $\lambda \in (-\infty, 0)$ and $f \in \mathcal{SH}^{\infty}_{L,0}[\Sigma_\omega]$, then*

$$s \mapsto (\lambda - s)^{-1} f(s) \in \mathcal{SH}^{\infty}_{L,0}[\Sigma_\omega]$$

and

$$\left((\lambda - s)^{-1} f(s)\right)(T) = (\lambda - T)^{-1} f(T) = S_L^{-1}(\lambda, T) f(T).$$

Similarly, if $\lambda \in (-\infty, 0)$ and $f \in \mathcal{SH}^{\infty}_{R,0}[\Sigma_\omega]$, then

$$s \mapsto f(s)(\lambda - s)^{-1} \in \mathcal{SH}^{\infty}_{R,0}[\Sigma_\omega]$$

and

$$\left(f(s)(\lambda - s)^{-1}\right)(T) = f(T)(\lambda - T)^{-1} = f(T) S_R^{-1}(\lambda, T).$$

Proof. Let $\lambda \in (-\infty, 0)$ and observe that, since λ is real, the S-resolvent equation turns into

$$\begin{aligned}(\lambda - T)^{-1} S_L^{-1}(s, T) &= S_R^{-1}(\lambda, T) S_L^{-1}(s, T) \\ &= \left(S_R^{-1}(\lambda, T) - S_L^{-1}(s, T)\right)(s - \lambda)^{-1}.\end{aligned}$$

If now $f \in \mathcal{SH}^{\infty}_{L,0}[\Sigma_\omega]$, then for suitable $\varphi \in (\omega, \pi)$ and $j \in \mathbb{S}$, we have

$$\begin{aligned}(\lambda\mathcal{I} - T)^{-1} f(T) &= \frac{1}{2\pi} \int_{\partial(\Sigma_\varphi \cap \mathbb{C}_j)} (\lambda\mathcal{I} - T)^{-1} S_L^{-1}(s, T)\, ds_j\, f(s) \\ &= \frac{1}{2\pi} \int_{\partial(\Sigma_\varphi \cap \mathbb{C}_j)} \left(S_R^{-1}(\lambda, T) - S_L^{-1}(s, T)\right)(s - \lambda)^{-1}\, ds_j\, f(s) \\ &= S_R^{-1}(\lambda, T) \frac{1}{2\pi} \int_{\partial(\Sigma_\varphi \cap \mathbb{C}_j)} ds_j\, (s - \lambda)^{-1} f(s) \\ &\quad + \frac{1}{2\pi} \int_{\partial(\Sigma_\varphi \cap \mathbb{C}_j)} S_L^{-1}(s, T)\, ds_j\, (\lambda - s)^{-1} f(s) \\ &= \left((\lambda - s)^{-1} f(s)\right)(T),\end{aligned}$$

where the last equality holds because $\frac{1}{2\pi} \int_{\partial(\Sigma_\varphi \cap \mathbb{C}_j)} ds_j\, (s - \lambda)^{-1} f(s) = 0$ by Cauchy's integral theorem. □

Similar to [165], we can extend the class of functions that are admissible for this functional calculus to the analogue of the extended Riesz class.

Definition 7.1.10. For $0 < \varphi < \pi$, we define

$$\mathcal{E}_L(\Sigma_\varphi) = \left\{ f(p) = \tilde{f}(p) + (1 + p)^{-1} a + b : \quad \tilde{f} \in \mathcal{SH}^{\infty}_{L,0}(\Sigma_\varphi), a, b \in \mathbb{H} \right\}$$

and similarly

$$\mathcal{E}_R(\Sigma_\varphi) = \left\{ f(p) = \tilde{f}(p) + a(1 + p)^{-1} + b : \quad \tilde{f} \in \mathcal{SH}^{\infty}_{R,0}(\Sigma_\varphi), a, b \in \mathbb{H} \right\}.$$

Finally, we define $\mathcal{E}(\Sigma_\varphi)$ as the set of all intrinsic functions in $\mathcal{E}_L(\Sigma_\varphi)$, i.e.,

$$\mathcal{E}(\Sigma_\varphi) = \left\{ f(p) = \tilde{f}(p) + (1 + p)^{-1} a + b : \quad \tilde{f} \in \mathcal{SH}^{\infty}_{0}(\Sigma_\varphi), a, b \in \mathbb{R} \right\}.$$

Keeping in mind the product rule of slice-hyperholomorphic functions, simple calculations as in the classical case show the following two corollaries, cf. [165, Lemma 2.2.3].

Corollary 7.1.11. *Let $0 < \varphi < \pi$.*

(i) *The set $\mathcal{E}_L(\Sigma_\varphi)$ is a quaternionic right vector space and it is closed under multiplication with functions in $\mathcal{E}(\Sigma_\varphi)$ from the left.*

(ii) *The set $\mathcal{E}_R(\Sigma_\varphi)$ is a quaternionic left vector space and it is closed under multiplication with functions in $\mathcal{E}(\Sigma_\varphi)$ from the right.*

(iii) *The set $\mathcal{E}(\Sigma_\varphi)$ is a real algebra.*

Corollary 7.1.12. *Let $0 < \varphi < \pi$. A function $f \in \mathcal{SH}_L(\Sigma_\varphi)$ (or $f \in \mathcal{SH}_R(\Sigma_\varphi)$ or $f \in \mathcal{N}(\Sigma_\varphi)$) belongs to $\mathcal{E}_L(\Sigma_\varphi)$ (resp. $\mathcal{E}_R(\Sigma_\varphi)$ or $\mathcal{E}(\Sigma_\varphi)$) if and only if it is bounded and has finite polynomial limits at 0 and infinity.*

Definition 7.1.13. For $\omega \in (0, \pi)$, we denote

$$\mathcal{E}_L[\Sigma_\omega] = \bigcup_{\omega<\varphi<\pi} \mathcal{E}_L(\Sigma_\varphi),$$
$$\mathcal{E}_R[\Sigma_\omega] = \bigcup_{\omega<\varphi<\pi} \mathcal{E}_R(\Sigma_\varphi),$$
$$\mathcal{E}[\Sigma_\omega] = \bigcup_{\omega<\varphi<\pi} \mathcal{E}(\Sigma_\varphi).$$

Definition 7.1.14. Let $T \in \operatorname{Sect}(\omega)$. We define for any function $f \in \mathcal{E}_L[\Sigma_\omega]$ with $f(s) = \tilde{f}(s) + (1+s)^{-1}a + b$ the bounded operator

$$f(T) := \tilde{f}(T) + (1+T)^{-1}a + \mathcal{I}b$$

and for any function $f \in \mathcal{E}_R[\Sigma_\omega]$ with $f(s) = \tilde{f}(s) + a(1+s)^{-1} + b$ the bounded operator

$$f(T) := \tilde{f}(T) + a(1+T)^{-1} + b\mathcal{I},$$

where $\tilde{f}(T)$ is intended in the sense of Definition 7.1.5.

Lemma 7.1.15. *Let $T \in \operatorname{Sect}(\omega)$ and let $f \in \mathcal{E}_L[\Sigma_\omega]$. If f is left slice hyperholomorphic at 0 and decays regularly at infinity, then*

$$f(T) = \frac{1}{2\pi} \int_{\partial(U(r)\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s), \tag{7.4}$$

with $j \in \mathbb{S}$ arbitrary and $U(r) = \Sigma_\varphi \cup B_r(0)$, where $\varphi \in (\omega, \pi)$ is such that $f \in \mathcal{E}_L(\Sigma_\varphi)$ and $r > 0$ is such that f is left slice hyperholomorphic on $\overline{B_r(0)}$. Moreover, if f is left slice hyperholomorphic both at 0 and at infinity, then

$$f(T) = f(\infty)\mathcal{I} + \frac{1}{2\pi} \int_{\partial(U(r,R)\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s), \tag{7.5}$$

with $j \in \mathbb{S}$ arbitrary and $U(r,R) = U(r) \cup (\mathbb{H} \setminus B_R(0))$, where $\varphi \in (\omega, \pi)$ is such that $f \in \mathcal{E}_L(\Sigma_\varphi)$, $r > 0$ is such that f is left slice hyperholomorphic on $\overline{B_r(0)}$ and $R > r$ is such that f is left slice-hyperholmorphic on $\mathbb{H} \setminus B_R(0)$.

Similarly, if $f \in \mathcal{E}_R[\Sigma_\omega]$, is right slice hyperholomorphic at 0 and decays regularly at infinity, then

$$f(T) = \frac{1}{2\pi} \int_{\partial(U(r)\cap\mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,T),$$

with $j \in \mathbb{S}$ arbitrary and $U(r)$ chosen as above. Moreover, if $f \in \mathcal{E}_R[\Sigma_\omega]$ is right slice hyperholomorphic both at 0 and at infinity, then

$$f(T) = f(\infty)\mathcal{I} + \frac{1}{2\pi} \int_{\partial(U(r,R)\cap\mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,T),$$

with $j \in \mathbb{S}$ arbitrary and $U(r,R)$ is chosen as above.

Proof. Let us first assume that $f \in \mathcal{E}_L[\Sigma_\omega]$ is left slice hyperholomorphic at 0 and regularly decaying at infinity. Then $f(s) = \tilde{f}(s) + (1+s)^{-1}a$, where $\tilde{f} \in \mathcal{SH}_{L,0}^\infty(\Sigma_{\varphi'})$ with $\omega < \varphi < \varphi'$, and the function $\tilde{f}$ is, moreover, left slice hyperholomorphic at 0. For $j \in \mathbb{S}$ and $\omega < \varphi < \varphi'$, we therefore have

$$\begin{aligned}&\frac{1}{2\pi} \int_{\partial(U(r)\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s)\\ &= \frac{1}{2\pi} \int_{\partial(U(r)\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, \tilde{f}(s) + \frac{1}{2\pi} \int_{\partial(U(r)\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, (1+s)^{-1}a.\end{aligned}$$

If $r' > r > 0$ is sufficiently small such that $\tilde{f}$ is left slice hyperholomorphic at $\overline{B_{r'}(0)}$, then Cauchy's integral theorem implies that the value of the first integral remains constant as r varies. Letting r tend to 0, we find that this integral equals $\tilde{f}(T)$ in the sense of Definition 7.1.5. For the second integral we find that

$$\begin{aligned}&\frac{1}{2\pi} \int_{\partial(U(r)\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, (1+s)^{-1}a\\ &= \lim_{R\to+\infty} \frac{1}{2\pi} \int_{\partial(U(r,R)\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, (1+s)^{-1}a = (1+T)^{-1}a,\end{aligned}$$

where the last identity can be deduced either from the compatibility of the S-functional calculus for closed operators with intrinsic polynomials in Lemma 3.5.3 and Theorem 3.5.1 or as in the complex case in [165, Lemma 2.3.2] from the residue theorem. In either way, we obtain (7.4).

If $f \in \mathcal{E}_L[\omega]$ is left slice hyperholomorphic both at 0 and at infinity, then $f(s) = \tilde{f}(s) + (1+s)^{-1}a + b$ where $\tilde{f} \in \mathcal{SH}_{L,0}^\infty(\Sigma_{\varphi'})$ with $\omega < \varphi' < \pi$ is left slice hyperholomorphic both at 0 and infinity and $a, b \in \mathbb{H}$. We therefore have

$$\begin{aligned} & f(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U(r,R)\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s) \\ &= \frac{1}{2\pi}\int_{\partial(U(r,R)\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,\tilde{f}(s) \\ &\quad + f(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U(r,R)\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,\left((1+s)^{-1}a+b\right). \end{aligned}$$

As before, because of the left slice hyperholomorphicity of $\tilde{f}$ at 0 and infinity, Cauchy's integral theorem allows us to vary the values of r and R for sufficiently small r and sufficiently large R without changing the value of the first integral. Letting r tend to 0 and R tend to ∞, we find that this integral equals $\tilde{f}(T)$ in the sense of Definition 7.1.5. Since $f(\infty) = b$, the remaining terms, however, equal $(1+T)^{-1}a + \mathcal{I}b$, which can again either be deduced by a standard application of the the residue theorem and Cauchy's integral theorem as in [165, Corollary 2.3.5] or from the properties of the S-functional calculus for closed operators since the function $s \mapsto (1+s)^{-1}a + b$ is left slice hyperholomorphic on the spectrum of T and at infinity. Altogether, we find that (7.5) holds true.

The right slice hyperholomorphic case follows by analogous arguments. □

Corollary 7.1.16. *The S-functional calculus for closed operators and the S-functional calculus for sectorial operators are compatible.*

Proof. Let $T \in \mathrm{Sect}(\omega)$. If $f \in \mathcal{E}_L[\Sigma_\omega]$ is admissible for the S-functional calculus for closed operators, then it is left slice hyperholomorphic at infinity such that (7.5) holds true. The set $U(r,R)$ in this representation is however a slice Cauchy domain and therefore admissible as a domain of integration in the S-functional calculus for closed operators. Hence, both approaches yield the same operator. □

Definition 7.1.14 is compatible with the algebraic structures of the underlying function classes.

Lemma 7.1.17. *If $T \in \mathrm{Sect}(\omega)$, then the following statements hold true.*

(i) If $f, g \in \mathcal{E}_L[\Sigma_\omega]$ and $a \in \mathbb{H}$, then $(fa+g)(T) = f(T)a + g(T)$. If $f, g \in \mathcal{E}_R[\Sigma_\omega]$ and $a \in \mathbb{H}$, then $(af+g)(T) = af(T) + g(T)$.

(ii) If $f \in \mathcal{E}[\Sigma_\omega]$ and $g \in \mathcal{E}_L[\Sigma_\omega]$, then $(fg)(T) = f(T)g(T)$. If $f \in \mathcal{E}_R[\Sigma_\omega]$ and $g \in \mathcal{E}[\Sigma_\omega]$, then also $(fg)(T) = f(T)g(T)$.

