Chapter 3

The direct approach to the S-functional calculus

The S-functional calculus can also be defined for unbounded operators $T : \mathcal{D}(T) \subset X \to X$, where X is a two-sided quaternionic Banach space X. In the papers [69,101] this calculus was defined using suitable transformations in order to reduce the problem to the case of bounded operators. The direct approach has been studied in the more recent paper [130] and it turned out that the two approaches, contrary to the complex setting, are not totally equivalent. In fact, in using the direct approach one can remove the assumption that the S-resolvent set contains a real point.

Definition 3.0.1. Let X be a two-sided quaternionic Banach space. A right linear operator $T : \mathcal{D}(T) \subset X \to X$ defined on a right-linear subspace $\mathcal{D}(T)$ of X is called closed if its graph is closed in $X \oplus X$. We denote the set of closed right linear operators $T : \mathcal{D}(T) \subset X \to X$ by $\mathcal{K}(X)$.

Remark 3.0.1. The notion of a closed right linear operator can also be considered on a right Banach space and does not necessarily require the existence of a left multiplication on X. However, for the reasons explained in Remark 2.2.7, one usually works on two-sided Banach spaces.

When we deal with closed operators, we have to pay attention to the domains on which they are defined. The powers of T are defined inductively as $T^0 = \mathcal{I}$ with $\mathcal{D}(T^0) = \mathcal{D}(\mathcal{I}) = X$ and $T^{n+1}v = T(T^n v)$ for $v \in \mathcal{D}(T^{n+1}) := \{v \in \mathcal{D}(T) : T^n \in \mathcal{D}(T)\}$. Polynomials of T with real coefficients are then defined as usual: if $P(s) = \sum_{\ell=0}^n a_\ell s^\ell$ with $a_\ell \in \mathbb{R}$, then $P(T)v = \sum_{\ell=0}^n a_\ell T^\ell v$ for $v \in \mathcal{D}(T^n)$. However, if the coefficients are not real, then we have to distinguish two cases: for a right slice hyperholomorphic polynomial $P(s) = \sum_{\ell=0}^n a_\ell s^\ell$ with $a_\ell \in \mathbb{H}$, we can again set $P(T)v = \sum_{\ell=0}^n a_\ell T^\ell v$ for $v \in \mathcal{D}(T^n)$. For a left slice hyperholomorphic polynomial $P(s) = \sum_{\ell=0}^n s^\ell a_\ell$ with $a_\ell \in \mathbb{H}$ it is, however, not always possible

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to set $P(T)v = \sum_{\ell=0}^{n} T^n a_\ell v$ for $v \in \mathcal{D}(T^n)$. Indeed, since T is right linear, the domain of $\mathcal{D}(T^\ell)$ is a right-linear but not necessarily a left-linear subspace of X. Hence, it might happen that $a_\ell v \notin \mathcal{D}(T^\ell)$ even though $v \in \mathcal{D}(T^n)$, so setting $P(T)v = \sum_{\ell=0}^{n} T^n a_\ell v$ is not meaningful.

3.1 Properties of the S-spectrum of a closed operator

For $T \in \mathcal{K}(X)$, we define

$$\mathcal{Q}_s(T) := T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I}, \quad \text{for } s \in \mathbb{H},$$

and the operator $\mathcal{Q}_s(T)$ is defined on $\mathcal{D}(T^2)$.

Definition 3.1.1. Let $T \in \mathcal{K}(X)$. We define the S-resolvent set of T as

$$\rho_S(T) := \left\{ s \in \mathbb{H} : \quad \mathcal{Q}_s(T)^{-1} \in \mathcal{B}(X) \right\}$$

and the S-spectrum of T as

$$\sigma_S(T) := \mathbb{H} \setminus \rho_S(T).$$

For $s \in \rho_S(T)$, the operator $\mathcal{Q}_s(T)^{-1}$ is called the pseudo-resolvent of T at s. Furthermore, we define the extended S-spectrum $\sigma_{SX}(T)$ as

$$\sigma_{SX}(T) := \begin{cases} \sigma_S(T) & \text{if } T \text{ is bounded,} \\ \sigma_S(T) \cup \{\infty\} & \text{if } T \text{ is unbounded.} \end{cases}$$

Before we study the properties of the S-spectrum of a closed operator, we need to investigate the differentiability properties of its pseudo-resolvent in detail. The correct tool for studying these properties is a series expansion of $\mathcal{Q}_s(T)^{-1}$, which was found in [52]. An heuristic approach for finding this expansion consists in considering the equation

$$\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_q(T)^{-1} = \mathcal{Q}_s(T)^{-1}(\mathcal{Q}_q(T) - \mathcal{Q}_s(T))\mathcal{Q}_q(T)^{-1} \tag{3.1}$$

and writing it as

$$\mathcal{Q}_s(T)^{-1} = \mathcal{Q}_q(T)^{-1} + \mathcal{Q}_s(T)^{-1}(\mathcal{Q}_q(T) - \mathcal{Q}_s(T))\mathcal{Q}_q(T)^{-1}.$$

Recursive application of this equation then yields the series expansion proved in the following, where we consider closed axially symmetric neighbourhoods, described by the function $d_S(s,q) = \max\left\{2|s_0 - q_0|, \left||q|^2 - |s|^2\right|\right\}$, which naturally rise from the series expansion of the pseudo-resolvent operator.

Theorem 3.1.2. *Let $T \in \mathcal{K}(X)$ and $q \in \rho_S(T)$ and let $s \in \mathbb{H}$. If the series*

$$\mathcal{J}(s) = \sum_{n=0}^{+\infty} \left(\mathcal{Q}_q(T) - \mathcal{Q}_s(T)\right)^n \mathcal{Q}_q(T)^{-(n+1)} \tag{3.2}$$

converges absolutely in $\mathcal{B}(X)$*, then* $s \in \rho_S(T)$ *and it equals the pseudo-resolvent* $\mathcal{Q}_s(T)^{-1}$ *of* T *at* s.

The series converges in particular uniformly on any of the closed axially symmetric neighbourhoods

$$C_\varepsilon(q) = \{s \in \mathbb{H} : \quad d_S(s,q) \leq \varepsilon\}$$

of q *with*

$$d_S(s,q) = \max\left\{2|s_0 - q_0|, \left||q|^2 - |s|^2\right|\right\}$$

and

$$\varepsilon < \frac{1}{\|T\mathcal{Q}_q(T)^{-1}\| + \|\mathcal{Q}_q(T)^{-1}\|}.$$

Proof. Let us first consider the question of the convergence of the series. The sets $C_\varepsilon(q)$ are obviously axially symmetric: if s_j belongs to the sphere $[s]$ associated to s, then $s_0 = \operatorname{Re}(s) = \operatorname{Re}(s_j)$ and $|s|^2 = |s_j|^2$. Thus, $d_S(s_j, q) = d_S(s,q)$ and in turn $s \in C_\varepsilon(q)$ if and only if $s_j \in C_\varepsilon(q)$. Moreover, since the map $s \mapsto d_S(s,q)$ is continuous, the sets $U_\varepsilon(q) := \{s \in \mathbb{H} : d_S(s,q) < \varepsilon\}$ are open in $\mathbb{H}$. Since $U_\varepsilon(q) \subset C_\varepsilon(q)$, the sets C_ε are actually neighbourhoods of q. In order to simplify the notation, we set

$$\Lambda(q,s) := \mathcal{Q}_q(T) - \mathcal{Q}_s(T) = 2(s_0 - q_0)T + (|q|^2 - |s|^2)\mathcal{I}.$$

Since $\mathcal{Q}_q(T)^{-1}$ maps X to $\mathcal{D}(T^2)$ and $\Lambda(q,s)$ commutes with $\mathcal{Q}_q(T)^{-1}$ on $\mathcal{D}(T^2)$, we have for any $s \in C_\varepsilon(q)$,

$$\begin{aligned}
&\sum_{n=0}^{+\infty} \left\|\Lambda(q,s)^n \mathcal{Q}_q(T)^{-(n+1)}\right\| \\
&= \sum_{n=0}^{+\infty} \left\|\left(\Lambda(q,s)\mathcal{Q}_q(T)^{-1}\right)^n \mathcal{Q}_q(T)^{-1}\right\| \\
&\leq \sum_{n=0}^{+\infty} \left\|\Lambda(q,s)\mathcal{Q}_q(T)^{-1}\right\|^n \left\|\mathcal{Q}_q(T)^{-1}\right\|.
\end{aligned}$$

We further have

$$\begin{aligned}
\left\|\Lambda(q,s)\mathcal{Q}_q(T)^{-1}\right\| &\leq 2|s_0 - q_0| \left\|T\mathcal{Q}_q(T)^{-1}\right\| + \left||q|^2 - |s|^2\right| \left\|\mathcal{Q}_q(T)^{-1}\right\| \\
&\leq d_S(s,q)\left(\left\|T\mathcal{Q}_q(T)^{-1}\right\| + \left\|\mathcal{Q}_q(T)^{-1}\right\|\right) \\
&\leq \varepsilon\left(\left\|T\mathcal{Q}_q(T)^{-1}\right\| + \left\|\mathcal{Q}_q(T)^{-1}\right\|\right) =: \varrho.
\end{aligned}$$

If now $\varepsilon < 1/\left(\left\|T\mathcal{Q}_q(T)^{-1}\right\| + \left\|\mathcal{Q}_q(T)^{-1}\right\|\right)$, then $0 < \varrho < 1$ and thus,

$$\sum_{n=0}^{+\infty} \left\|\Lambda(q,s)^n \mathcal{Q}_q(T)^{-(n+1)}\right\| \leq \left\|\mathcal{Q}_q(T)^{-1}\right\| \sum_{n=0}^{+\infty} \varrho^n < +\infty$$

and the series converges uniformly in $\mathcal{B}(X)$ on $C_\varepsilon(q)$.

