Chapter 6

The Phillips functional calculus

In Chapter 3, we have introduced the direct approach to the S-functional calculus for unbounded operators, which only requires the operator T to be closed and have a nonempty S-resolvent set. As in the complex case, the price for these weak requirements on the operator are the relatively strong assumptions that one has to make on the class of admissible functions, namely that they are slice hyperholomorphic on the S-spectrum of T and at infinity. However, similar to the classical case, additional knowledge about the operator allows us to extend the class of admissible functions.

In this chapter we shall assume that T is the infinitesimal generator of a strongly continuous group $\{\mathcal{U}_T(t)\}_{t\in\mathbb{R}}$ and we let $\omega \geq 0$ and $M > 0$ be the constants from Theorem 4.3.1 such that

$$\sigma_S(T) \subset \{s \in \mathbb{H} : \quad -\omega \leq \operatorname{Re}(s) \leq \omega\} \quad \text{and} \quad \|\mathcal{U}_T(t)\| \leq M e^{\omega|t|}, \ t \in \mathbb{R}.$$

If f is the quaternionic Laplace–Stieltjes transform of a quaternion-valued measure μ on $\mathbb{R}$, that is,

$$f(s) = \int_{\mathbb{R}} d\mu(t)\, e^{-st}, \quad -(\omega+\varepsilon) < \operatorname{Re}(s) < \omega+\varepsilon,$$

then we can formally replace the exponential in the above integral by the group $\mathcal{U}_T(t)$, which formally corresponds to e^{tT}, and define

$$f(T) := \int_{\mathbb{R}} d\mu(t) \mathcal{U}_T(-t).$$

The function f is now slice hyperholomorphic on $\sigma_S(T)$, but not necessarily at infinity.

In the complex setting the above procedure yields the Phillips functional calculus, which was introduced in [35, 185]. In this chapter we introduce its quaternionic counterpart and study its properties and its relation with the S-functional calculus following the treatise in [110]. The presented results were published in [12].

F. Colombo, J. Gantner, *Quaternionic Closed Operators, Fractional Powers and Fractional Diffusion Processes*, Operator Theory: Advances and Applications 274,
https://doi.org/10.1007/978-3-030-16409-6_6

6.1 Preliminaries on quaternionic measure theory

Before we are able to define the Phillips functional calculus for quaternionic linear operators, we have to recall some facts about quaternion-valued measures and investigate their product measures. These results will be essential when we study the properties of the quaternionic Laplace–Stieltjes transform.

In [5, Section 3], the authors showed that quaternion-valued measures have properties similar to the properties of complex-valued measures. In particular, it is possible to define their variation, which has analogous properties as the variation of a complex measure, and find that the Radon–Nikodỳm theorem also holds true in this setting. We recall the results that we will need in the sequel taken from [5]. Since some of these results follow by adapting the classical case, we just recall them. We will add the proofs of the results related to the product of quaternionic measures that are taken from [12].

Definition 6.1.1. Let $(\Omega, \mathcal{A})$ be a measurable space. A quaternionic measure is a function $\mu \colon \mathcal{A} \to \mathbb{H}$ that satisfies

$$\mu\left(\bigcup\nolimits_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n),$$

for any sequence of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. We denote the set of all quaternionic measures on $\mathcal{A}$ by $\mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$ or simply by $\mathcal{M}(\Omega, \mathbb{H})$ or $\mathcal{M}(\Omega)$ if there is no possibility of confusion.

Corollary 6.1.2. *Let $(\Omega, \mathcal{A})$ be a measurable space. The set $\mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$ is a two-sided quaternionic vector space with the operations*

$$(\mu + \nu)(A) := \mu(A) + \nu(A), \quad (a\mu)(A) := a\mu(A), \quad (\mu a)(A) := \mu(A)a,$$

for $\mu, \nu \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$, $a \in \mathbb{H}$ and $A \in \mathcal{A}$.

Remark 6.1.1. Let $j, i \in \mathbb{S}$ with $j \perp i$. Since $\mathbb{H} = \mathbb{C}_j + i\mathbb{C}_j$, it is immediate that a mapping $\mu : \mathcal{A} \to \mathbb{H}$ is a quaternionic measure if and only if there exist two $\mathbb{C}_j$-valued complex measures μ_1, μ_2 such that $\mu(A) = \mu_1(A) + i\mu_2(A)$ for any $A \in \mathcal{A}$. Moreover, since $\mathbb{H} = \mathbb{C}_j + \mathbb{C}_j i$, there exist $\mathbb{C}_j$-valued measures $\tilde{\mu}_1, \tilde{\mu}_2$ such that $\mu(A) = \tilde{\mu}_1(A) + \tilde{\mu}_2(A)i$ for any $A \in \mathcal{A}$.

Definition 6.1.3. Let $\mu \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$. For all $A \in \mathcal{A}$, we denote by $\Pi(A)$ the set of all countable partitions π of A into pairwise disjoint, measurable sets $A_\ell, \ell \in \mathbb{N}$. The total variation of μ is the set function

$$|\mu|(A) := \sup\left\{ \sum_{A_\ell \in \pi} |\mu(A_\ell)| : \quad \pi \in \Pi(A) \right\} \qquad \text{for all } A \in \mathcal{A}.$$

From the definition, we easily obtain the following lemma.

Lemma 6.1.4. *The total variation $|\mu|$ of a measure $\mu \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$ is a finite positive measure on Ω. Moreover, $|a\mu| = |\mu a| = |\mu||a|$ and $|\mu + \nu| \leq |\mu| + |\nu|$ for any $\mu, \nu \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$ and $a \in \mathbb{H}$.*

Recall that a measure μ is called absolutely continuous with respect to a positive measure ν if $\mu(A) = 0$ for any $A \in \mathcal{A}$ with $\nu(A) = 0$. In this case, we write $\mu \ll \nu$. We denote by $L^1(\Omega, \mathcal{A}, \nu, \mathbb{H})$ the Banach space of all $\mathbb{H}$-valued functions on Ω that are integrable with respect to the positive measure ν.

Theorem 6.1.5 (Radon–Nikodỳm theorem for quaternionic measures)**.** *Let ν be a σ-finite positive measure on $(\Omega, \mathcal{A})$. A quaternionic measure $\mu \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$ is absolutely continuous with respect to ν if and only if there exists a function $f \in L^1(\Omega, \mathcal{A}, \nu, \mathbb{H})$ such that*

$$\mu(A) = \int_A f(x)\, d\nu(x), \quad \text{for all } A \in \mathcal{A}.$$

Moreover, f is unique and we have

$$|\mu|(A) = \int_A |f(x)|\, d\nu(x), \quad \text{for all } A \in \mathcal{A}. \tag{6.1}$$

The identity (6.1) follows as in the classical case, cf. [192, Theorem 6.13].

Corollary 6.1.6. *Let $\mu \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$. Then there exists an $\mathcal{A}$-measurable function $h : \Omega \to \mathbb{H}$ such that $|h(x)| = 1$ for any $x \in \Omega$ and such that $\mu(A) = \int_A h(x)\, d|\mu|(x)$ for any $A \in \mathcal{A}$.*

In order to define the quaternionic Laplace–Stieltjes transform and the Phillips functional calculus for quaternionic linear operators, we define integrals with respect to quaternionic-valued measures as in [12]. Let us again consider a quaternionic two-sided Banach space X and let ν be a positive measure. We recall that

(i) X becomes a real Banach space if we restrict the scalar multiplication to the real numbers.

(ii) $\mathbb{H}$ itself is a quaternionic two-sided Banach space.

So, let ν be a positive measure. Recall that in Bochner's integration theory, a function f with values in X is called ν-measurable if there exists a sequence of functions $f_n(x) = \sum_{\ell=1}^{n} a_i \chi_{A_\ell}(x)$, where $a_\ell \in X$ and χ_{A_ℓ} is the characteristic function of a measurable set A_ℓ, such that $f_n(x) \to f(x)$ as $n \to +\infty$ for ν-almost all $x \in \Omega$. The next lemma follows as a simple application of the Pettis measurability theorem, see [107].

Lemma 6.1.7. *Let X be a quaternionic two-sided Banach space and let ν be a positive measure on $(\Omega, \mathcal{A})$. If $f : \Omega \to X$ and $g : \Omega \to \mathbb{H}$ are ν-measurable, then the functions fg and gf are ν-measurable.*

Let ν be a positive measure on $(\Omega, \mathcal{A})$. Recall that a ν-measurable function on Ω with values in a real Banach space is called Bochner-integrable, if

$$\int_\Omega \|f(x)\| d\mu(x) < +\infty.$$

Definition 6.1.8. Let X be a two-sided quaternionic Banach space, let $\mu \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$ and let $h : \Omega \to \mathbb{H}$ be the function with $|h| = 1$ such that $d\mu(x) = h(x)\, d|\mu|(x)$. We call two μ-measurable functions $f : \Omega \to X$ and $g : \Omega \to \mathbb{H}$ a *μ-integrable pair*, if

$$\int_\Omega \|f\| \|g\|\, d|\mu| < +\infty.$$

In this case, we define

$$\int_\Omega f\, d\mu\, g := \int_\Omega fhg\, d|\mu| \tag{6.2}$$

and

$$\int_\Omega g\, d\mu\, f = \int_\Omega ghf\, d|\mu|, \tag{6.3}$$

as the integrals of a function with values in a real Banach space in the sense of Bochner.