Proof. The compatibility with the respective vector space structure is trivial. In order to show the product rule, consider $f \in \mathcal{E}[\Sigma_\omega]$ and $g \in \mathcal{E}_L[\Sigma_\omega]$ with $f(s) = \tilde{f}(s) + (1+s)^{-1}a + b$ with $\tilde{f} \in \mathcal{SH}_0^\infty[\Sigma_\omega]$ and $a, b \in \mathbb{R}$ and $g(s) = \tilde{g}(s) + (1+s)^{-1}c + d$ with $\tilde{g} \in \mathcal{SH}_{L,0}^\infty[\Sigma_\omega]$ and $c, d \in \mathbb{H}$. By Lemma 7.1.7, Lemma 7.1.9 and the identity

$$(\mathcal{I}+T)^{-2} = (\mathcal{I}+T)^{-1} - T(\mathcal{I}+T)^{-2},$$

we then have

$$\begin{aligned} f(T)g(T) &= \tilde{f}(T)\tilde{g}(T) + \tilde{f}(T)(\mathcal{I}+T)^{-1}c + \tilde{f}(T)d + (\mathcal{I}+T)^{-1}\tilde{g}(T)a \\ &\quad + (\mathcal{I}+T)^{-2}ac + (\mathcal{I}+T)^{-1}ad + \tilde{g}(T)b + (\mathcal{I}+T)^{-1}bc + bd\mathcal{I} \\ &= \left(\tilde{f}\tilde{g} + \tilde{f}(1+s)^{-1}c + \tilde{f}d + (1+s)^{-1}\tilde{g}a + \tilde{g}b\right)(T) \\ &\quad - T(\mathcal{I}+T)^{-2}ac + (\mathcal{I}+T)^{-1}(ad+ac+bc) + bd\mathcal{I}. \end{aligned}$$

Since $-s(1+s)^{-2} \in \mathcal{E}_L[\Sigma_\omega]$ is left slice hyperholomorphic at zero and infinity, Corollary 7.1.16 and the properties of the S-functional calculus imply

$$(-s(1+s)^2)(T) = -T(\mathcal{I}+T)^{-2}$$

such that

$$\begin{aligned} f(T)g(T) &= \left[\tilde{f}\tilde{g} + \tilde{f}(1+s)^{-1}c + \tilde{f}d + (1+s)^{-1}\tilde{g}a + \tilde{g}b - s(1+s)^{-2}ac\right](T) \\ &\quad + (\mathcal{I}+T)^{-1}(ad+ac+bc) + bd\mathcal{I} = (fg)(T) \end{aligned}$$

since

$$\begin{aligned} (fg)(s) &= \tilde{f}(s)\tilde{g}(s) + \tilde{f}(s)(1+s)^{-1}c + \tilde{f}(s)d + (1+s)^{-1}\tilde{g}(s)a \\ &\quad + \tilde{g}(s)b - s(1+s)^{-2}ac + (1+s)^{-1}(ad+ac+bc) + bd. \end{aligned}$$

The product rule in the right slice hyperholomorphic case can be shown with analogous arguments. □

Lemma 7.1.18. *If $T \in \mathrm{Sect}(\omega)$, then the following statements hold true.*

(i) *We have $(s(1+s)^{-1})(T) = T(\mathcal{I}+T)^{-1}$.*

(ii) *If A is closed and commutes with $\mathcal{Q}_s(T)^{-1}$ and $T\mathcal{Q}_s(T)^{-1}$ for all $s \in \rho_S(T)$, then A commutes with $f(T)$ for any $f \in \mathcal{E}[\Sigma_\omega]$. In particular, T commutes with $f(T)$ for any $f \in \mathcal{E}[\Sigma_\omega]$.*

(iii) *If $y \in \ker(T)$ and $f \in \mathcal{E}_R[\Sigma_\omega]$, then $f(A)y = f(0)y$. In particular, this holds true if $f \in \mathcal{E}[\Sigma_\omega]$.*

Proof. The first statement holds as

$$\left(s(1+s)^{-1}\right)(T) = (1-(1+s)^{-1})(T) = \mathcal{I} - (\mathcal{I}+T)^{-1} = T(\mathcal{I}+T)^{-1}$$

and the second one follows from Lemma 7.1.8. Finally, if $y \in \ker(T)$, then

$$\mathcal{Q}_s(T)y = \left(T^2 - 2s_0T + |s|^2\mathcal{I}\right)y = |s|^2 y$$

and hence

$$S_R^{-1}(s,T)y = (\overline{s}\mathcal{I} - T)\mathcal{Q}_s(T)^{-1}y = \overline{s}|s|^{-2}y = s^{-1}y.$$

For $\tilde{f} \in \mathcal{SH}^\infty_{R,0}[\Sigma_\varphi]$, we hence have

$$\begin{aligned}\tilde{f}(T)y &= \frac{1}{2\pi}\int_{\partial(\Sigma_\varphi\cap\mathbb{C}_j)} \tilde{f}(s)\,ds_j\,S_R^{-1}(s,T)y\\ &= \frac{1}{2\pi}\int_{\partial(\Sigma_\varphi\cap\mathbb{C}_j)} \tilde{f}(s)\,ds_j\,s^{-1}y = 0\end{aligned}$$

by Cauchy's integral theorem such that for

$$f(s) = \tilde{f}(s) + a(1+s)^{-1} + b$$

and $y \in \ker(T)$

$$f(T)y = \tilde{f}(T)y + a(\mathcal{I}+T)^{-1}y + b\mathcal{I}y = ay + by = f(0)y. \qquad \square$$

Remark 7.1.2. If $f \in \mathcal{E}_L(\Sigma_\omega)$, then we cannot expect (iii) in Lemma 7.1.18 to hold true. In this case

$$\tilde{f}(T)y = \frac{1}{2\pi}\int_{\partial(\Sigma_\varphi\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s)y,$$

but y and $ds_j\,f(s)$ do not commute. So we cannot exploit the fact that $y \in \ker(T)$ to simplify $S_L^{-1}(s,T)y = s^{-1}y$. Indeed, this identity does not necessarily hold true as

$$S_L^{-1}(s,T) = \mathcal{Q}_s(T)^{-1}(\overline{s}-T)y = \mathcal{Q}_s(T)^{-1}\overline{s}y$$

for $y \in \ker(T)$. But the kernel of T is in general not a left linear subspace of T and hence we cannot assume $\overline{s}y \in \ker(T)$. The simplification $\mathcal{Q}_s(T)^{-1}\overline{s}y = |s|^2\overline{s}y = s^{-1}y$ is not possible.

7.2 The H^∞-functional calculus

The H^∞-functional calculus for complex linear sectorial operators in [165] applies to meromorphic functions that are regularizable. Properly defining the orders of zeros poles of slice-hyperholomorphic functions is not our goal and goes beyond the scope of this book. Hence we use the following simple definition, which is sufficient for our purposes.

Definition 7.2.1. Let $s \in \mathbb{H}$ and let f be left slice hyperholomorphic on an axially symmetric neighborhood $[B_r(s)] \setminus [s]$ of $[s]$ with

$$[B_r(s)] = \{q \in \mathbb{H} : \quad \operatorname{dist}([s],q) < r\}$$

and assume that f does not have a left slice hyperholomorphic continuation to all of $[B_r(s)]$. We say that f has a pole at the sphere $[s]$ if there exists $n \in \mathbb{N}$ such that

$$q \mapsto \mathcal{Q}_s(q)^n f(q)$$

has a left slice hyperholomorphic continuation to $[B_r(s)]$ if $s \notin \mathbb{R}$ (resp. if there exists $n \in \mathbb{N}$ such that $q \mapsto (q-s)^{-n} f(q)$ has a left slice hyperholomorphic continuation to $[B_r(s)]$ if $s \in \mathbb{R}$).

Remark 7.2.1. If $[s]$ is a pole of f and q_n is a sequence with $\lim_{n\to+\infty} \operatorname{dist}(q_n, [s]) = 0$, then not necessarily $\lim_{n\to+\infty} |f(q_n)| = +\infty$. One can see this easily if f is restricted to one of the complex planes $\mathbb{C}_j$. If $j, i \in \mathbb{S}$ with $i \perp j$, then the function $f_j := f|_{[B_r(s)]\cap\mathbb{C}_j}$ a meromorphic function with values in the complex (left) vector space $\mathbb{H} \cong \mathbb{C}_j + \mathbb{C}_j i$ over $\mathbb{C}_j$. It must have a pole at $s_j = s_0 + js_1$ or $\overline{s_j} = s_0 - js_1$. Otherwise, we could extend f_j to a holomorphic function on $B_r(s) \cap \mathbb{C}_j$. The representation formula would allow us then to define a slice hyperholomorphic extension of f to $B_r(s)$. However, s_j and $\overline{s_j}$ are not necessarily both poles of f_j. Consider for instance the function

$$f(q) = S_L^{-1}(s,q) = (q^2 - 2s_0 q + |s|^2)^{-1}(\overline{s} - q),$$

which is defined on $U = \mathbb{H} \setminus [s]$. If we choose $j = j_s$, then $f|_{U\cap\mathbb{C}_j} = (s-q)^{-1}$, which obviously does not have a pole at $\overline{s}$. Hence, if $q_n \in \mathbb{C}_j$ tends to $\overline{s}$, then $|f(q_n)|$ remains bounded.

However, the representation formula implies that there exists at most one complex plane $\mathbb{C}_j$ such that only one of the points $\overline{s_j}$ and s_j is a pole of f_j. Otherwise we could use it again to find a slice hyperholomorphic extension of f to $B_r(0)$. For intrinsic functions both points s_j and $\overline{s_j}$ always need to be poles of f_j as in this case $f_j(\overline{q}) = \overline{f_j(q)}$. In general we therefore do not have

$$\lim_{\operatorname{dist}(q,[s])\to 0} |f(q)| = +\infty,$$

but at least for the limit superior, the equality

$$\limsup_{\operatorname{dist}(q,[s])\to 0} |f(q)| = +\infty$$

holds. If f is intrinsic, then even $\lim_{\operatorname{dist}(q,[s])\to 0} |f(q)| = +\infty$ holds true.

Definition 7.2.2. Let $U \subset \mathbb{H}$ be axially symmetric. A function f is said to be left meromorphic on U if there exist isolated spheres $[q_n] \subset U$ for $n \in \Theta$, where Θ is a subset of $\mathbb{N}$, such that $f|_{\tilde{U}} \in \mathcal{SH}_L(\tilde{U})$ with

$$\tilde{U} = U \setminus \bigcup_{n\in\Theta} [q_n]$$

and such that each sphere $[q_n]$ is a pole of f. We denote the set of all such functions by $\mathcal{M}_L(U)$ and the set of all such functions that are intrinsic by $\mathcal{M}(U)$.

For $U = \Sigma_\omega$ with $0 < \omega < \pi$, we furthermore denote

$$\mathcal{M}_L[\Sigma_\omega] = \bigcup_{\omega<\varphi<\pi} \mathcal{M}_L(\Sigma_\varphi) \quad \text{and} \quad \mathcal{M}[\Sigma_\omega] = \bigcup_{\omega<\varphi<\pi} \mathcal{M}(\Sigma_\varphi).$$

Definition 7.2.3. Let $T \in \mathrm{Sect}(\omega)$.

(i) A left slice hyperholomorphic function f is said to be regularisable if $f \in \mathcal{M}_L(\Sigma_\varphi)$ for some $\omega < \varphi < \pi$ and there exists $e \in \mathcal{E}(\Sigma_\varphi)$ such that $e(T)$ defined in the sense of Definition 7.1.14 is injective and $ef \in \mathcal{E}_L(\Sigma_\varphi)$. In this case we call e a regulariser for f.

(ii) We denote the set of all regularisable functions by $\mathcal{M}_L[\Sigma_\omega]_T$. Furthermore, we denote the subset of intrinsic functions in $\mathcal{M}_L[\Sigma_\omega]_T$ by $\mathcal{M}[\Sigma_\omega]_T$.

Lemma 7.2.4. *Let $T \in \mathrm{Sect}(\omega)$.*

(i) *If $f, g \in \mathcal{M}_L[\Sigma_\omega]_T$ and $a \in \mathbb{H}$, then $fa + g \in \mathcal{M}_L[\Sigma_\omega]_T$. If furthermore $f \in \mathcal{M}[\Sigma_\omega]_T$, then also $fg \in \mathcal{M}_L[\Sigma_\omega]_T$.*

(ii) *The space $\mathcal{M}[\Sigma_\omega]_T$ is a real algebra.*

Proof. If e_1 is a regulariser for f and e_2 is a regulariser for g, then $e = e_1e_2$ is a regulariser for $fa + g$ and also for fg if f is intrinsic. Hence the statement follows. □

Definition 7.2.5 (H^∞-functional calculus)**.** Let $T \in \mathrm{Sect}(\omega)$. For $f \in \mathcal{M}_L[\Sigma_\omega]_T$, we define

$$f(T) := e(T)^{-1}(ef)(T),$$

where $e(T)^{-1}$ is the closed inverse of $e(T)$ and $(ef)(T)$ is intended in the sense of Definition 7.1.14.

Theorem 7.2.6. *The operator $f(T) := e(T)^{-1}(ef)(T)$ is independent of the regulariser e and hence well-defined.*

Proof. If $\tilde{e}$ is a different regulariser, then e and $\tilde{e}$ commute because they both belong to $\mathcal{E}[\Sigma_\omega]$. Hence,

$$\tilde{e}(T)e(T) = (\tilde{e}e)(T) = (e\tilde{e})(T) = e(T)\tilde{e}(T)$$

by Lemma 7.1.17. Inverting this equality yields $e(T)^{-1}\tilde{e}(T)^{-1} = \tilde{e}(T)^{-1}e(T)$ so

$$\begin{aligned} e(T)^{-1}(ef)(T) &= e(T)^{-1}\tilde{e}(T)^{-1}\tilde{e}(T)(ef)(T) \\ &= e(T)^{-1}\tilde{e}(T)^{-1}(\tilde{e}ef)(T) \\ &= \tilde{e}(T)^{-1}e(T)^{-1}(e\tilde{e}f)(T) \\ &= \tilde{e}(T)^{-1}e(T)^{-1}e(T)(\tilde{e}f)(T) = \tilde{e}(T)^{-1}(\tilde{e}f)(T). \end{aligned}$$

If $f \in \mathcal{E}_L[\Sigma_\omega]$, then we can use the constant function 1 with $1(T) = \mathcal{I}$ as a regulariser in order to see that Definition 7.2.5 is consistent with Definition 7.1.14. □

Remark 7.2.2. Since we are considering right linear operators, Definition 7.2.5 is not possible for right slice hyperholomorphic functions. Right slice hyperholomorphic functions maintain slice hyperholomorphicity under multiplication with intrinsic functions from the right. A regulariser of a function f would hence be a function e such that $e(T)$ is injective and $fe \in \mathcal{E}_R(\Sigma_\varphi)$. The operator $f(T)$ would then be defined as $(fe)(T)e(T)^{-1}$, but this operator is only defined on $\operatorname{ran} e(T)$ and can hence not be independent of the choice of e. If we consider left linear operators, the situation is of course vice versa, which is a common phenomenon in quaternionic operator theory, cf. Remark 7.1.2.

The next lemma shows that the function f needs to have a proper limit behaviour at 0 if T is not injective.

Lemma 7.2.7. *Let $T \in \operatorname{Sect}(\omega)$ and $f \in \mathcal{M}_L[\Sigma_\omega]_T$. If T is not injective, then f has finite polynomial limit $f(0) \in \mathbb{H}$ in Σ_ω at 0. If furthermore f is intrinsic, then $f(T)y = f(0)y$ for any $y \in \ker(T)$.*

Proof. Assume that T is not injective and let e be a regulariser for f. Since $e(T)y = e(0)y$ for all $y \in \ker(T)$ because of (iii) in Lemma 7.1.18, we have $e(0) \neq 0$ as $e(T)$ is injective. The limit

$$e(0)f(0) := \lim_{p\to 0} e(p)f(p)$$

of $e(p)f(p)$ as p tends to 0 in Σ_ω exists and is finite because $ef \in \mathcal{E}_L(\Sigma_\omega)$. Hence, the respective limit of $f(p) = e(p)^{-1}(e(p)f(p))$ exists too and is finite. Indeed, it is

$$f(0) = \lim_{p\to 0} f(p) = e(0)^{-1}(e(0)f(0)).$$

We find that

$$f(p) - f(0) = e(p)^{-1}\left[(e(p)f(p) - e(0)f(0)) - (e(p) - e(0))\, f(0)\right] = O(|p|^\alpha)$$

as p tends to 0 in Σ_ω because both ef and e have polynomial limit at 0. Hence, f has polynomial limit $f(0)$ at 0 in Σ_ω.

If f is intrinsic, then ef is intrinsic too and $e(0)$, $(ef)(0)$ and $f(0)$ are all real. Hence, for any $y \in \ker(T)$, we have $(ef)(0)y = y(ef)(0) \in \ker(T)$. As $\ker(T)$ is a right linear subspace of X, we conclude that also $(ef)(0)y \in \ker(T)$ and so (iii) in Lemma 7.1.18 yields

$$f(T)y = e(T)^{-1}(ef)(T)y = e(T)^{-1}(ef)(0)y = e(0)^{-1}(ef)(0)y = f(0)y. \qquad \square$$

Remark 7.2.3. If T is injective, then f does not need to have finite polynomial limit at 0 in Σ_ω. Indeed, the function $p \mapsto p(1+p)^{-2}$ or the function $p \mapsto p\left(1+p^2\right)^{-1}$ and their powers can then serve as regularisers that may compensate a singularity at 0. Choosing the latter as a specific regulariser yields exactly the approach chosen in [30], where the H^∞-functional calculus was first introduced for quaternionic linear operators.

The proof of the following lemma is analogous to the complex proofs of Proposition 1.2.2 and Corollary 1.2.4 in [165], and does not employ any specific quaternionic techniques. For the convenience of the reader, we nevertheless give the detailed proof as this result turns out to be crucial for what follows.