Now assume that the series (3.2) converges and observe that $\mathcal{Q}_s(T)$, $\mathcal{Q}_q(T)$ and $\mathcal{Q}_q(T)^{-1}$ commute on $\mathcal{D}(T^2)$. Hence, we have for $y \in \mathcal{D}(T^2)$ that

$$\begin{aligned}
\mathcal{J}(s)\mathcal{Q}_s(T)y &= \sum_{n=0}^{+\infty} \Lambda(q,s)^n \mathcal{Q}_q(T)^{-(n+1)} \mathcal{Q}_s(T)y \\
&= \sum_{n=0}^{+\infty} \Lambda(q,s)^n \mathcal{Q}_q(T)^{-(n+1)} \left[-\Lambda(q,s) + \mathcal{Q}_q(T)\right] y \\
&= -\sum_{n=0}^{+\infty} \Lambda(q,s)^{n+1} \mathcal{Q}_q(T)^{-(n+1)} y \\
&\quad + \sum_{n=0}^{+\infty} \Lambda(q,s)^n \mathcal{Q}_q(T)^{-n} y = y.
\end{aligned}$$

On the other hand,

$$y_N := \sum_{n=0}^{N} \Lambda(q,s)^n \mathcal{Q}_q(T)^{-(n+1)} y = \mathcal{Q}_q(T)^{-1} \sum_{n=0}^{N} \Lambda(q,s)^n \mathcal{Q}_q(T)^{-n} y$$

belongs to $\mathcal{D}(T^2)$ for any $y \in X$ and we have

$$\begin{aligned}
\mathcal{Q}_s(T)y_N &= (-\Lambda(q,s) + \mathcal{Q}_q(T)) \sum_{n=0}^{N} \Lambda(q,s)^n \mathcal{Q}_q(T)^{-(n+1)} y \\
&= -\sum_{n=0}^{N} \Lambda(q,s)^{n+1} \mathcal{Q}_q(T)^{-(n+1)} y + \sum_{n=0}^{N} \Lambda(q,s)^n \mathcal{Q}_q(T)^{-n} y \\
&= -\Lambda(q,s)^{N+1} \mathcal{Q}_p(T)^{-(n+1)} y + y.
\end{aligned}$$

Now observe that

$$\Lambda(q,s) = 2(s_0 - q_0)T + (|q|^2 - |s|^2)\mathcal{I}$$

is defined on $\mathcal{D}(T)$ and maps $\mathcal{D}(T^2)$ to $\mathcal{D}(T)$. Hence, $\Lambda(q,s)^2 \mathcal{Q}_q(T)^{-1}$ belongs to $\mathcal{B}(X)$ and for $N \geq 1$

$$\begin{aligned}
&\left\| -\Lambda(q,s)^{N+1} \mathcal{Q}_p(T)^{-(n+1)} y \right\| \\
&= \left\| -\Lambda(q,s)^{N-1} \mathcal{Q}_p(T)^{-N} \Lambda(q,s)^2 \mathcal{Q}_q(T)^{-1} y \right\| \\
&\leq \left\| -\Lambda(q,s)^{N-1} \mathcal{Q}_p(T)^{-N} \right\| \left\| \Lambda(q,s)^2 \mathcal{Q}_q(T)^{-1} y \right\| \overset{N\to\infty}{\longrightarrow} 0
\end{aligned}$$

because the series (3.2) converges in the norm of $\mathcal{B}(X)$ by assumption. Thus, $\mathcal{Q}_s(T)y_N \to y$ and $y_N \to y_\infty := \mathcal{J}(s)y$ as $N \to \infty$. Since $\mathcal{Q}_s(T)$ is closed, we obtain that

$$\mathcal{J}(s)y \in \mathcal{D}(\mathcal{Q}_s(T)) = \mathcal{D}(T^2) \quad \text{and} \quad \mathcal{Q}_s(T)\mathcal{J}(s)y = y.$$

Hence, $\mathcal{J}(s) = \mathcal{Q}_s(T)^{-1}$ and in turn $s \in \rho_S(T)$. □

Lemma 3.1.3. *Let $T \in \mathcal{K}(X)$. The functions $s \to \mathcal{Q}_s(T)^{-1}$ and $s \to T\mathcal{Q}_s(T)^{-1}$, which are defined on $\rho_S(T)$ and take values in $\mathcal{B}(X)$, are continuous.*

Proof. Let $q \in \rho_S(T)$. Then $\mathcal{Q}_s(T)^{-1}$ can be represented by the series (3.2), which converges uniformly on a neighborhood of q. Hence, we have

$$\begin{aligned}\lim_{s\to q} \mathcal{Q}_s(T)^{-1} &= \sum_{n=0}^{+\infty} \lim_{s\to q} \left(2(s_0 - q_0)T + \left(|q|^2 - |s|^2\right)\mathcal{I}\right)^n \mathcal{Q}_q(T)^{-(n+1)} \\ &= \mathcal{Q}_q(T)^{-1},\end{aligned}$$

because each term in the sum is a polynomial in s_0 and s_1 (since $s = s_0 + js_1$ for $j \in \mathbb{S}$) with coefficients in $\mathcal{B}(X)$ and thus, continuous. Indeed

$$\begin{aligned}&\left((s_0 - q_0)T + \left(|q|^2 - |s|^2\right)\mathcal{I}\right)^n \mathcal{Q}_q(T)^{-(n+1)} \\ &= \sum_{k=0}^{n} \binom{n}{k} (s_0 - q_0)^k \left(|q|^2 - |s|^2\right)^{n-k} T^k \mathcal{Q}_q(T)^{-(n+1)}\end{aligned}$$

and the coefficients $T^k\mathcal{Q}_q(T)^{-(n+1)}$ belong to $\mathcal{B}(X)$ because $\mathcal{Q}_q(T)^{-(n+1)}$ maps X to $\mathcal{D}(T^{2(n+1)})$ and $k < 2(n+1)$. The function $s \mapsto T\mathcal{Q}_s(T)^{-1}$ is continuous because the identity (3.1) implies

$$\begin{aligned}&\lim_{h\to 0} \left\|T\mathcal{Q}_{s+h}(T)^{-1} - T\mathcal{Q}_s(T)^{-1}\right\| \\ &= \lim_{h\to 0} \left\|T\mathcal{Q}_{s+h}(T)^{-1}(\mathcal{Q}_s(T) - \mathcal{Q}_{s+h}(T))\mathcal{Q}_s(T)^{-1}\right\|.\end{aligned}$$

The operator $\mathcal{Q}_s(T)^{-1}$ maps X to $\mathcal{D}(T^2)$ and so

$$(\mathcal{Q}_s(T) - \mathcal{Q}_{s+h}(T))\mathcal{Q}_s(T)^{-1} = (2h_0T + (|s|^2 - |s+h|^2)\mathcal{I})\mathcal{Q}_s(T)^{-1}$$

maps $X$to $\mathcal{D}(T)$. Since T and $\mathcal{Q}_{s+h}(T)^{-1}$ commute on $\mathcal{D}(T)$ we thus have

$$\begin{aligned}&\lim_{h\to 0} \left\|T\mathcal{Q}_{s+h}(T)^{-1} - T\mathcal{Q}_s(T)^{-1}\right\| \\ &= \lim_{h\to 0} \left\|\mathcal{Q}_{s+h}(T)^{-1}\left(2h_0T^2 + \left(|s|^2 - |s+h|^2\right)T\right)\mathcal{Q}_s(T)^{-1}\right\| \\ &\leq \lim_{h\to 0} \left\|\mathcal{Q}_{s+h}(T)^{-1}\right\| \lim_{h\to 0} 2h_0 \left\|T^2\mathcal{Q}_s(T)^{-1}\right\| \\ &\quad + \lim_{h\to 0} \left\|\mathcal{Q}_{s+h}(T)^{-1}\right\| \lim_{h\to 0} \left(|s|^2 - |s+h|^2\right) \left\|T\mathcal{Q}_s(T)^{-1}\right\| = 0.\end{aligned}$$

□

Lemma 3.1.4. *Let $T \in \mathcal{K}(X)$ and $s \in \rho_S(T)$. The pseudo-resolvent $\mathcal{Q}_s(T)^{-1}$ is continuously real differentiable with*

$$\frac{\partial}{\partial s_0}\mathcal{Q}_s(T)^{-1} = (2T - 2s_0\mathcal{I})\mathcal{Q}_s(T)^{-2} \quad \text{and} \quad \frac{\partial}{\partial s_1}\mathcal{Q}_s(T)^{-1} = -2s_1\mathcal{Q}_s(T)^{-2}.$$

Proof. Let us first compute the partial derivative of $\mathcal{Q}_s(T)^{-1}$ with respect to the real part s_0. Applying equation (3.1), we have

$$\begin{aligned}\frac{\partial}{\partial s_0}\mathcal{Q}_s(T)^{-1} &= \lim_{\mathbb{R}\ni h\to 0}\frac{1}{h}\left(\mathcal{Q}_{s+h}(T)^{-1} - \mathcal{Q}_s(T)^{-1}\right)\\ &= \lim_{\mathbb{R}\ni h\to 0}\frac{1}{h}\mathcal{Q}_{s+h}(T)^{-1}\left(\mathcal{Q}_s(T) - \mathcal{Q}_{s+h}(T)\right)\mathcal{Q}_s(T)^{-1}\\ &= \lim_{\mathbb{R}\ni h\to 0}\mathcal{Q}_{s+h}(T)^{-1}\left(2T - 2s_0\mathcal{I} - h\mathcal{I}\right)\mathcal{Q}_s(T)^{-1},\end{aligned}$$

where $\lim_{\mathbb{R}\ni h\to 0} f(h)$ denotes the limit of a function f as h tends to 0 in $\mathbb{R}$. Since the composition and the multiplication with scalars are continuous operations on $\mathcal{B}(X)$, we further have

$$\begin{aligned}\frac{\partial}{\partial s_0}\mathcal{Q}_s(T)^{-1} &= \lim_{\mathbb{R}\ni h\to 0}\mathcal{Q}_{s+h}(T)^{-1}\lim_{\mathbb{R}\ni h\to 0}\left((2T - 2s_0\mathcal{I})\mathcal{Q}_s(T)^{-1} - h\mathcal{Q}_s(T)^{-1}\right)\\ &= \mathcal{Q}_s(T)^{-1}(2T - 2s_0\mathcal{I})\mathcal{Q}_s(T)^{-1}\\ &= (2T - 2s_0\mathcal{I})\mathcal{Q}_s(T)^{-2},\end{aligned}$$

where the last equation holds true because $\mathcal{Q}_s(T)^{-1}$ maps X to $\mathcal{D}(T^2) \subset \mathcal{D}(T)$ and T and $\mathcal{Q}_s(T)^{-1}$ commute on $\mathcal{D}(T)$. Observe that $\frac{\partial}{\partial s_0}\mathcal{Q}_s(T)^{-1}$ is even continuous because it is the sum and product of continuous functions by Lemma 3.1.3.

