Chapter 5

Perturbations of the generator of a group

In the applications, it is in general not trivial to verify the conditions of the Hille–Phillips–Yosida theorem. So an other aspect that we will investigate is the generation by perturbation. Precisely, given a closed operator T that generates the evolution operator $\mathcal{U}_T(t)$, we are interested in finding under which conditions a closed operator P is such that $T+P$ generates the evolution operator $\mathcal{U}_{T+P}(t)$. In the sequel, we will consider only right linear quaternionic operators even though the theory can be developed for left linear quaternionic operators following similar lines.

5.1 A series expansion of the S-resolvent operator

For right linear operators, we are also in need of the notion of left resolvent set and of left spectrum.

Definition 5.1.1. Let X be a two-sided quaternionic Banach space and let T be a right linear closed quaternionic operator.

We define the left resolvent set of T and denote it by $\rho_L(T)$ as

$$\rho_L(T) = \{\lambda \in \mathbb{H} : \quad (\lambda\mathcal{I} - T)^{-1} \in \mathcal{B}(X)\},$$

where the notation $\lambda\mathcal{I}$ in $\mathcal{B}(X)$ means that $(\lambda\mathcal{I})(v) = \lambda v$.

The operator $(\lambda\mathcal{I} - T)^{-1}$ is called the left resolvent operator.

We define the left spectrum of T as

$$\sigma_L(T) = \mathbb{H} \setminus \rho_L(T).$$

Since the operator T is assumed to be right linear, then the left resolvent operator $(\lambda\mathcal{I} - T)^{-1}$ in Definition 5.1.1 is right linear. The S-spectrum and the

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left spectrum are not, in general, related and we point out that there is no notion of holomorphicity over the quaternions such that $(\lambda\mathcal{I}-T)^{-1}$ turns out to be hyperholomorphic on the resolvent set $\rho_L(T)$. In the sequel, we will need the following expansion of the S-resolvent operator.

Proposition 5.1.2 (Expansion of the S-resolvent operator)**.** *Let X be a two-sided quaternionic Banach space. Let $T : \mathcal{D}(T) \subset X \to X$ and $P : \mathcal{D}(P) \subset X \to X$ be right linear closed quaternionic operators and assume that*

(a) $\lambda \in \rho_S(T)$,

(b) $\mathcal{D}(T) \subset \mathcal{D}(P)$,

(c) $B_\lambda \mathcal{Q}_\lambda^{-1}(T) : \ X \mapsto X, \quad \text{for all } \lambda \in \rho_S(T)$,

(d) $\|B_\lambda \mathcal{Q}_\lambda^{-1}(T)\| < 1, \quad \text{for some} \ \ \lambda \in \rho_S(T)$,

where

$$\mathcal{Q}_\lambda^{-1}(T) := (T^2 - 2\lambda_0 T + |\lambda|^2)^{-1}$$

and

$$B_\lambda := 2\lambda_0 P - P^2 - TP - PT. \tag{5.1}$$

Then $\lambda \in \rho_S(T+P)$ and

$$\mathcal{Q}_\lambda^{-1}(T+P) = \sum_{m=0}^{\infty} \mathcal{Q}_\lambda^{-1}(T)(B_\lambda \mathcal{Q}_\lambda^{-1}(T))^m; \tag{5.2}$$

moreover, $S_R^{-1}(\lambda, T+P)$ is given by

$$S_R^{-1}(\lambda, T+P)v = (\overline{\lambda}\mathcal{I} - T - P) \sum_{m=0}^{\infty} \mathcal{Q}_\lambda^{-1}(T)(B_\lambda \mathcal{Q}_\lambda^{-1}(T))^m v, \quad v \in X. \tag{5.3}$$

Proof. Let $\lambda \in \rho_S(T)$, so we have

$$\begin{aligned}
\mathcal{Q}_\lambda^{-1}(T+P) &= [(T+P)^2 - 2\lambda_0(T+P) + |\lambda|^2]^{-1} \\
&= [T^2 - 2\lambda_0 T + |\lambda|^2 - (2\lambda_0 P - P^2 - TP - PT)]^{-1} \\
&= \sum_{m=0}^{\infty} (\mathcal{Q}_\lambda^{-1}(T)(2\lambda_0 P - P^2 - TP - PT))^m \mathcal{Q}_\lambda^{-1}(T) \\
&= \sum_{m=0}^{\infty} \mathcal{Q}_\lambda^{-1}(T)((2\lambda_0 P - P^2 - TP - PT)\mathcal{Q}_\lambda^{-1}(T))^m.
\end{aligned}$$

Recalling (5.1), this concludes the proof. Finally (5.3) follows from the definition of $S_R^{-1}(\lambda, T+P)$ and from (5.2). □

Remark 5.1.1. In the case the operators T and P anti-commute, then the operator B_λ depends only on P, in fact it is

$$B_\lambda = 2\lambda_0 P - P^2.$$

In this case, the operator B_λ depends just on the perturbation P.

Proposition 5.1.3. *If* $\overline{\lambda} \in \rho_L(T)$ *and* $\mathrm{Re}(\lambda) > \omega_0$, *where*

$$\omega_0 := \lim_{t \to +\infty} \frac{1}{t} \ln \|\mathcal{U}_T(t)\|, \tag{5.4}$$

then we have

$$\mathcal{Q}_\lambda^{-1}(T)v = (\overline{\lambda}\mathcal{I} - T)^{-1} \int_0^\infty e^{-t\lambda}\, \mathcal{U}_T(t)\, v\; dt, \quad v \in X. \tag{5.5}$$

Proof. Since $\overline{\lambda} \in \rho_L(T)$, then $(\overline{\lambda}\mathcal{I} - T)^{-1}$ is a bounded linear operator. From Theorem 4.2.2, we can write $S_R^{-1}(\lambda, T)$ as the Laplace transform of the evolution operator, since $(\overline{\lambda}\mathcal{I} - T)Q_\lambda(T)v = S_R^{-1}(\lambda, T)v$, for $v \in X$. □

5.2 The class of operators $\mathbf{A}(T)$ and some properties

We now introduce a class of closed operators which will be useful in the sequel.