Lemma 7.2.8. *Let $T \in \operatorname{Sect}(\omega)$.*

(i) *If $A \in \mathcal{B}(X)$ commutes with T, then A commutes with $f(T)$ for any function $f \in \mathcal{M}[\Sigma_\omega]_T$. Moreover, if $f \in \mathcal{M}[\Sigma_\omega]_T$ and $f(T) \in \mathcal{B}(X)$, then $f(T)$ commutes with T.*

(ii) *If $f, g \in \mathcal{M}_L[\Sigma_\omega]_T$, then*

$$f(T) + g(T) \subset (f+g)(T).$$

If furthermore $f \in \mathcal{M}[\Sigma_\omega]_T$, then

$$f(T)g(T) \subset (fg)(T)$$

with $\mathcal{D}(f(T)g(T)) = \mathcal{D}((fg)(T)) \cap \mathcal{D}(g(T))$. In particular, the above inclusion turns into an equality if $g(T) \in \mathcal{B}(X)$.

(iii) *Let $f \in \mathcal{M}[\Sigma_\omega]_T$ and $g \in \mathcal{M}[\Sigma_\omega]$ be such that $fg \equiv 1$. Then $g \in \mathcal{M}[\Sigma_\omega]_T$ if and only if $f(T)$ is injective. In this case $f(T) = g(T)^{-1}$.*

Proof. If $A \in \mathcal{B}(X)$ commutes with T, then it commutes with $\mathcal{Q}_s(T)^{-1}$ and $T\mathcal{Q}_s(T)^{-1}$ for any $s \in \rho_S(T)$. Hence, it also commutes with $e(T)$ for any $e \in \mathcal{E}[\Sigma_\omega]$ by Lemma 7.1.18. If $f \in \mathcal{M}[\Sigma_\varphi]_T$ and e is a regulariser for f, we thus have

$$\begin{aligned} Af(T) &= Ae(T)^{-1}(ef)(T) \\ &\subset e(T)^{-1}A(ef)(T) \\ &= e(T)^{-1}(ef)(T)A = f(T)A \end{aligned}$$

such that the first assertion in (i) holds true. Because of (i) in Lemma 7.1.18, the function $(1+p)^{-1}$ regularizes the identity function $p \mapsto p$ and we have $p(T) = T$. Once we have shown (ii), we can hence obtain the second assertion in (i) from

$$f(T)T \subset (f(p)p)(T) = (pf(p))(T) = Tf(T).$$

In order to show (ii) assume that $f, g \in \mathcal{M}_L[\Sigma_\omega]_T$ and let e_1 be a regulariser for f and e_2 be a regulariser for g. Then $e = e_1e_2$ regularises both f and g and hence also $f+g$ such that

$$\begin{aligned} f(T) + g(T) &= e(T)^{-1}(ef)(T) + e(T)^{-1}(eg)(T) \\ &\subset e(T)^{-1}\left[(ef)(T) + (eg)(T)\right] \\ &= e(T)^{-1}(e(f+g))(T) = (f+g)(T). \end{aligned}$$

Applying this relation to the functions $f+g$ and $-g$, we find that

$$(f+g)(T) - g(T) \subset f(T)$$

and so

$$(f+g)(T) = f(T) + g(T)$$

if $g(T)$ is bounded. If even $f \in \mathcal{E}[\Sigma_\omega]_T$, then f and e_2 are both intrinsic and hence commute. Thus

$$e(fg) = (e_1 f)(e_2 g) \in \mathcal{E}_L[\Sigma_\omega|_T$$

by Corollary 7.1.11 and so e regularises fg. Because of (ii) in Lemma 7.1.18, the operator $(e_1 f)(T)$ commutes with $e_2(T)$ and hence also with the inverse $e_2(T)^{-1}$. Because of Lemma 7.1.17, we thus find that

$$\begin{aligned} f(T)g(T) &= e_1(T)^{-1}(e_1 f)(T) e_2(T)^{-1}(e_2 g)(T) \\ &\subset e_1(T)^{-1} e_2(T)^{-1}(e_1 f)(T)(e_2 g)(T) \\ &= \left[e_2(T) e_1(T)\right]^{-1}(e_1 f e_2 g)(T) \\ &= e(T)^{-1}(efg)(T) = (fg)(T). \end{aligned}$$

In order to prove the statement about the domains, we consider

$$y \in \mathcal{D}((fg)(T)) \cap \mathcal{D}(g(T)).$$

Then $w := (e_2 g)(T) y \in \mathcal{D}\left(e_2(T)^{-1}\right)$. Since $(e_1 f)(T)$ commutes with $e_2(T)^{-1}$, we conclude that also $(e_1 f)(T) w \in \mathcal{D}\left(e_2(T)^{-1}\right)$. Since $y \in \mathcal{D}((fg)(T))$ and

$$(fg)(T) y = e(T)^{-1}(efg)(T) y,$$

we further have $(efg)(T) y \in \mathcal{D}(e(T)^{-1})$. As

$$e(T)^{-1} = e_1(T)^{-1} e_2(T)^{-1}$$

this implies

$$e_2(T)^{-1}(efg)(T) y \in \mathcal{D}(e_1(T)^{-1}).$$

From the identity

$$\begin{aligned} (e_1 f)(T) g(T) y &= (e_1 f)(T) e_2(T)^{-1} w \\ &= e_2(T)^{-1}(e_1 f)(T) w = e_2(T)^{-1}(efg)(T) y \end{aligned}$$

we conclude that $(e_1 f)(T) g(T) y \in \mathcal{D}\left(e_1(T)^{-1}\right)$. Thus, $g(T) y \in \mathcal{D}(f(T))$ and in turn $y \in \mathcal{D}(f(T)g(T))$. Therefore,

$$\mathcal{D}(f(T)g(T)) \supset \mathcal{D}((fg)(T)) \cap \mathcal{D}(g(T)).$$

The other inclusion is trivial such that altogether we find equality. If $g(T)$ is bounded, then $\mathcal{D}(g(T)) = X$ and we find

$$\mathcal{D}(f(T)g(T)) = \mathcal{D}((fg)(T))$$

such that both operators agree.

We show now the statement (iii) and assume that $f, g \in \mathcal{M}[\Sigma_\omega]$ with $fg \equiv 1$ and that f is regularisable. If g is regularisable too, then (iii) implies $g(T)f(T) \subset (gf)(T) = 1(T) = \mathcal{I}$ with

$$\mathcal{D}(g(T)f(T)) = \mathcal{D}(\mathcal{I}) \cap \mathcal{D}(f(T)) = \mathcal{D}(f(T)).$$

Hence, $f(T)$ is injective, and interchanging the role of f and g shows that $f(T)g(T) = \mathcal{I}$ on $\mathcal{D}(g(T))$ such that actually $f(T) = g(T)^{-1}$. Conversely, if $f(T)$ is injective and e is a regulariser for f, then

$$(fe)g = e(fg) = e \in \mathcal{E}[\Sigma_\omega]_T.$$

Moreover, $(fe)(T)$ is injective as $f(T)$ and $e(T)$ are both injective and $(fe)(T) = f(T)e(T)$ by (ii). Thus, fe is a regulariser for g, i.e., $g \in \mathcal{M}[\Sigma_\omega]_T$. □

Intrinsic polynomials of an operator T are defined as $P[T] = \sum_{k=0}^{n} T^k a_k$ with $\mathcal{D}\left(P[T]\right) = \mathcal{D}\left(T^n\right)$ for any polynomial $P(q) = \sum_{k=0}^{n} q^k a_k$. We use the squared brackets to indicate that the operator $P[T]$ is defined via this functional calculus and not via the H^∞-functional calculus. However, as the next lemma shows, both approaches are consistent.

Lemma 7.2.9. *The H^∞-functional calculus is compatible with intrinsic rational functions. More precisely, if $r(p) = P(p)Q(p)^{-1}$ is an intrinsic rational function with intrinsic polynomials P and Q such that the zeros of Q lie in $\rho_S(T)$, then $r \in \mathcal{M}[\Sigma_\omega]_T$ and the operator $r(T)$ is given by $r(T) = P[T]Q[T]^{-1}$.*

Proof. We first prove compatibility with intrinsic polynomials. For intrinsic polynomials of degree 1 this follows from the linearity of the H^∞-functional calculus and from (i) in Lemma 7.1.18, which shows that $(1+p)^{-1}$ regularises the identity function $p \mapsto p$ and that

$$p(T) = \left((1+p)^{-1}(T)\right)^{-1} (p(1+p)^{-1})(T) = (\mathcal{I}+T)T(\mathcal{I}+T)^{-1} = T.$$

Let us now generalize the statement by induction and let us assume that it holds for intrinsic polynomials of degree n. If P is a polynomial of degree $n+1$, let us write $P(q) = Q(q)q + a$ with $a \in \mathbb{R}$ and an intrinsic polynomial Q of degree n. The induction hypothesis implies that $Q \in \mathcal{M}[\Sigma_\omega]_T$, that $Q(T) = Q[T]$, and that $\mathcal{D}(Q(T)) = \mathcal{D}(T^n)$. Since $\mathcal{M}[\Sigma_\omega]_T$ is a real algebra, we find that P also belongs to $\mathcal{M}[\Sigma_\omega]_T$ and we deduce from (iii) in Lemma 7.2.8 that

$$P(T) \supset Q(T)T + a\mathcal{I} = Q[T]T + a\mathcal{I} = P[T]$$

with

$$\mathcal{D}(P[T]) = \mathcal{D}\left(T^{n+1}\right) = \mathcal{D}(Q(T)T) = \mathcal{D}(P(T)) \cap \mathcal{D}(T).$$

Hence, if we show that $\mathcal{D}(T) \supset \mathcal{D}(P(T))$, the induction is complete. In order to do this, we consider $y \in \mathcal{D}(P(T))$. Then $(\mathcal{I}+T)^{-1}y$ also belongs to $\mathcal{D}(P(T))$ because

$$(\mathcal{I}+T)^{-1}P(T) \subset P(T)(\mathcal{I}+T)^{-1}$$

by (i) in Lemma 7.2.8. But obviously also $(\mathcal{I}+T)^{-1}y \in \mathcal{D}(T)$ and hence

$$(\mathcal{I}+T)^{-1}y \in \mathcal{D}(P(T)) \cap \mathcal{D}(T) = \mathcal{D}\left(T^{n+1}\right),$$

which implies $y \in \mathcal{D}(T^n) \subset \mathcal{D}(T)$. We conclude $\mathcal{D}(T) \supset \mathcal{D}(P(T))$.

Let us now turn to arbitrary intrinsic rational functions. If $s \in \rho_S(T)$ is not real, then $\mathcal{Q}_s(T)$ is injective because $\mathcal{Q}_s(T)^{-1} \in \mathcal{B}(X)$ and hence $\mathcal{Q}_s(p)^{-1} \in \mathcal{M}[\Sigma_\omega]_T$ by (iii) in Lemma 7.2.8. Similarly, if $s \in \rho_S(T)$ is real, then

$$q \mapsto (s-q)^{-1} \in \mathcal{M}[\Sigma_\omega]_T$$

because $(s-q)(T) = (s\mathcal{I} - T)$ is injective as $(s\mathcal{I}-T)^{-1} = S_L^{-1}(s,T) \in \mathcal{B}(X)$. If now $r(q) = P(q)Q(q)^{-1}$ is an intrinsic rational function with poles in $\rho_S(T)$, then we can write $Q(q)$ as the product of such factors, namely

$$Q(q) = \prod_{\ell=1}^{N}(\lambda_\ell - q)^{n_\ell} \prod_{\kappa=1}^{M} \mathcal{Q}_{s_\kappa}(q)^{m_\kappa},$$

where $\lambda_1, \ldots, \lambda_N \in \rho_S(T)$ are the real zeros of Q and $[s_1], \ldots, [s_M] \subset \rho_S(T)$ are the spherical zeros of Q and n_ℓ and m_κ are the orders of λ_ℓ (resp $[s_\kappa]$). Since $\mathcal{M}[\Sigma_\omega]_T$ is a real algebra, we conclude that $Q \in \mathcal{M}[\Sigma_\omega]_T$ and because of (iii) we find

$$Q^{-1}(T) = Q(T)^{-1} = Q[T]^{-1}.$$

Moreover, (ii) in Lemma 7.2.8 implies

$$Q^{-1}(T) = \prod_{\ell=1}^{N}(\lambda_\ell\mathcal{I} - T)^{-n_\ell} \prod_{\kappa=1}^{M} \mathcal{Q}_{s_\kappa}(T)^{-m_\kappa} \in \mathcal{B}(X)$$

because each of the factors in this product is bounded. Finally, we deduce from the boundedness of $Q^{-1}(T)$ and (ii) in Lemma 7.2.8 that

$$r(T) = \left(PQ^{-1}\right)(T) = P(T)Q^{-1}(T) = P[T]Q[T]^{-1} = r[T]. \qquad \square$$

7.3 The composition rule

Let us now turn our attention to the composition rule, which will occur at several occasions when we consider fractional powers of sectorial operators. As always in the quaternionic setting, we can only expect such a rule to hold true if the inner function is intrinsic since the composition of two slice hyperholomorphic functions is slice hyperholomorphic only if the inner function is intrinsic.

Theorem 7.3.1 (The Composition Rule)**.** *Let $T \in \mathrm{Sect}(\omega)$ and $g \in \mathcal{M}[\Sigma_\omega]_T$ be such that $g(T) \in \mathrm{Sect}(\omega')$. Furthermore, assume that for any $\varphi' \in (\omega', \pi)$, there exists some $\varphi \in (\omega, \pi)$ such that $g \in \mathcal{M}(\Sigma_\varphi)$ and $g(\Sigma_\varphi) \subset \overline{\Sigma_{\varphi'}}$. Then $f \circ g \in \mathcal{M}[\Sigma_\omega]_T$ for any $f \in \mathcal{M}_L[\Sigma_{\omega'}]_{g(T)}$ and*

$$(f \circ g)(T) = f(g(T)).$$

Proof. Let us first assume that $g \equiv c$ is constant. In this case $g(T) = c\mathcal{I}$. Since g is intrinsic, we have $\overline{c} = \overline{g(s)} = g(\overline{s}) = c$ and so $c \in \mathbb{R}$. Since g maps Σ_φ into $\overline{\Sigma_{\varphi'}}$ for suitable $\varphi \in (\omega, \pi)$ and $\varphi' \in (\omega', \pi)$, we further find

$$c \in \overline{\Sigma_{\varphi'}} \cap \mathbb{R} = [0, +\infty).$$

If $c \neq 0$, then $(f \circ g)(p) \equiv f(c)$ and we deduce easily, for instance from Corollary 7.1.16, that

$$(f \circ g)(T) = f(c)\mathcal{I} = f(g(T)).$$

If on the other hand $c = 0$, then Lemma 7.2.7 implies that $f(0) := \lim_{p\to 0} f(p)$ as p tends to 0 in Σ_ω exists. Hence $f \circ g$ is well defined. It is the constant function $f \circ g \equiv f(0)$ and so $(f \circ g)(T) = f(0)\mathcal{I}$. If f is intrinsic, then Lemma 7.2.7 implies

$$f(g(T)) = f(0)\mathcal{I} = (f \circ g)(T).$$

If f is not intrinsic, then $f = f_0 + \sum_{\ell=1}^3 f_\ell e_\ell$ with intrinsic components f_ℓ. Since $\ker g(T) = \ker(0\mathcal{I}) = X$, for any vector y, also the vectors $e_\ell y$, $\ell = 1, 2, 3$, belong to $\ker g(T)$, then we conclude, again from Lemma 7.2.7, that

$$\begin{aligned} f(g(T))y &= f_0(g(T))y + \sum_{\ell=1}^3 f_\ell(g(T))e_\ell y \\ &= f_0(0)y + \sum_{\ell=1}^3 f_\ell(0)e_\ell y \\ &= \left(f_0(0) + \sum_{\ell=1}^3 f_\ell(0)e_\ell\right) y \\ &= f(0)y = (f \circ g)(T)y. \end{aligned}$$

In the following, we shall thus assume that g is not constant.