Remark 6.1.2. Note that in the definition of the integrals in (6.2) and (6.3), we can replace the variation of μ by any σ-finite positive measure ν with $\mu \ll \nu$. If h_ν is the density of μ with respect to ν and $\rho_{|\mu|}$ and ρ_ν are the real-valued densities of $|\mu|$ and ν with respect to $|\mu| + \nu$. Then we have

$$\mu = h|\mu| = h\rho_{|\mu|}(|\mu| + \nu) \quad \text{and} \quad \mu = h_\nu \nu = h_\nu \rho_\nu (|\mu| + \nu).$$

Theorem 6.1.5 implies $h\rho_{|\mu|} = h_\nu \rho_\nu$ in $L^1(|\mu| + \nu)$. Therefore,

$$\begin{aligned}
\int_\Omega f h_\nu g\, d\nu &= \int_\Omega \int_\Omega f h_\nu g \rho_\nu\, d(|\mu| + \nu) \\
&= \int_\Omega \int_\Omega f h_\nu \rho_\nu g\, d(|\mu| + \nu) \\
&= \int_\Omega f h \rho_{|\mu|} g\, d(|\mu| + \nu) \\
&= \int_\Omega f h g \rho_{|\mu|}\, d(|\mu| + \nu) \\
&= \int_\Omega f h g\, d|\mu| \\
&= \int_\Omega f\, d\mu\, g.
\end{aligned}$$

Hence, the integral is linear in the measure: if $\mu, \nu \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$, then μ and ν are absolutely continuous with respect to $\tau = |\mu| + |\nu|$. If ρ_μ and ρ_ν are the densities of μ and ν with respect to τ, then

$$\begin{aligned}\int_\Omega f\, d(\mu + \nu)\, g &= \int_\Omega f(\rho_\mu + \rho_\nu) g\, d\tau \\ &= \int_\Omega f \rho_\mu g\, d\tau + \int_\Omega f \rho_\nu g\, d\tau \\ &= \int_\Omega f\, d\mu\, g + \int_\Omega f\, d\nu\, g.\end{aligned}$$

Similarly, if $a \in \mathbb{H}$ and $\mu = \rho|\mu|$, then $a\mu = a\rho|\mu|$ and so

$$\int_\Omega f\, d(a\mu)\, g = \int_\Omega f(a\rho) g\, d|\mu| = \int_\Omega (fa)\rho g\, d|\mu| = \int_\Omega (fa)\, d\mu\, g.$$

In the same way, one can see that $\int_\Omega f\, d(\mu a) g = \int_\Omega f\, d\mu\, (ag)$.

We finally define the product measure and the convolution of two quaternionic measures as in [12]. Also these concepts will be essential when we discuss the product rule of the quaternionic Phillips functional calculus.

Lemma 6.1.9. *Let $\mu \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{H})$ and $\nu \in \mathcal{M}(\Upsilon, \mathcal{B}, \mathbb{H})$. Then there exists a unique measure $\mu \times \nu$ on the product measurable space $(\Omega \times \Upsilon, \mathcal{A} \otimes \mathcal{B})$ such that*

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B), \tag{6.4}$$

for all $A \in \mathcal{A}, B \in \mathcal{B}$. We call $\mu \times \nu$ the product measure *of μ and ν.*

Proof. Let $j, i \in \mathbb{S}$ with $j \perp i$ and let $\mu = \mu_1 + i\mu_2$ with $\mu_1, \mu_2 \in \mathcal{M}(\Omega, \mathcal{A}, \mathbb{C}_j)$ and $\nu = \nu_1 + \nu_2 i$ with $\nu_1, \nu_2 \in \mathcal{M}(\Upsilon, \mathcal{B}, \mathbb{C}_j)$. Then, there exist unique complex product measures $\mu_\ell \times \nu_\kappa \in M(\Omega_1 \times \Omega_2, \mathcal{A} \otimes \mathcal{B}, \mathbb{C}_j)$ of μ_ℓ and ν_ℓ for $\ell, \kappa = 1, 2$. If we set

$$\mu \times \nu = \mu_1 \times \nu_1 + i\mu_2 \times \nu_1 + \mu_1 \times \nu_2 i + J\mu_2 \times \nu_2 i,$$

then $\mu \times \nu$ is a quaternionic measure on $(\Omega \times \Upsilon, \mathcal{A} \otimes \mathcal{B})$ and

$$\begin{aligned}\mu(A)\nu(B) &= \mu_1(A)\nu_1(B) + i\mu_2(A)\nu_1(B) \\ &\quad + \mu_1(A)\nu_2(B)i + i\mu_2(A)\nu_2(B)i \\ &= \mu_1 \times \nu_1(A \times B) + i\mu_2 \times \nu_1(A \times B) \\ &\quad + \mu_1 \times \nu_2(A \times B)i + i\mu_2 \times \nu_2(A \times B)i = (\mu \times \nu)(A \times B).\end{aligned}$$

In order to prove the uniqueness of the product measure, assume that two quaternionic measures $\rho = \rho_1 + \rho_2 i$ and $\tau = \tau_1 + \tau_2 i$ on $(\Omega \times \Upsilon, \mathcal{A} \times \mathcal{B})$ satisfy $\rho(A \times B) = \tau(A \times B)$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then $\rho_1(A \times B) = \tau_1(A \times B)$ and $\rho_2(A \times B) = \tau_2(A \times B)$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Since two complex measures on the product space $(\Omega \times \Upsilon, \mathcal{A} \otimes \mathcal{B})$ are equal if and only if they coincide on sets of the form $A \times B$, we obtain $\rho_1 = \tau_1$ and $\rho_2 = \tau_2$ and, in turn, $\rho = \rho_1 + \rho_2 i = \tau_1 + \tau_2 i = \tau$. Therefore, $\mu \times \nu$ is uniquely determined by (6.4). □

Remark 6.1.3. Note that it is also possible to define a commuted product measure $\mu \times_c \nu$ on $(\Omega \times \Upsilon, \mathcal{A} \otimes \mathcal{B})$ that satisfies

$$(\mu \times_c \nu)(A \times B) = \nu(B)\mu(A), \quad \forall A \in \mathcal{A}, B \in \mathcal{B}.$$

This measure is different from the measure $\nu \times \mu$ that is defined on $(\Upsilon \times \Omega, \mathcal{B} \otimes \mathcal{A})$ and satisfies

$$(\nu \times \mu)(B \times A) = \nu(B)\mu(A), \quad \forall B \in \mathcal{B}, A \in \mathcal{A}.$$

Lemma 6.1.10. *Let $(\Omega, \mathcal{A}, \mu)$ and $(\Upsilon, \mathcal{B}, \nu)$ be quaternionic measure spaces. Then*

$$|\mu \times \nu| = |\mu| \times |\nu|.$$

Moreover, if $d\mu(x) = f(x)\, d|\mu|(x)$ and $\nu(x) = g(x)\, d|\nu|(x)$ as in Corollary 6.1.6, then, for any $C \in \mathcal{A} \times \mathcal{B}$,

$$(\mu \times \nu)(C) = \int_C f(s)g(t)\, d|\mu \times \nu|(s,t).$$

Proof. Let $f \colon \Omega \to \mathbb{H}$ and $g \colon \Upsilon \to \mathbb{H}$ with $|f| = 1$ and $|g| = 1$ be functions as in Corollary 6.1.6 such that

$$\mu(A) = \int_A f(t)\, d|\mu|(t)$$

and

$$\nu(B) = \int_B g(s)\, d|\nu|(s),$$

for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Moreover, let $r = (t,s)$ and let $h(r) = f(t)g(s)$. Then the function $C \mapsto \int_C h(r)\, d(|\mu| \times |\nu|)(r)$ defines a measure on $(\Omega \times \Upsilon, \mathcal{A} \times \mathcal{B})$ and Fubini's theorem for positive measures implies

$$\begin{aligned}\int_{A \times B} h(r)\, d|\mu| \times |\nu|(r) &= \int_A \int_B f(t)g(s)\, d|\mu|(t)\, d|\nu|(s)\\ &= \int_A f(t)\, d|\mu|(t) \int_B g(s)\, d|\nu|(s) = \mu(A)\nu(B).\end{aligned}$$

The uniqueness of the product measure implies $\mu \times \nu(C) = \int_C h(r)\, d(|\mu| \times |\nu|)(r)$, for any $C \in \mathcal{A} \times \mathcal{B}$. Since $|h| = |f|\,|g| = 1$, we deduce from (6.1) that

$$|\mu \times \nu|(C) = \int_C |h(r)|\, d(|\mu| \times |\nu|)(r) = (|\mu| \times |\nu|)(C),$$

for all $C \in \mathcal{A} \times \mathcal{B}$. □

Splitting the measure μ into two complex components and applying the respective result for complex measures, we obtain the transformation rule for integrals with respect to a pushforward measure stated in the following lemma.

Lemma 6.1.11. *Let $(\Omega, \mathcal{A}, \mu)$ be a quaternionic measure space, let $(\Upsilon, \mathcal{B})$ be a measurable space and let $\phi : \Omega \to \Upsilon$ be a measurable function. If a function $f : \Upsilon \to X$ with values in a quaternionic Banach space X is integrable with respect to the image measure $\mu^{\phi}(B) := \mu\left(\phi^{-1}(B)\right)$ and $f \circ \phi$ is integrable with respect to μ, then*

$$\int_{\Upsilon} f \, d\mu^{\phi} = \int_{\Omega} f \circ \phi \, d\mu. \tag{6.5}$$

Definition 6.1.12. We denote the Borel sets on a topological space X by $\mathsf{B}(X)$. In particular, we denote the Borel sets on $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ by $\mathsf{B}(\mathbb{R})$, $\mathsf{B}(\mathbb{C})$ and $\mathsf{B}(\mathbb{H})$, respectively.