If we write $s = s_0 + j_s s_1$, then we can argue in a similar way to show that the derivative of $\mathcal{Q}_s(T)^{-1}$ with respect to s_1 is

$$\begin{aligned}\frac{\partial}{\partial s_1}\mathcal{Q}_s(T)^{-1} &= \lim_{\mathbb{R}\ni h\to 0}\frac{1}{h}\left(\mathcal{Q}_{s+hj_s}(T)^{-1} - \mathcal{Q}_s(T)^{-1}\right)\\ &= \lim_{\mathbb{R}\ni h\to 0}\frac{1}{h}\mathcal{Q}_{s+hj_s}(T)^{-1}\left(\mathcal{Q}_s(T) - \mathcal{Q}_{s+hj_s}(T)\right)\mathcal{Q}_s(T)^{-1}\\ &= \lim_{\mathbb{R}\ni h\to 0}\mathcal{Q}_{s+hj_s}(T)^{-1}\left(-2s_1 - h\right)\mathcal{Q}_s(T)^{-1}\\ &= \lim_{\mathbb{R}\ni h\to 0}\mathcal{Q}_{s+hj_s}(T)^{-1}\lim_{\mathbb{R}\ni h\to 0}\left(-2s_1\mathcal{Q}_s(T)^{-1} - h\mathcal{Q}_s(T)^{-1}\right)\\ &= -2s_1\mathcal{Q}_s(T)^{-2}.\end{aligned}$$

Again this derivative is continuous as it is the product of two continuous functions by Lemma 3.1.3.

Finally, we easily obtain that $\mathcal{Q}_s(T)^{-1}$ is continuously real differentiable from the fact that $\mathcal{Q}_s(T)^{-1}$ is continuously differentiable in the variables s_0 and s_1. If we write s in terms of its four real coordinates as $s = \xi_0 + \sum_{\ell=1}^{3}\xi_\ell e_\ell$, then the partial derivative with respect to ξ_0 corresponds to the partial derivative with respect to s_0 and thus, exists and is continuous. The partial derivative with respect to ξ_ℓ for $1 \leq \ell \leq 3$ on the other hand exists and is continuous for $s_1 \neq 0$ because $\mathcal{Q}_s(T)^{-1}$ can be considered as the composition of the continuously differentiable functions $s \mapsto s_1 = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$ and $s_1 \to \mathcal{Q}_{s+js_1}(T)^{-1}$ with fixed $j \in \mathbb{S}$. We find

$$\frac{\partial}{\partial \xi_\ell} \mathcal{Q}_s(T)^{-1} = -2s_1 \mathcal{Q}_s(T)^{-2} \frac{\partial}{\partial \xi_\ell} s_0 = -2\xi_\ell \mathcal{Q}_s(T)^{-2}.$$

For $s_1 = 0$ (that is for $s \in \mathbb{R}$), we can simply choose $j = e_\ell$ and then the partial derivative with respect to ξ_ℓ agrees with the partial derivative with respect to s_1. In particular, we see that also the partial derivatives with respect to the real coordinates $\xi_0, \ldots, \xi_3$ are continuous. □

Lemma 3.1.5. *Let $T \in \mathcal{K}(X)$ and $s \in \rho_S(T)$. The function $s \mapsto T\mathcal{Q}_s(T)^{-1}$ is continuously real differentiable with*

$$\frac{\partial}{\partial s_0} T\mathcal{Q}_s(T)^{-1} = (2T^2 - 2s_0T)\mathcal{Q}_s(T)^{-2}$$

and

$$\frac{\partial}{\partial s_1} T\mathcal{Q}_s(T)^{-1} = -2s_1 T\mathcal{Q}_s(T)^{-2}.$$

Proof. If $\lim_{\mathbb{R}\ni h\to 0} f(h)$ denotes again the limit of a function f as h tends to 0 in $\mathbb{R}$, then we obtain from (3.1) that

$$\begin{aligned}\frac{\partial}{\partial s_0} T\mathcal{Q}_s(T)^{-1} &= \lim_{\mathbb{R}\ni h\to 0} \frac{1}{h}\left(T\mathcal{Q}_{s+h}(T)^{-1} - T\mathcal{Q}_s(T)^{-1}\right)\\ &= \lim_{\mathbb{R}\ni h\to 0} \frac{1}{h} T\mathcal{Q}_{s+h}(T)^{-1}\left(\mathcal{Q}_s(T) - \mathcal{Q}_{s+h}(T)\right)\mathcal{Q}_s(T)^{-1}\\ &= \lim_{\mathbb{R}\ni h\to 0} \frac{1}{h} T\mathcal{Q}_{s+h}(T)^{-1}\left(2hT - 2hs_0\mathcal{I} - h^2\mathcal{I}\right)\mathcal{Q}_s(T)^{-1}\\ &= \lim_{\mathbb{R}\ni h\to 0} \mathcal{Q}_{s+h}(T)^{-1}\left(2T^2 - 2s_0T - hT\right)\mathcal{Q}_s(T)^{-1},\end{aligned}$$

because $\left(2hT - 2hs_0\mathcal{I} - h^2\mathcal{I}\right)\mathcal{Q}_s(T)^{-1}$ maps X to $\mathcal{D}(T)$ and T and $\mathcal{Q}_{s+h}(T)^{-1}$ commute on $\mathcal{D}(T)$. Since the composition and the multiplication with scalars are continuous operations on the space $\mathcal{B}(X)$ and since the pseudo-resolvent is continuous by Lemma 3.1.3, we get

$$\begin{aligned}\frac{\partial}{\partial s_0} T\mathcal{Q}_s(T)^{-1} &= \lim_{\mathbb{R}\ni h\to 0} \mathcal{Q}_{s+h}(T)^{-1} \lim_{\mathbb{R}\ni h\to 0}\left(\left(2T^2 - 2s_0T\right)\mathcal{Q}_s(T)^{-1} - hT\mathcal{Q}_s(T)^{-1}\right)\\ &= \mathcal{Q}_s(T)^{-1}(2T^2 - 2s_0T)\mathcal{Q}_s(T)^{-1}\\ &= (2s_0T - 2T^2)\mathcal{Q}_s(T)^{-2}.\end{aligned}$$

This function is continuous because we can write it as the product of functions that are continuous by Lemma 3.1.3.

The derivative with respect to s_1 can be computed using similar arguments via

$$
\begin{aligned}
\frac{\partial}{\partial s_1} T\mathcal{Q}_s(T)^{-1} &= \lim_{\mathbb{R}\ni h\to 0} \frac{1}{h}\left(T\mathcal{Q}_{s+hj_s}(T)^{-1} - T\mathcal{Q}_s(T)^{-1}\right)\\
&= \lim_{\mathbb{R}\ni h\to 0} \frac{1}{h} T\mathcal{Q}_{s+hj_s}(T)^{-1}\left(\mathcal{Q}_s(T) - \mathcal{Q}_{s+hj_s}(T)\right)\mathcal{Q}_s(T)^{-1}\\
&= \lim_{\mathbb{R}\ni h\to 0} \frac{1}{h} T\mathcal{Q}_{s+hj_s}(T)^{-1}\left(-2hs_1 - h^2\right)\mathcal{Q}_s(T)^{-1}\\
&= \lim_{\mathbb{R}\ni h\to 0} \mathcal{Q}_{s+hj_s}(T)^{-1} \lim_{\mathbb{R}\ni h\to 0}\left(-2s_1 T\mathcal{Q}_s(T)^{-1} - hT\mathcal{Q}_s(T)^{-1}\right)\\
&= -2s_1 T\mathcal{Q}_s(T)^{-2}.
\end{aligned}
$$

Also this derivative is continuous because

$$\frac{\partial}{\partial s_1} T\mathcal{Q}_s(T)^{-1} = -2s_1\left(T\mathcal{Q}_s(T)^{-1}\right) Q_s(T)^{-1}$$

is the product of functions that are continuous by Lemma 3.1.3.

Finally, we see as in the proof of Lemma 3.1.4 that $T\mathcal{Q}_s(T)^{-1}$ is continuously differentiable in the four real coordinates by considering it as the composition of the two continuously real differentiable functions $s \mapsto (s_0, s_1)$ and $(s_0, s_1) \mapsto T\mathcal{Q}_{s+js_1}(T)^{-1}$ choosing j_s appropriately if $s \in \mathbb{R}$. □

Let us return now to studying the S-spectrum of T. As we show in the next theorem, it has properties that are analogue to the properties of the usual spectrum of a complex linear operator.

Theorem 3.1.6. *Let $T \in \mathcal{K}(X)$.*

(i) *The S-spectrum $\sigma_S(T)$ of T is axially symmetric. It contains the set of right eigenvalues $\sigma_R(T)$ of T and if X has finite dimension, then it equals $\sigma_R(T)$.*

(ii) *The S-spectrum $\sigma_S(T)$ is a closed subset of $\mathbb{H}$ and the extended S-spectrum $\sigma_{SX}(T)$ is a closed and compact subset of $\mathbb{H}_\infty := \mathbb{H} \cup \{\infty\}$.*

(iii) *If T is bounded, then $\sigma_S(T)$ is nonempty and bounded by the norm of T.*

Proof. We have $q \in [s]$ if and only if $\mathrm{Re}(q) = \mathrm{Re}(s)$ and $|q| = |s|$. In this case

$$\mathcal{Q}_s(T) = T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I} = T^2 - 2\mathrm{Re}(q) + |q|^2\mathcal{I} = \mathcal{Q}_q(T)$$

and $s \in \rho_S(T)$ if and only if $q \in \rho_S(T)$. Hence, $\rho_S(T)$ and $\sigma_S(T)$ are both axially symmetric. Furthermore,

$$\mathcal{Q}_s(T)v = T(Tv - vs) - (Tv - vs)\overline{s}. \tag{3.3}$$

If $s \in \sigma_R(T)$, then there exists a right eigenvector $v \in X \setminus \{0\}$ associated with s, that is $Tv - vs = 0$, and hence $\mathcal{Q}_s(T)v = 0$ because of (3.3). Therefore $\mathcal{Q}_s(T)v$ is not invertible and so $s \in \sigma_S(T)$.