Let φ' and φ be a couple of angles as in the assumptions of the theorem. Since g is intrinsic, $g|_{\mathbb{C}_j \cap \Sigma_\varphi}$ is a non-constant holomorphic function on $\mathbb{C}_j \cap \Sigma_\varphi$. Hence, it maps the open set $g(\Sigma_\varphi \cap \mathbb{C}_j)$ to an open set. The set

$$g(\Sigma_\varphi) = [g(\Sigma_\varphi \cap \mathbb{C}_j)]$$

is therefore also open and so actually contained in $\Sigma_{\varphi'}$, not only in $\overline{\Sigma_{\varphi'}}$. In particular, we find that $f \circ g$ is defined and slice hyperholomorphic on Σ_φ.

We assume for the moment that $f \in \mathcal{E}_L(\Sigma_{\varphi'})$ with $\varphi' \in (\omega', \pi)$ and we choose $\varphi \in (\omega, \pi)$ such that $g \in \mathcal{M}(\Sigma_\varphi)$ and $g(\Sigma_\varphi) \subset \Sigma_{\varphi'}$. Since f is bounded on $\Sigma_{\varphi'}$, the function $f \circ g$ is a bounded function in $\mathcal{SH}_L(\Sigma_\varphi)$. If T is injective, then

$$e(q) = q(1+q)^{-2} \in \mathcal{E}(\Sigma_\varphi)$$

such that $e(T)T(\mathcal{I}+T)^{-2}$ is injective. Moreover, the function $q \mapsto e(q)(f \circ g)(q)$ decays regularly at 0 and infinity in Σ_φ and hence belongs to $\mathcal{E}_L(\Sigma_\varphi)$. In other words, e is a regulariser for $f \circ g$ and hence

$$f \circ g \in \mathcal{M}_L[\Sigma_\omega]_T.$$

If T is not injective, then g has polynomial limit $g(0)$ at 0 by Lemma 7.2.7. Since g is intrinsic, it only takes real values on the real line and so $g(0) \in \mathbb{R}$. It furthermore maps Σ_φ to $\Sigma_{\varphi'}$ and so

$$g(0) \in \overline{\Sigma_{\varphi'}} \cap \mathbb{R} = [0, +\infty).$$

Therefore f has polynomial limit at $g(0)$: if $g(0) = 0$ this follows from Corollary 7.1.12, otherwise it follows from the Taylor expansion of f at $g(0) \in (0, \infty)$, cf. Theorem 2.1.12. As a consequence, $f \circ g$ has polynomial limit at 0. Therefore the function

$$q \mapsto (1+q)^{-1}(f \circ g)(q)$$

belongs to $\mathcal{E}_L(\Sigma_\varphi)$. Since $(\mathcal{I}+T)^{-1}$ is injective because $-1 \in \rho_S(T)$, we find that $(1+q)^{-1}$ is a regularizer for $f \circ g$ and hence $f \circ g \in \mathcal{M}_L[\Sigma_\omega]_T$.

We have

$$f(q) = \tilde{f}(q) + (1+q)^{-1}a + b$$

with $\tilde{f} \in \mathcal{SH}^\infty_{L,0}(\Sigma_{\varphi'})$ and $a, b \in \mathbb{H}$. Because of the additivity of the functional calculus, we can treat each of these pieces separately. The case that $f \equiv b$ has already been considered above. For $f(q) = (1+q)^{-1}a$, the identity

$$(f \circ g)(T) = (\mathcal{I} + g(T))^{-1}$$

follows from (iii) in Lemma 7.2.8 because $p \mapsto 1 + g(p)$ and

$$p \mapsto (f \circ g)(p) = (1 + g(p))^{-1}$$

do both belong to $\mathcal{M}_L[\Sigma_\omega]_T$. Hence, let us assume that

$$f = \tilde{f} \in \mathcal{SH}^\infty_{L,0}(\Sigma_{\varphi'})$$

with $\varphi' \in (\omega', \pi)$.

We choose $\theta' \in (\omega', \varphi')$ and $j \in \mathbb{S}$ and set

$$\Gamma_p = \partial(\Sigma_{\theta'} \cap \mathbb{C}_j).$$

We furthermore choose $\rho' \in (\omega', \theta')$ and by our assumptions on g, we can find $\varphi \in (\omega, \pi)$ such that $g(\Sigma_\varphi) \subset \Sigma_{\rho'} \subsetneq \Sigma_{\theta'}$. We choose $\theta \in (\omega, \varphi)$ and set $\Gamma_s = \partial(\Sigma_\theta \cap \mathbb{C}_j)$. The subscripts s and p in Γ_s and Γ_p refer to the corresponding variable of integration in the following computations. For any $p \in \Gamma_p$, the functions

$$s \mapsto \mathcal{Q}_p(g(s))^{-1} = (g(s)^2 - 2p_0 g(s) + |p|^2)^{-1}$$

and

$$s \mapsto S_L^{-1}(p, g(s))$$

do then belong to $\mathcal{E}_L(\Sigma_\varphi)$ and we have

$$\left[\mathcal{Q}_p(g(\cdot))^{-1}\right](T) = \mathcal{Q}_p(g(T))^{-1}$$

and

$$\left[S_L^{-1}(p, g(\cdot))\right](T) = S_L^{-1}(p, g(T)).$$

Indeed, by (ii) in Lemma 7.2.8, we have

$$\begin{aligned} \left[\mathcal{Q}_p(g(\cdot))\right](T) &= (g^2 - 2p_0 g + |p|^2)(T) \\ &\supset g(T)^2 - 2p_0 g(T) + |p|^2 \mathcal{I} = \mathcal{Q}_p(g(T)). \end{aligned} \tag{7.6}$$

Taking the closed inverses of these operators, we deduce from (iii) in Lemma 7.2.8 that

$$\left[\mathcal{Q}_p(g(\cdot))^{-1}\right](T) = \left[\mathcal{Q}_p(g(\cdot))\right](T)^{-1} \supset \mathcal{Q}_p(g(T))^{-1}. \tag{7.7}$$

Since $p \in \rho_S(T)$, the operator $\mathcal{Q}_p(g(T))^{-1}$ is bounded and so already defined on all of X. Hence, the inclusion $\supset$ in (7.7) and (7.6) is actually an equality and we find

$$\left[\mathcal{Q}_p(g(\cdot))^{-1}\right](T) = \mathcal{Q}_p(g(T))^{-1}.$$

From (ii) we further conclude that also

$$\begin{aligned} \left[S_L^{-1}(p, g(\cdot))\right](T) &= \left[\mathcal{Q}_p(g(\cdot))^{-1}\overline{p} - g(\cdot)\mathcal{Q}_p(g(\cdot))^{-1}\right](T) \\ &= \mathcal{Q}_p(g(T))^{-1}\overline{p} - g(T)\mathcal{Q}_p(g(T))^{-1} = S_L^{-1}(p, g(T)). \end{aligned}$$

We hence have

$$f(g(T)) = \frac{1}{2\pi} \int_{\Gamma_p} S_L^{-1}(p, g(T))\, dp_j\, f(p) = \frac{1}{2\pi} \int_{\Gamma_p} \left[S_L^{-1}(p, g(\cdot))\right](T)\, dp_j\, f(p).$$

Let us first assume that T is injective. Since f and in turn also $f \circ g$ are bounded, we can use $e(q) = q(\mathcal{I} + q)^{-2}$ as a regulariser for $f \circ g$. As e decays regularly at 0 and infinity, also the functions $s \mapsto e(s) S_L^{-1}(p, g(s))$ decays regularly at 0 and infinity for any $p \in \Gamma_p$. Hence it belongs to $\mathcal{SH}^{\infty}_{L,0}(\Sigma_\varphi)$ and so

$$
\begin{aligned}
&f(g(T)) = e(T)^{-1}e(T)f(g(T))\\
&= e(T)^{-1}\frac{1}{2\pi}\int_{\Gamma_p} e(T)S_L^{-1}(p,g(T))\,dp_j\,f(p)\\
&= e(T)^{-1}\frac{1}{2\pi}\int_{\Gamma_p} \left[e(\cdot)S_L^{-1}(p,g(\cdot))\right](T)\,dp_j f(p)\\
&= e(T)^{-1}\frac{1}{(2\pi)^2}\int_{\Gamma_p}\left(\int_{\Gamma_s} S_L^{-1}(s,T)\,ds_j\,s(1+s)^{-2}S_L^{-1}(p,g(s))\right)dp_j f(p).
\end{aligned}
\tag{7.8}
$$

We can now apply Fubini's theorem in order to exchange the order of integration: estimating the resolvent using (7.1), we find that the integrand in the above integral is bounded by the function

$$
F(s,p) := C_\theta\left|pS_L^{-1}(p,g(s))\right|\frac{1}{|1+s|^2}\frac{|f(p)|}{|p|}. \tag{7.9}
$$

Since p, s and $g(s)$ belong to the same complex plane as g is intrinsic, we have due to (2.26) that

$$
\left|pS_L^{-1}(p,g(s))\right| \le \max_{\tilde{s}\in[s]}\frac{|p|}{|p-g(\tilde{s})|} = \max\left\{\frac{1}{|1-p^{-1}g(s)|},\frac{1}{|1-p^{-1}g(\overline{s})|}\right\}. \tag{7.10}
$$

Since $g(\Gamma_s)\subset\Sigma_{\rho'}\cap\mathbb{C}_j\subsetneq\Sigma_{\theta'}\cap\mathbb{C}_j$ and $\Gamma_p=\partial(\Sigma_{\theta'}\cap\mathbb{C}_j)$, these expressions are bounded by a constant depending on θ' and ρ' but neither on p nor on s. Hence, $\left|pS_L^{-1}(p,g(s))\right|$ is uniformly bounded on $\Gamma_s\times\Gamma_p$ and $F(s,p)$ is in turn integrable on $\Gamma_p\times\Gamma_s$ because f has polynomial limit 0 both at 0 and infinity.

After exchanging the order of integration in (7.8), we deduce from Cauchy's integral formula that

$$
\begin{aligned}
f(g(T)) &= e(T)^{-1}\frac{1}{(2\pi)^2}\int_{\Gamma_s} S_L^{-1}(s,T)\,ds_j\,s(1+s)^{-2}\left(\int_{\Gamma_p} S_L^{-1}(p,g(s))\,dp_j f(p)\right)\\
&= e(T)^{-1}\frac{1}{2\pi}\int_{\Gamma_s} S_L^{-1}(s,T)\,ds_j\,e(s)f(g(s))\\
&= e(T)^{-1}e(T)(f\circ g)(T) = (f\circ g)(T).
\end{aligned}
$$

Let us now consider the case that T is not injective. By Lemma 7.2.7, the function g has then finite polynomial limit $g(0)\in\mathbb{R}$ in Σ_φ and hence the function $\tilde{g}(p) = g(p)-g(0)\in\mathcal{M}(\Sigma_\varphi)_T$ has finite polynomial limit 0 in at 0. Let us choose a regulariser e for $\tilde{g}$ with polynomial limit 0 at infinity. (This is always possible: if $\tilde{e}$ is an arbitrary regulariser for $\tilde{g}$, we can choose for instance $e(s) = (1+s)^{-1}\tilde{e}(s)$.) We have then $e\tilde{g}\in\mathcal{SH}^{\infty}_{L,0}(\Sigma_\varphi)$. Since $g(0)$ is real, we have $S_L^{-1}(p,g(0)) = (p-g(0))^{-1}$.

Moreover $g(s)$ and $\mathcal{Q}_p(g(s))^{-1}$ commute for any $s \in \Gamma_s$. For $p \notin \overline{\Sigma_{\rho'}}$ we find thus

$$
\begin{aligned}
& e(s)S_L^{-1}(p, g(s)) - e(s)S_L^{-1}(p, g(0)) \\
& = e(s)\mathcal{Q}_p(g(s))^{-1} \left[(\overline{p} - g(s))(p - g(0)) - \mathcal{Q}_p(g(s))\right](p - g(0))^{-1} \\
& = e(s)\mathcal{Q}_p(g(s))^{-1} \Big[(\overline{p} - g(s))p - g(0)(\overline{p} - g(s)) \\
& \qquad\qquad\qquad + g(s)(\overline{p} - g(s)) - (\overline{p} - g(s))p\Big](p - g(0))^{-1} \\
& = e(s)(g(s) - g(0))S_L^{-1}(p, g(s))(p - g(0))^{-1} \\
& = e(s)\tilde{g}(s)S_L^{-1}(p, g(s))S_L^{-1}(p, g(0)).
\end{aligned}
\tag{7.11}
$$

Hence, e regularises also the function $s \mapsto S_L^{-1}(p, g(s)) - S_L^{-1}(p, g(0))$ and the function $e(\cdot)\left(S_L^{-1}(p, g(\cdot)) - S_L^{-1}(p, g(0))\right)$ does even belong to $\mathcal{SH}_{L,0}^\infty(\Sigma_\varphi)$. We thus have

$$
\begin{aligned}
f(g(T)) &= e(T)^{-1}e(T)f(g(T)) \\
&= e(T)^{-1}\frac{1}{2\pi}\int_{\Gamma_p} e(T)S_L^{-1}(p, g(T))\, dp_j\, f(p) \\
&= e(T)^{-1}\frac{1}{2\pi}\int_{\Gamma_p} \left[e(\cdot)S_L^{-1}(p, g(\cdot))\right](T)\, dp_j\, f(p) \\
&= e(T)^{-1}\frac{1}{2\pi}\int_{\Gamma_p} \left[e(\cdot)\tilde{g}(\cdot)S_L^{-1}(p, g(\cdot))S_L^{-1}(p, g(0))\right](T)\, dp_j f(p) \\
&\quad + e(T)^{-1}\frac{1}{2\pi}\int_{\Gamma p} e(T)S_L^{-1}(p, g(0))\, dp_j f(p).
\end{aligned}
$$

For the second integral, Cauchy's integral formula yields

$$
\begin{aligned}
& e(T)^{-1}\frac{1}{2\pi}\int_{\Gamma p} e(T)S_L^{-1}(p, g(0))\, dp_j f(p) \\
& = e(T)^{-1}e(T)f(g(0)) = f(g(0))\mathcal{I},
\end{aligned}
$$

as f decays regularly at infinity in Σ_θ. For the first integral, we have

$$
\begin{aligned}
& e(T)^{-1}\frac{1}{2\pi}\int_{\Gamma_p} \left[e(\cdot)\tilde{g}(\cdot)S_L^{-1}(p, g(\cdot))S_L^{-1}(p, g(0))\right](T)\, dp_j f(p) \\
& = e(T)^{-1}\frac{1}{(2\pi)^2}\int_{\Gamma_p} \bigg(\int_{\Gamma_s} S_L^{-1}(s, T)\, ds_j\, e(s)\tilde{g}(s) \\
& \qquad\qquad\qquad \cdot S_L^{-1}(p, g(s))S_L^{-1}(p, g(0))\bigg)\, dp_j f(p)
\end{aligned}
$$

$$\overset{(A)}{=} e(T)^{-1}\frac{1}{(2\pi)^2}\int_{\Gamma_s} S_L^{-1}(s,T)\,ds_j \cdot\left(\int_{\Gamma_p} e(s)\tilde{g}(s)S_L^{-1}(p,g(s))S_L^{-1}(p,g(0))\,dp_j\,f(p)\right)$$

$$\overset{(B)}{=} e(T)^{-1}\frac{1}{(2\pi)^2}\int_{\Gamma_s} S_L^{-1}(s,T)\,ds_j\,e(s) \cdot\left(\int_{\Gamma_p} S_L^{-1}(p,g(s)) - S_L^{-1}(p,g(0))\,dp_j\,f(p)\right)$$

$$\overset{(C)}{=} e(T)^{-1}\frac{1}{2\pi}\int_{\Gamma_s} S_L^{-1}(s,T)\,ds_j\,(e(s)f(g(s)) - f(g(0)))$$

$$= e(T)^{-1}(e(T)f\circ g(T) - e(T)f(g(0))\mathcal{I}) = f\circ g(T) - f(g(0))\mathcal{I}, \tag{7.12}$$

where the identity (A) follows from Fubini's theorem, the identity (B) follows from (7.11) and the identity (C) finally follows from Cauchy's integral formula. Altogether, we have

$$f(g(T)) = f\circ g(T) - f(g(0))\mathcal{I} + f(g(0))\mathcal{I} = f\circ g(T).$$

In order to justify the application of Fubini's theorem in (A), we observe that the integrand is bounded by the function

$$F(s,p) = C_\theta\left|pS_L^{-1}(p,g(s))\right|\frac{|e(s)\tilde{g}(s)|}{|s|}\frac{|f(p)|}{|p|}\frac{1}{|p-g(0)|},$$

where we used (7.1) in order to estimate the S-resolvent $S_L^{-1}(s,T)$.