We recall that, for any Borel set $E \in \mathsf{B}(\mathbb{R})$, the set

$$P(E) := \{(u, v) \in \mathbb{R}^2 : \quad u + v \in E\}$$

is a Borel subset of $\mathbb{R}^2$.

Definition 6.1.13. Let μ, ν be quaternionic measures on $\mathsf{B}(\mathbb{R})$. The convolution $\mu * \nu$ of μ and ν is the image measure of $\mu \times \nu$ under the mapping $\phi : \mathbb{R}^2 \to \mathbb{R}, (u, v) \mapsto u + v$, that is, for any $E \in \mathsf{B}(\mathbb{R})$, we set

$$(\mu * \nu)(E) := (\mu \times \nu)(P(E)).$$

The following corollary is immediate.

Corollary 6.1.14. *Let $\mu, \nu, \rho \in \mathcal{M}(\mathbb{R}, \mathsf{B}(\mathbb{R}), \mathbb{H})$ and let $a, b \in \mathbb{H}$. Then*

(i) $(\mu + \nu) * \rho = \mu * \rho + \nu * \rho$ *and* $\mu * (\nu + \rho) = \mu * \nu + \mu * \rho$ *and*

(ii) $(a\mu) * \nu = a(\mu * \nu)$ *and* $\mu * (\nu a) = (\mu * \nu)a$.

Then we prove the following results.

Corollary 6.1.15. *Let $\mu, \nu \in \mathcal{M}(\mathbb{R}, \mathsf{B}(\mathbb{R}), \mathbb{H})$. Then the estimate*

$$|\mu * \nu|(E) \leq (|\mu| * |\nu|)(E)$$

holds true for all $E \in \mathsf{B}(\mathbb{R})$.

Proof. Let $E \in \mathsf{B}(\mathbb{R})$ and let $\pi \in \Pi(E)$ be a countable measurable partition of E. Then

$$\begin{aligned}\sum_{E_\ell \in \pi} |(\mu * \nu)(E_\ell)| &= \sum_{E_\ell \in \pi} |(\mu \times \nu)(P(E_\ell))| \\ &\leq \sum_{E_\ell \in \pi} |\mu \times \nu|(P(E_\ell)) = |\mu \times \nu|(P(E)),\end{aligned}$$

and taking the supremum over all possible partitions $\pi \in \Pi(E)$ yields

$$|\mu * \nu|(E) \leq |\mu \times \nu|(P(E)) = (|\mu| \times |\nu|)(P(E)) = (|\mu| * |\nu|)(E). \qquad \square$$

Corollary 6.1.16. *Let $\mu, \nu \in \mathcal{M}(\mathbb{R}, \mathsf{B}(\mathbb{R}), \mathbb{H})$ and let $F\colon \mathbb{R} \to X$ be integrable with respect to $\mu * \nu$ and such that $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \|F(s+t)\|\, d|\mu|(s)\, d|\nu|(t) < +\infty$. Then*

$$\int_{\mathbb{R}} F(r)\, d(\mu * \nu)(r) = \int_{\mathbb{R}}\int_{\mathbb{R}} F(s+t)\, d\mu(s)\, d\nu(t).$$

Proof. Our assumptions and Definition 6.1.13 allow us to apply Lemma 6.1.11 with $\phi(s,t) = s+t$. If $\mu(A) = \int_A f(t)\, d|\mu|(t)$ and $\nu(A) = \int_A g(s)\, d|\nu|(s)$, then the product measure satisfies

$$(\mu \times \nu)(B) = \int_B f(s)g(t)\, d(|\mu| \times |\nu|)(s,t)$$

by Lemma 6.1.10. Applying Fubini's theorem yields

$$\begin{aligned} \int_{\mathbb{R}} F(r)\, d(\mu * \nu)(r) &= \int_{\mathbb{R}} F(\phi(s,t))\ d(\mu \times \nu)(s,t) \\ &= \int_{\mathbb{R}} F(\phi(s,t)) f(s) g(t)\ d|\mu \times \nu|(s,t) \\ &= \int_{\mathbb{R}} F(s+t) f(s) g(t)\ d|\mu|(s)\, d|\nu|(t) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} F(s+t)\, d\mu(s)\, d\nu(t). \end{aligned}$$

□

6.2 Functions of the generator of a strongly continuous group

In the following we assume that $T \in \mathcal{K}(X)$ is the infinitesimal generator of the strongly continuous group $\{\mathcal{U}_T(t)\}_{t\in\mathbb{R}}$ of operators on a two-sided Banach space X. By Theorem 4.3.1, there exist positive constants $M > 0$ and $\omega \geq 0$ such that $\|\mathcal{U}_T(t)\| \leq Me^{\omega|t|}$ and such that the S-spectrum of the infinitesimal generator T lies in the strip

$$W_\omega := \{s \in \mathbb{H} : \quad -\omega < \mathrm{Re}(s) < \omega\}.$$

Moreover, we have

$$S_R^{-1}(s,T) = \int_0^{+\infty} e^{-ts}\, \mathcal{U}_T(t)\, dt, \quad \mathrm{Re}(s) > \omega$$

and

$$S_R^{-1}(s,T) = -\int_{-\infty}^{0} e^{-ts}\, \mathcal{U}_T(t)\, dt, \quad \mathrm{Re}(s) < -\omega.$$

Definition 6.2.1. We denote by $\mathbf{S}(T)$ the family of all quaternionic measures μ on $\mathsf{B}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} d|\mu|(t)\, e^{(\omega+\varepsilon)|t|} < +\infty$$

for some $\varepsilon = \varepsilon(\mu) > 0$. The function

$$\mathcal{L}(\mu)(s) = \int_{\mathbb{R}} d\mu(t)\, e^{-st}$$

with domain

$$W_{\omega+\varepsilon} := \{s \in \mathbb{H} : \quad -(\omega+\varepsilon) < \operatorname{Re}(s) < (\omega+\varepsilon)\}$$

is called the quaternionic two-sided (right) Laplace–Stieltjes transform of μ.

Definition 6.2.2. We denote by $\mathbf{X}(T)$ the set of quaternionic two-sided Laplace–Stieltjes transforms of measures in $\mathbf{S}(T)$.

Lemma 6.2.3. *Let $\mu, \nu \in \mathbf{S}(T)$ and $a \in \mathbb{H}$.*

(i) *The measures $a\mu$ and $\mu + \nu$ belong to $\mathbf{S}(T)$ and*

$$\mathcal{L}(a\mu) = a\mathcal{L}(\mu), \quad \mathcal{L}(\mu+\nu) = \mathcal{L}(\mu) + \mathcal{L}(\nu).$$

(ii) *The measures $\mu * \nu$ belongs to $\mathbf{S}(T)$. If ν is real-valued, then*

$$\mathcal{L}(\mu * \nu) = \mathcal{L}(\mu)\mathcal{L}(\nu).$$

Proof. Let $\varepsilon = \min\{\varepsilon(\mu), \varepsilon(\nu)\}$. Lemma 6.1.4 implies

$$\int_{\mathbb{R}} d|a\mu|\, e^{|t|(\omega+\varepsilon)} = |a| \int_{\mathbb{R}} d|\mu|\, e^{|t|(\omega+\varepsilon)} < +\infty$$

and

$$\int_{\mathbb{R}} d|\mu+\nu|\, e^{|t|(\omega+\varepsilon)} \leq \int_{\mathbb{R}} d|\mu|\, e^{|t|(\omega+\varepsilon)} + \int_{\mathbb{R}} d|\nu|\, e^{|t|(\omega+\varepsilon)} < +\infty.$$

Thus, $a\mu$ and $\mu + \nu$ belong to $\mathbf{S}(T)$. The relations $\mathcal{L}(a\mu) = a\mathcal{L}(\mu)$ and $\mathcal{L}(\mu+\nu) = \mathcal{L}(\mu) + \mathcal{L}(\nu)$ follow from the left linearity of the integral in the measure.