If furthermore X is finite-dimensional, then $\mathcal{Q}_s(T)$ is invertible if and only if $\ker \mathcal{Q}_s(T) \neq \{0\}$. Hence, if $s \in \sigma_S(T)$, then there exists $v \in X \setminus \{0\}$ with $\mathcal{Q}_s(T)v = 0$. If $Tv = vs$, then we already see that $s \in \sigma_R(T)$. Otherwise, we see from (3.3) that $\tilde{v} := Tv - vs \neq 0$ is a right eigenvector of T associated with $\overline{s}$. If $s = u + jv$ with $j \in \mathbb{S}$, we can choose $i \in \mathbb{S}$ with $i \perp j$. Then $ji = -ij$ and in turn $si = i\overline{s}$ so that

$$T(\tilde{v}i) = T(\tilde{v})i = (\tilde{v}\overline{s})i = (\tilde{v}i)s.$$

Hence, $\tilde{v}$ is a right eigenvector of T associated with s and so $s \in \sigma_R(T)$. Thus, (i) holds true.

If $s \in \rho_S(T)$, then Theorem 3.1.2 shows that there exists an axially symmetric neighborhood of s that also belongs to $\rho_S(T)$. Hence, $\rho_S(T)$ is an open subset of $\mathbb{H}$ and $\sigma_S(T) = \mathbb{H} \setminus \rho_S(T)$ is in turn a closed subset of $\mathbb{H}$. If $\sigma_S(T)$ is bounded in $\mathbb{H}$, then it is also closed in $\mathbb{H}_\infty$. Hence, if T is bounded, then $\sigma_{SX}(T) = \sigma_S(T)$ is closed in $\mathbb{H}_\infty$. Similarly, if T is unbounded and $\sigma_S(T)$ is bounded, then $\sigma_{SX}(T) = \sigma_S(T) \cup \{\infty\}$ is the union of two closed subsets of $\mathbb{H}_\infty$ and hence bounded itself. Finally, if $\sigma_S(T)$ is unbounded, then T must be unbounded and we find that $\sigma_{SX}(T)$ is closed as

$$\sigma_{SX}(T) = \sigma_S(T) \cup \{\infty\} = \overline{\sigma_S(T)}^{\mathbb{H}_\infty}.$$

Hence, (ii) holds true. Finally, (iii) is part of the statement of Theorem 2.2.11. □

Definition 3.1.7. Let $T \in \mathcal{K}(X)$.

(i) We call $s \in \mathbb{R}$ an S-eigenvalue of T if $(T - s\mathcal{I})x = 0$ for some $x \in X \setminus \{0\}$.

(ii) Let $s \in \mathbb{H} \setminus \mathbb{R}$. We call $[s]$ an eigensphere of T if $\mathcal{Q}_s(T)x = 0$ for some $x \in X \setminus \{0\}$.

In both cases, the respective vector x is called an S-eigenvector associated with the S-eigenvalue s (resp. the eigensphere $[s]$).

The next theorem clarifies the relation between the S-spectrum and the classical spectrum known from the theory of complex linear operators. The quaternionic Banach space X also carries, for any $j \in \mathbb{S}$, the structure of a Banach space over the complex field $\mathbb{C}_j$. We only have to restrict the multiplication of vectors with quaternionic scalars from the right to the complex plane $\mathbb{C}_j$ and obtain a complex Banach space over $\mathbb{C}_j$. We denote this $\mathbb{C}_j$-complex Banach space by X_j. (Observe that $\mathbb{C}_j$-complex multiples of the identity $\mathcal{I}_{X_j}$ on X_j act as $(\lambda\mathcal{I}_{X_j})y = y\lambda$ for $\lambda \in \mathbb{C}_j$ and $y \in X_j$.) Any quaternionic right linear operator T on X is then also a $\mathbb{C}_j$-linear operator on X_j. We denote the resolvent set and the spectrum of T as a complex linear operator on X_j by $\rho_{\mathbb{C}_j}(T)$ and $\sigma_{\mathbb{C}_j}(T)$.

Theorem 3.1.8. *Let $T \in \mathcal{K}(X)$ and choose $j \in \mathbb{S}$. The spectrum $\sigma_{\mathbb{C}_j}(T)$ of T considered as a closed complex linear operator on X_j equals $\sigma_S(T) \cap \mathbb{C}_j$, i.e.,*

$$\sigma_{\mathbb{C}_j}(T) = \sigma_S(T) \cap \mathbb{C}_j. \tag{3.4}$$

For any λ in the resolvent set $\rho_{\mathbb{C}_j}(T)$ of T as a complex linear operator on X_j, the $\mathbb{C}_j$-linear resolvent of T is given by $R_\lambda(T) = \left(\overline{\lambda}\mathcal{I}_{X_j} - T\right)\mathcal{Q}_\lambda(T)^{-1}$, i.e.,

$$R_\lambda(T)y := \mathcal{Q}_\lambda(T)^{-1}y\overline{\lambda} - T\mathcal{Q}_\lambda(T)^{-1}y. \tag{3.5}$$

For any $i \in \mathbb{S}$ with $j \perp i$, we, moreover, have

$$R_{\overline{\lambda}}(T)y = -[R_\lambda(T)(yi)]i. \tag{3.6}$$

Finally, if $s = u + iv \in \rho_S(T)$, we can set $s_j = u + jv$ and find

$$Q_s(T)^{-1} = R_{s_j}(T)R_{\overline{s_j}}(T). \tag{3.7}$$

Proof. Let $\lambda \in \rho_S(T)\cap\mathbb{C}_j$. The resolvent $(\lambda\mathcal{I}_{X_j} - T)^{-1}$ of T as a $\mathbb{C}_j$-linear operator on X_j is then given by (3.5). Indeed, since T and $Q_\lambda(T)^{-1}$ commute, we have for $y \in \mathcal{D}(T)$ that

$$\begin{aligned}
&R_\lambda(T)(\lambda\mathcal{I}_{X_j} - T)y\\
&= (\overline{\lambda}\mathcal{I}_{X_j} - T)\mathcal{Q}_\lambda(T)^{-1}(y\lambda - Ty)\\
&= (\overline{\lambda}\mathcal{I}_{X_j} - T)\left(\mathcal{Q}_\lambda(T)^{-1}y\lambda - T\mathcal{Q}_\lambda(T)^{-1}y\right)\\
&= \mathcal{Q}_\lambda(T)^{-1}y\lambda\overline{\lambda} - T\mathcal{Q}_\lambda(T)^{-1}y\overline{\lambda} - T\mathcal{Q}_\lambda(T)^{-1}y\lambda + T^2\mathcal{Q}_\lambda(T)^{-1}y\\
&= (|\lambda|^2\mathcal{I}_{X_j} - 2\lambda_0T + T^2)\mathcal{Q}_\lambda(T)^{-1}y = y.
\end{aligned}$$

Similarly, for $y \in X_j$, we have

$$\begin{aligned}
&(\lambda\mathcal{I}_{X_j} - T)R_\lambda(T)y\\
&= (\lambda\mathcal{I}_{X_j} - T)\left(\mathcal{Q}_\lambda(T)^{-1}y\overline{\lambda} - T\mathcal{Q}_\lambda(T)^{-1}y\right)\\
&= \mathcal{Q}_\lambda(T)^{-1}y\overline{\lambda}\lambda - T\mathcal{Q}_\lambda(T)^{-1}y\lambda - T\mathcal{Q}_\lambda(T)^{-1}y\overline{\lambda} + T^2\mathcal{Q}_\lambda(T)^{-1}y\\
&= (|\lambda|^2\mathcal{I}_{X_j} - 2\lambda_0T + T^2)\mathcal{Q}_\lambda(T)^{-1}y = y.
\end{aligned}$$

Since $\mathcal{Q}_\lambda(T)^{-1}$ maps X_j to $\mathcal{D}(T^2) \subset \mathcal{D}(T)$, we find that the operator $R_\lambda(T) = (\lambda\mathcal{I}_{X_j} - T)\mathcal{Q}_\lambda(T)^{-1}$ is bounded and so λ belongs to the resolvent set $\rho_{\mathbb{C}_j}(T)$ of T considered as a $\mathbb{C}_j$-linear operator on X_j. Hence, $\rho_S(T)\cap\mathbb{C}_j \subset \rho_{\mathbb{C}_j}(T)$ and in turn $\sigma_{\mathbb{C}_j}(T) \subset \sigma_S(T)\cap\mathbb{C}_j$. Together with the axial symmetry of the S-spectrum, this further implies

$$\sigma_{\mathbb{C}_j}(T) \cup \overline{\sigma_{\mathbb{C}_j}(T)} \subset (\sigma_S(T)\cap\mathbb{C}_j) \cup (\overline{\sigma_S(T)\cap\mathbb{C}_j}) = \sigma_S(T)\cap\mathbb{C}_j, \tag{3.8}$$

where $\overline{A} = \{\overline{z} : z \in A\}$.