Definition 5.2.1 (The class $\mathbf{A}(T)$)**.** Let X be a two-sided quaternionic Banach space and let $\mathcal{U}_T(t)$ be the strongly continuous quaternionic semigroup generated by T where $T : \mathcal{D}(T) \subset X \to X$ is a right linear closed quaternionic operator. We denote by $\mathbf{A}(T)$ the class of closed right linear quaternionic operators A that satisfy the conditions

(1) $\mathcal{D}(A) \supseteq \mathcal{D}(T)$.

(2) For every $t > 0$ there exists a positive constant $C(t)$ such that

$$\|Ae^{-\lambda t}\mathcal{U}_T(t)v\| \leq C(t)\|v\|$$

for $v \in \mathcal{D}(T)$ and for $\mathrm{Re}(\lambda) > \omega_0$, where ω_0 is defined in (5.4).

(3) The constant $C(t)$ can be chosen such that $\int_0^1 C(t)dt$ exists and is finite.

Lemma 5.2.2. *Let X be a two-sided quaternionic Banach space and let $\mathcal{U}_T(t)$ be the strongly continuous quaternionic semigroup generated by T where $T : \mathcal{D}(T) \subset X \to X$ is a right linear closed quaternionic operator. Assume*

(1) $\overline{\lambda} \in \rho_L(T + P)$, *and* $\mathrm{Re}(\lambda) > \omega_0$, *where* ω_0 *is defined in* (5.4),

(2) $B_\lambda : \mathcal{D}(T^2) \to X$, *and*

(3) $B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1} \in \mathbf{A}(T)$,

where B_λ *is defined in* (5.1)*. Then we have*

(a) $\mathcal{D}(B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-t\lambda}) \supseteq \bigcup_{t>0} \mathcal{U}_T(t)X$.

(b) *The map* $v \mapsto B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-t\lambda}\mathcal{U}_T(t)v$ *for* $v \in \mathcal{D}(T)$ *has a unique extension to a bounded quaternionic operator defined on all* X*. (We will denote the extension with the same symbol.)*

(c) $B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-t\lambda}\mathcal{U}_T(t)v$ *is continuous in* t *for* $t > 0$ *and for every* $v \in X$*. Moreover, if* ω_0 *is defined in* (5.4)*, then*

$$\limsup_{t\to\infty} \frac{\ln \|B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-t\lambda}\mathcal{U}_T(t)\|}{t} \leq \omega_0,$$

for $\mathrm{Re}(\lambda) > 0$*.*

(d) *Since* $\mathrm{Re}(\lambda) > \omega_0$*, then*

$$B_\lambda \mathcal{Q}_\lambda^{-1}(T)v = \int_0^\infty B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}\ e^{-t\lambda}\ \mathcal{U}_T(t)\, v\ dt, \quad v \in X.$$

Proof. To prove (a) let $v_0 \in X$ such that $v_0 = \lim_{n\to\infty} v_n$, where $v_n \in \mathcal{D}(T)$ we can make this choice since $\mathcal{D}(T)$ is dense in X thanks to Lemma 4.1.9. Then $\mathcal{U}_T(t)v_n \to \mathcal{U}_T(t)v_0$ and

$$B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-t\lambda}\mathcal{U}_T(t)v_n \to B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-t\lambda}\mathcal{U}_T(t)v_0.$$

Since $B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}$ is closed because it belongs to $\mathbf{A}(T)$, we have

$$e^{-t\lambda}\mathcal{U}_T(t)v_0 \in \mathcal{D}(B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1})$$

for $t \geq 0$, $\mathrm{Re}(\lambda) > 0$ and

$$B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-t\lambda}\left(\mathcal{U}_T(t)v_0\right) = \left(B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-t\lambda}(\mathcal{U}_T(t)\right) v_0.$$

Point (b) follows from condition (2) in Definition 5.2.1 and the Principle of extension by continuity (see Theorem 12.0.10 below).

To prove Point (c) let $0 < \delta < t$. The continuity follows from the semigroup properties since

$$B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-t\lambda}\mathcal{U}_T(t)v = B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-t\lambda}\mathcal{U}_T(\delta)\mathcal{U}_T(t-\delta)v.$$

The second part follows from

$$\ln \|B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-t\lambda}\mathcal{U}_T(t)\| \leq \ln \|B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-t\lambda}\mathcal{U}_T(\delta)\| + \ln \|\mathcal{U}_T(t-\delta)\|$$

so

$$\begin{aligned}
&\lim_{t\to\infty}\sup \frac{\ln \|B_\lambda(\overline{\lambda}\mathcal{I}-T-P)^{-1}e^{-t\lambda}\mathcal{U}_T(t)\|}{t}\\
&\leq \lim_{t\to\infty} \frac{\ln \|B_\lambda(\overline{\lambda}\mathcal{I}-T-P)^{-1}e^{-t\lambda}\mathcal{U}_T(\delta)\|}{t}\\
&\quad + \lim_{t\to\infty}\frac{\ln\|\mathcal{U}_T(t-\delta)\|}{t}\\
&=\omega_0,
\end{aligned}$$

where we have used the fact that

$$\lim_{t\to\infty} \frac{\ln \|B_\lambda(\overline{\lambda}\mathcal{I}-T-P)^{-1}e^{-t\lambda}\mathcal{U}_T(\delta)\|}{t}=0$$

since $\mathrm{Re}(\lambda)>\omega_0$. Statement (d) follows from Theorem 12.0.18. □

Lemma 5.2.3. *Let X be a two-sided quaternionic Banach space and let $\mathcal{U}_T(t)$ be the strongly continuous quaternionic semigroup generated by T where $T:\mathcal{D}(T)\subset X\to X$ is a right linear closed quaternionic operator. Let us assume that*