If $g(0)\neq 0$, then $|p-g(0)|^{-1}$ is uniformly bounded in p. Just as before, $\left|pS_L^{-1}(p,g(s))\right|$ is uniformly bounded on $\Gamma_s\times\Gamma_p$. Since $\tilde{g}$ decays regularly at 0, since e decays regularly at infinity and since f decays regularly both at 0 and infinity, the function F is hence integrable on $\Gamma_s\times\Gamma_p$ and we can apply Fubini's theorem.

If on the other hand $g(0)=0$, then $g=\tilde{g}$ and we can write

$$\begin{aligned} F(s,p) &= C_\theta\left|S_L^{-1}(p,g(s))\right|\frac{|e(s)\tilde{g}(s)|}{|s|}\frac{|f(p)|}{|p|} \\ &= C_\theta\left|p^\alpha S_L^{-1}(p,g(s))g(s)^{1-\alpha}\right|\frac{|e(s)g(s)^\alpha|}{|s|}\frac{|f(p)|}{|p|^{1+\alpha}}, \end{aligned} \tag{7.13}$$

with $\alpha\in(0,1)$ such that $|f(p)|/|p|^{1+\alpha}$ is integrable. This is possible because f decays regularly at 0. Just as in (7.10), we can estimate the first factor in (7.13) by

$$\begin{aligned} &\left|p^\alpha S_L^{-1}(p,g(s))g(s)^{1-\alpha}\right| \\ &\leq \max\left\{\frac{|g(s)|^{1-\alpha}}{|p|^{1-\alpha}}\frac{1}{|1-p^{-1}g(s)|}, \frac{|g(\overline{s})|^{1-\alpha}}{|p|^{1-\alpha}}\frac{1}{|1-p^{-1}g(\overline{s})|}\right\}, \end{aligned}$$

where we applied $|g(s)| = \left|\overline{g(\overline{s})}\right| = |g(\overline{s})|$ because g is intrinsic. As before, this expression is uniformly bounded on $\Gamma_s \times \Gamma_p$ because $g(\Gamma_s) \subset \Sigma_{\rho'} \cap \mathbb{C}_j$. Hence, F is again integrable and it is actually possible to apply Fubini's theorem.

Altogether, we have shown that $f(g(T)) = (f \circ g)(T)$ for any $f \in \mathcal{E}_L[\Sigma_{\omega'}]$. Finally, we consider a general function $f \in \mathcal{M}_L[\Sigma_{\omega'}]_{g(T)}$ that does not necessarily belong to $\mathcal{E}_L[\Sigma_{\omega'}]$. If e is a regulariser for f, then e and ef both belong to $\mathcal{E}_L[\omega']$. By what we have just shown, we have $e_g := e \circ g \in \mathcal{M}[\Sigma_\omega]_T$ and $(ef)_g := (ef) \circ g \in \mathcal{M}_L[\Sigma_\omega]_T$ with $e_g(T) = e(g(T))$ and $(ef)_g(T) = (ef)(g(T))$.

Let τ_1 and τ_2 be regularizers for e_g and $(ef)_g$. Then $\tau = \tau_1\tau_2$ regularizes both of them and hence

$$e_g(T) = \tau^{-1}(T)(\tau e_g)(T).$$

Since $e_g(T) = (e \circ g)(T) = e(g(T))$ is injective because e is a regulariser for f, the operator $(\tau e_g)(T)$ is injective too. Moreover, for $f_g := f \circ g$, we find $(\tau e_g) f_g = \tau(e_g f_g) = \tau(ef)_g \in \mathcal{E}_L[\omega]$ because τ was chosen to regularize both e_g and $(ef)_g$. Therefore, τe_g is a regulariser for f_g and hence $f_g \in \mathcal{M}_L[\Sigma_\omega]_T$. Finally, we deduce from Lemma 7.2.8 that

$$\begin{aligned} f(g(T)) &= e(g(T))^{-1}(ef)(g(T)) = (e_g)(T)^{-1}((ef)_g)(T) \\ &= (e_g)(T)^{-1}\tau(T)^{-1}\tau(T)((ef)_g)(T) \\ &= (\tau e_g)(T)^{-1}((\tau e)_g f_g)(T) = f_g(T) = (f \circ g)(T). \end{aligned}$$ □

Corollary 7.3.2. *Let $T \in \operatorname{Sect}(\omega)$ be injective and let $f \in \mathcal{M}_L[\Sigma_\omega]$. Then we have $f \in \mathcal{M}_L[\Sigma_\omega]_T$ if and only if $p \mapsto f(p^{-1}) \in \mathcal{M}_L[\Sigma_\omega]_{T^{-1}}$ and in this case*

$$f(T) = f(p^{-1})(T^{-1}).$$

Proof. Since T is injective, the function p^{-1} belongs to $\mathcal{M}[\Sigma_\omega]_T$ and the statement follows from Theorem 7.3.1. □

7.4 Extensions according to spectral conditions

As in the complex case, cf. [165, Section 2.5], one can extend the H^∞-functional calculus for sectorial operators to a larger class of functions if the operator satisfies additional spectral conditions. We shall mention the following three cases, which are relevant in the proof of the spectral mapping theorem in Section 7.5. In order to explain them, we introduce the notation

$$\Sigma_{\varphi,r,R} = (\Sigma_\varphi \cap B_R(0)) \setminus B_r(0)$$

for $0 \leq r < R \leq \infty$. (We set $B_\infty(0) = \mathbb{H}$ for $R = \infty$.)

(i) If the operator $T \in \mathrm{Sect}(\omega)$ has a bounded inverse, then $B_\varepsilon(0) \subset \rho_S(T)$ for sufficiently small $\varepsilon > 0$. We can thus define the class

$$\mathcal{E}_L{}^\infty(\Sigma_\varphi) = \{f = \tilde{f} + a \in \mathcal{SH}_L(\Sigma_\varphi) : \quad a \in \mathbb{H},\ \tilde{f} \in \mathcal{SH}_L(\Sigma_\varphi) \text{ dec. reg. at } \infty\},$$

and $\mathcal{E}^\infty(\Sigma_\varphi)$ as the set of all intrinsic functions in $\mathcal{E}_L{}^\infty(\Sigma_\varphi)$, where dec. reg. is short for decays regularly. For any function $f \in \mathcal{E}_L{}^\infty(\Sigma_\varphi)$ with $\varphi > 0$, we can define $f(T)$ as

$$f(T) = \frac{1}{2\pi} \int_{\partial(\Sigma_{\varphi,r,\infty} \cap \mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s) + a\mathcal{I},$$

with $0 < r < \varepsilon$ arbitrary. It follows as in Lemma 7.1.15 from Cauchy's integral theorem that this approach is consistent with the usual one if $f \in \mathcal{E}_L(\Sigma_\varphi)$, but the class of admissible functions $\mathcal{E}_L{}^\infty(\Sigma_\varphi)$ is now larger. We can further extend this functional calculus by calling a function $e \in \mathcal{E}_L{}^\infty(\Sigma_\varphi)$ a regulariser for a function $f \in \mathcal{M}_L(\Sigma_\varphi)$, if $e(T)$ is injective and $ef \in \mathcal{E}_L{}^\infty(\Sigma_\varphi)$. In this case, we define

$$f(T) = e(T)^{-1}(ef)(T).$$

Clearly, all the results shown so far still hold for this extended functional calculus since the respective proofs can be carried out in this setting with marginal and obvious modifications. Only in the case of the composition rule we have to consider several conditions, just as in the complex case, namely the combinations

a) T is sectorial and $g(T)$ is invertible and sectorial,

b) T is invertible and sectorial and $g(T)$ is sectorial,

c) T and $g(T)$ are both invertible and sectorial.

In a) and c), one needs the additional assumption $0 \notin \overline{g(\Sigma_\omega)}$ on the function g.

(ii) If the operator $T \in \mathrm{Sect}(\omega)$ is bounded, then $\mathbb{H} \setminus B_\rho(0) \subset \rho_S(T)$ for sufficiently large $\rho > 0$. We can thus define the class

$$\mathcal{E}_L{}^0(\Sigma_\varphi) = \{f = \tilde{f} + a \in \mathcal{SH}_L(\Sigma_\varphi) : \quad a \in \mathbb{H}, \tilde{f} \in \mathcal{SH}_L(\Sigma_\varphi) \text{ dec. reg. at } 0\}$$

and $\mathcal{E}^0(\Sigma_\varphi)$ as the set of all intrinsic functions in $\mathcal{E}_L{}^0(\Sigma_\varphi)$. For any function $f \in \mathcal{E}_L{}^0(\Sigma_\varphi)$ with $\varphi > 0$, we can define $f(T)$ as

$$f(T) = \frac{1}{2\pi} \int_{\partial(\Sigma_{\varphi,0,R} \cap \mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s) + a\mathcal{I},$$

with $0 < \rho < R$ arbitrary. As before, this approach is consistent with the usual one if $f \in \mathcal{E}_L(\Sigma_\varphi)$, but the class of admissible functions $\mathcal{E}_L{}^0(\Sigma_\varphi)$ is again larger than $\mathcal{E}_L(\Sigma_\varphi)$. We can further extend this functional calculus by calling $e \in \mathcal{E}_L{}^0(\Sigma_\varphi)$ a regulariser for $f \in \mathcal{M}_L(\Sigma_\varphi)$, if $e(T)$ is injective and $ef \in \mathcal{E}_L{}^0(\Sigma_\varphi)$ and define again $f(T) = e(T)^{-1}(ef)(T)$ for such f.

As before, all results shown so far hold for this extended functional calculus because the respective proofs can be carried out in this setting with marginal and obvious modifications. For the composition rule, we have to consider again several cases and distinguish the following situations:

a) T is sectorial and $g(T)$ is bounded and sectorial,

b) T is invertible and sectorial and $g(T)$ is bounded and sectorial,

c) T and $g(T)$ are both bounded and sectorial,

d) T is bounded and sectorial and $g(T)$ is sectorial,

e) T is bounded and sectorial and $g(T)$ is invertible and sectorial.

In the cases a), b) and c), one needs the additional assumption $\infty \notin \overline{g(\Sigma_\omega)}^{\mathbb{H}_\infty}$ and in the case e) one needs the additional assumption $0 \notin \overline{g(\Sigma_\omega)}$ on the function g.

(iii) If finally $T \in \mathrm{Sect}(\omega)$ is bounded and has a bounded inverse, then we can set $\mathcal{E}_L{}^{0,\infty}(\Sigma_\varphi) = \mathcal{SH}_L(\Sigma_\varphi)$ and $\mathcal{E}^{0,\infty}(\Sigma_\varphi)$ and define for such functions

$$f(T) = \frac{1}{2\pi} \int_{\partial(\Sigma_{\varphi,r,R} \cap \mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s)$$

for sufficiently small r and sufficiently large R. Choosing regularizers in $\mathcal{E}^{0,\infty}(\Sigma_\varphi)$ gives again an extension of the H^∞-functional calculus and of the two extended functional calculi presented in (i) and (ii). All the results presented so far still hold for this extended functional calculus, where the composition rule can be shown again under suitable conditions on the function g.

7.5 The spectral mapping theorem

Let us now show the spectral mapping theorem for the H^∞-functional calculus. We point out that a substantial technical difficulty appears here that does not occur in the classical situation: the proof of the spectral mapping theorem in the complex setting makes use of the fact that $f\left(T|_{X_\sigma}\right) = f(T)|_{X_\sigma}$ if σ is a spectral set and X_σ is the invariant subspace associated with σ. However, subspaces that are invariant under right linear operators are in general only right linear subspaces, but not necessarily left linear subspaces. Hence, they are not two-sided Banach spaces and we cannot define $f\left(T|_{X_\sigma}\right)$ with the techniques presented in this book because the S-functional calculus as defined in Chapter 3 requires the operator to act on a two-sided Banach space. The S-resolvents can otherwise not be defined.

Instead of using the properties of the S-functional calculus for $T|_{X_\sigma}$, we thus have to find a workaround and prove several steps directly, which is essentially done in Lemma 7.5.5.

We start with two technical lemmas that are necessary in order to show the spectral inclusion theorem.

Lemma 7.5.1. *Let $T \in \mathrm{Sect}(\omega)$ and let $s \in \mathbb{H}$. If $\mathcal{Q}_s(T)$ is injective and there exist $e \in \mathcal{M}[\Sigma_\omega]_T$ and $c \in \mathbb{H}$, $c \neq 0$ such that*

$$f(q) := \mathcal{Q}_c(e(q))\, \mathcal{Q}_s(q)^{-1} \in \mathcal{M}[\Sigma_\omega]_T$$

and such that $e(T)$ and $f(T)$ are bounded, then $e(T)\mathcal{Q}_s(T)^{-1} = \mathcal{Q}_s(T)^{-1}e(T)$.

Proof. By assumption, the operator $\mathcal{Q}_s(T)$ is injective and hence (iii) in Lemma 7.2.8 implies that $\mathcal{Q}_s^{-1} \in \mathcal{M}[\omega]_T$. Since $e(T)$ is bounded, it commutes with T and so also with $\mathcal{Q}_s(T)^{-1}$. We thus have

$$e(T)\mathcal{Q}_s(T)^{-1} \subset \mathcal{Q}_s(T)^{-1}e(T).$$

In order to show that this relation is actually an equality, it is sufficient to show that $y \in \mathcal{D}\left(\mathcal{Q}_s(T)^{-1}\right)$ for any $y \in X$ with $e(T)y \in \mathcal{D}(\mathcal{Q}_s(T)^{-1})$. This is indeed the case: if $e(T)y$ belongs to $\mathcal{D}(\mathcal{Q}_s(T)^{-1})$, then there exists $x \in \mathcal{D}(\mathcal{Q}_s(T))$ with $e(T)y = \mathcal{Q}_s(T)x$. Hence,

$$\begin{aligned} \mathcal{Q}_c(e(T))y &= e(T)^2y - 2c_0e(T)y + |c|^2y \\ &= e(T)\mathcal{Q}_s(T)x - 2c_0\mathcal{Q}_s(T)x + |c|^2y \\ &= \mathcal{Q}_s(T)(e(T)x - 2c_0x) + |c|^2y, \end{aligned} \tag{7.14}$$

where the last identity follows again from (i) in Lemma 7.2.8 because $e(T)$ is bounded and commutes with T and in turn also with $\mathcal{Q}_s(T)$. Since $f(T) \in \mathcal{B}(X)$, we conclude on the other hand from (ii) of Lemma 7.2.8 that

$$\mathcal{Q}_c(e(T)) = \mathcal{Q}_s(T)\left[\mathcal{Q}_c(e(\cdot))\mathcal{Q}_s(\cdot)^{-1}\right](T) = \mathcal{Q}_s(T)f(T).$$

Due to (7.14), we then find

$$\begin{aligned} y &= \frac{1}{|c|^2}\left(\mathcal{Q}_c(e(T))y - \mathcal{Q}_s(T)(e(T)x - 2c_0x)\right) \\ &= \mathcal{Q}_s(T)\frac{1}{|c|^2}\left(f(T)y - e(T)x + 2c_0x\right). \end{aligned}$$

Hence, y belongs to $\mathcal{D}(\mathcal{Q}_s(T)^{-1})$ and the statement follows. □

Lemma 7.5.2. *Let $T \in \mathrm{Sect}(\omega)$ and let $f \in \mathcal{M}_L[\Sigma_\omega]_T$. For any $s \in \overline{\Sigma_\omega} \setminus \{0\}$ there exists a regulariser e for f with $e(s) \neq 0$.*