The variation of the convolution of μ and ν satisfies $|\mu*\nu|(E) \leq (|\mu| * |\nu|)(E)$ for any Borel set $E \in \mathsf{B}(\mathbb{R})$, cf. Corollary 6.1.15. In view of Corollary 6.1.16, we have

$$\begin{aligned}\int_{\mathbb{R}} d|\mu * \nu|(r) e^{(w+\varepsilon)|r|} &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} d|\mu|(s)\, d|\nu|(t) e^{(w+\varepsilon)|s+t|} \\ &\leq \int_{\mathbb{R}} d|\mu|(s)\, e^{(w+\varepsilon)|s|} \int_{\mathbb{R}} d|\nu|(t)\, e^{(w+\varepsilon)|t|} < +\infty.\end{aligned}$$

Therefore, $\mu*\nu \in \mathbf{S}(T)$. If ν is real-valued, then ν commutes with e^{-st} and Fubini's theorem implies for $s \in \mathbb{H}$ with $-(\omega+\varepsilon) < \operatorname{Re}(s) < \omega + \varepsilon$

$$\begin{aligned}\mathcal{L}(\mu * \nu)(s) &= \int_{\mathbb{R}} d(\mu*\nu)(r)\, e^{-sr} = \int_{\mathbb{R}} \int_{\mathbb{R}} d\mu(t)\, d\nu(u)\, e^{-s(t+u)} \\ &= \int_{\mathbb{R}} d\mu(t)\, e^{-st} \int_{\mathbb{R}} d\nu(u)\, e^{-su} = \mathcal{L}(\mu)(s)\, \mathcal{L}(\nu)(s).\end{aligned}$$

□

Theorem 6.2.4. *Let $f \in \mathbf{X}(T)$ with $f(s) = \int_{\mathbb{R}} d\mu(t)\, e^{-st}$, for any s in the strip*

$$W_{\omega+\varepsilon} := \{s \in \mathbb{H} : \quad -(\omega+\varepsilon) < \operatorname{Re}(s) < \omega+\varepsilon\}.$$

(i) *The function f is right slice hyperholomorphic on the strip $W_{\omega+\varepsilon}$.*

(ii) *For any $n \in \mathbb{N}$, the measure μ^n defined by*

$$\mu^n(E) = \int_E d\mu(t)\, (-t)^n, \quad \text{for } E \in \mathsf{B}(\mathbb{R})$$

belongs to $\mathbf{S}(T)$ and, for s with $-(\omega+\varepsilon) < \operatorname{Re}(s) < \omega+\varepsilon$, we have

$$\partial_S{}^n f(s) = \int_{\mathbb{R}} d\mu^n(t)\, e^{-st} = \int_{\mathbb{R}} d\mu(t)\, (-t)^n e^{-st}, \tag{6.6}$$

where $\partial_S f$ denotes the slice derivative of f.

Proof. In the proof we will make use of the same kind of arguments as in [110, Lemma 2, p. 642]. For every $n \in \mathbb{N}$ and every $0 < \varepsilon_1 < \varepsilon$ there exists a constant K such that

$$|t|^n e^{(\omega+\varepsilon_1)|t|} \leq K e^{(\omega+\varepsilon)|t|}, \quad t \in \mathbb{R}.$$

Since $\mu \in \mathbf{S}(T)$, we have

$$\int_{\mathbb{R}} d|\mu^n|(t)\, e^{(\omega+\varepsilon_1)|t|} = \int_{\mathbb{R}} d|\mu|(t)\, |t|^n e^{(\omega+\varepsilon_1)|t|} \leq K \int_{\mathbb{R}} d|\mu|(t)\, e^{(\omega+\varepsilon)|t|} < +\infty$$

and so $\mu^n \in \mathbf{S}(T)$. The function f is a right slice function as, for $s = u + jv$,

$$f(s) = \int_{\mathbb{R}} d\mu(t) e^{-t(u+jv)} = \int_{\mathbb{R}} d\mu(t) e^{-tu}\cos(v) - \int_{\mathbb{R}} d\mu(t) e^{-tu}\sin(v) j$$

when we set

$$\alpha(u,v) := \int_{\mathbb{R}} d\mu(t) e^{-tu}\cos(v)$$

and

$$\beta(u,v) := -\int_{\mathbb{R}} d\mu(t) e^{-tu}\sin(v)$$

satisfy the compatibility condition (2.4). For any $s = u + jv \in W_{\omega+\varepsilon}$, we have

$$\lim_{\mathbb{C}_j \ni p \to s} (f_j(p) - f_j(s))(p-s)^{-1} = \lim_{\mathbb{C}_j \ni p \to s} \int_{\mathbb{R}} d\mu(t)\, \frac{e^{-pt} - e^{-st}}{p-s}.$$

If p is sufficiently close to s such that also $p \in W_{\omega+\varepsilon}$, then the simple calculation

$$|e^{-pt} - e^{-st}| = \left| \int_0^1 e^{-ts - t\xi(p-s)} t(p-s)\, d\xi \right| \leq |t| e^{(\omega+\varepsilon)|t|} |p-s|,$$

yields the estimate

$$\frac{|e^{-pt} - e^{-st}|}{|p-s|} \le |t| e^{(\omega+\varepsilon)|t|},$$

which allows us to apply Lebesgue's theorem of dominated convergence in order to exchange limit and integration. We obtain

$$\lim_{\mathbb{C}_j \ni p \to s} (f_j(p) - f_j(s))(p-s)^{-1} = \int_{\mathbb{R}} d\mu(t)\,(-t)e^{-st} = \int_{\mathbb{R}} d\mu^1(t)\,e^{-st}. \tag{6.7}$$

Consequently, the restriction f_j of f to the complex plane $\mathbb{C}_j$ is right holomorphic and, by Lemmas 2.1.5 and 2.1.9, the function f is in turn right slice hyperholomorphic on the strip $\{s \in \mathbb{H} : -(\omega+\varepsilon) < \mathrm{Re}(s) < \omega+\varepsilon\}$. Moreover, (6.7) implies

$$\partial_S f(s) = \int_{\mathbb{R}} d\mu^1(t)\,e^{-st}$$

for $-(\omega+\varepsilon) < \mathrm{Re}(s) < \omega+\varepsilon$. By induction we get (6.6). □

Definition 6.2.5 (The Phillips functional calculus)**.** Let $T \in \mathcal{K}(X)$ be the infinitesimal generator of the strongly continuous group $\{\mathcal{U}_T(t)\}_{t\in\mathbb{R}}$ of operators on a two-sided Banach space X. For $f \in \mathbf{X}(T)$ with

$$f(s) = \int_{\mathbb{R}} d\mu(t)\,e^{-st} \quad \text{for } -(\omega+\varepsilon) < \mathrm{Re}(s) < \omega+\varepsilon,$$

and $\mu \in \mathbf{S}(T)$, we define the right linear operator $f(T)$ on X by

$$f(T) = \int_{\mathbb{R}} d\mu(t)\,\mathcal{U}_T(-t). \tag{6.8}$$

Remark 6.2.1. In particular for $p \in \mathbb{H}$ with $\mathrm{Re}(p) < -\omega$ the function $s \mapsto S_R^{-1}(p,s)$ belongs to $\mathbf{S}(T)$. Set $d\mu_p(t) = -\chi_{[0,+\infty)}(t)e^{tp}\,dt$, where χ_A denotes the characteristic function of a set A. If $\mathrm{Re}(p) < \mathrm{Re}(s)$, then

$$\mathcal{L}(\mu_p)(s) = \int_{\mathbb{R}} d\mu_p(t)\,e^{-ts} = -\int_0^{+\infty} e^{tp}e^{-ts}\,dt = -S_L^{-1}(s,p) = S_R^{-1}(p,s)$$

and

$$\begin{aligned}\mathcal{L}(\mu_p)(T) &= \int_{\mathbb{R}} d\mu_p(t)\,\mathcal{U}(-t)\\ &= -\int_0^{+\infty} e^{tp}\,\mathcal{U}(-t)\,dt\\ &= -\int_{-\infty}^{0} e^{-tp}\,\mathcal{U}(t)\,dt = S_R^{-1}(p,T).\end{aligned}$$

For $p \in \mathbb{H}$ with $\omega < \mathrm{Re}(p)$ set $d\mu_p(t) = \chi_{(-\infty,0]}(t)e^{tp}\,dt$. Similar computations show that also in this case $S_R^{-1}(p,s) = \mathcal{L}(\mu_p)(s) \in \mathbf{S}(T)$ if $\mathrm{Re}(s) < \mathrm{Re}(p)$ and $\mathcal{L}(\mu_p)(T) = S_R^{-1}(p,T)$.

Theorem 6.2.6. *For any $f \in \mathbf{X}(T)$, the operator $f(T)$, defined in (6.8), is bounded.*

Proof. Let $f(s) = \int_{\mathbb{R}} d\mu(t)\, e^{-st} \in \mathbf{X}(T)$ with $\mu \in \mathbf{S}(T)$. Since $\|\mathcal{U}_T(t)\| \leq Me^{w|t|}$, we have

$$\|f(T)\| \leq \int_{\mathbb{R}} d|\mu|(t)\, \|\mathcal{U}_T(-t)\| \leq M \int_{\mathbb{R}} d|\mu|(t)\, e^{w|t|} < +\infty. \qquad \square$$

Lemma 6.2.7. *Let $f(T)$ be the operator defined in (6.8). Let $f = \mathcal{L}(\mu)$ and $g = \mathcal{L}(\nu)$ belong to $\mathbf{X}(T)$ and let $a \in \mathbb{H}$.*

(i) *We have $(af)(T) = af(T)$ and $(f+g)(T) = f(T) + g(T)$.*

(ii) *If g is an intrinsic function, then ν is real-valued and $(fg)(T) = f(T)g(T)$.*

Proof. The statement (i) follows immediately from Lemma 6.2.3 and the left linearity of the integral (6.8) in the measure. In order to show (ii), we assume that $g = \mathcal{L}(\nu)$ is intrinsic. Then the measure ν is real-valued and Lemma 6.2.3 gives $fg = \mathcal{L}(\mu * \nu) \in \mathbf{X}(T)$. We find

$$\begin{aligned}(fg)(T)v &= \int_{\mathbb{R}} d(\mu * \nu)(r)\, \mathcal{U}_T(-r)\\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} d\mu(s)\, d\nu(t)\, \mathcal{U}_T(-(s+t))\\ &= \int_{\mathbb{R}} d\mu(s)\, \mathcal{U}_T(-s) \int_{\mathbb{R}} d\nu(t)\, \mathcal{U}_T(-t) = f(T)g(T),\end{aligned}$$

where we use that $\mathcal{U}_T(-s)$ and ν commute because ν is real-valued. $\square$

6.3 Comparison with the S-Functional Calculus

A natural question that arises is regarding the relation between the Phillips functional calculus introduced in Definition 6.2.5 and the S-functional calculus for closed operators. In this section we show that the two functional calculi coincide if the function f is slice hyperholomorphic at infinity. In order to prove this, we need a specialized version of the Residue theorem that fits into our setting.