(1) *$h\in C((0,\infty),X)\cap L^1((0,\infty),X)$ and*

(2) *$P:\mathcal{D}(P)\subset X\to X$ is a right linear closed quaternionic operator such that $B_\lambda(\overline{\lambda}\mathcal{I}-T-P)^{-1}$, for $\overline{\lambda}\in\rho_L(T+P)$ and $\mathrm{Re}(\lambda)>\omega_0$, belongs to the class $\mathbf{A}(T)$, where B_λ is defined in (5.1).*

If we define

$$g(t):=\int_0^t e^{-\lambda(t-s)}\mathcal{U}_T(t-s)h(s)ds,\quad t\geq 0,\ \mathrm{Re}(\lambda)>0, \tag{5.6}$$

then $g\in\mathcal{D}(B_\lambda(\overline{\lambda}\mathcal{I}-T-P)^{-1})$ and we have

$$B_\lambda(\overline{\lambda}\mathcal{I}-T-P)^{-1}g(t)=\int_0^t B_\lambda(\overline{\lambda}\mathcal{I}-T-P)^{-1}e^{-\lambda(t-s)}\mathcal{U}_T(t-s)h(s)ds. \tag{5.7}$$

Moreover, g and $B_\lambda(\overline{\lambda}\mathcal{I}-T-P)^{-1}g(t)$ are continuous functions of t for $t>0$.

Proof. The integral that defines g exists for every $t\geq 0$ since $\|\mathcal{U}_T(t)\|$ is bounded in every finite interval by Proposition 4.1.4. For all $s<t$, the function

$$s\mapsto e^{-\lambda(t-s)}\mathcal{U}_T(t-s)h(s)$$

belongs to $\mathcal{D}(B_\lambda(\overline{\lambda}\mathcal{I}-T-P)^{-1})$ by point (a) in Lemma 5.2.2.

Thus by Theorem 12.0.18, we will show that $\int_0^t e^{-\lambda(t-s)}\mathcal{U}_T(t-s)h(s)ds$ belongs to $\mathcal{D}(B_\lambda(\overline{\lambda}\mathcal{I}-T-P)^{-1})$ and it will also prove the formula (5.7) when we show that the function

$$s\mapsto B_\lambda(\overline{\lambda}\mathcal{I}-T-P)^{-1}e^{-\lambda(t-s)}\mathcal{U}_T(t-s)h(s)$$

is integrable over the interval $[0, t]$. Moreover, observe that by definition for $\overline{\lambda} \in \rho_L(T+P)$ the operator $(\overline{\lambda}\mathcal{I} - T - P)^{-1}$ is continuous. From the Principle of Uniform Boundedness, see Theorem 12.0.9, and from Lemma 5.2.2 (b) it follows that $\|B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda t}\mathcal{U}_T(t)\|$ is bounded on every interval that does not contain the origin. Let $0 < t_1 < t$ so that the function

$$s \mapsto \|B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda(t-s)}\mathcal{U}_T(t-s)\|$$

is bounded and $\|h(\cdot)\|$ is integrable on the interval $0 \le s \le t_1$ while $\|h(\cdot)\|$ is bounded and

$$s \mapsto \|B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda(t-s)}\mathcal{U}_T(t-s)\|$$

is integrable on the interval $t_1 \le s \le t$ by Proposition 4.1.4 and Definition 5.2.1 (3).

To see that $B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}g(t)$ is continuous for $t > 0$, assume $0 < 2\delta < t_0$ and set

$$M_1 = \sup_{t_0 - 2\delta \le s \le t_0 + \delta} \|B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda s}\mathcal{U}_T(s)\|.$$

Then

$$\|B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda(t-s)}\mathcal{U}_T(t-s)h(s)\| \le M_1\|h(s)\|,$$

if $|t - t_0| \le \delta$. Consequently from Lebesgue dominated convergence theorem (see, for example, [110]),

$$\lim_{t \to t_0} \int_0^\delta B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda(t-s)}\mathcal{U}_T(t-s)h(s)ds$$
$$= \int_0^\delta B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda(t_0-s)}\mathcal{U}_T(t_0-s)h(s)ds.$$

We can write

$$\int_\delta^t B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda(t-s)}\mathcal{U}_T(t-s)h(s)ds$$
$$= \int_0^{t_0} B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda s}\mathcal{U}_T(s)h(t-s)\chi_{[0,t-\delta]}(s)ds$$

where $\chi_{[0,t-\delta]}$ is the characteristic function of the interval $[0, t-\delta]$ and if we set

$$M_2 := \sup_{\delta \le s \le t_0 + \delta} \|h(s)\|,$$

we obtain that the norm of the integral on the right satisfies the estimate

$$\|\int_0^{t_0} B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda s}\mathcal{U}_T(s)h(t-s)\chi_{[0,t-\delta]}(s)ds\|$$
$$\le M_2\|B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda s}\mathcal{U}_T(s)\|.$$

Thus,

$$\lim_{t \to t_0} \int_\delta^t B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}\mathcal{U}_T(t-s)e^{-\lambda(t-s)}h(s)ds$$
$$= \int_\delta^{t_0} B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}\mathcal{U}_T(t_0-s)e^{-\lambda(t_0-s)}h(s)ds.$$

Combining this result with the limit above, we see that $B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}g(t)$ is continuous at the arbitrary point $t_0 > 0$. The result just proved, if applied to the case when $B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}$ is replaced by the identity operator $\mathcal{I}$, shows that g is continuous. □

5.3 Perturbation of the generator

We define some operators that will be useful in the sequel.

Definition 5.3.1. For $\overline{\lambda} \in \rho_L(T+P)$ let us define the operator

$$W_0(t) := (\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda t}\mathcal{U}_T(t)$$

and the convolution

$$(W_0 * B_\lambda W_0)(t) := \int_0^t W_0(t-s)B_\lambda W_0(s)\, ds,$$

where B_λ is defined in (5.1).