Proof. Let $\tilde{e}$ be an arbitrary regulariser of f such that $\tilde{e} \in \mathcal{E}[\Sigma_\omega]$, $\tilde{e}f \in \mathcal{E}_L[\Sigma_\omega]$ and $\tilde{e}(T)$ is injective. If $\tilde{e}(s) \neq 0$, then we can set $e = \tilde{e}$ and we are done. Otherwise, recall that $[s]$ is a spherical zero of $\tilde{e}$ and that its order is a finite number $n \in \mathbb{N}$ since $e \not\equiv 0$ as $e(T)$ is injective. We define now $e(q) := \mathcal{Q}_s^{-n}(q)e(q)$ with $\mathcal{Q}_s(q) = q^2 - 2s_0 q + |s|^2$. Then $e \in \mathcal{E}[\Sigma_\omega]$ with $e(s) \neq 0$ and $ef = \mathcal{Q}_s^{-n}\tilde{e}f \in \mathcal{E}_L[\Sigma_\omega]$. Furthermore, by (ii) in Lemma 7.2.8, we have $\tilde{e}(T) = \mathcal{Q}_s(T)e(T)$. Since $\tilde{e}(T)$ is injective, we deduce that also $e(T)$ is injective. Hence, e is a regulariser for f with $e(s) \neq 0$. □

Lemma 7.5.3. *Let $T \in \mathrm{Sect}(\omega)$ and let $s \in \overline{\Sigma_\omega}$ with $s \neq 0$. If $f(T)$ has a bounded inverse for some $f \in \mathcal{M}[\Sigma_\omega]_T$ with $f(s) = 0$, then $s \in \rho_S(T)$.*

Proof. Let f be as above and let us first show that $\mathcal{Q}_s(T) = T^2 - 2s_0T + |s|^2\mathcal{I}$ is injective and hence invertible as a closed operator. By Lemma 7.5.2, there exists a regulariser e for f with $c := e(s) \neq 0$. We have $ef \in \mathcal{E}[\Sigma_\omega]$ with $(ef)(s) = 0$. Since all zeros of intrinsic functions are spherical zeros, we find that also $h = ef\mathcal{Q}_s^{-1} = \mathcal{Q}_s^{-1}ef \in \mathcal{E}[\Sigma_\omega]$. The product rule (ii) in Lemma 7.2.8 implies therefore

$$h(T)\mathcal{Q}_s(T) \subset (h\mathcal{Q}_s)(T) = (ef)(T) = (fe)(T) = f(T)e(T),$$

where $ef = fe$ because both functions are intrinsic. Since $e(T)$ and $f(T)$ are both injective, we find that $\mathcal{Q}_s(T)$ is injective. Moreover, e is also a regulariser for $\mathcal{Q}_s^{-1}f$. Now observe that the function

$$g(q) := \mathcal{Q}_c(e(q))\mathcal{Q}_s(q)^{-1} = (e(q)^2 - 2c_0e(q) + |c|^2)(q^2 - 2s_0q + |s|^2)^{-1}$$

belongs to $\mathcal{E}[\Sigma_\omega]$. Indeed, by Corollary 7.1.11, the space $\mathcal{E}[\Sigma_\omega]$ is a real algebra such $\mathcal{Q}_c(e(q)) = e(q)^2 - 2c_0e(q) + |c|^2$ belongs to it as e does. The function $\mathcal{Q}_c(e(q))$ however has a spherical zero at s because $e(s) = c$ such that $g(q) = \mathcal{Q}_c(e(q))\mathcal{Q}_s^{-1}(q)$ is bounded and hence belongs to $\mathcal{E}[\Sigma_\omega]$ by Corollary 7.1.12. In particular, this implies that $g(T)$ is bounded.

We deduce from Lemma 7.5.1 that $e(T)\mathcal{Q}_s(T)^{-1} = \mathcal{Q}_s(T)^{-1}e(T)$ and inverting both sides of this equation yields $\mathcal{Q}_s(T)e(T)^{-1} = e(T)^{-1}\mathcal{Q}_s(T)$. The product rule in (ii) of Lemma 7.2.8, the boundedness of $h(T) = (e\mathcal{Q}_s^{-1}f)(T)$ and the fact that $\mathcal{Q}_s^{-1}$ and e commute because both are intrinsic functions imply

$$\begin{aligned}
f(T) &= e(T)^{-1}(ef)(T)\\
&= e(T)^{-1}\left(\mathcal{Q}_s e\mathcal{Q}_s^{-1}f\right)(T)\\
&= e(T)^{-1}\mathcal{Q}_s(T)\left(e\mathcal{Q}_s^{-1}f\right)(T)\\
&= \mathcal{Q}_s(T)e(T)^{-1}(e\mathcal{Q}_s^{-1}f)(T)\\
&= \mathcal{Q}_s(T)(\mathcal{Q}_s^{-1}f)(T).
\end{aligned}$$

Since $f(T)$ is surjective, we find that $\mathcal{Q}_s(T)$ is surjective too. Hence, $\mathcal{Q}_s(T)^{-1}$ is an everywhere defined closed operator and thus bounded by the closed graph theorem. Consequently $s \in \rho_S(T)$. □

Proposition 7.5.4. *If $T \in \mathrm{Sect}(\omega)$ and $f \in \mathcal{M}[\Sigma_\omega]_T$, then*

$$f(\sigma_S(T) \setminus \{0\}) \subset \sigma_{SX}(f(T)).$$

Proof. Let $s \in \sigma_S(T) \setminus \{0\}$ and set $c := f(s)$. If $c \neq \infty$, then Lemma 7.5.3 implies that

$$\mathcal{Q}_c(f(T))^2 = f(T)^2 - 2c_0 f(T) + |c|^2 \mathcal{I}$$

does not have a bounded inverse because $g = f^2 - 2c_0 f + |c|^2$ belongs to $\mathcal{M}[\Sigma_\omega]_T$ and satisfies $g(c) = 0$. Hence, $c = f(s)$ belongs to $\sigma_S(f(T))$ for $s \in \sigma_S(T) \setminus \{0\}$ with $f(s) \neq \infty$.

If on the other hand $c = \infty$, then suppose that $c \notin \sigma_{SX}(f(T))$, i.e., that $f(T)$ is bounded. In this case there exists $p \in \mathbb{H}$ such that $\mathcal{Q}_p(f(T))$ has a bounded inverse. By (iii) in Lemma 7.2.8, this implies $g(q) = \mathcal{Q}_p(f(q))^{-1} \in \mathcal{M}[\Sigma_\omega]_T$. The operator $g(T)$ is invertible as $g(T)^{-1} = \mathcal{Q}_p(f(T))$ belongs to $\mathcal{B}(X)$ because $f(T)$ is bounded. Moreover, since $g(s) = 0$ as $f(s) = \infty$, another application of Lemma 7.5.3 yields $s \in \rho_S(T)$. But this contradicts our assumption $s \in \sigma_S(T) \setminus \{0\}$. Hence, we must have $c \in \sigma_{SX}(f(T))$. □

We have so far shown the spectral inclusion theorem for spectral values not equal to 0 nor ∞. These two values need a special treatment. They also need additional assumptions on the function f for a spectral inclusion theorem to hold as we shall see in the following. (The assumptions presented here might, however, not be the most general ones that are possible, cf. [165].)

First, we have to show a technical lemma. We recall that if $\sigma \subset \sigma_{SX}(T)$ is a spectral set, then $E_\sigma := \chi_\sigma(T)$ is by Theorem 3.7.8 a projection that commutes with T, i.e., it is a projection onto a right-linear subspace of X that is invariant under T. If $\infty \notin \sigma$, then we can choose a bounded slice Cauchy domain $U_\sigma \subset \mathbb{H}$ such that $\sigma \subset U_\sigma$ and such that $(\sigma_S(T) \setminus \sigma) \cap U_\sigma = \emptyset$. The projection E_σ is then given by

$$E_\sigma = \frac{1}{2\pi} \int_{\partial(U_\sigma \cap \mathbb{C}_j)} ds_j \, S_R^{-1}(s,T) = \frac{1}{2\pi} \int_{\partial(U_\sigma \cap \mathbb{C}_j)} S_L^{-1}(p,T)\, dp_j. \tag{7.15}$$

If on the other hand $\infty \in \sigma$, then we can choose an unbounded slice Cauchy domain $U_\sigma \subset \mathbb{H}$ such that $\sigma \subset U_\sigma$ and such that $(\sigma_S(T) \setminus \sigma) \cap U_\sigma = \emptyset$. The projection E_σ is then given by

$$E_\sigma = \mathcal{I} + \frac{1}{2\pi} \int_{\partial(U_\sigma \cap \mathbb{C}_j)} ds_j \, S_R^{-1}(s,T) = \mathcal{I} + \frac{1}{2\pi} \int_{\partial(U_\sigma \cap \mathbb{C}_j)} S_L^{-1}(p,T)\, dp_j.$$

Lemma 7.5.5. *Let $T \in \mathrm{Sect}(\omega)$ be unbounded and assume that $\sigma_S(T)$ is bounded. Furthermore, let E_∞ be the spectral projection onto the invariant subspace associated with ∞. If $f \in \mathcal{M}[\Sigma_\omega]_T$ has polynomial limit 0 at infinity, then*

$$\{f(T)\}_\infty := f(T) E_\infty$$

is a bounded operator that is given by the slice hyperholomorphic Cauchy integral

$$\{f(T)\}_\infty = \int_{\partial(\Sigma_\varphi \setminus B_r(0))\cap\mathbb{C}_j} f(s)\, ds_j\, S_R^{-1}(s,T), \tag{7.16}$$

where $B_r(0)$ is the ball centered at 0 with $r > 0$ sufficiently large such that it contains $\sigma_S(T)$ and any singularity of f. Moreover, for two such functions, we have

$$\{f(T)\}_\infty \{g(T)\}_\infty = \{(fg)(T)\}_\infty. \tag{7.17}$$

Proof. Let us first assume that $f \in \mathcal{E}[\Sigma_\omega]$, i.e., $f \in \mathcal{E}(\Sigma_\varphi)$ with $\omega < \varphi < \pi$. Since f decays regularly at infinity, it is of the form $f(s) = \tilde{f}(s) + a(1+s)^{-1}$ with $a \in \mathbb{R}$ and $f \in \mathcal{SH}_0^\infty(\Sigma_\varphi)$. The operator $\tilde{f}(T)$ is given by the slice hyperholomorphic Cauchy integral

$$\tilde{f}(T) = \frac{1}{2\pi}\int_{\partial(\Sigma_{\varphi'}\cap\mathbb{C}_j)} \tilde{f}(s)\, ds_j\, S_R^{-1}(s,T) \tag{7.18}$$

with $j \in \mathbb{S}$ and $\varphi' \in (\omega,\varphi)$. Let now $r_1 < r_2$ be such that $\sigma_S(T) \subset B_r(0)$. Cauchy's integral theorem allows us to replace the path of integration in (7.18) by the union of $\Gamma_{s,1} = \partial(\Sigma_{\varphi'} \cap B_{r_1}(0)) \cap \mathbb{C}_j$ and $\Gamma_{s,2} = \partial(\Sigma_{\varphi'} \setminus B_{r_2}(0)) \cap \mathbb{C}_j$ such that

$$\tilde{f}(T) = \frac{1}{2\pi}\int_{\Gamma_{s,1}} \tilde{f}(s)\, ds_j\, S_R^{-1}(s,T) + \frac{1}{2\pi}\int_{\Gamma_{s,2}} \tilde{f}(s)\, ds_j\, S_R^{-1}(s,T). \tag{7.19}$$

Let us choose $R \in (r_1, r_2)$. Since $\sigma_{SX}(T) = \sigma_S(T) \cup \{\infty\}$, we have $E_\infty = \mathcal{I} - E_{\sigma_S(T)}$ and the spectral projection $E_{\sigma_S(T)}$ is given by the slice hyperholomorphic Cauchy integral (7.15) along $\Gamma_p = \partial(B_R(0) \cap \mathbb{C}_j)$. The subscripts s and p in $\Gamma_{s,1}$, $\Gamma_{s,2}$ and Γ_p are chosen in order to indicate the corresponding variable of integration in the following computation.

If we write the operators $\tilde{f}(T)$ and $E_{\sigma_S(T)}$ in terms of the slice hyperholomorphic Cauchy integrals defined above, we find that

$$\begin{aligned}\tilde{f}(T)E_{\sigma_s(T)} &= \frac{1}{2\pi}\int_{\Gamma_{s,1}} \tilde{f}(s)\, ds_j\, S_R^{-1}(s,T) \frac{1}{2\pi}\int_{\Gamma_p} S_L^{-1}(p,T)\, dp_j \\ &\quad + \frac{1}{2\pi}\int_{\Gamma_{s,2}} \tilde{f}(s)\, ds_j\, S_R^{-1}(s,T) \frac{1}{2\pi}\int_{\Gamma_p} S_L^{-1}(p,T)\, dp_j.\end{aligned} \tag{7.20}$$

If we apply the S-resolvent equation in the first integral, which we denote by Ψ_1

for neatness, we find

$$\begin{aligned}\Psi_1 = &\frac{1}{(2\pi)^2}\int_{\Gamma_{s,1}} \tilde{f}(s)\,ds_j\, S_R^{-1}(s,T)\int_{\Gamma_p} p\left(p^2-2s_0p+|s|^2\right)^{-1}\,dp_j\\ &-\frac{1}{(2\pi)^2}\int_{\Gamma_{s,1}} \tilde{f}(s)\,ds_j\,\overline{s}S_R^{-1}(s,T)\int_{\Gamma_p}\left(p^2-2s_0p+|s|^2\right)^{-1}\,dp_j\\ &-\frac{1}{(2\pi)^2}\int_{\Gamma_p}\Bigg(\int_{\Gamma_{s,1}} \tilde{f}(s)\,ds_j\,(S_L^{-1}(p,T)p\\ &\qquad -\overline{s}S_L^{-1}(p,T))\left(p^2-2s_0p+|s|^2\right)^{-1}\Bigg)\,dp_j.\end{aligned} \tag{7.21}$$

For $s\in\Gamma_s$, the functions

$$p\mapsto\left(p^2-2s_0p+|s|^2\right)^{-1}\quad\text{and}\quad p\mapsto p\left(p^2-2s_0p+|s|^2\right)^{-1}$$

are rational functions on $\mathbb{C}_j$ that have two singularities, namely $s=s_0+js_1$ and $\overline{s}=s_0-js_1$. Since we chose $r_1<R$, these singularities lie inside of $B_R(0)$ for any $s\in\Gamma_s$. As $\Gamma_p=\partial(B_R(0)\cap\mathbb{C}_j)$, the residue theorem yields

$$\begin{aligned}&\frac{1}{2\pi}\int_{\Gamma_p} p\left(p^2-2s_0p+|s|^2\right)^{-1}dp_j\\ &=\lim_{\mathbb{C}_j\ni p\to s} p(p-\overline{s})^{-1}+\lim_{\mathbb{C}_j\ni p\to\overline{s}} p(p-s)^{-1}=1\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2\pi}\int_{\Gamma_p}\left(p^2-2s_0p+|s|^2\right)^{-1}dp_j\\ &=\lim_{\mathbb{C}_j\ni p\to s}(p-\overline{s})^{-1}+\lim_{\mathbb{C}_j\ni p\to\overline{s}}(p-\overline{s})^{-1}=0,\end{aligned}$$

where $\lim_{\mathbb{C}_j\ni p\to s}\tilde{f}(p)$ denotes the limit of $\tilde{f}(p)$ as p tends to s in $\mathbb{C}_j$. If we apply the identity (2.49) with $B=S_L^{-1}(p,T)$ in the third integral in (7.21), it turns into

$$\begin{aligned}&\frac{1}{(2\pi)^2}\int_{\Gamma_p}\left(\int_{\Gamma_{s,1}}\tilde{f}(s)\,ds_j\,\left(s^2-2p_0s+|p|^2\right)^{-1}sS_L^{-1}(p,T)\right)dp_j\\ &\quad-\frac{1}{(2\pi)^2}\int_{\Gamma_p}\left(\int_{\Gamma_{s,1}}\tilde{f}(s)\,ds_j\,\left(s^2-2p_0s+|p|^2\right)^{-1}S_L^{-1}(p,T)\overline{p}\right)dp_j=0.\end{aligned}$$