If λ and $\overline{\lambda}$ both belong to $\rho_{\mathbb{C}_j}(T)$, then $[\lambda] \subset \rho_S(T)$ because

$$\begin{aligned}
&(\lambda\mathcal{I}_{X_j} - T)(\overline{\lambda}\mathcal{I}_{X_j} - T)y\\
&= (y\overline{\lambda})\lambda - (Ty)\lambda - T(y\overline{\lambda}) + T^2y\\
&= (T^2 - 2\lambda_0T + |\lambda|^2)y
\end{aligned}$$

and hence $\mathcal{Q}_\lambda(T)^{-1} = R_\lambda(T)R_{\overline{\lambda}}(T) \in \mathcal{B}(X)$. Thus, $\rho_S(T) \cap \mathbb{C}_j \supset \rho_{\mathbb{C}_j}(T) \cap \overline{\rho_{\mathbb{C}_j}(T)}$ and in turn

$$\sigma_S(T) \cap \mathbb{C}_j \subset \sigma_{\mathbb{C}_j}(T) \cup \overline{\sigma_{\mathbb{C}_j}(T)}. \tag{3.9}$$

The two relations (3.8) and (3.9) yield together

$$\sigma_S(T) \cap \mathbb{C}_j = \sigma_{\mathbb{C}_j}(T) \cup \overline{\sigma_{\mathbb{C}_j}(T)}. \tag{3.10}$$

What remains to show is that $\rho_{\mathbb{C}_j}(T)$ and $\sigma_{\mathbb{C}_j}(T)$ are symmetric with respect to the real axis, which then implies

$$\sigma_S(T) \cap \mathbb{C}_j = \sigma_{\mathbb{C}_j}(T) \cup \overline{\sigma_{\mathbb{C}_j}(T)} = \sigma_{\mathbb{C}_j}(T). \tag{3.11}$$

Let $\lambda \in \rho_{\mathbb{C}_j}(T)$ and choose $i \in \mathbb{S}$ with $j \perp i$. We show that $R_{\overline{\lambda}}(T)$ equals the mapping $Ay := -\left[R_\lambda(T)(yi)\right] i$. As $\lambda i = i\overline{\lambda}$ and $i\lambda = \overline{\lambda} i$, we have for $y \in \mathcal{D}(T)$ that

$$\begin{aligned}
A\left(\overline{\lambda}\mathcal{I}_{X_j} - T\right) y &= A\left(y\overline{\lambda} - Ty\right)\\
&= -\left[R_\lambda(T)\left((y\overline{\lambda})i - (Ty)i\right)\right] i\\
&= -\left[R_\lambda(T)((yi)\lambda - T(yi))\right] i\\
&= -\left[R_\lambda(T)(\lambda\mathcal{I}_{X_j} - T)(yi)\right] i = -yii = y.
\end{aligned}$$

Similarly, for arbitrary $y \in X_j = X$, we have

$$\begin{aligned}
\left(\overline{\lambda}\mathcal{I}_{X_j} - T\right) Ay &= (Ay)\,\overline{\lambda} - T\,(Ay)\\
&= -\left[R_\lambda(T)(yi)\right] i\overline{\lambda} + T\left(\left[R_\lambda(T)(yi)\right] i\right)\\
&= -\left[R_\lambda(T)(yi)\lambda - T(R_\lambda(T)(yi))\right] i\\
&= -\left[(\lambda\mathcal{I}_{X_j} - T)R_\lambda(T)(yi)\right] i = -yii = y.
\end{aligned}$$

Hence, if $\lambda \in \rho_{\mathbb{C}_j}(T)$, then $R_{\overline{\lambda}}(T) = -\left[R_\lambda(T)(yi)\right] i$ such that in particular $\overline{\lambda} \in \rho_{\mathbb{C}_j}(T)$. Consequently $\rho_{\mathbb{C}_j}(T)$ and in turn also $\sigma_{\mathbb{C}_j}(T)$ are symmetric with respect to the real axis such that (3.11) holds true. □

Remark 3.1.1. The relations (3.10) and (3.7) had been observed in [159]. Also the relation $R_\lambda(T)R_{\overline{\lambda}}(T) = \mathcal{Q}_\lambda(T)^{-1}$, which is a consequence of (3.5), was understood in that paper. The complete statement, in particular the fact that for a quaternionic linear operator T always $\sigma_{\mathbb{C}_j}(T) = \overline{\sigma_{\mathbb{C}_j}(T)}$ due to (3.6), was finally established in [131]. For unitary operators, this symmetry was already understood in [196], but the correct notion of spectrum for quaternionic operators had not yet been developed so it was impossible to see the full picture.

3.2 The S-resolvent of a closed operator

For closed operators, the definition of the S-resolvent operators needs a little modification. If we define the left S-resolvent operator as in the case of bounded operators, we obtain

$$S_L^{-1}(s,T)x := -\mathcal{Q}_s(T)^{-1}(T-\overline{s}\mathcal{I})x, \tag{3.12}$$

which is only defined for $x \in \mathcal{D}(T)$ and not on all of X. However, for $x \in \mathcal{D}(T)$, we have $\mathcal{Q}_s(T)^{-1}Tx = T\mathcal{Q}_s(T)^{-1}x$ and so we can commute T and $\mathcal{Q}_s(T)^{-1}$ in order to obtain an operator that is defined on all of X.

Definition 3.2.1 (The S-resolvent operators of a closed operator)**.** Let $T \in \mathcal{K}(X)$. For $s \in \rho_S(T)$, we define the left S-resolvent operator of T at s as

$$S_L^{-1}(s,T)x := \mathcal{Q}_s(T)^{-1}\overline{s}x - T\mathcal{Q}_s(T)^{-1}x, \quad \text{for all } x \in X, \tag{3.13}$$

and the right S-resolvent operator of T at s as

$$S_R^{-1}(s,T)x := -(T-\mathcal{I}\overline{s})\mathcal{Q}_s(T)^{-1}x, \quad \text{for all } x \in X. \tag{3.14}$$

Remark 3.2.1. For $s \in \rho_S(T)$, the operator $\mathcal{Q}_s(T)^{-1}$ maps X to $\mathcal{D}(T^2)$. Hence, $T\mathcal{Q}_s(T)^{-1}$ is a bounded operator and so $S_L^{-1}(s,T)$ and $S_R^{-1}(s,T)$ are bounded, too. The converse, however, is not necessarily true. As the next example shows, there might exist points $s \in \mathbb{H}$ that belong to $\sigma_S(T)$ even though $S_L^{-1}(s,T)$ or $S_R^{-1}(s,T)$ are bounded operators. In order to determine the S-spectrum of an operator T one therefore always has to work with the operator $\mathcal{Q}_s(T)^{-1}$ even though, as we will see later on, the S-resolvent $S_L^{-1}(s,T)$ and $S_R^{-1}(s,T)$ and not the pseudo-resolvent $\mathcal{Q}_s(T)^{-1}$ appear in the S-functional calculus.

If a sphere $[s] = u + \mathbb{S}v$ belongs to $\sigma_S(T)$, then the S-resolvents can be bounded at most at one point in $[s]$. We will prove in the following that the right S-resolvent is left slice hyperholomorphic in s. If $S_R^{-1}(s_k,T)$ with $s_k = u+kv$ and $S_R^{-1}(s_i,T)$ with $s_i = u+iv$ are bounded, then (2.11) in Corollary 2.1.8 implies for any $s_j = u+jv$ with $j \in \mathbb{S}\setminus\{i,k\}$ that

$$\begin{aligned}\left\|S_R^{-1}(s_j,T)\right\| \leq {} & \left|(i-k)^{-1}i + j(k-i)^{-1}\right| \left\|S_R^{-1}(s_i,T)\right\| \\ & + \left|(k-i)^{-1}k + j(k-i)^{-1}\right| \left\|S_R^{-1}(s_k,T)\right\| < +\infty.\end{aligned}$$

Hence, if $S_R^{-1}(s,T)$ is bounded at two points in $[s]$, then it is bounded at any $s \in [s]$. The estimates that we will show in Lemma 3.2.8 imply then that $[s] \subset \rho_S(T)$. For the left S-resolvent, we can argue similarly.

Example 3.2.2. Let $\ell^2(\mathbb{H})$ be the quaternionic Hilbert space of all square-summable sequences in $\mathbb{H}$ and let $i \in \mathbb{S}$. On this space, we consider the operator

$$T : \begin{cases} \ell^2(\mathbb{H}) & \to & \ell^2(\mathbb{H}) \\ (a_n)_{n\in\mathbb{N}} & \mapsto & \left(\frac{n-1}{n} i a_n\right)_{n\in\mathbb{N}} . \end{cases}$$

This operator is obviously bounded with $\|T\| = 1$ and if $e_n = (\delta_{n,m})_{m,\in\mathbb{N}}$, where $\delta_{n,m} = 1$ if $m = n$ and $\delta_{n,m} = 0$ if $m \neq n$, then $Te_n = e_n\frac{n-1}{n}i$. Hence, we conclude from Theorem 3.1.6 that

$$\sigma_S(T) \supset \overline{\bigcup_{n\in\mathbb{N}} \frac{n-1}{n}\mathbb{S}} = \mathbb{S} \cup \bigcup_{n\in\mathbb{N}} \frac{n-1}{n}\mathbb{S}. \tag{3.15}$$

Straightforward computations show, that we even have equality in (3.15) since $\mathcal{Q}_s(T)^{-1}$ is bounded for any $s \notin \mathbb{S} \cup \bigcup_{n\in\mathbb{N}} \frac{n-1}{n}\mathbb{S}$.

Let us now consider the point $-j$, which obviously belongs to $\sigma_S(T)$. The pseudo-resolvent of T at $-j$ applied to $(a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{H})$ is

$$\mathcal{Q}_{-j}(T)(a_n)_{n\in\mathbb{N}} = (T^2 + \mathcal{I})^{-1}(a_n)_{n\in\mathbb{N}} = \left(\frac{n^2}{2n-1}a_n\right)_{n\in\mathbb{N}}.$$

As expected, this is an unbounded operator on $\ell^2(\mathbb{H})$, because $\frac{n^2}{2n-1} \to +\infty$. The left S-resolvent at $-i$ on the other hand is

$$S_L^{-1}(-i,T)(a_n)_{n\in\mathbb{N}} = \mathcal{Q}_{-i}(T)^{-1}(-i\mathcal{I} - T)(a_n)_{n\in\mathbb{N}} = \left(\frac{n}{2n-1}ia_n\right)_{n\in\mathbb{N}},$$

and this is a bounded operator because

$$\|S_L^{-1}(-i,T)\| = \sup_{n\in\mathbb{N}} \left|\frac{n}{2n-1}i\right| < +\infty.$$

Hence, $S_L^{-1}(-i,T)$ is bounded even though $-i \notin \rho_S(T)$.

A second difference between the left and the right S-resolvent operators is that the right S-resolvent equation only holds true on $\mathcal{D}(T)$.