Theorem 5.3.2. *Let X be a two-sided quaternionic Banach space and let $T : \mathcal{D}(T) \subset X \to X$ be the generator of the strongly continuous semigroup $\{\mathcal{U}_T(t)\}_{t\geq 0}$. Let $P : \mathcal{D}(P) \subset X \to X$ be a quaternionic closed operator and let B_λ be the operator defined in (5.1). We assume that*

(1) $\overline{\lambda} \in \rho_L(T+P)$, *and* $\mathrm{Re}(\lambda) > \omega_0$, *where* ω_0 *is defined in* (5.4),

(2) $\mathcal{D}(P) \supseteq \mathcal{D}(T)$,

(3) $B_\lambda : \mathcal{D}(T^2) \to X$,

(4) $B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1} \in \mathbf{A}(T)$,

(5) *there exists a positive function* $\mathcal{K}_\lambda$ *such that* $\|(\overline{\lambda}\mathcal{I} - T - P)^{-1}\| \leq \mathcal{K}_\lambda$, *for* $\mathrm{Re}(\lambda) > \omega_0$, *and*

(6) $\|B_\lambda \mathcal{Q}_\lambda^{-1}(T)\| < 1$, *for some* $\lambda \in \rho_S(T)$, *where* $\mathcal{Q}_\lambda^{-1}(T)$ *is the pseudo-resolvent operator.*

Then $T+P$, defined on $\mathcal{D}(T)$, is closed and it is the infinitesimal generator of the semigroup $\mathcal{U}_{T+P}(t)$. Moreover, we have the following representation

$$\mathcal{U}_{T+P}(t)v = e^{\lambda t}(\overline{\lambda}\mathcal{I} - T - P)W(t)v, \quad v \in X, \tag{5.8}$$

where

$$W(t)v = \sum_{m=0}^{\infty} W_m(t)v, \quad v \in X, \tag{5.9}$$

$$W_0(t)v = (\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda t}\mathcal{U}_T(t)v, \quad v \in X, \tag{5.10}$$

$$W_m(t)v := W_0 * B_\lambda W_{m-1}(t)v, \quad m \in \mathbb{N},\ v \in X. \tag{5.11}$$

Proof. We break the proof into several steps.

Consider the inductive construction (5.10)–(5.11) and write (5.11) explicitly as

$$W_m(t)v = \int_0^t W_0(t-\tau)B_\lambda W_{m-1}(\tau)v d\tau, \quad v \in X. \tag{5.12}$$

We define the functions

$$\chi(t) := \|(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda t}\mathcal{U}_T(t)\| \tag{5.13}$$

and

$$\psi(t) := \|B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda t}\mathcal{U}_T(t)\|. \tag{5.14}$$

By Proposition 4.1.4 the function $\chi(t)$ is measurable and $\psi(t)$ is measurable thanks to conditions (2) and (3) in Definition 5.2.1 and by Proposition 4.1.4. Thanks to Proposition 4.1.6, if $\omega > \omega_0$, where ω_0 is defined in (5.4), there exists an $M_\omega < \infty$ such that

$$\|\mathcal{U}_T(t)\| \leq M_\omega e^{\omega t}.$$

By assumption (5), we get

$$\begin{aligned}\chi(t) = \|(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda t}\mathcal{U}_T(t)\| &\leq \|(\overline{\lambda}\mathcal{I} - T - P)^{-1}\|\,|e^{-\lambda t}|\,\|\mathcal{U}_T(t)\| \\ &\leq \mathcal{K}_\lambda e^{-Re(\lambda t)}M_\omega e^{\omega t}.\end{aligned}$$

By Proposition 4.1.4 and Definition 5.2.1 (3) we have that

$$\int_0^\beta \psi(t)dt < \infty$$

for every $\beta > 0$.

Now put inductively

$$\psi^{(1)}(t) = \psi(t), \quad \psi^{(n)}(t) = (\psi^{(n-1)} * \psi)(t).$$

By Lemma 12.0.17 (c) we see inductively that all the functions $\psi^{(n)}(t)$ are Lebesgue integrable over every finite interval of the real positive axis. Set

$$\chi^{(0)}(t) = \chi(t), \quad \chi^{(n)}(t) = (\chi * \psi^{(n)})(t).$$

By Lemma 12.0.17 (c) the functions $\chi^{(n)}(t)$ are Lebesgue integrable over every finite interval contained in the real positive axis.

Step 1. We show that the inductive construction (5.10)-(5.11) is well defined in terms of function spaces. Indeed, for every $m \in \mathbb{N}$ and $v \in X$, we show that

(I) $W_m(t)v \in \mathcal{D}(B_\lambda)$,

(II) $W_m(t)v$ is continuous in t for $t > 0$,

(III) $\|W_m(t)\| \le \chi^{(m)}(t)$,

(IV) $B_\lambda W_m(t)v$ is continuous in t for $t > 0$, and

(V) $\|B_\lambda W_m(t)\| \le \psi^{(m+1)}(t)$.

For $m = 0$ conditions (I)–(V) follow from Lemmas 5.2.2 and 5.2.3. We now suppose that they hold for $m = k$. Then from (I), (IV) and (V), we have that the integral in (5.12) exists for all $t \in (0, \infty)$ and can be used to define $W_{k+1}(t)$. The properties (I), (II) and (IV), when $m = k + 1$, follow from Lemma 5.2.3. We observe that we have the estimates

$$\begin{aligned}\|W_{k+1}(t)v\| &= \|\int_0^t W_0(t-\tau)B_\lambda W_k(\tau)v d\tau\| \\ &\le \|v\| \int_0^t \chi(t-\tau)\psi^{(k+1)}(\tau)d\tau \\ &= \|v\|\chi^{(k+1)}(t)\end{aligned}$$

and by Lemma 5.2.3 we also have

$$\begin{aligned}\|B_\lambda W_{k+1}(t)v\| &= \|\int_0^t B_\lambda W_0(t-\tau)B_\lambda W_k(\tau)v d\tau\| \\ &\le \|v\| \int_0^t \psi(t-\tau)\psi^{(k+1)}(\tau)d\tau \\ &= \|v\|\psi^{(k+2)}(t)\end{aligned}$$

which proves (III) and (V) for the case $m = k+1$. Consequently, the five conditions (I)–(V) are proved inductively for all m.