The last identity follows from Cauchy's integral theorem because $\tilde{f}(s)$ is right slice hyperholomorphic and the functions $s\mapsto(s^2-2p_0s+|p|^2)^{-1}S_L^{-1}(p,T)$ and

$s \mapsto s(s^2 - 2p_0 s + |p|^2)^{-1} S_L^{-1}(p,T)$ are left slice hyperholomorphic on $\Sigma_{\varphi'} \cap B_{r_1}(0)$ for any $p \in \Gamma_p$ as we chose $R > r_1$. Hence, we find

$$\Psi_1 = \frac{1}{2\pi} \int_{\Gamma_{s,1}} \tilde{f}(s)\, ds_j\, S_R^{-1}(p,T).$$

The second integral in (7.20), which we denote by Ψ_2 for neatness, turns after an application of the S-resolvent equation into

$$\begin{aligned} \Psi_2 = & \frac{1}{(2\pi)^2} \int_{\Gamma_{s,2}} \tilde{f}(s)\, ds_j\, S_R^{-1}(s,T) \int_{\Gamma_p} p \left(p^2 - 2s_0 p + |s|^2\right)^{-1} dp_j \\ & - \frac{1}{(2\pi)^2} \int_{\Gamma_{s,2}} \tilde{f}(s)\, ds_j\, \overline{s} S_R^{-1}(s,T) \int_{\Gamma_p} \left(p^2 - 2s_0 p + |s|^2\right)^{-1} dp_j \\ & - \frac{1}{(2\pi)^2} \int_{\Gamma_{s,2}} \left(\int_{\Gamma_p} \tilde{f}(s)\, ds_j\, (S_L^{-1}(p,T)p - \right. \\ & \qquad \left. - \overline{s} S_L^{-1}(p,T)) \left(p^2 - 2s_0 p + |s|^2\right)^{-1} \right) dp_j. \end{aligned} \tag{7.22}$$

Since we chose $R < r_2$, the singularities of $p \mapsto \left(p^2 - 2s_0 p + |s|^2\right)^{-1}$ and $p \mapsto p\left(p^2 - 2s_0 p + |s|^2\right)^{-1}$ lie outside of $\overline{B_R(0)}$ for any $s \in \Gamma_{s,2}$. Hence, these functions are right slice hyperholomorphic on $\overline{B_R(0)}$ and so Cauchy's integral theorem implies that the first two integrals in (7.22) vanish. Since $\tilde{f}$ decays regularly at infinity, since (7.1) holds true and since Γ_p is a path of finite length, we can apply Fubini's theorem and exchange the order of integration in the third integral of (7.22). After applying the identity (2.49), we find

$$\begin{aligned} \Psi_2 = \frac{1}{(2\pi)^2} \int_{\Gamma_p} \Bigg(\int_{\Gamma_{s,2}} & \tilde{f}(s) ds_j \left(s^2 - 2p_0 s + |p|^2\right)^{-1} \\ & \cdot \left(s S_L^{-1}(p,T) - S_L^{-1}(p,T)\overline{p} \right) \Bigg) dp_j. \end{aligned}$$

However, this integral also vanishes: as f decays regularly at infinity, the integrand decays sufficiently fast so that we can use Cauchy's integral theorem to transform the path of integration and write

$$\begin{aligned} & \int_{\Gamma_{s,2}} \tilde{f}(s) ds_j \left(s^2 - 2p_0 s + |p|^2\right)^{-1} \left(s S_L^{-1}(p,T) - S_L^{-1}(p,T)\overline{p} \right) \\ & = \lim_{\rho \to +\infty} \int_{\partial(U_\rho \cap \mathbb{C}_j)} \tilde{f}(s)\, ds_j \left(s^2 - 2p_0 s + |p|^2\right)^{-1} \left(s S_L^{-1}(p,T) - S_L^{-1}(p,T)\overline{p} \right) = 0 \end{aligned}$$

where $U_\rho = (\Sigma_\varphi \setminus B_{r_2(0)}) \cap B_\rho(0)$ for $\rho > r_2$. The last identity follows again from Cauchy's integral theorem because the singularities p and $\overline{p}$ of the functions

$s \mapsto (s^2 - 2p_0 s + |p|^2)^{-1}$ and $s \mapsto (s^2 - 2p_0 s + |p|^2)^{-1} s$ lie outside of $\overline{U_\rho}$ because we chose $R < r_2$.

Putting these pieces together, we find that

$$\tilde{f}(T) E_{\sigma_S(T)} = \frac{1}{2\pi} \int_{\Gamma_{s,1}} \tilde{f}(p)\, dp_j\, S_R^{-1}(p,T). \tag{7.23}$$

We therefore deduce from (7.19) and $E_\infty = \mathcal{I} - E_{\sigma_S(T)}$ that

$$\tilde{f}(T) E_\infty = \tilde{f}(T) - \tilde{f}(T) E_{\sigma_S(T)} = \frac{1}{2\pi} \int_{\Gamma_{s,2}} \tilde{f}(p)\, dp_j\, S_R^{-1}(p,T). \tag{7.24}$$

Let us now consider the operator $a(\mathcal{I}+T)^{-1}$. Since it is slice hyperholomorphic on $\sigma_S(T)$ and at infinity, it is admissible for the S-fuctional calculus. If we set $\chi_{\{\infty\}}(s) := \chi_{\mathbb{H}\setminus U_R(0)}$—that is $\chi_{\{\infty\}}(s) = 1$ if $s \notin U_R(0)$ and $\chi_{\{\infty\}}(s) = 0$ if $s \in \overline{U_R(0)}$—then $\chi_{\{\infty\}}(T) = E_\infty$ via the S-functional calculus. The product rule of the S-functional calculus yields $a(\mathcal{I}+T)^{-1}E_\infty = g(T)$ with $g(s) = a(1+s)\chi_{\{\infty\}}(s)$. If we set

$$U_{\rho,1} := (\Sigma_\varphi \setminus B_{r_2}(0)) \cup (\mathbb{H} \setminus B_\rho(0)) \quad \text{and} \quad U_{\rho,2} = (\Sigma_\varphi \cap B_{r_1}(0)) \cup B_\varepsilon(0)$$

with $0 < \varepsilon < 1$ sufficiently small, then $U_\rho = U_{\rho,1} \cup U_{\rho,2}$ is an unbounded slice Cauchy domain that contains $\sigma_S(T)$ and such that g is slice hyperholomorphic on $\overline{U_\rho}$. Hence,

$$\begin{aligned} a(\mathcal{I}+T)^{-1}E_\infty &= g(\infty)\mathcal{I} + \frac{1}{2\pi} \int_{\partial(U_\rho \cap \mathbb{C}_j)} dp_j\, S_R^{-1}(p,T) \\ &= \frac{1}{2\pi} \int_{\partial(U_{\rho,1} \cap \mathbb{C}_j)} a(1+s)\, dp_j\, S_R^{-1}(p,T) \end{aligned}$$

and letting ρ tend to infinity, we finally find

$$a(\mathcal{I}+T)^{-1}E_\infty = \frac{1}{2\pi} \int_{\Gamma_{s,2}} a(1+s)\, dp_j\, S_R^{-1}(p,T). \tag{7.25}$$

Adding (7.24) and (7.25), we find that (7.16) holds true for $f \in \mathcal{E}[\Sigma_\omega]$.

Now let f be an arbitrary function in $\mathcal{M}[\Sigma_\omega]_T$ that decays regularly at infinity and let e be a regulariser for f. We can assume that e decays regularly at infinity; otherwise, we can replace e by $s \mapsto (1+s)^{-1}e(s)$, which is a regulariser for f with this property. We expect that

$$\begin{aligned} f(T)E_\infty &= e^{-1}(T)(ef)(T)E_\infty \\ &= e^{-1}(T)\{(ef)(T)\}_\infty \\ &\overset{(*)}{=} e^{-1}(T)\{e(T)\}_\infty\{f(T)\}_\infty \\ &= e^{-1}(T)e(T)E_\infty\{f(T)\}_\infty \\ &= E_\infty\{f(T)\}_\infty \overset{(**)}{=} \{f(T)\}_\infty \end{aligned} \tag{7.26}$$

such that (7.16) holds true. Then, the boundedness of $f(T)E_\infty$ also follows from the boundedness of the integral $\{f(T)\}_\infty$. The second and the fourth of the above equalities follow from the above arguments since ef and e both belong to $\mathcal{E}[\Sigma_\varphi]$ and decay regularly at infinity. The equalities marked with $(*)$ and $(**)$ however remain to be shown.

Let $\omega < \varphi_2 < \varphi_1 < \varphi$ be such that $e, f \in \mathcal{E}(\Sigma_\varphi)$ and let $r_1 < r_2$ be such that $B_{r_1}(0)$ contains $\sigma_S(T)$ and any singularity of f. We set $U_s = \Sigma_{\varphi_1} \setminus B_{r_1}(0)$ and $U_p = \Sigma_{\varphi_2} \setminus B_{r_2}(0)$, where the subscripts s and p indicate again the respective variable of integration in the following computation. An application of the S-resolvent equation shows then that, using the notation $\mathcal{Q}_s(p)^{-1} = (p^2 - 2s_0p + |s|^2)^{-1}$,

$$
\begin{aligned}
&\{e(T)\}_\infty \{f(T)\}_\infty \\
&= \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} e(s)\, ds_j\, S_R^{-1}(s,T) \frac{1}{2\pi} \int_{\partial(U_p \cap \mathbb{C}_j)} S_L^{-1}(p,T)\, dp_j\, f(p) \\
&= \frac{1}{(2\pi)^2} \int_{\partial(U_s \cap \mathbb{C}_j)} e(s)\, ds_j S_R^{-1}(s,T) \int_{\partial(U_p \cap \mathbb{C}_j)} p\mathcal{Q}_s(p)^{-1}\, dp_j f(p) \\
&\quad + \frac{1}{(2\pi)^2} \int_{\partial(U_s \cap \mathbb{C}_j)} e(s)\, ds_j S_R^{-1}(s,T) \int_{\partial(U_p \cap \mathbb{C}_j)} \mathcal{Q}_s(p)^{-1}\, dp_j f(p) \\
&\quad + \frac{1}{(2\pi)^2} \int_{\partial(U_s \cap \mathbb{C}_j)} e(s)\, ds_j \left(\overline{s} S_L^{-1}(p,T) - S_L^{-1}(p,T)p\right) \\
&\qquad\qquad \cdot \int_{\partial(U_p \cap \mathbb{C}_j)} \mathcal{Q}_s(p)^{-1}\, dp_j f(p).
\end{aligned}
$$

Because of our choice of U_s and U_p, the singularities of $p \mapsto (p^2 - s_0p + |s|^2)^{-1}$ lie outside U_p for any $s \in \partial(U_s \cap \mathbb{C}_j)$ such that $p \mapsto (p^2 - 2s_0p + |s|)^{-1}$ and $p \mapsto p(p^2 - 2s_0p + |s|)^{-1}$ are right slice hyperholomorphic on $\overline{U_p}$ for any such s. Since f also decays regularly in U_p at infinity, Cauchy's integral theorem implies that the first two of the above integrals equal zero. The fact that e and f decay polynomially at infinity allows us to exchange the order of integration in the third integral, such that

$$
\begin{aligned}
\{e(T)\}_\infty \{f(T)\}_\infty = \frac{1}{(2\pi)^2} \int_{\partial(U_p \cap \mathbb{C}_j)} \Bigg[\int_{\partial(U_s \cap \mathbb{C}_j)} & e(s)\, ds_j \\
& \cdot \left(\overline{s} S_L^{-1}(p,T) - S_L^{-1}(p,T)p\right) \mathcal{Q}_s(p)^{-1} \Bigg] dp_j f(p).
\end{aligned}
$$

If $p \in \partial(U_p \cap \mathbb{C}_j)$, then p lies for sufficiently large ρ in the bounded axially symmetric Cauchy domain $U_{s,\rho} = U_s \cap B_\rho(0)$. Since f is an intrinsic function on $\overline{U_{s,\rho}}$, Lemma 2.2.24 implies

$$\begin{aligned}&\frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_j)} e(s)\,ds_j\left(\overline{s}S_L^{-1}(p,T)-S_L^{-1}(p,T)p\right)\mathcal{Q}_s(p)^{-1}\\&=\lim_{\rho\to\infty}\frac{1}{2\pi}\int_{\partial(U_s\cap B_\rho(0)\cap\mathbb{C}_j)} e(s)\,ds_j\left(\overline{s}S_L^{-1}(p,T)-S_L^{-1}(p,T)p\right)\mathcal{Q}_s(p)^{-1}\\&=S_L^{-1}(p,T)e(p).\end{aligned}$$

Recalling the equivalence of right and left slice hyperholomorphic Cauchy integrals for intrinsic functions, cf. Remark 3.4.2, we finally find that

$$\{e(T)\}_\infty\{f(T)\}_\infty=\frac{1}{2\pi}\int_{\partial(U_p\cap\mathbb{C}_j)} S_L^{-1}(p,T)\,dp_j e(p)f(p)=\{(ef)(T)\}_\infty.$$

Hence, the identity $(*)$ in (7.26) is true.

Similar arguments show that also $(**)$ holds true. We choose $0<R<r$ such that $B_R(0)$ contains $\sigma_S(T)$ and all singularities of $f(T)$ and we choose $\omega<\varphi'<\varphi$ such that $f\in\mathcal{E}(\Sigma_{\varphi'})$ and set $U_p:=\Sigma_{\varphi'}\setminus B_r(0)$. An application of the S-resolvent equation shows that

$$\begin{aligned}&E_{\sigma_s(T)}\{f(T)\}_\infty\\&=\frac{1}{2\pi}\int_{\partial(B_R(0)\cap\mathbb{C}_j)} ds_j\,S_R^{-1}(s,T)\frac{1}{2\pi}\int_{\partial(U_p\cap\mathbb{C}_j)} S_L^{-1}(p,T)\,dp_j\,f(p)\\&=\frac{1}{(2\pi)^2}\int_{\partial(B_R(0)\cap\mathbb{C}_j)} ds_j\,S_R^{-1}(s,T)\int_{\partial(U_p\cap\mathbb{C}_j)} p\mathcal{Q}_s(p)^{-1}\,dp_j\,f(p)\\&\quad-\frac{1}{(2\pi)^2}\int_{\partial(B_R(0)\cap\mathbb{C}_j)} ds_j\,\overline{s}S_R^{-1}(s,T)\int_{\partial(U_p\cap\mathbb{C}_j)}\mathcal{Q}_s(p)^{-1}\,dp_j\,f(p)\\&\quad+\frac{1}{(2\pi)^2}\int_{\partial(B_R(0)\cap\mathbb{C}_j)}\left[\int_{\partial(U_p\cap\mathbb{C}_j)} ds_j\right.\\&\qquad\qquad\left.\cdot\left(\overline{s}S_L^{-1}(p,T)-S_L^{-1}(p,T)p\right)\mathcal{Q}_s(p)^{-1}\right]dp_j\,f(p).\end{aligned}$$

Again, the first two integrals vanish as a consequence of Cauchy's integral theorem because the poles of the function $p\mapsto(p^2-2s_0p+|s|^2)^{-1}$ lie outside of $\overline{U_p}$ for any $s\in\partial(B_R(0)\cap\mathbb{C}_j)$ and f decays regularly at infinity. Because of (7.1) and the regular decay of f at infinity, we can, however, apply Fubini's theorem to exchange the order of integration in the third integral and find

$$\begin{aligned}E_{\sigma_S(T)}\{f(T)\}_\infty=&\frac{1}{(2\pi)^2}\int_{\partial(U_p\cap\mathbb{C}_j)}\left[\int_{\partial(B_R(0)\cap\mathbb{C}_j)} ds_j\right.\\&\left.\mathcal{Q}_s(p)^{-1}\left(sS_L^{-1}(p,T)-S_L^{-1}(p,T)\overline{p}\right)\right]dp_j\,f(p).\end{aligned}$$