Theorem 3.2.3 (The S-resolvent equations)**.** *Let $T \in \mathcal{K}(X)$. For $s \in \rho_S(T)$, the left S-resolvent operator satisfies the identity*

$$S_L^{-1}(s,T)sx - TS_L^{-1}(s,T)x = x, \quad \textit{for all } x \in X. \tag{3.16}$$

Moreover, the right S-resolvent operator satisfies the identity

$$sS_R^{-1}(s,T)x - S_R^{-1}(s,T)Tx = x, \quad \textit{for all } x \in \mathcal{D}(T). \tag{3.17}$$

Proof. We have for $x \in \mathcal{D}(T)$ that

$$\begin{aligned} &sS_R^{-1}(s,T)x - S_R^{-1}(s,T)Tx \\ &= -s(T - \mathcal{I}\bar{s})\mathcal{Q}_s(T)^{-1}x + (T - \mathcal{I}\bar{s})\mathcal{Q}_s(T)^{-1}Tx \\ &= (-sT + |s|^2\mathcal{I})\mathcal{Q}_s(T)^{-1}x + (T^2 - \bar{s}T)\mathcal{Q}_s(T)^{-1}x \\ &= (T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})\mathcal{Q}_s(T)^{-1}x = x. \end{aligned}$$

Similar computations show (3.16). □

Remark 3.2.2. We can extend (3.17) to an equation that holds on the entire space X, similarly to how we could extend (3.12) to a bounded operator on the entire space X. This equation is

$$sS_R^{-1}(s,T)x + (T^2 - \bar{s}T)\mathcal{Q}_s(T)^{-1}x = x, \quad \text{for all } x \in X.$$

Theorem 3.2.4 (S-resolvent equation)**.** *Let $T \in \mathcal{K}(X)$. If $s, q \in \rho_S(T)$ with $s \notin [q]$, then*

$$S_R^{-1}(s,T)S_L^{-1}(q,T) = \big[[S_R^{-1}(s,T) - S_L^{-1}(q,T)]q - \overline{s}[S_R^{-1}(s,T) - S_L^{-1}(q,T)]\big](q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}. \quad (3.18)$$

Proof. As in the case of bounded operators, the S-resolvent equation is deduced from the left and the right S-resolvent equation. However, we have to pay attention to being consistent with the domains of definition of every operator that appears in the following. We show that, for every $x \in X$, one has

$$S_R^{-1}(s,T)S_L^{-1}(q,T)(q^2 - 2s_0q + |s|^2)x = [S_R^{-1}(s,T) - S_L^{-1}(q,T)]qx - \overline{s}[S_R^{-1}(s,T) - S_L^{-1}(q,T)]x. \quad (3.19)$$

We then obtain (3.18) by replacing x by $(q^2 - 2s_0q + |s|^2)^{-1}x$. For $w \in X$, the left S-resolvent equation (3.16) implies

$$S_R^{-1}(s,T)S_L^{-1}(q,T)qw = S_R^{-1}(s,T)TS_L^{-1}(q,T)w + S_R^{-1}(s,T)w.$$

The pseudo-resolvent $\mathcal{Q}_s(T)^{-1}$ maps X onto $\mathcal{D}(T^2)$. Therefore the left S-resolvent operator $S_L^{-1}(s,T) = \mathcal{Q}_s(T)^{-1}\overline{s} - T\mathcal{Q}_s(T)^{-1}$ maps X to $\mathcal{D}(T)$ and so $S_L^{-1}(q,T)w \in \mathcal{D}(T)$. The right S-resolvent equation (3.17) yields

$$\begin{aligned} &S_R^{-1}(s,T)S_L^{-1}(q,T)qw \\ &= sS_R^{-1}(s,T)S_L^{-1}(q,T)w - S_L^{-1}(q,T)w + S_R^{-1}(s,T)w. \end{aligned} \quad (3.20)$$

If we apply this identity with $w = qx$ we get

$$\begin{aligned} &S_R^{-1}(s,T)S_L^{-1}(q,T)(q^2 - 2s_0q + |s|^2)x \\ &= S_R^{-1}(s,T)S_L^{-1}(q,T)q^2x - 2s_0S_R^{-1}(s,T)S_L^{-1}(q,T)qx \\ &\quad + |s|^2S_R^{-1}(s,T)S_L^{-1}(q,T)x \\ &= sS_R^{-1}(s,T)S_L^{-1}(q,T)qx - S_L^{-1}(q,T)qx + S_R^{-1}(s,T)qx \\ &\quad - 2s_0S_R^{-1}(s,T)S_L^{-1}(q,T)qx + |s|^2S_R^{-1}(s,T)S_L^{-1}(q,T)x. \end{aligned}$$

Applying identity (3.20) again with $w = x$ gives

$$\begin{aligned} &S_R^{-1}(s,T)S_L^{-1}(q,T)(q^2 - 2s_0q + |s|^2)x \\ &= s^2S_R^{-1}(s,T)S_L^{-1}(q,T)x - sS_L^{-1}(q,T)x + sS_R^{-1}(s,T)x \\ &\quad - S_L^{-1}(q,T)qx + S_R^{-1}(s,T)qx \\ &\quad - 2s_0sS_R^{-1}(s,T)S_L^{-1}(q,T)x + 2s_0S_L^{-1}(q,T)x - 2s_0S_R^{-1}(s,T)x \\ &\quad + |s|^2S_R^{-1}(s,T)S_L^{-1}(q,T)x \\ &= (s^2 - 2s_0s + |s|^2)S_R^{-1}(s,T)S_L^{-1}(q,T)x \\ &\quad - (2s_0 - s)[S_R^{-1}(s,T)x - S_L^{-1}(q,T)x] \\ &\quad + [S_R^{-1}(s,T) - S_L^{-1}(q,T)]qx. \end{aligned}$$

The identity $2s_0 = s + \overline{s}$ implies $s^2 - 2s_0 s + |s|^2 = 0$ and $2s_0 - s = \overline{s}$ and hence we obtain the desired equation (3.19). □

We want to show now the slice hyperholomorphicity of the S-resolvent operators of a closed quaternionic operator. The fact that they are differentiable follows from the series expansion of the pseudo-resolvent that was found in Theorem 3.1.2.

Lemma 3.2.5. *Let $T \in \mathcal{K}(X)$ and $s \in \rho_S(T)$. The left and the right S-resolvents of T are continuously real differentiable.*

Proof. The S-resolvents are sums of functions that are continuously real differentiable by Lemma 3.1.4 and Lemma 3.1.5 and hence continuously real differentiable themselves. □

Theorem 3.2.6. *Let $T \in \mathcal{K}(X)$. The left S-resolvent $S_L^{-1}(s,T)$ is right slice hyperholomorphic and the right S-resolvent $S_R^{-1}(s,T)$ is left slice hyperholomorphic in the variable s.*

Proof. We consider only the case of the left S-resolvent, the other one works with analogous arguments. We have

$$S_L^{-1}(s,T) = \alpha(s_0,s_1) + j_s\beta(s_0,s_1)$$

with

$$\alpha(s_0,s_1) = \mathcal{Q}_s(T)^{-1}s_0 - T\mathcal{Q}_s(T) \quad \text{and} \quad \beta(s_0,s_1) = -\mathcal{Q}_s(T)^{-1}s_1.$$

Obviously α and β satisfy the compatibility condition (2.4) and hence $S_L^{-1}(s,T)$ is a right slice function in s.

Applying Lemma 3.1.4 and Lemma 3.1.5, we have

$$\begin{aligned}
\frac{\partial}{\partial s_0} S_L^{-1}(s,T) &= \frac{\partial}{\partial s_0}\mathcal{Q}_s(T)^{-1}\overline{s} - \frac{\partial}{\partial s_0} T\mathcal{Q}_s(T)^{-1} \\
&= (2T - 2s_0\mathcal{I})\mathcal{Q}_s(T)^{-2}\overline{s} + \mathcal{Q}_s(T)^{-1} - \left(2T^2 - 2s_0T\right)\mathcal{Q}_s(T)^{-2} \\
&= (2T - 2s_0\mathcal{I})\mathcal{Q}_s(T)^{-2}\overline{s} + \left(-T^2 + |s|^2\mathcal{I}\right)\mathcal{Q}_s(T)^{-2}.
\end{aligned}$$

Since s_0 and $|s|^2$ are real, they commute with $\mathcal{Q}_s(T)^{-2}$. If we apply the identities $2s_0 = s + \overline{s}$ and $|s|^2 = s\overline{s}$, we obtain

$$\frac{\partial}{\partial s_0} S_L^{-1}(s,T) = -T^2\mathcal{Q}_s(T)^{-2} + 2T\mathcal{Q}_s(T)^{-2}\overline{s} - \mathcal{Q}_s(T)^{-2}\overline{s}^2.$$

For the partial derivative with respect to s_1, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial s_1} S_L^{-1}(s,T) = \frac{\partial}{\partial s_1}\mathcal{Q}_s(T)^{-1}\overline{s} - \frac{\partial}{\partial s_1} T\mathcal{Q}_s(T)^{-1} \\
&= -2s_1\mathcal{Q}_s(T)^{-2}\overline{s} - \mathcal{Q}_s(T)^{-1}j_s + 2s_1T\mathcal{Q}_s(T)^{-2} \\
&= -2s_1\mathcal{Q}_s(T)^{-2}\overline{s} - (T^2 - 2s_0T + |s|^2\mathcal{I})\mathcal{Q}_s(T)^{-2}j_s + 2s_1T\mathcal{Q}_s(T)^{-2}.
\end{aligned}$$

We can again commute $2s_0$, $2s_1$ and $|s|^2$ with $\mathcal{Q}_s(T)^{-1}$ because they are real. By exploiting the identities $2s_0 = s+\overline{s}$, $-2s_1 = (s-\overline{s})j_s$ and $|s|^2 = s\overline{s}$, we obtain the formula

$$\frac{\partial}{\partial s_1} S_L^{-1}(s,T) = \left(-T^2\mathcal{Q}_s(T)^{-2} + 2T\mathcal{Q}_s(T)^{-2}\overline{s} - \mathcal{Q}_s(T)^{-2}(T)\overline{s}^2\right) j_s.$$

So the function $s \mapsto S_L^{-1}(s,T)$ is right slice hyperholomorphic as

$$\frac{1}{2}\left(\frac{\partial}{\partial s_0} S_L^{-1}(s,T) + \frac{\partial}{\partial s_1} S_L^{-1}(s,T) j_s\right) = 0. \qquad \square$$

In Section 8.3 we will need the fact that the S-resolvent set is the maximal domain of slice hyperholomorphicity of the S-resolvent operators such that they do not have a slice hyperholomorphic continuation. In the complex case this is guaranteed by the well-known estimate

$$\|R(z,A)\| \geq \frac{1}{\mathrm{dist}(z,\sigma(A))}, \tag{3.21}$$

where $R(z,A)$ denotes the resolvent operator and $\sigma(A)$ the spectrum of the complex linear operator A. This estimate assures that $\|R(z,A)\| \to +\infty$ as z approaches $\sigma(A)$ and in turn that the resolvent does not have any holomorphic continuation to a larger domain, see [177, 191].