Step 2. We estimate the series $\sum_{m=0}^{\infty} \|W_m(t)\|$.
Because of (III) we have

$$\sum_{m=0}^{\infty} \|W_m(t)\| \le \sum_{m=0}^{\infty} \chi^{(m)}(t).$$

By Lemma 5.2.2 (c) for every $\omega > \omega_0$, where ω_0 is defined in (5.4), there exists a constant $M_\omega < \infty$ such that

$$\psi(t) = \|B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1} e^{-\lambda t}\mathcal{U}_T(t)\| < M_\omega e^{\omega t}$$

for t sufficiently large. On the other hand, the function $\psi(t)$ is integrable over every finite interval of $\mathbb{R}^+$ (see Proposition 4.1.4). So if we choose ω_1 sufficiently large we have

$$\int_0^\infty e^{-\omega_1 t}\psi(t)dt < \infty.$$

From the Lebesgue Dominated Convergence Theorem we get, for $p \in \mathbb{R}^+$

$$\lim_{p\to\infty}\int_0^\infty e^{-pt}\psi(t)dt = \lim_{p\to\infty}\int_0^\infty e^{-(p-\omega_1)t}e^{-\omega_1 t}\psi(t)dt = 0$$

so that if $\omega > \omega_1$ is chosen sufficiently large, it is

$$\int_0^\infty e^{-\omega t}\psi(t)dt = \gamma < 1.$$

Since, using the notation $\mathrm{Re}(\lambda) = \lambda_0$,

$$\chi(t) = \chi^{(0)}(t) \le \mathcal{K}_\lambda M_\omega e^{-(\lambda_0-\omega)t},$$

we will now show by induction that

$$\chi^{(m)}(t) \le M_\omega e^{(\omega-\lambda_0)t}\gamma_1^m, \quad \text{for some } \gamma_1 < 1.$$

For $m = 0$ and ω sufficiently large so that $\omega - \lambda_0 > \omega_1$, it is

$$\begin{aligned}\chi^{(1)}(t) &= \int_0^t \chi^{(0)}(t-\tau)\psi(\tau)v d\tau\\ &\le \mathcal{K}_\lambda M_\omega e^{-(\lambda_0-\omega)t}\int_0^t e^{(\lambda_0-\omega)\tau}\psi(\tau)d\tau\\ &\le \mathcal{K}_\lambda M_\omega e^{(\omega-\lambda_0)t}\gamma_1\end{aligned}$$

for some $\gamma_1 < 1$. Assume that it holds for a given m, then, by Lemma 12.0.17 we have

$$\begin{aligned}\chi^{(m+1)}(t) &= \int_0^t \chi^{(m)}(t-\tau)\psi(\tau)v d\tau\\ &\le \mathcal{K}_\lambda M_\omega e^{-(\lambda_0-\omega)t}\gamma_1^m\int_0^t e^{-(\omega-\lambda_0)\tau}\psi(\tau)d\tau\\ &\le \mathcal{K}_\lambda M_\omega e^{(\omega-\lambda_0)t}\gamma_1^{m+1}\end{aligned}$$

for $t > 0$. Since

$$\begin{aligned}\chi^{(m)}(t) &= \int_0^t \chi(t-\tau)\psi^{(m)}(\tau)d\tau\\ &\le \int_0^t \mathcal{K}_\lambda M_\omega e^{-(\lambda_0-\omega)(t-\tau)}\psi^{(m)}(\tau)d\tau\\ &\le \mathcal{K}_\lambda M_\omega e^{-(\lambda_0-\omega)t}\int_0^t e^{(\lambda_0-\omega)\tau}\psi^{(m)}(\tau)d\tau,\end{aligned}$$

it is clear that $\chi^{(m)}(t) \to 0$ as $t \to 0$ for $m \ge 1$. Thus, since $\|W_m(t)\| \le \chi^{(m)}(t)$, it is also clear that $W_m(t) \to 0$ for $m \ge 1$. Recall that in the strong operator topology, we have $\lim_{t\to 0}\mathcal{U}_T(t) = \mathcal{I}$. Hence, if we put

$$W_0(0) = (\overline{\lambda}\mathcal{I} - T - P)^{-1}$$

and

$$W_m(0) = 0 \quad \text{for } m \geq 1,$$

then $W_m(t)v$ will be continuous in t for $t \geq 0$ and for every $v \in X$. Moreover, we will clearly have

$$\|W_m(t)\| \leq \mathcal{K}_\lambda M_\omega e^{-(\lambda_0-\omega)t}\gamma_1^m \quad \text{for } t \geq 0 \text{ and } m \geq 0.$$

So it follows that the series $\sum_{m=0}^{\infty} \|W_m(t)\|$ converges absolutely and uniformly in each finite interval $[a, b]$, and

$$\sum_{m=0}^{\infty} \|W_m(t)\| \leq (1-\gamma_1)^{-1}\mathcal{K}_\lambda M_\omega e^{(\omega-\lambda_0)t}.$$

Since each of the terms of the series (5.9),

$$W(t)v = \sum_{m=0}^{\infty} W_m(t)v, \quad \text{for } t \geq 0,$$

is strongly continuous for $t \geq 0$, $W(t)$ is also strongly continuous and, furthermore, we have the important estimate

$$\|W(t)\| \leq (1-\gamma_1)^{-1}\mathcal{K}_\lambda M_\omega e^{(\omega-\lambda_0)t}.$$

Step 3. To conclude the proof, we show that

$$\mathcal{Q}_\lambda^{-1}(T+P) = \int_0^\infty W(t)dt$$

because this implies that

$$S_R^{-1}(\lambda, T+P)v = (\overline{\lambda}\mathcal{I} - T - P)\int_0^\infty W(t)dtv, \quad v \in X. \tag{5.15}$$

Thanks to Proposition 5.1.3 and the fact that $(\overline{\lambda}\mathcal{I} - T - P)^{-1}$ is continuous for $\overline{\lambda} \notin \sigma_L(T)$, it is

$$\mathcal{Q}_\lambda^{-1}(T)v = \int_0^\infty (\overline{\lambda}\mathcal{I} - T - P)^{-1}\, e^{-t\lambda}\, \mathcal{U}_T(t)\, v\, dt, \quad v \in X. \tag{5.16}$$