As the functions $s \mapsto (s^2 - 2p_0 s + |p|^2)^{-1}$ and $s \mapsto (s^2 - 2p_0 s + |p|^2)^{-1} s$ are right slice hyperholomorphic on $\overline{B_R(0)}$ for any $p \in \partial(U_p \cap \mathbb{C}_j)$, this integral also vanishes due to Cauchy's integral theorem. Consequently, the identity $(**)$ in (7.26) holds also true as

$$E_\infty\{f(T)\}_\infty = \{f(T)\}_\infty - E_\sigma\{f(T)\}_\infty = \{f(T)\}_\infty.$$

Finally, we point out that the above computations, which proved that

$$\{(ef)(T)\}_\infty = \{e(T)\}_\infty \{f(T)\}_\infty,$$

did not require that $e \in \mathcal{E}[\Sigma_\omega]$. They also work if e belongs to $\mathcal{M}[\Sigma_\omega]_T$ and decays regularly at infinity. Hence the same calculations show that (7.17) holds true. $\square$

Theorem 7.5.6. *Let $T \in \operatorname{Sect}(\omega)$ and $s \in \{0, \infty\}$. If $f \in \mathcal{M}[\Sigma_\omega]_T$ has polynomial limit c at s and $s \in \sigma_{SX}(T)$, then $c \in \sigma_{SX}(f(T))$.*

Proof. If $c \neq \infty$, then $c \in \mathbb{R}$ because, as an intrinsic function, f takes only real values on the real line. We can hence consider the function $f - c$ instead of f because

$$\sigma_{SX}(f(T)) = \sigma_{SX}(f(T) - c\mathcal{I}) + c$$

so that it is sufficient to consider the cases $c = 0$ or $c = \infty$.
Let us start with the case $c = 0$ and $s = \infty$. If $\infty \in \overline{\sigma_S(T) \setminus \{0\}}^{\mathbb{H}_\infty}$, then

$$0 \in \overline{f(\sigma_S(T) \setminus \{0\})}^{\mathbb{H}_\infty} \subset \sigma_{SX}(f(T))$$

because $f(\sigma_S(T) \setminus \{0\}) \subset \sigma_{SX}(f(T))$ by Proposition 7.5.4 and the latter is a closed subset of $\mathbb{H}_\infty$. In case $\infty \notin \overline{\sigma_S(T)}^{\mathbb{H}_\infty}$, we show that $0 \notin \sigma_{SX}(f(T))$ implies that T is bounded so that even $\infty \notin \sigma_{SX}(T)$. Let us hence assume that $\infty \notin \overline{\sigma_S(T)}^{\mathbb{H}_\infty}$ and that $0 \notin \sigma_{SX}(f(T))$. In this case, there exists $R > 0$ such that $\sigma_S(T)$ is contained in the open ball $B_R(0)$ of radius R centered at zero. The integral

$$E_{\sigma_S(T)} := \frac{1}{2\pi} \int_{\partial(B_R(0) \cap \mathbb{C}_j)} ds_j \, S_R^{-1}(s, T)$$

defines then a bounded projection that commutes with T, namely the spectral projection associated with the spectral set $\sigma_S(T) \subset \sigma_{SX}(T)$ that is obtained from the S-functional calculus. The compatibility of the S-functional calculus with polynomials in T moreover implies

$$TE_{\sigma_S(T)} = (s\chi_{\sigma_S(T)})(T) = \frac{1}{2\pi} \int_{\partial(B_R(0) \cap \mathbb{C}_j)} s \, ds_j \, S_R^{-1}(s, T) \in \mathcal{B}(X),$$

where $\chi_{\sigma_S(T)}(s)$ denotes the characteristic function of an arbitrary axially symmetric bounded set that contains $\overline{B_R(0)}$.

Set $E_\infty := \mathcal{I} - E_{\sigma_S(T)}$ and let $X_\infty := E_\infty X$ be the range of E_∞. Since T commutes with E_∞, the operator $T_\infty := T|_{X_\infty}$ is a closed operator on X_∞ with domain $\mathcal{D}(T_\infty) = \mathcal{D}(T) \cap X_\infty$. Moreover, we conclude from the properties of the projections that

$$\sigma_{SX}(T_\infty) = \sigma_{SX}(T) \setminus \sigma_S(T) \subset \{\infty\}$$

and so in particular

$$\sigma_S(T_\infty) = \sigma_{SX}(T_\infty) \setminus \{\infty\} = \emptyset. \tag{7.27}$$

Now observe that $f(T)$ commutes with E_∞ because of (i) in Lemma 7.2.8. Hence, $f(T)$ leaves X_∞ invariant and $f(T)_\infty := f(T)|_{X_\infty}$ defines a closed operator on X_∞ with domain $\mathcal{D}(f(T)_\infty) = \mathcal{D}(f(T)) \cap X_\infty$. (Note that $f(T)_\infty$ intuitively corresponds to $f(T_\infty)$. The S-functional calculus as introduced in this book is, however, only defined on two-sided Banach spaces. As X_∞ is only a right-linear subspace of X and hence not a two-sided Banach space, we can not define the operator $f(T_\infty)$, cf. the remark at the beginning of Section 7.5.) Since $f(T)$ is invertible because we assumed $0 \notin \sigma_{SX} f(T)$, the operator $f(T)_\infty$ is invertible too and its inverse is $f(T)^{-1}|_{X_\infty} \in \mathcal{B}(X_\infty)$.

Our goal is now to show that T_∞ is bounded. Since any bounded operator on a nontrivial Banach space has non-empty S-spectrum, we can conclude from (7.27) that $X_\infty = \{0\}$. Since f decays regularly at infinity, there exists $n \in \mathbb{N}$ such that $sf^n(s) \in \mathcal{M}[\omega]_T$ decays regularly at infinity too. Because of Lemma 7.2.8, the operators $Tf^n(T)$ and $(sf^n)(T)$ both commute with E_∞. Hence, they leave X_∞ invariant and we find, again because of Lemma 7.2.8, that

$$Tf^n(T)|_{X_\infty} \subset (sf^n)(T)|_{X_\infty} \in \mathcal{B}(X_\infty)$$

with

$$\begin{aligned}\mathcal{D}\left(Tf^n(T)|_{X_\infty}\right) &= \mathcal{D}\left(Tf^n(T)\right) \cap X_\infty \\ &= \mathcal{D}\left((sf^n)(T)\right) \cap \mathcal{D}\left(f^n(T)\right) \cap X_\infty \\ &= \mathcal{D}\left((sf^n)(T)|_{X_\infty}\right) \cap \mathcal{D}\left(f^n(T)|_{X_\infty}\right).\end{aligned}$$

But since sf^n and f^n both decay regularly at infinity in Σ_φ, Lemma 7.5.5 implies that $f^n(T)|_{X_\infty}$ and $(sf^n)(T)|_{X_\infty}$ are both bounded linear operators on X_∞. Hence, their domain is the entire space X_∞ and we find that

$$Tf^n(T)|_{X_\infty} = (sf^n)(T)|_{X_\infty} \in \mathcal{B}(X_\infty).$$

Finally, observe that Lemma 7.5.5 also implies that $f^n(T)|_{X_\infty} = (f(T)|_{X_\infty})^n$. As $f(T)|_{X_\infty}$ has a bounded inverse on X_∞, namely $f(T)^{-1}|_{X_\infty}$, we find that $T_\infty \in \mathcal{B}(X_\infty)$ too. As pointed out above, this implies $X_\infty = \{0\}$.

Altogether we find that $X = X_{\sigma_S(T)} := E_{\sigma_S(T)}X$ such that $T = T|_{X_{\sigma_S(T)}}$ belongs to $\mathcal{B}(X_{\sigma_S(T)}) = \mathcal{B}(X)$ and in turn

$$\infty \notin \sigma_{SX}(T) \quad \text{if } 0 = f(\infty) \notin \sigma_{SX}(f(T)).$$

Now let us consider the case that $s = 0$ and $c = 0$, that is $f(0) = 0$. If 0 does not belong to $\sigma_{SX}(f(T))$, then $f(T)$ has a bounded inverse. Let e be a regulariser for f such that $ef \in \mathcal{E}[\Sigma_\omega]$. Since $f(T) = e(T)^{-1}(ef)(T)$ is injective, the operator $(ef)(T)$ must be injective too. As the function ef has polynomial limit 0 at 0, we conclude from Lemma 7.2.7 that even T is injective. If we define $\tilde{f}(q) := f(q^{-1})$, then $\tilde{f}$ has polynomial limit 0 at ∞ and $\tilde{f}(T^{-1})$ is invertible as $\tilde{f}(T^{-1}) = f(T)$ by Corollary 7.3.2. Hence, $0 = \tilde{f}(\infty) \notin \sigma_{SX}(\tilde{f}(T^{-1}))$ and arguments as the ones above show that $\infty \notin \sigma_{SX}(T^{-1})$ such that $T^{-1} \in \mathcal{B}(X)$. Thus, T has a bounded inverse and in turn $0 \notin \sigma_{SX}(T)$ if $0 = f(0) \notin \sigma_{SX}(f(T))$.

Finally, let us consider the case $c = f(s) = \infty$ with $s = 0$ or $s = \infty$ and let us assume that $\infty \notin \sigma_{SX}(f(T))$, that is that $f(T)$ is bounded. If we choose $a \in \mathbb{R}$ with $|a| > \|f(T)\|$, then $a \in \rho_S(f(T))$ and hence $a\mathcal{I} - f(T)$ has a bounded inverse. By (iii) in Lemma 7.2.8, the function $g(q) := (a - f(q))^{-1}$ belongs to $\mathcal{M}[\Sigma_\omega]_T$. Moreover, $g(T)$ is invertible and $g(T)$ has polynomial limit 0 at s. As we have shown above, this implies $s \notin \sigma_{SX}(T)$, which concludes the proof. □

Combining Proposition 7.5.4 and Theorem 7.5.6, we arrive at the following theorem.

Theorem 7.5.7. *Let $T \in \operatorname{Sect}(\omega)$. If $f \in \mathcal{M}[\Sigma_\omega]_T$ and f has polynomial limits at $\sigma_{SX}(T) \cap \{0, \infty\}$, then*

$$f(\sigma_{SX}(T)) \subset \sigma_{SX}(f(T)).$$

Let us now consider the inverse inclusion. We start with the following auxiliary lemma.

Lemma 7.5.8. *Let $T \in \operatorname{Sect}(\omega)$ and let $f \in \mathcal{M}[\Sigma_\omega]_T$ have finite polynomial limits at $\{0, \infty\} \cap \sigma_{SX}(T)$ in Σ_φ for some $\varphi \in (\omega, \pi)$. Furthermore, assume that all poles of f are contained in $\rho_S(T)$.*

(i) *If $\{0, \infty\} \subset \sigma_{SX}(T)$, then $f(T)$ is defined by the H^∞-functional calculus for sectorial operators.*

(ii) *If $0 \in \sigma_{SX}(T)$ but $\infty \notin \sigma_{SX}(T)$, then $f(T)$ is defined by the extended H^∞-functional calculus for bounded sectorial operators.*

(iii) *If $\infty \in \sigma_{SX}(T)$ but $0 \notin \sigma_{SX}(T)$, then $f(T)$ is defined by the extended H^∞-functional calculus for invertible sectorial operators.*

(iv) *If $0, \infty \notin \sigma_{SX}(T)$, then $f(T)$ is defined by the H^∞-functional calculus for bounded and invertible sectorial operators.*

In all of these cases $f(T) \in \mathcal{B}(X)$.

Proof. Let us first consider the case (i), i.e., we assume that $\{0, \infty\} \subset \sigma_{SX}(T)$. Since f has polynomial limits at 0 and ∞ in Σ_ω, the function f has only finitely many poles $[s_1], \dots, [s_n]$ in $\overline{\Sigma_\omega}$. Moreover, for suitably large $m_1 \in \mathbb{N}$, the function $f_1(q) = (1+q)^{-2m_1} \mathcal{Q}_{s_1}(q)^{m_1} f(q)$ has also polynomial limits at 0 and ∞ and poles at $[s_2], \dots, [s_n]$ but it does not have a pole at $[s_1]$. Moreover, if we set $r_1(q) =$

$(1+q)^{-2m_1}\mathcal{Q}_{s_1}(q)^{m_1}$, then $r_1(T)$ is bounded and injective because $[s_1] \subset [\rho_S(T))]$. We can now repeat this argument and find inductively $m_2,\dots,m_n$ such that, after setting $r_\ell(q) = (1+q)^{-2m_\ell}\mathcal{Q}_{s_\ell}(q)^{m_\ell}$ for $\ell = 2,\dots,n$ and $r := r_n \cdots r_1$, the function $\tilde{f} = rf$ belongs to $\mathcal{M}[\Sigma_\omega]_T$, has polynomial limits at 0 and ∞ and does not have any poles in $\overline{\Sigma_\omega}$. Hence, it belongs to $\mathcal{E}[\Sigma_\omega]$. Moreover, r belongs to $\mathcal{E}[\Sigma_\omega]$ too and since $r(T) = r_n(T)\cdots r_1(T)$ is the product of invertible operators, it is invertible itself. Hence, r regularises f such that $f(T)$ is defined in terms of the H^∞-functional calculus. Moreover, $f(T) = r(T)^{-1}\tilde{f}(T)$ is bounded as it is the product of two bounded operators.

Similar arguments show the other cases: in (ii), for example, the function f has polynomial limit at 0 but not at ∞, such that the poles of f may accumulate at ∞. However, we integrate along the boundary of $\Sigma_{\omega,0,R} = \Sigma_\omega \cap B_R(0)$ in $\mathbb{C}_j$ for sufficiently large R when we define the H^∞-functional calculus for bounded sectorial operators. Hence, only finitely many poles are contained in $\Sigma_{\omega,0,R}$ and therefore relevant. Thus, we can apply the above strategy again in order to show that f is regularised by a rational intrinsic function and that $f(T)$ is defined and a bounded operator. Similar, we can argue for (iii) and (iv), where the poles of f may accumulate at 0 (resp. at 0 and ∞), but only finitely many of them are relevant. □

Proposition 7.5.9. *Let $T \in \operatorname{Sect}(\omega)$. If $f \in \mathcal{M}[\Sigma_\omega]_T$ has polynomial limits at any point in $\sigma_S(T) \cap \{0,\infty\}$, then*

$$f(\sigma_{SX}(T)) \supset \sigma_{SX}(f(T)).$$

Proof. Let $s \in \mathbb{H}$ with $s \notin f(\sigma_{SX}(T))$. The function $q \mapsto \mathcal{Q}_s(f(q))^{-1}$ belongs then to $\mathcal{M}[\Sigma_\omega]_T$ and has finite polynomial limits at $\sigma_{SX}(T) \cap \{0,\infty\}$. Moreover, the set of poles of $\mathcal{Q}_s(f(\cdot))$ as an element of $\mathcal{M}[\Sigma_\omega]$ consists of those spheres $[q]$ in $\overline{\Sigma_\omega} \setminus \{0\}$ for which $f([q]) = [f(q)] = [s]$ and it is contained in the S-resolvent set of T as we chose $s \notin f(\sigma_{SX}(T))$. From Lemma 7.5.8 we therefore deduce that $\mathcal{Q}_s(f(T))^{-1}$ is defined and belongs to $\mathcal{B}(X)$. Hence, $\mathcal{Q}_s(f(T))$ has a bounded inverse and so $s \in \sigma_{SX}(f(T))$.

If finally $s = \infty \notin f(\sigma_{SX}(T))$, then the poles of f are contained in the S-resolvent set of T. Hence, Lemma 7.5.8 implies that $f(T)$ is a bounded operator and in turn $s = \infty \notin \sigma_{SX}(f(T))$. □

Combining Theorem 7.5.7 and Proposition 7.5.9, we obtain the following spectral mapping theorem

Theorem 7.5.10 (Spectral Mapping Theorem)**.** *Let $T \in \operatorname{Sect}(\omega)$ and let $f \in \mathcal{M}[\Sigma_\omega]_T$ have polynomial limits at $\{0,\infty\} \cap \sigma_{SX}(T)$. Then*

$$f(\sigma_{SX}(T)) = \sigma_{SX}(f(T)).$$