In the quaternionic setting, an estimate similar to (3.21) cannot hold true. We can for example consider the operator $T = \lambda\mathcal{I}$ on a two-sided Banach space X for some $\lambda = \lambda_0 + j_\lambda\lambda_1$ with $\lambda_1 > 0$. Its S-spectrum $\sigma_S(T)$ coincides with the sphere $[\lambda]$ associated with λ and its left S-resolvent is

$$S_L^{-1}(s,T) = (\lambda^2 - 2s_0\lambda + |s|^2)^{-1}(\overline{s}-\lambda)\mathcal{I}.$$

If $s \in \mathbb{C}_{j_\lambda}$, then λ and s commute so that the left S-resolvent reduces to

$$S_L^{-1}(s,T) = (s-\lambda)^{-1}\mathcal{I}$$

with $\|S_L^{-1}(s,T)\| = 1/|s-\lambda|$. If s tends to $\overline{\lambda}$ in $\mathbb{C}_{j_\lambda}$, then $\mathrm{dist}(s,\sigma_S(T)) \to 0$ because $\overline{\lambda} \in \sigma_S(T)$. But at the same time

$$\|S_L^{-1}(s,T)\| \to 1/|\lambda - \overline{\lambda}| = 1/(2\lambda_1) < +\infty.$$

Nevertheless, although (3.21) does not have a pointwise counterpart in the quaternionic setting, we can show that the norms of the S-resolvents explode near the S-spectrum. As it happens often in quaternionic operator theory, this requires that we work with spectral spheres of associated quaternions instead of single spectral values.

Lemma 3.2.7. *Let $T \in \mathcal{K}(X)$ and $s \in \rho_S(T)$. Then*

$$\|\mathcal{Q}_s(T)^{-1}\| + \|T\mathcal{Q}_s(T)^{-1}\| \geq \frac{1}{d_S(s, \sigma_S(T))}, \tag{3.22}$$

where

$$d_S(s, \sigma_S(T)) = \inf_{q \in \sigma_S(T)} d_S(s, q)$$

and $d_S(s, x)$ is defined as in Lemma 3.1.2.

Proof. Set $C_s := \|\mathcal{Q}_s(T)^{-1}\| + \|T\mathcal{Q}_s(T)^{-1}\|$. If $d_S(s, q) < 1/C_s$, then $x \in \rho_S(T)$ by Lemma 3.1.2. Thus, $d_S(s, q) \geq 1/C_s$ for any $q \in \sigma_S(T)$. If we take the infimum over all $q \in \sigma_S(T)$, this inequality still holds true and we obtain $d_S(s, \sigma_S(T)) \geq 1/C_s$, which is equivalent to (3.22). □

Lemma 3.2.8. *Let $T \in \mathcal{K}(X)$ and $s \in \rho_S(T)$. Then*

$$\sqrt{2\,\|\mathcal{Q}_s(T)^{-1}\|} \leq \left\|S_L^{-1}(s, T)\right\| + \left\|S_L^{-1}(\overline{s}, T)\right\|$$

and in turn

$$\sqrt{\|\mathcal{Q}_s(T)^{-1}\|} \leq \sqrt{2} \sup_{s_j \in [s]} \left\|S_L^{-1}(s_j, T)\right\|.$$

Analogous estimates hold for the right S-resolvent operator.

Proof. Observe that $\mathcal{Q}_s(T)^{-1} = \mathcal{Q}_{\overline{s}}(T)^{-1}$ for $s \in \rho_S(T)$. Because of $2s_0 = s + \overline{s}$, we have

$$\begin{aligned}
&S_L^{-1}(s,T)S_L^{-1}(s,T) + S_L^{-1}(s,T)S_L^{-1}(\overline{s},T)\\
&= \left(\mathcal{Q}_s(T)^{-1}\overline{s} - T\mathcal{Q}_s(T)^{-1}\right)\left(\mathcal{Q}_s(T)^{-1}\overline{s} - T\mathcal{Q}_s(T)^{-1}\right)\\
&\quad + \left(\mathcal{Q}_s(T)^{-1}\overline{s} - T\mathcal{Q}_s(T)^{-1}\right)\left(\mathcal{Q}_s(T)^{-1}s - T\mathcal{Q}_s(T)^{-1}\right)\\
&= \left(\mathcal{Q}_s(T)^{-1}\overline{s} - T\mathcal{Q}_s(T)^{-1}\right) 2\left(s_0\mathcal{I} - T\right)\mathcal{Q}_s(T)^{-1}
\end{aligned}$$

and similarly

$$\begin{aligned}
&S_L^{-1}(\overline{s},T)S_L^{-1}(s,T) + S_L^{-1}(\overline{s},T)S_L^{-1}(\overline{s},T)\\
&= \left(\mathcal{Q}_s(T)^{-1}s - T\mathcal{Q}_s(T)^{-1}\right) 2\left(s_0\mathcal{I} - T\right)\mathcal{Q}_s(T)^{-1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&S_L^{-1}(s,T)S_L^{-1}(s,T) + S_L^{-1}(s,T)S_L^{-1}(\overline{s},T)\\
&\quad + S_L^{-1}(\overline{s},T)S_L^{-1}(s,T) + S_L^{-1}(\overline{s},T)S_L^{-1}(\overline{s},T)\\
&= \left(\mathcal{Q}_s(T)^{-1}\overline{s} - T\mathcal{Q}_s(T)^{-1}\right) 2\left(s_0\mathcal{I} - T\right)\mathcal{Q}_s(T)^{-1}\\
&\quad + \left(\mathcal{Q}_s(T)^{-1}s - T\mathcal{Q}_s(T)^{-1}\right) 2\left(s_0\mathcal{I} - T\right)\mathcal{Q}_s(T)^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= 2\left(s_0\mathcal{I} - T\right)\mathcal{Q}_s(T)^{-1} 2\left(s_0\mathcal{I} - T\right)\mathcal{Q}_s(T)^{-1} \\
&= 4(T^2 - 2s_0T + s_0^2\mathcal{I})\mathcal{Q}_s(T)^{-2} = 4\mathcal{Q}_s(T)^{-1} - 4s_1^2\mathcal{Q}_s(T)^{-2},
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
4\mathcal{Q}_s(T)^{-1} = {} & S_L^{-1}(s,T)S_L^{-1}(s,T) + S_L^{-1}(s,T)S_L^{-1}(\overline{s},T) \\
& + S_L^{-1}(\overline{s},T)S_L^{-1}(s,T) + S_L^{-1}(\overline{s},T)S_L^{-1}(\overline{s},T) + 4s_1^2\mathcal{Q}_s(T)^{-2}.
\end{aligned}
$$

Thus, we can estimate

$$
\begin{aligned}
& 4\left\|\mathcal{Q}_s(T)^{-1}\right\| \\
&= \left\|S_L^{-1}(s,T)\right\|\left\|S_L^{-1}(s,T)\right\| + \left\|S_L^{-1}(s,T)\right\|\left\|S_L^{-1}(\overline{s},T)\right\| \\
&\quad + \left\|S_L^{-1}(\overline{s},T)\right\|\left\|S_L^{-1}(s,T)\right\| + \left\|S_L^{-1}(\overline{s},T)\right\|\left\|S_L^{-1}(\overline{s},T)\right\| \\
&\quad + 4\left\|s_1^2\mathcal{Q}_s(T)^{-2}\right\| \\
&= \left(\left\|S_L^{-1}(s,T)\right\| + \left\|S_L^{-1}(\overline{s},T)\right\|\right)^2 + \left\|2s_1\mathcal{Q}_s(T)^{-1}\right\|\left\|2s_1\mathcal{Q}_s(T)^{-1}\right\|. \qquad (3.23)
\end{aligned}
$$

Finally observe that

$$
\begin{aligned}
2\mathcal{Q}_s(T)^{-1}s_1j_s = {} & T\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T)^{-1}(s_0 - j_ss_1) \\
& - \left(T\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T)^{-1}(s_0 + j_ss_1)\right) = S_L^{-1}(s,T) - S_L^{-1}(\overline{s},T)
\end{aligned}
$$

and hence

$$
\left\|2s_1\mathcal{Q}_s(T)^{-1}\right\| = \left\|2\mathcal{Q}_s(T)^{-1}s_1j_s\right\| \leq \left\|S_L^{-1}(s,T)\right\| + \left\|S_L^{-1}(\overline{s},T)\right\|.
$$

Combining this estimate with (3.23), we finally obtain

$$
2\left\|\mathcal{Q}_s(T)^{-1}\right\| \leq \left(\left\|S_L^{-1}(s,T)\right\| + \left\|S_L^{-1}(\overline{s},T)\right\|\right)^2
$$

and hence the statement for the left S-resolvent operator. The estimates for the right S-resolvent operator can be shown with similar computations. □

From the above results we get:

Lemma 3.2.9. *Let $T \in \mathcal{K}(X)$. If $(s_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\rho_S(T)$ with*

$$
\lim_{n\to\infty} \operatorname{dist}(s_n, \sigma_S(T)) = 0,
$$

then

$$
\lim_{n\to\infty} \sup_{s\in[s_n]} \left\|S_L^{-1}(s,T)\right\| = +\infty \quad \textit{and} \quad \lim_{n\to\infty} \sup_{s\in[s_n]} \left\|S_R^{-1}(s,T)\right\| = +\infty.
$$