Using the expansion of $\mathcal{Q}_\lambda^{-1}(T+P)$ in Proposition 5.1.2 we get

$$\mathcal{Q}_\lambda^{-1}(T+P) = \sum_{m=0}^{\infty} ((\mathcal{Q}_\lambda^{-1}(T)B_\lambda)^m \mathcal{Q}_\lambda^{-1}(T). \tag{5.17}$$

Let us reason on the second term in the expansion (5.17). Using Theorem 5.2.3, we can take B_λ, under the integral so

$$\mathcal{Q}_\lambda^{-1}(T)B_\lambda\mathcal{Q}_\lambda^{-1}(T) = \int_0^\infty (\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda t}\mathcal{U}_T(t)\,dt \\ \times \int_0^\infty B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda s}\mathcal{U}_T(s)\,ds$$

and also, for the Fubini theorem, we obtain

$$\mathcal{Q}_\lambda^{-1}(T)B_\lambda\mathcal{Q}_\lambda^{-1}(T) = \int_0^\infty dt\int_0^\infty ds(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda t}\mathcal{U}_T(t) \\ \times B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda s}\mathcal{U}_T(s),$$

so with a change of variable $t \to (t-s)$ we get

$$\mathcal{Q}_\lambda^{-1}(T)B_\lambda\mathcal{Q}_\lambda^{-1}(T) = \int_0^\infty dt\int_0^t (\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda(t-s)}(t-s) \\ \times B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1}e^{-\lambda s}\mathcal{U}_T(s)\,ds.$$

Using the functions introduced in Definition 5.3.1 we have

$$\mathcal{Q}_\lambda^{-1}(T) = \int_0^\infty W_0(t)\,dt$$

and

$$\mathcal{Q}_\lambda^{-1}(T)B_\lambda\mathcal{Q}_\lambda^{-1}(T) = \int_0^\infty W_0 * B_\lambda W_0(t)\,dt.$$

With these notations we get the series

$$\mathcal{Q}_\lambda^{-1}(T+P)v = \int_0^\infty W_0(t)\,dtv \\ + \int_0^\infty W_0 * B_\lambda W_0(t)\,dt \\ + \int_0^\infty W_0 * (B_\lambda W_0) * B_\lambda W_0)(t)\,dt + \dots .$$

We observe that $W_m(t)$, introduced in Step 1, is given by

$$W_m(t) = (W_0 * (B_\lambda W_0)^{*m})(t), \quad m = 1, 2, 3, \dots,$$

where the symbol $*m$ stands for m times the convolution of $B_\lambda W_0$ with itself. With the position

$$W(t) = \sum_{m=0}^\infty W_m(t),$$

we have

$$\mathcal{Q}_\lambda^{-1}(T+P) = \int_0^\infty W(t)dt = \sum_{m=0}^\infty \int_0^\infty W_m(t)dt, \qquad (5.18)$$

where we have used Step 2 for the uniform convergence of the series.

Now we prove (5.15) from (5.18). Take $v \in X$ and consider $(\overline{s}\mathcal{I} - T - P)\int_0^\infty W(t)dtv$. Since $\mathcal{D}(T) \subset \mathcal{D}(P)$, we have $\mathcal{D}(T+P) = \mathcal{D}(T)$ and $S_R^{-1}(\lambda, T)X = \mathcal{D}(T)$. The relation

$$((T+P)^2 - 2\lambda_0(T+P) + |\lambda|^2\mathcal{I})(\overline{\lambda}\mathcal{I} - T - P)^{-1}(\overline{\lambda}\mathcal{I} - T - P)\int_0^\infty W(t)dtv = v$$

for $\lambda \in \rho_S(T)$ and for $v \in X$, holds true because

$$((T+P)^2 - 2\lambda_0(T+P) + |\lambda|^2\mathcal{I})\int_0^\infty W(t)dtv = v, \quad v \in X$$

is a consequence of (5.18). Now consider $v \in \mathcal{D}(T)$ and

$$(\overline{\lambda}\mathcal{I} - T - P)\int_0^\infty W(t)dt\ ((T+P)^2 - 2\lambda_0(T+P) + |\lambda|^2\mathcal{I})(\overline{\lambda}\mathcal{I} - T - P)^{-1}v = v,$$

for $v \in \mathcal{D}(T)$. Since

$$\int_0^\infty W(t)dt\ ((T+P)^2 - 2\lambda_0(T+P) + |\lambda|^2\mathcal{I}) = \mathcal{I} : \mathcal{D}(T^2) \to \mathcal{D}(T),$$

because we have assumed that $\overline{\lambda} \in \rho_L(T+P)$, we have that

$$(\overline{\lambda}\mathcal{I} - T - P)(\overline{\lambda}\mathcal{I} - T - P)^{-1}v = v, \quad v \in \mathcal{D}(T).$$

So we have (5.15). □

5.4 Comparison with the complex setting

We recall the complex version of the generation result in order to compare it with the one in the quaternionic setting. Let X be a complex Banach space and let A be the (complex) infinitesimal generator of a strongly continuous semigroup $U_A(t)$.

Definition 5.4.1. We denote by $\mathbb{P}(A)$ the class of closed operators P that satisfies the conditions

(1) $\mathcal{D}(P) \supseteq \mathcal{D}(A)$.

(2) For every $t > 0$ there exists a positive constant $C(t)$ such that

$$\|PU_A(t)x\| \leq C(t)\|x\|, \quad \text{for } x \in \mathcal{D}(A).$$

(3) The constant $C(t)$ can be chosen such that $\int_0^1 C(t)dt$ exists and is finite.