Lemma 6.3.1. *Let $O \subset \mathbb{H}$ be an axially symmetric open set, let $f : O \setminus [p] \to \mathbb{H}$ be right slice hyperholomorphic and let $g : O \to X$ be left slice hyperholomorphic such that $p = u + jv \in O$ is a pole of order $n_f \geq 0$ of the $\mathbb{H}$-valued right holomorphic function $f_j := f|_{O \cap \mathbb{C}_j}$. If $\varepsilon > 0$ is such that $\overline{B_\varepsilon(p) \cap \mathbb{C}_j} \subset O$, then*

$$\frac{1}{2\pi} \int_{\partial(B_\varepsilon(p)\cap\mathbb{C}_j)} f(s)\, ds_j\, g(s) = \sum_{k=0}^{n_f - 1} \frac{1}{k!} \mathrm{Res}_p \left(f_j(s)(s-p)^k\right) \left(\partial_S{}^k g(p)\right).$$

Proof. Since f is right slice hyperholomorphic, its restriction f_j is a vector-valued holomorphic function on $O \cap \mathbb{C}_j$ if we consider $\mathbb{H}$ as a vector space over $\mathbb{C}_j$ by restricting the multiplication with quaternions on the right to $\mathbb{C}_j$. Similarly, since g is left slice hyperholomorphic, its restriction $g_j := g|_{O\cap\mathbb{C}_j}$ is an X-valued holomorphic function if we consider X as a complex vector space over $\mathbb{C}_j$ by restricting the left scalar multiplication to $\mathbb{C}_j$. Consequently, if we set $\rho = \operatorname{dist}(p, \partial(O \cap \mathbb{C}_j))$, then

$$f_j(s) = \sum_{k=-n_f}^{+\infty} a_k(s-p)^k \quad \text{and} \quad g_j(s) = \sum_{k=0}^{+\infty}(s-p)^k b_k \tag{6.9}$$

for $s \in (B_\rho(p)\cap\mathbb{C}_j)\setminus\{p\}$ with $a_k \in \mathbb{H}$ and $b_k \in X$. These series converge uniformly on $\partial(B_\varepsilon(p) \cap \mathbb{C}_j)$ for any $0 < \varepsilon < \rho$. Thus,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial(B_\varepsilon(p)\cap\mathbb{C}_j)} f(s)\, ds_j\, g(s)\\
&= \frac{1}{2\pi}\int_{\partial(B_\varepsilon(p)\cap\mathbb{C}_j)} \left(\sum_{k=0}^{+\infty} a_{k-n_f}(s-p)^{k-n_f}\right) ds_j \left(\sum_{j=0}^{+\infty}(s-p)^j b_j\right)\\
&= \sum_{k=0}^{+\infty}\sum_{j=0}^{k} a_{k-j-n_f}\left(\frac{1}{2\pi}\int_{\partial(B_\varepsilon(p)\cap\mathbb{C}_j)}(s-p)^{k-j-n_f}\, ds_j\,(s-p)^j\right) b_j\\
&= \sum_{j=0}^{n_f-1} a_{-(j+1)} b_j,
\end{aligned}$$

since $\frac{1}{2\pi}\int_{\partial(B_\varepsilon(p)\cap\mathbb{C}_j)}(s-p)^{k-n_f}\, ds_j$ equals 1 if $k - n_f = -1$ and 0 otherwise. Finally, we observe that $a_{-k} = \operatorname{Res}_p\left(f_j(s)(s-p)^{k-1}\right)$ and $b_k = \frac{1}{k!}\partial_S{}^k g_j(p)$ by their definition in (6.9). □

In order to compute the integral in the S-functional calculus, we recall the definition of the strip

$$W_c := \{s \in \mathbb{H} : \quad -c < \operatorname{Re}(s) < c\} \quad \text{for } c > 0$$

and we introduce the set $\partial(W_c\cap\mathbb{C}_j)$ for $j \in \mathbb{S}$. It consists of the two lines $s = c+j\tau$ and $s = -c - j\tau$ with $\tau \in \mathbb{R}$.

Proposition 6.3.2. *Let α and c be real numbers such that $\omega < c < |\alpha|$. For any vector $y \in \mathcal{D}(T^2)$, we have*

$$\mathcal{U}_T(t)y = \frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_j)} e^{ts}(\alpha - s)^{-2}\, ds_j\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2 y. \tag{6.10}$$

Proof. We recall that

$$S_R^{-1}(s,T) = \int_0^\infty e^{-ts}\,\mathcal{U}_T(t)\, dt, \quad \operatorname{Re}(s) > \omega.$$

Since $\|\mathcal{U}_T(t)\| \leq Me^{\omega|t|}$, we get a bound for the S-resolvent operator by

$$\|S_R^{-1}(s,T)\| = M\int_0^\infty e^{(\omega-\mathrm{Re}(s))t}\,dt, \quad \mathrm{Re}(s) > \omega \tag{6.11}$$

which assures that $\|S_R^{-1}(s,T)\|$ is uniformly bounded on $\{s \in \mathbb{H} : \mathrm{Re}(s) > \omega + \varepsilon\}$ for any $\varepsilon > 0$. A similar consideration gives a uniform bound of $\|S_R^{-1}(s,T)\|$ on $\{s \in \mathbb{H} : \mathrm{Re}(s) < -(\omega+\varepsilon)\}$. Thanks to such bound, the integral in (6.10) is well defined since the $(\alpha - s)^{-2}$ goes to zero with order $1/|s|^2$ as $s \to \infty$. We set

$$F(t)y = \frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_j)} e^{ts}(\alpha-s)^{-2}\,ds_j\, S_R^{-1}(s,T)(\alpha\mathcal{I}-T)^2y$$

for $y \in \mathcal{D}(T^2)$. We show that $F(t)y = \mathcal{U}_T(t)y$ using the Laplace transform and we first assume $t > 0$. If $\mathrm{Re}(p) > c$, then

$$\begin{aligned}
&\int_0^\infty e^{-pt}F(t)y\,dt\\
&= \frac{1}{2\pi}\int_0^\infty e^{-pt}\int_{\partial(W_c\cap\mathbb{C}_j)} e^{ts}(\alpha-s)^{-2}\,ds_j\, S_R^{-1}(s,T)(\alpha\mathcal{I}-T)^2y\,dt\\
&= \frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_j)}\left(\int_0^{+\infty} e^{-pt}e^{ts}dt\right)(\alpha-s)^{-2}\,ds_j\, S_R^{-1}(s,T)(\alpha\mathcal{I}-T)^2y.
\end{aligned}$$

Now observe that

$$\int_0^\infty e^{-pt}e^{ts}\,dt = S_R^{-1}(p,s),$$

so we have

$$\begin{aligned}
&\int_0^\infty e^{-pt}F(t)y\,dt\\
&= \frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_j)} S_R^{-1}(p,s)(\alpha-s)^{-2}\,ds_j\, S_R^{-1}(s,T)(\alpha\mathcal{I}-T)^2y.
\end{aligned}$$

We point out that the function $s \mapsto S_R^{-1}(p,s)(\alpha-s)^{-2}$ is right slice hyperholomorphic for $s \notin [p]\cup\{\alpha\}$ and that the function $s \mapsto S_R^{-1}(s,T)(\alpha\mathcal{I}-T)^2y$ is left slice hyperholomorphic on $\rho_S(T)$. Observe that the integrand is such that $(\alpha-s)^{-2}$ goes to zero with order $1/|s|^2$ as $s \to \infty$. By applying Cauchy's integral theorem, we can replace the path of integration by small negatively oriented circles of radius $\delta > 0$ around the singularities of the integrand in the plane $\mathbb{C}_j$. These singularities are α, $p_j = p_0 + jp_1$ and $\overline{p}_j$ if $j \neq \pm j_p$. We obtain

$$\begin{aligned}
&\int_0^\infty e^{-pt}F(t)y\,dt \\
&= -\frac{1}{2\pi}\int_{\partial(U_\delta(\alpha)\cap\mathbb{C}_j)} S_R^{-1}(p,s)(\alpha-s)^{-2}\,ds_j\,S_R^{-1}(s,T)(\alpha\mathcal{I}-T)^2y \\
&\quad -\frac{1}{2\pi}\int_{\partial(U_\delta(p_j)\cap\mathbb{C}_j)} S_R^{-1}(p,s)(\alpha-s)^{-2}\,ds_j\,S_R^{-1}(s,T)(\alpha\mathcal{I}-T)^2y \\
&\quad -\frac{1}{2\pi}\int_{\partial(U_\delta(\overline{p_j})\cap\mathbb{C}_j)} S_R^{-1}(p,s)(\alpha-s)^{-2}\,ds_j\,S_R^{-1}(s,T)(\alpha\mathcal{I}-T)^2y.
\end{aligned}$$