Theorem 5.3.2 extends the following classical result to the quaternionic setting, see [110, p. 630]:

Theorem 5.4.2. *Let A be the infinitesimal generator of a strongly continuous semigroup $U_A(t)$ on X. If $P \in \mathbb{P}(A)$ then $A+P$ defined on $\mathcal{D}(A)$ is closed and is the infinitesimal generator of the semigroup $U_{A+P}(t)$. Moreover, an explicit construction of the semigroup $U_{A+P}(t)$ is given by*

$$U_{A+P}(t) = \sum_{n\geq 0} \mathcal{R}_n(t), \quad t \geq 0 \tag{5.19}$$

where

$$\mathcal{R}_0(t)x = U_A(t)x, \quad \mathcal{R}_n(t)\,x = (U_A * P\mathcal{R}_{n-1})(t)\,x, \quad x \in X,\ n = 1, 2, 3, \dots,$$

and

$$(U_A * P\mathcal{R}_{n-1})(t)\,x := \int_0^t U_A(t-s)P\mathcal{R}_{n-1}(s)\,x\,ds.$$

The series (5.19) converges uniformly for $t \in [0, \tau]$ where τ is a positive fixed real number. The function $t \to \mathcal{R}_n(t)\,x$, for fixed $n \in \mathbb{N}$ and $x \in X$, is continuous for $t \geq 0$.

For the ensuing comments, it is useful to write the first terms in the expansion of the semigroups in both the complex and the quaternionic case, which are

$$U_{A+P}(t) = U_A(t) + (U_A * PU_A)(t) + \dots$$

and

$$\mathcal{U}_{T+P}(t) = \mathcal{U}_T(t) + \mathcal{U}_T(t) * B_\lambda(\overline{\lambda}\mathcal{I} - T - P)^{-1} e^{-\lambda t}\mathcal{U}_T(t) + \dots,$$

respectively.

Remark 5.4.1. Note that, in the complex case, the expansion of $U_{A+P}(t)$ involves just the semigroup $U_A(t)$ and the perturbation operator P. This expansion is based on the fact that the classical resolvent operator

$$R(\lambda, A+P) := (\lambda I - A - P)^{-1}$$

for $A+P$, for $\|PR(\lambda, A)\| < 1$, is given by

$$R(\lambda, A+P) = R(\lambda, A) \sum_{n=0}^{\infty} (PR(\lambda, A))^n \tag{5.20}$$

and the main point of the matter is that the resolvent operator $R(\lambda, A)$ is the Laplace transform of $U_A(t)$.

Remark 5.4.2. In the quaternionic case, the expansion of (5.20) has to be replaced by the expansion of the pseudo-resolvent operator (5.2), namely

$$\mathcal{Q}_\lambda^{-1}(T+P) = \sum_{m=0}^{\infty} \mathcal{Q}_\lambda^{-1}(T)(B_\lambda \mathcal{Q}_\lambda^{-1}(T))^m,$$

where $B_\lambda := 2\lambda_0 P - P^2 - TP - PT$ and $\|B_\lambda \mathcal{Q}_\lambda^{-1}(T)\| < 1$. Thus, the S-resolvent operator $S_R^{-1}(\lambda, T+P)$ can be written as (see (5.3))

$$S_R^{-1}(\lambda, T+P)v = (\overline{\lambda}\mathcal{I} - T - P)\sum_{m=0}^{\infty} \mathcal{Q}_\lambda^{-1}(T)(B_\lambda \mathcal{Q}_\lambda^{-1}(T))^m v, \quad v \in X.$$

Note that the relation between $S_R^{-1}(\lambda, T+P)$ and the Laplace transform of the quaternionic evolution operator $\mathcal{U}_T(t)$, see Remark 5.1.3, involves also the left resolvent operator, in fact

$$\mathcal{Q}_\lambda^{-1}(T)v = (\overline{\lambda}\mathcal{I} - T)^{-1} \int_0^\infty e^{-t\lambda}\, \mathcal{U}_T(t)\, v\; dt, \quad v \in X.$$

Thus, in the quaternionic setting, two spectral problems are involved.

Remark 5.4.3. We point out that one can also use the consistency of quaternionic spectral theory with complex spectral theory in order to develop a different approach to the perturbation theory of generators of strongly continuous semi groups. If T is the quaternionic infinitesimal generator of the quaternionic semigroup $\mathcal{U}_T(T)$, then we can choose $j \in \mathbb{S}$ and consider T as a $\mathbb{C}_j$-linear operator. Then T is the infinitesimal generator of the complex semigroup obtained from considering $\mathcal{U}_T(t)$ as a $\mathbb{C}_j$-linear operator for each $t \geq 0$.

Now let P be a quaternionic linear operator that satisfies conditions analogue to those required in the complex case in Definition 5.4.1, that is,

(i) $\mathcal{D}(T) \subset \mathcal{D}(P)$.

(ii) For each $t > 0$, there exists $C(t)$ such that $\|P\mathcal{U}_T(t)y\| \leq C(t)\|y\|$ for all $y \in \mathcal{D}(T)$.

(iii) The constants $C(t)$ can be chosen such that $\int_0^1 C(t)\, dt$ exists and is finite.

If we consider P also as a $\mathbb{C}_j$-linear operator, then Theorem 5.4.2 implies that $T+P$ is the generator of a strongly continuous semigroup $\mathcal{U}_{T+P}(t)$ of $\mathbb{C}_j$-complex linear operators. However, since T and P are quaternionic linear, the operator $T+P$ is quaternionic linear and $\mathcal{U}_{T+P}(t)$ consists of quaternionic linear operators. (To show this, we can use quaternionic linearity since it survives the Yosida approxiation procedure.) Therefore $T+P$ generates a strongly continuous quaternionic semigroup under the above assumptions, which is furthermore given by the series (5.19). This approach does not, however, allow to obtain the central result of this chapter, namely the series expansion (5.9) that contains the quaternionic parameter λ.

Remark 5.4.4. We finally observe that when λ is a real number the expansion (5.8) becomes

$$\mathcal{U}_{T+P}(t) = \mathcal{U}_T(t) + \mathcal{U}_T(t) * B_\lambda(\lambda\mathcal{I} - T - P)^{-1}\mathcal{U}_T(t) + \dots$$

where B_λ is defined in (5.1).

5.5 An application

As an application, we study a quaternionic differential equation in the space of quaternionic-valued continuous functions. Consider $Y : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{H}$ and the Cauchy problem

$$\frac{\partial}{\partial t}Y(t,x) = \frac{\partial^2}{\partial x^2}Y(t,x) + h(x)\frac{\partial}{\partial x}Y(t,x),$$

$$\lim_{t\to 0^+} Y(t,x) = y_0(x), \quad \text{uniformly in } x \in \mathbb{R},$$

where

$$y_0(x) = y_0(x) + y_1(x)e_1 + y_2(x)e_2 + y_3(x)e_3 : \mathbb{R} \to \mathbb{H}$$

and

$$h = h_0(x) + h_1(x)e_1 + h_2(x)e_2 + h_3(x)e_3 : \mathbb{R} \to \mathbb{H}$$

are given functions. We now need some general facts on the group of translations that can be obtained in the quaternionic setting by adapting the arguments in [110, p. 629], with obvious modifications. The group of translations defined by

$$U_A(t)Z(\tau) = Z(t+\tau)$$

is a strongly continuous group on $X = C(\overline{\mathbb{R}}, \mathbb{H})$ where $\overline{\mathbb{R}} = [-\infty, +\infty]$ and its infinitesimal generator is $A = \frac{d}{d\tau}$ with domain

$$\mathcal{D}(A) = \{Z \in C(\overline{\mathbb{R}}, \mathbb{H}) : \quad Z' \in C(\overline{\mathbb{R}}, \mathbb{H})\}.$$

To determine the S-resolvent set of A we observe that, for $Z \in C(\overline{\mathbb{R}}, \mathbb{H})$, the quaternionic differential equation

$$\lambda Z(\tau) - Z'(\tau) = X(\tau)$$

must have a unique solution in $Z \in C(\overline{\mathbb{R}}, \mathbb{H})$. But $\lambda \in \mathbb{H}$ and A commute with λ, since A does not contain any imaginary units, so the S-resolvent operator reduces to $(\lambda\mathcal{I} - A)^{-1}$. The linear quaternionic differential equation for Z reduces to a linear system of differential equations for the components of Z, so it follows that the S-spectrum is given by

$$\sigma_S(A) = \{uj : \quad u \in \mathbb{R},\ j \in \mathbb{S}\}.$$

Consider now the operator $A^2 = \frac{\partial^2}{\partial \tau^2}$ with domain

$$\mathcal{D}(A^2) = \{Z \in C(\overline{\mathbb{R}}, \mathbb{H}) : \quad Z', Z'' \in C(\overline{\mathbb{R}}, \mathbb{H})\}.$$

With similar considerations as in Theorem XII9.7 in [110] we have that operator A^2 is closed and $\mathcal{D}(A^2)$ is dense in $C(\overline{\mathbb{R}}, \mathbb{H})$. Since

$$\sigma_S(A) = \{uj : \quad u \in \mathbb{R}, \ for \ j \in \mathbb{S}\},$$

by the spectral mapping theorem it follows that

$$\sigma_S(A^2) = \{u \in \mathbb{R} : \quad u < 0\}.$$

Since A^2 commutes with the quaternion λ, it is

$$\begin{aligned} S_R^{-1}(\lambda, A^2) Z(\tau) &= (\lambda \mathcal{I} - A^2)^{-1} Z(\tau) \\ &= \int_{\mathbb{R}} \frac{e^{-|\theta|\sqrt{\lambda}}}{2\sqrt{\lambda}} Z(\theta + \tau) d\theta, \quad \tau \in \mathbb{R}. \end{aligned}$$

With computations similar to those in [110, p. 640], we have an explicit formula for the evolution operator:

$$U_{A^2}(t) Z(\tau) = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} e^{-\theta^2/4t} Z(\theta + \tau) d\theta, \quad t > 0. \tag{5.21}$$

Let us consider $h \in C(\overline{\mathbb{R}}, \mathbb{H})$ and the operator P whose domain is

$$\mathcal{D}(P) = \Big\{ y \in C(\overline{\mathbb{R}}, \mathbb{H}) \text{ such that } y' \text{ is continuous in a neighborhood of each point } \tau_0 \text{ for which } h(\tau_0) \neq 0 \text{ and such that } hy' \in C(\overline{\mathbb{R}}, \mathbb{H}) \Big\}$$

and it is defined by

$$(Py)(\tau) = h(\tau) y'(\tau), \quad y \in \mathcal{D}(P).$$

The operator P is closed and with some computations we have the estimate

$$\begin{aligned} \|P U_{A^2}(t) Z(\tau)\| &\leq \|h\| \| \frac{\partial}{\partial \tau} \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} e^{-\theta^2/4t} Z(\theta + \tau) d\theta \| \\ &\leq \frac{\|h\| \|Z\|}{\sqrt{\pi t}}. \end{aligned}$$

The above estimate shows that the conditions (2) and (3) in Definition 5.2.1 are fulfilled since they are the conditions (2) and (3) in Definition 5.4.1. This is due to the fact that, in this case, A^2 commute with the quaternions. In view of theorem of generation by perturbation, the solution of the Cauchy problem is

$$Y(t, x) = U_{A^2 + P}(t) y_0(x).$$

We conclude this chapter pointing out that the above result, which has been obtained by adapting the scalar case in [110], shows that this generation theorem is useful also in quaternionic quantum mechanics, see [4, p. 38], since the quaternionic version of Schrödinger equation is of the form

$$\frac{\partial}{\partial t}\psi(t,x) = -H(x)\psi(t,x)$$

where the Hamiltonian is given by

$$H(x) = H_0(x) + e_1 H_1(x) + e_2 H_2(x) + e_3 H_3(x)$$

and ψ is the quaternionic wave function. Even if it is nontrivial in concrete applications of physical interest, one may consider the operator $e_1 H_1(x) + e_2 H_2(x) + e_3 H_3(x)$ as a perturbation of $H_0(x)$.