Observe that the integrand has a pole of order 2 at α and poles of order 1 at p_j and $\overline{p_j}$ (except if $j = \pm j_p$). Applying Lemma 6.3.1 with $f(s) = S_R^{-1}(p,s)(\alpha-s)^{-2}$ and $g(s) = S_R^{-1}(s,T)(\alpha\mathcal{I}-T)^2y$ yields therefore

$$\begin{aligned}
\int_0^\infty e^{-pt}F(t)y\,dt = &-\mathrm{Res}_\alpha\left(S_R^{-1}(p,s)(\alpha-s)^{-2}\right)S_R^{-1}(\alpha,T)(\alpha\mathcal{I}-T)^2y \\
&-\mathrm{Res}_\alpha\left(S_R^{-1}(p,s)(s-\alpha)^{-1}\right)\left(\partial_S S_R^{-1}(\alpha,T)(\alpha\mathcal{I}-T)^2y\right) \\
&-\mathrm{Res}_{p_j}\left(S_R^{-1}(p,s)(\alpha-s)^{-2}\right)S_R^{-1}(p_j,T)(\alpha\mathcal{I}-T)^2y \\
&-\mathrm{Res}_{\overline{p_j}}\left(S_R^{-1}(p,s)(\alpha-s)^{-2}\right)S_R^{-1}(\overline{p_j},T)(\alpha\mathcal{I}-T)^2y.
\end{aligned}$$

We calculate the residues of the function $f(s) = S_R^{-1}(p,s)(\alpha-s)^{-2}$. Since it has a pole of order two at α, we have

$$\mathrm{Res}_\alpha(f_j) = \lim_{\mathbb{C}_j\ni s\to\alpha}\frac{\partial}{\partial s}f_j(s)(s-\alpha)^2 = \lim_{\mathbb{C}_j\ni s\to\alpha}\frac{\partial}{\partial s}S_R^{-1}(p,s) = (p-\alpha)^{-2},$$

where the last identity holds because α is real, and

$$\mathrm{Res}_\alpha(f_j(s)(s-\alpha)) = \lim_{\mathbb{C}_j\ni s\to\alpha} f_j(s)(s-\alpha)^2 = S_R^{-1}(p,\alpha).$$

The point $p_j = p_0 + jp_1$ is a pole of order 1. Thus, setting $s_{j_p} = s_0 + j_p s_1 \in \mathbb{C}_{j_p}$ for $s = s_0 + js_1 \in \mathbb{C}_j$, we deduce from the representation formula that

$$\begin{aligned}
\mathrm{Res}_{p_j}(f_j) &= \lim_{\mathbb{C}_j\ni s\to p_j} f_j(s)(s-p_j) = \lim_{\mathbb{C}_j\ni s\to p_j} S_R^{-1}(p,s)(\alpha-s)^{-2}(s-p_j) \\
&= \lim_{\mathbb{C}_j\ni s\to p_j}\left[S_R^{-1}(p,s_{j_p})(1-j_pj)\frac{1}{2} + S_R^{-1}(p,\overline{s_{j_p}})(1+j_pj)\frac{1}{2}\right](s-p_j)(\alpha-s)^{-2} \\
&= \lim_{\mathbb{C}_j\ni s\to p_j}(p-s_{j_p})^{-1}(1-j_pj)(s-p_j)\frac{1}{2}(\alpha-p_j)^{-2} \\
&\quad + \lim_{\mathbb{C}_j\ni s\to p_j}(p-\overline{s_{j_p}})^{-1}(1+j_pj)(s-p_j)\frac{1}{2}(\alpha-p_j)^{-2} \\
&= \left[\lim_{\mathbb{C}_j\ni s\to p_j}(p-s_{j_p})^{-1}(1-j_pj)(s-p_j)\right]\frac{1}{2}(\alpha-p_j)^{-2}.
\end{aligned}$$

We compute

$$\begin{aligned}
&\lim_{\mathbb{C}_j\ni s\to p_j}(p-s_{j_p})^{-1}(1-j_pj)(s-p_j)\\
&=\lim_{\mathbb{C}_j\ni s\to p_j}(p-s_{j_p})^{-1}(1-j_pj)(s_0-p_0)+(p-s_{j_p})^{-1}(1-j_pj)j(s_1-p_1)\\
&=\lim_{\mathbb{C}_j\ni s\to p_j}(p-s_{j_p})^{-1}(s_0-p_0)(1-j_pj)+(p-s_{j_p})^{-1}(s_1-p_1)(j+j_p)\\
&=\lim_{\mathbb{C}_j\ni s\to p_j}(p-s_{j_p})^{-1}(s_0-p_0)(1-j_pj)+(p-s_{j_p})^{-1}(s_1-p_1)j_p(-j_pj+1)\\
&=\lim_{\mathbb{C}_j\ni s\to p_j}(p-s_{j_p})^{-1}(s_0-p_0+j_p(s_1-p_1))(1-j_pj)\\
&=\lim_{\mathbb{C}_j\ni s\to p_j}(p-s_{j_p})^{-1}(s_{j_p}-p)(1-j_pj)=-(1-j_pj)
\end{aligned}$$

and finally obtain

$$\mathrm{Res}_{p_j}(f_j)=-\frac{1}{2}(1-j_pj)(\alpha-p_j)^{-2}.$$

Replacing j by $-j$ in this formula yields

$$\mathrm{Res}_{\overline{p_j}}(f_j)=-\frac{1}{2}(1+j_pj)(\alpha-\overline{p_j})^{-2}.$$

Note that these formulas also hold true if $j=\pm j_p$. In this case either $\mathrm{Res}_{p_j}(f_j)=-(\alpha-p_j)^{-2}$ and $\mathrm{Res}_{\overline{p_j}}(f_j)=0$ because $\overline{p_j}$ is a removable singularity of f_j, or vice versa. Moreover,

$$S_R^{-1}(\alpha,T)(\alpha\mathcal{I}-T)^2y=(\alpha\mathcal{I}-T)^{-1}(\alpha\mathcal{I}-T)^2y=(\alpha\mathcal{I}-T)y$$

and

$$\partial_S S_R^{-1}(\alpha,T)(\alpha\mathcal{I}-T)^2y=-(\alpha\mathcal{I}-T)^{-2}(\alpha\mathcal{I}-T)^2y=-y$$

because α is real and so $S_R^{-1}(\alpha,T)=(\alpha\mathcal{I}-T)^{-1}$. Putting these pieces together, we get

$$\begin{aligned}
\int_0^\infty e^{-pt}F(t)y\,dt&=-(p-\alpha)^{-2}S_R^{-1}(\alpha,T)(\alpha\mathcal{I}-T)^2y\\
&\quad+S_R^{-1}(p,\alpha)S_R^{-2}(\alpha,T)(\alpha\mathcal{I}-T)^2y\\
&\quad+\frac{1}{2}(1-j_pj)(\alpha-p_j)^{-2}S_R^{-1}(p_j,T)(\alpha\mathcal{I}-T)^2y\\
&\quad+\frac{1}{2}(1+j_pj)(\alpha-\overline{p_j})^{-2}S_R^{-1}(\overline{p_j},T)(\alpha\mathcal{I}-T)^2y\\
&=-(p-\alpha)^{-2}(\alpha\mathcal{I}-T)y+(p-\alpha)^{-1}y\\
&\quad+(p-\alpha)^{-2}S_R^{-1}(p,T)(\alpha\mathcal{I}-T)^2y,
\end{aligned}$$

where that last identity follows from representation formula because the mapping

$$p\mapsto(\alpha-p)^{-2}S_R^{-1}(p,T)(\alpha\mathcal{I}-T)^2y$$

is left slice hyperholomorphic. We factor out $(p-\alpha)^{-2}$ on the left and obtain

$$\begin{aligned}\int_0^\infty e^{-pt}F(t)y\,dt &= (p-\alpha)^{-2}\left(-(\alpha\mathcal{I}-T)y+(p-\alpha)y+S_R^{-1}(p,T)(\alpha\mathcal{I}-T)^2y\right)\\ &= (p-\alpha)^{-2}\left(py-2\alpha y+Ty+S_R^{-1}(p,T)(\alpha\mathcal{I}-T)^2y\right).\end{aligned}$$

Recall that we assumed that $y\in\mathcal{D}(T^2)$. Hence, $Ty\in\mathcal{D}(T)$ and so we can apply the right S-resolvent equation twice to obtain

$$\begin{aligned}&S_R^{-1}(p,T)(\alpha\mathcal{I}-T)^2y\\ &\quad= S_R^{-1}(p,T)(T^2y-2\alpha Ty+\alpha^2y)\\ &\quad= pS_R^{-1}(p,T)Ty-Ty-2\alpha pS_R^{-1}(p,T)y+2\alpha y+\alpha^2S_R^{-1}(p,T)y\\ &\quad= p^2S_R^{-1}(p,T)y-py-Ty-2\alpha pS_R^{-1}(p,T)y+2\alpha y+\alpha^2S_R^{-1}(p,T)y\\ &\quad= (p-\alpha)^2S_R^{-1}(p,T)y-py+2\alpha y-Ty.\end{aligned}$$

So finally

$$\int_0^\infty e^{-pt}F(t)y\,dt = (p-\alpha)^{-2}(p-\alpha)^2S_R^{-1}(p,T)y = S_R^{-1}(p,T)y.$$

Hence,

$$\int_0^\infty e^{-pt}F(t)y\,dt = S_R^{-1}(p,T)y = \int_0^\infty e^{-pt}\,\mathcal{U}_T(t)y\,dt,$$

for $\operatorname{Re}(p)>c$, which implies $F(t)y=\mathcal{U}_T(t)y$ for $y\in D(T^2)$ and $t\geq 0$ as a consequence of the quaternionic version of the Hahn–Banach theorem (see Corollary 12.0.7).

Applying the same reasoning to the semigroup $\{\mathcal{U}(-t)\}_{t\geq 0}$, with infinitesimal generator $-T$, we see that

$$\begin{aligned}\mathcal{U}(-t)y &= \frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_j)} e^{ts}(\alpha-s)^{-2}\,ds_j\,S_R^{-1}(s,-T)(\alpha\mathcal{I}+T)^2y\\ &= \frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_j)} e^{-ts}(\alpha+s)^{-2}\,ds_j\,S_R^{-1}(s,T)(\alpha\mathcal{I}+T)^2y,\end{aligned}$$

where the second equality follows by substitution of s by $-s$ because

$$S_R^{-1}(-s,-T) = -S_R^{-1}(s,T).$$

Replacing α by $-\alpha$ and $-t$ by t, we finally find

$$\mathcal{U}(t)y = \frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_j)} e^{ts}(\alpha-s)^{-2}\,ds_j\,S_R^{-1}(s,T)(\alpha\mathcal{I}-T)^2y$$

also for $t<0$. □

Proposition 6.3.3. *Let α and c be real numbers such that $\omega < c < |\alpha|$. If $f \in \mathbf{X}(T)$ is right slice hyperholomorphic on $\overline{W_c}$, then, for any $y \in D(T^2)$, we have*

$$f(T)y = \frac{1}{2\pi} \int_{\partial(W_c \cap \mathbb{C}_j)} f(s)(\alpha - s)^{-2}\, ds_j\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2 y. \tag{6.12}$$

Proof. We recall that f can be represented as

$$f(s) = \int_{\mathbb{R}} d\mu(t)\, e^{-st}$$

with $\mu \in \mathbf{S}(T)$. Using Proposition 6.3.2, we obtain

$$\begin{aligned}
&\frac{1}{2\pi} \int_{\partial(W_c \cap \mathbb{C}_j)} f(s)(\alpha - s)^{-2}\, ds_j\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2 y \\
&= \frac{1}{2\pi} \int_{\partial(W_c \cap \mathbb{C}_j)} \int_{\mathbb{R}} d\mu(t)\, e^{-st}(\alpha - s)^{-2}\, ds_j\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2 y \\
&= \int_{\mathbb{R}} d\mu(t) \left(\frac{1}{2\pi} \int_{\partial(W_c \cap \mathbb{C}_j)} e^{-st}(\alpha - s)^{-2}\, ds_j\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2 y \right) \\
&= \int_{\mathbb{R}} d\mu(t)\, \mathcal{U}_T(-t) y = f(T)y.
\end{aligned}$$

Note that Fubini's theorem allows us to exchange the order of integration as the S-resolvent $S_R^{-1}(s,T)$ is uniformly bounded on $\partial(W_c \cap \mathbb{C}_j)$ because of (6.11). So there exists a constant $K > 0$ such that

$$\begin{aligned}
&\frac{1}{2\pi} \int_{\partial(W_c \cap \mathbb{C}_j)} \int_{\mathbb{R}} \left\| d\mu(t)\, e^{-st}(\alpha - s)^{-2}\, ds_j\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2 y \right\| \\
&\leq \frac{1}{2\pi} \int_{\partial(W_c \cap \mathbb{C}_j)} \int_{\mathbb{R}} d|\mu|(t)\, e^{-\mathrm{Re}(s)t} \frac{1}{|\alpha - s|^{-2}} \|S_R^{-1}(s,T)\| \|(\alpha\mathcal{I} - T)^2 y\| ds \\
&\leq K \int_{\partial(W_c \cap \mathbb{C}_j)} \int_{\mathbb{R}} d|\mu|(t)\, e^{c|t|} \frac{1}{(1 + |s|)^2}\, ds.
\end{aligned}$$

This integral is finite because, as $\mu \in \mathbf{S}(T)$, we have

$$\int_{\mathbb{R}} d|\mu|(t)\, e^{c|t|} < +\infty.$$

□

Theorem 6.3.4. *Let $f \in \mathbf{X}(T)$ and suppose that f is right slice hyperholomorphic at infinity. Then the operator $f(T)$ defined using the Laplace transform equals the operator $f[T]$ obtained from the S-functional calculus.*

Proof. Consider $\alpha \in \mathbb{R}$ with $c < |\alpha|$ and observe that the function

$$g(s) := f(s)(\alpha - s)^{-2}$$

is right slice hyperholomorphic and, since f is right slice hyperholomorphic at infinity, tends to zero with order $1/|s|^2$ as s tends to infinity. The S-functional calculus for unbounded operators thus satisfies

$$g[T] = g(\infty)\mathcal{I} + \int_{\partial(W_c\cap\mathbb{C}_j)} g(s)\, ds_j\, S_R^{-1}(s,T).$$

By Theorem 3.5.1, we have for $y \in X$ that

$$f[T](\alpha\mathcal{I} - T)^{-2}y = \frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_j)} f(s)(\alpha - s)^2\, ds_j\, S_R^{-1}(s,T)y.$$

But by Proposition 6.3.3, it is

$$f(T)x = \frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_j)} f(s)(\alpha - s)^2\, ds_j\, S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2 x,$$

for $x \in D(T^2)$. Setting $y = (\alpha\mathcal{I} - T)^2 x$, we conclude

$$f[T]y = f(T)y, \quad \text{for } y \in D(T^2).$$

Since $D(T^2)$ is dense in X and since the operators $f[T]$ and $f(T)$ are bounded we get $f[T] = f(T)$. □

6.4 The Inversion of the Operator $f(T)$

We study the inversion of the operator $f(T)$ defined by the Phillips functional calculus via approximation with polynomials P_n such that $\lim_{n\to\infty} P_n(s)f(s) = 1$. In general, the pointwise product $P_n(s)f(s)$ is not slice hyperholomorphic and therefore we must limit ourselves to intrinsic functions. The main goal of this section is to deduce sufficient conditions such that

$$\lim_{n\to+\infty} P_n(T)f(T)y = y, \quad \text{for every } y \in X.$$

Lemma 6.4.1. *Let $T \in \mathcal{K}(X)$ such that $\rho_S(T)\cap\mathbb{R} \neq \emptyset$. If $\mathcal{D}(T)$ is closed, then $\mathcal{D}(T^n)$ is dense in X for every $n \in \mathbb{N}$.*

Proof. If $\alpha \in \rho_S(T)\cap\mathbb{R}$, then

$$\mathcal{D}(T^n) = \mathcal{D}((\alpha\mathcal{I} - T)^n) = (\alpha\mathcal{I} - T)^{-n}X.$$

Therefore, a continuous right linear functional $y^* \in X^*$ on X vanishes on $\mathcal{D}(T^n)$ if and only if the functional $y^*(\alpha\mathcal{I} - T)^{-n}$, which is defined as

$$\langle y^*(\alpha\mathcal{I} - T)^{-n}, y\rangle := \langle y^*, (\alpha\mathcal{I} - T)^{-n}y\rangle$$

for $y \in X$, vanishes on the entire space X. We prove the statement by induction. It is obviously true for $n = 0$, so let us choose $n \in \mathbb{N}$ and let us assume that it holds for $n-1$. By the above arguments, a functional $y^* \in X^*$ vanishes on $\mathcal{D}(T^n)$ if and only if

$$y^*(\alpha\mathcal{I} - T)^{-n} = y^*(\alpha\mathcal{I} - T)^{-(n-1)}(\alpha\mathcal{I} - T)^{-1}$$

vanishes on X, which is in turn equivalent to $y^*(\alpha\mathcal{I}-T)^{-(n-1)}$ vanishing on $\mathcal{D}(T)$. Now observe that by assumption $\mathcal{D}(T)$ is dense in X. Hence, Corollary 12.0.8 implies that a functional $x^* \in X^*$ vanishes on $\mathcal{D}(T)$ if and only if it vanishes on all of X. We conclude that $y^* \in X^*$ vanishes on $\mathcal{D}(T^n)$ if and only if $y^*(\alpha\mathcal{I}-T)^{-(n-1)}$ vanishes on all of X^*, which is in turn equivalent to y^* vanishing on $\mathcal{D}(T^{n-1})$. Since $\mathcal{D}(T^{n-1})$ is dense in X by the induction hypothesis, Corollary 12.0.8 implies again that a functional $x^* \in X^*$ vanishes on $\mathcal{D}(T^{n-1})$ if and only if it vanishes on all of X. Therefore, we finally find that y^* vanishes on $\mathcal{D}(T^n)$ if and only if it vanishes on all of X^* and a final application of Corollary 12.0.8 yields that $\mathcal{D}(T^n)$ is dense in X. □

Lemma 6.4.2. *Let P be an intrinsic polynomial of degree m and let f and $P_n f$ both belong to $\mathbf{X}(T)$. Then $f(T)X \subseteq \mathcal{D}(T^m)$ and*

$$P(T)f(T)y = (Pf)(T)y, \quad \textit{for all } y \in X.$$

Proof. We first consider the case $y \in \mathcal{D}\left(T^{m+2}\right)$. Let $\alpha, c \in \mathbb{R}$ with $w < c < |\alpha|$ and let $j \in \mathbb{S}$. The function Pf is the product of two intrinsic functions and therefore intrinsic itself. By Proposition 6.3.3, Lemma 6.2.7 and Remark 6.2.1, we have

$$\begin{aligned}&(\alpha\mathcal{I} - T)^{-m}(Pf)(T)y\\ &= \frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_j)}(\alpha - s)^{-m}P(s)f(s)(\alpha - s)^{-2}\,ds_j\,S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2 y.\end{aligned}$$

We write the polynomial P in the form $P(s) = \sum_{k=0}^{m} a_k(\alpha - s)^k$ with $a_k \in \mathbb{R}$. In view of Proposition 6.3.3, Lemma 6.2.7 and Remark 6.2.1 we obtain again

$$\begin{aligned}&(\alpha\mathcal{I} - T)^{-m}(Pf)(T)y\\ &= \sum_{k=0}^{m} a_k \frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_j)}(\alpha - s)^{-m+k}f(s)(\alpha - s)^{-2}\,ds_j\,S_R^{-1}(s,T)(\alpha\mathcal{I} - T)^2 y\\ &= \sum_{k=0}^{m} a_k(\alpha\mathcal{I} - T)^{-m+k}f(T)u = (\alpha\mathcal{I} - T)^{-m}\sum_{k=0}^{m} a_k(\alpha\mathcal{I} - T)^k f(T)y\\ &= (\alpha\mathcal{I} - T)^{-m}P(T)f(T)y.\end{aligned}$$

Consequently, $(Pf)(T)y = P(T)f(T)y$ for $y \in \mathcal{D}\left(T^{m+2}\right)$.

Now let $y \in X$ be arbitrary. Since $\mathcal{D}(T^{m+2})$ is dense in X by Lemma 6.4.1, there exists a sequence $y_n \in \mathcal{D}(T^{m+2})$ with $\lim_{n\to+\infty} y_n = y$. Then $f(T)y_n \to$

$f(T)y$ and $P(T)f(T)y_n = (Pf)(T)y_n \to (Pf)(T)y$ as $n \to +\infty$. Since $P(T)$ is closed with domain $\mathcal{D}(T^m)$, it follows that $f(T)y \in \mathcal{D}(T^m)$ and $P(T)f(T)y = (Pf)(T)y$. □

Definition 6.4.3. A sequence of intrinsic polynomials $\{P_n\}_{n\in\mathbb{N}}$ is called an inverting sequence for an intrinsic function $f \in \mathbf{X}(T)$ if

(i) $P_n f \in \mathbf{X}(T)$,

(ii) $|P_n(s)f(s)| \le M$, $n \in \mathbb{N}$ for some constant $M > 0$ and

$$\lim_{n\to+\infty} P_n(s)f(s) = 1$$

in a strip $W_{\omega+\varepsilon} = \{s \in \mathbb{H} : -(\omega+\varepsilon) < \mathrm{Re}(s) \le \omega+\varepsilon\}$,

(iii) $\|(P_n f)(T)\| \le M$, $n \in \mathbb{N}$ for some constant $M > 0$.

Theorem 6.4.4. *If $\{P_n\}_{n\in\mathbb{N}}$ is an inverting sequence for an intrinsic function $f \in \mathbf{X}(T)$, then*

$$\lim_{n\to+\infty} P_n(T)f(T)y = y, \quad \forall y \in X.$$

Proof. First consider $y \in \mathcal{D}(T^2)$ and choose $\alpha \in \mathbb{R}$ with $\omega < |\alpha|$. Then Proposition 6.3.3 and Lemma 6.4.2 imply

$$\begin{aligned} P_n(T)f(T)y &= (P_n f)(T)y \\ &= \frac{1}{2\pi}\int_{\partial(W_{c_n}\cap\mathbb{C}_j)} P_n(s)f(s)(\alpha-s)^{-2}\,ds_j\,S_R^{-1}(s,T)(\alpha\mathcal{I}-T)^2 y, \end{aligned}$$

for arbitrary $j \in \mathbb{S}$ and $c_n \in \mathbb{R}$ with $w < c_n < |\alpha|$ such that $P_n f$ is right slice hyperholomorphic on $\overline{W_{c_n}}$. However, we have assumed that there exists a constant M such that $|P_n(s)f(s)| \le M$ for any $n \in \mathbb{N}$ on a strip $-(\omega+\varepsilon) \le \mathrm{Re}(s) \le \omega+\varepsilon$. Moreover, because of (6.11), the right S-resolvent is uniformly bounded on any set $\{s \in \mathbb{C}_j : |\mathrm{Re}(s)| > \omega+\varepsilon'\}$ with $\varepsilon' > 0$. Applying Cauchy's integral theorem we can therefore replace $\partial(W_{c_n}\cap\mathbb{C}_j)$ for any $n \in \mathbb{N}$ by $\partial(W_c\cap\mathbb{C}_j)$ where c is a real number with $\omega < c < \min\{|\alpha|, \omega+\varepsilon\}$. In particular, we can choose c independent of n. Lebesgue's dominated convergence theorem allows us to exchange limit and integration and we obtain

$$P_n(T)f(T)y = \frac{1}{2\pi}\int_{\partial(W_c\cap\mathbb{C}_j)} (\alpha-s)^{-2}\,ds_I\,S_R^{-1}(s,T)(\alpha\mathcal{I}-T)^2 y = y.$$

If $y \in X$ does not belong to $\mathcal{D}(T^2)$, then we can choose for any $\varepsilon > 0$ a vector $y_\varepsilon \in \mathcal{D}(T^2)$ with $\|y-y_\varepsilon\| < \varepsilon$. Since the mappings $(P_n f)(T)$ are uniformly bounded by a constant $M > 0$, we get

$$\begin{aligned} &\|(P_n f)(T)y - y\| \\ &\le \|(P_n f)(T)y - (P_n f)(T)y_\varepsilon\| + \|(P_n f)(T)y_\varepsilon - y_\varepsilon\| + \|y_\varepsilon - y\| \\ &\le M\|y-y_\varepsilon\| + \|(P_n f)(T)y_\varepsilon - y_\varepsilon\| + \|y_\varepsilon - y\| \\ &\overset{n\to+\infty}{\longrightarrow} M\|y-y_\varepsilon\| + \|y_\varepsilon - y\| \le (M+1)\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we deduce $\lim_{n\to+\infty} \|(P_n f)(T)y - y\| = 0$ even for arbitrary $y \in X$. $\square$

Corollary 6.4.5. *Let X be reflexive and let P_n be an inverting sequence for an intrinsic function $f \in \mathbf{S}(T)$. A vector y belongs to the range of $f(T)$ if and only if it is in $\mathcal{D}(P_n(T))$ for all $n \in \mathbb{N}$ and the sequence $\{P_n(T)y\}_{n\in\mathbb{N}}$ is bounded.*

Proof. If $y \in \operatorname{ran} f(T)$ with $y = f(T)x$ then Lemma 6.4.2 implies $y \in \mathcal{D}(P_n(T))$ for all $n \in \mathbb{N}$. Theorem 6.4.4 states $\lim_{n\to+\infty} P_n(T)y = x$, which implies that the sequence $(P_n(T)y)_{n\in\mathbb{N}}$ is bounded.

To prove the converse statement consider $y \in X$ such that $\{P_n(T)y\}_{n\in\mathbb{N}}$ is bounded. Since X is reflexive the set $\{P_n(T)y\}_{n\in\mathbb{N}}$ is weakly sequentially compact. (The proof that a set E in a reflexive quaternionic Banach space X is weakly sequentially compact if and only if E is bounded can be completed similarly to the classical case when X is a complex Banach space, see [110, Theorem II.28].) Hence, there exists a subsequence $\{P_{n_k}(T)y\}_{k\in\mathbb{N}}$ and a vector $x \in X$ such that

$$\langle y^*, P_{n_k}(T)y\rangle \to \langle y^*, x\rangle$$

as $k \to +\infty$ for any $y^* \in X^*$. We show $y = f(T)x$. For any functional $y^* \in X^*$, the mapping $y^* f(T)$, which is defined by

$$\langle y^* f(T), w\rangle = \langle y^*, f(T)w\rangle,$$

also belongs to X^*. Hence,

$$\langle y^*, f(T)P_{n_k}(T)y\rangle = \langle y^* f(T), P_{n_k}(T)y\rangle \to \langle y^* f(T), x\rangle = \langle y^*, f(T)x\rangle.$$

Recall that the measure μ with $f = \mathcal{L}(\mu)$ is real-valued since f is intrinsic. Therefore it commutes with the operator $P_{n_k}(T)$. Recall also that if $w \in \mathcal{D}(T^n)$ for some $n \in \mathbb{N}$, then $\mathcal{U}_T(t)w \in \mathcal{D}(T^n)$ for any $t \in \mathbb{R}$ and $\mathcal{U}_T(t)T^n w = T^n \mathcal{U}_T(t)w$. Thus,

$$P_{n_k}(T)\mathcal{U}_T(t)y = \mathcal{U}_T(t)P_{n_k}(T)y$$

because P_{n_k} has real coefficients. Moreover, we can therefore exchange the integral with the unbounded operator $P_{n_k}(T)$ in the following computation

$$\begin{aligned} f(T)P_{n_k}(T)y &= \int_{\mathbb{R}} d\mu(t)\,\mathcal{U}_T(-t)P_{n_k}(T)y \\ &= P_{n_k}(T)\int_{\mathbb{R}} d\mu(t)\,\mathcal{U}_T(-t)y = P_{n_k}(T)f(T)y. \end{aligned}$$

Theorem 6.4.4 implies for any $y^* \in X^*$

$$\langle y^*, y\rangle = \lim_{k\to\infty}\langle y^*, P_{n_k}(T)f(T)y\rangle = \lim_{k\to\infty}\langle y^*, f(T)P_{n_k}(T)y\rangle = \langle x^*, f(T)x\rangle$$

and so $y = f(T)x$ follows from Corollary 12.0.8. $\square$