Proof. First of all observe that $\operatorname{dist}(s_n, \sigma_S(T)) \to 0$ if and only if $d_S(s_n, \sigma_S(T)) \to 0$ because $\sigma_S(T)$ is axially symmetric. Indeed, for any $n \in \mathbb{N}$, there exits $x_n \in \sigma_S(T)$ such that

$$|s_n - x_n| < \operatorname{dist}(s_n, \sigma_S(T)) + 1/n.$$

If $\operatorname{dist}(s_n, \sigma_S(T)) \to 0$, then $|s_n - x_n| \to 0$ and hence $|s_{n,0} - x_{n,0}| \to 0$. Since the sequence s_n is bounded, the sequence x_n is bounded too and we also have

$$\left||s_n|^2 - |x_n|^2\right| \leq |s_n||\overline{s_n} - \overline{x_n}| + |s_n - x_n||\overline{x_n}| \to 0$$

and in turn

$$0 < d_S(s_n, \sigma_S(T)) \leq d_S(s_n, x_n) = \max\left\{|s_{n,0} - x_{n,0}|, \left||s_n|^2 - |x_n|^2\right|\right\} \longrightarrow 0.$$

If on the other hand $d_S(s_n, \sigma_S(T))$ tends to zero, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\sigma_S(T)$ such that

$$d_S(s_n, x_n) < d_S(s_n, \sigma_S(T)) + 1/n$$

and in turn $d_S(s_n, x_n) \to 0$. Since $\sigma_S(T)$ is axially symmetric and $d(s_n, x_{n,j}) = d(s_n, x_n)$ for any $x_{n,j} \in [x_n]$, we can, moreover, assume that $j_{x_n} = j_{s_n}$. Then

$$0 \leq |s_{n,0} - x_{n,0}| \leq d_S(s_n, x_n) \to 0.$$

Since s_n and in turn also x_n are bounded, this implies $|s_{n,0}^2 - x_{n,0}^2| \to 0$, from which we deduce that also $|s_{n,1}^2 - x_{n,1}^2| \to 0$ because

$$0 \leq \left|s_{n,0}^2 - x_{n,0}^2 + s_{n,1}^2 - x_{n,1}^2\right| = \left||s_n|^2 - |x_n|^2\right| \leq d_S(s_n, x_n) \to 0.$$

Since $s_{n,1} \geq 0$ and $x_{n,1} \geq 0$, we conclude that $s_{n,1} - x_{n,1} \to 0$ and, since $j_s = j_x$, also

$$0 < \operatorname{dist}(s_n, \sigma_S(T)) \leq |s_n - x_n| = \sqrt{(s_{n,0} - x_{n,0})^2 + (s_{n,1} - x_{n,1})^2} \to 0.$$

Now assume that $s_n \in \rho_S(T)$ with $\operatorname{dist}(s_n, \sigma_S(T)) \to 0$. By the above considerations and (3.22), we have

$$\|\mathcal{Q}_{s_n}(T)^{-1}\| + \|T\mathcal{Q}_{s_n}(T)^{-1}\| \to +\infty. \tag{3.24}$$

We show now that every subsequence $(s_{n_k})_{k\in\mathbb{N}}$ has a subsequence $(s_{n_{k_j}})_{j\in\mathbb{N}}$ such that

$$\lim_{j\to+\infty} \sup_{s\in[s_{n_{k_j}}]} \|S_L^{-1}(s, T)\| = +\infty, \tag{3.25}$$

which implies $\lim_{n\to+\infty} \sup_{s\in[s_n]} \|S_L^{-1}(s, T)\| = +\infty$. We consider an arbitrary subsequence $(s_{n_k})_{k\in\mathbb{N}}$ of $(s_n)_{n\in\mathbb{N}}$. If this subsequence has a subsequence $(s_{n_{k_j}})_{j\in\mathbb{N}}$ such that $\|Q_{s_{n_{k_j}}}(T)\| \to +\infty$, then Lemma 3.2.8 implies (3.25). Otherwise,

$$\|\mathcal{Q}_{s_{n_k}}(T)^{-1}\| \leq C$$

for some constant $C > 0$ and we deduce from (3.24) that $\|T\mathcal{Q}_{s_{n_k}}(T)^{-1}\| \to +\infty$. Observe that

$$T\mathcal{Q}_{s_{n_k}}(T)^{-1} = -\frac{1}{2}S_L^{-1}(s_{n_k},T) - \frac{1}{2}S_L^{-1}(\overline{s_{n_k}},T) + s_{n_k,0}\mathcal{Q}_{s_{n_k}}(T)^{-1},$$

from which we obtain the estimate

$$\begin{aligned}\left\|T\mathcal{Q}_{s_{n_k}}(T)^{-1}\right\| &\leq \sup_{s\in[s_{n_k}]}\left\|S_L^{-1}(s_{n_k},T)\right\| + |s_{n_k,0}|\left\|\mathcal{Q}_{s_{n_k}}(T)^{-1}\right\| \\ &\leq \sup_{s\in[s_{n_k}]}\left\|S_L^{-1}(s_{n_k},T)\right\| + CM\end{aligned}$$

with $M = \sup_{n\in\mathbb{N}}|s_n| < +\infty$. Since the left-hand side tends to infinity as $k \to +\infty$, we obtain that also

$$\sup_{s\in[s_{n_k}]}\left\|S_L^{-1}(s_{n_k},T)\right\| \to +\infty$$

and thus, the statement holds true. The case of the right S-resolvent can be shown with analogous arguments. □

Definition 3.2.10 (Slice hyperholomorphic continuation)**.** Let f be a left (or right) slice hyperholomorphic function defined on an axially symmetric open set U. A left (or right) slice hyperholomorphic function g defined on an axially symmetric open set U' with $U \subsetneq U'$ is called a slice hyperholomorphic continuation of f if $f(s) = g(s)$ for all $s \in U$. It is called nontrivial if $V = U' \setminus U$ cannot be separated from U, i.e. if $U' \neq U \cup V$ for some open set V with $\overline{V} \cap \overline{U} = \emptyset$.

Theorem 3.2.11. *Let $T \in \mathcal{K}(X)$. There does not exist any nontrivial slice hyperholomorphic continuation of the left or of the right S-resolvent operators.*

Proof. Assume that there exists a nontrivial continuation f of $S_L^{-1}(s,T)$ to an axially symmetric open set U with $\rho_S(T) \subsetneq U$. Then there exists a point $s \in U \cap \partial(\rho_S(T))$ and a sequence $s_n \in \rho_S(T)$ with $\lim_{n\to+\infty} s_n = s$ such that

$$\lim_{n\to+\infty}\left\|S_L^{-1}(s_n,T)\right\| = \lim_{n\to+\infty}\|f(s_n)\| = \|f(s)\| < +\infty.$$

Moreover, $\overline{s_n} \to \overline{s}$ as $n \to +\infty$ and in turn

$$\lim_{n\to+\infty}\left\|S_L^{-1}(\overline{s_n},T)\right\| = \lim_{n\to+\infty}\|f(\overline{s_n})\| = \|f(\overline{s})\| < +\infty.$$

From the representation formula (2.1.7) we then deduce

$$\lim_{n\to+\infty}\sup_{s\in[s_n]}\left\|S_L^{-1}(s,T)\right\| \leq \lim_{n\to+\infty}\left\|S_L^{-1}(s_n,T)\right\| + \left\|S_L^{-1}(\overline{s_n},T)\right\| < +\infty.$$

On the other hand the sequence s_n is bounded and

$$\operatorname{dist}(s_n,\sigma_S(T)) \leq |s_n - s| \to 0.$$

Lemma 3.2.9 therefore implies

$$\lim_{n\to+\infty}\sup_{s\in[s_n]}\left\|S_L^{-1}(s,T)\right\| = +\infty,$$

which is a contradiction. Thus, the analytic continuation (f,U) cannot exist. For the right S-resolvent, we argue analogously. □

One could suspect that it might be possible to improve the above results by finding an estimate of the form (3.21) for the pseudo-resolvent $\mathcal{Q}_s(T)^{-1}$ instead of the S-resolvents. This is, however, not possible as the following example shows.

Example 3.2.12. We consider for $p \in [1,+\infty)$ the space $\ell^p(\mathbb{N})$ of p-summable sequences with quaternionic entries. Any sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n \in \mathbb{H}$ obviously defines a right linear, densely defined and closed operator on $\ell^p(\mathbb{N})$ via $T(y) = (\lambda_n v_n)_{n\in\mathbb{N}}$ for $y = (v_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})$. If $(\lambda_n)_{n\in\mathbb{N}}$ is unbounded, then T is unbounded. Otherwise, we have

$$\|T\| = \sup_{n\in\mathbb{N}}|\lambda_n| = \|(\lambda_n)_{n\in\mathbb{N}}\|_\infty.$$

Indeed,

$$\|T(y)\|_p = \sqrt[p]{\sum_{n\in\mathbb{N}}|\lambda_n v_n|^p} \le \|(\lambda_n)_{n\in\mathbb{N}}\|_\infty \sqrt[p]{\sum_{n\in\mathbb{N}}|v_n|^p} = \|(\lambda_n)_{n\in\mathbb{N}}\|_\infty \|y\|_p$$

such that $\|T\| \le \|(\lambda_n)_{n\in\mathbb{N}}\|_\infty$. On the other hand, with $e_m = (\delta_{m,n})_{n\in\mathbb{N}}$ where $\delta_{m,n}$ is the Kronecker delta,

$$|\lambda_m| = \sqrt[p]{\sum_{n\in\mathbb{N}}|\lambda_n\delta_{m,n}\|^p} = \|T(e_m)\| \le \|T\|,$$

for any $m\in\mathbb{N}$ such that also $\|(\lambda_n)_{n\in\mathbb{N}}\|_\infty \le \|T\|$. The S-spectrum of T is

$$\sigma_S(T) = \overline{\bigcup_{n\in\mathbb{N}}[\lambda_n]} \tag{3.26}$$

as one can see easily: any λ_n is a right eigenvalue of T since $T(e_n) = e_n\lambda_n$ and hence the relation $\supset$ in (3.26) holds true by Theorem 3.1.6. If on the other hand s does not belong to the right hand side of (3.26), then

$$\delta_s = \inf_{n\in\mathbb{N}}\operatorname{dist}(s,[\lambda_n]) = \inf_{n\in\mathbb{N}}|s_{j_{\lambda_n}} - \lambda_n| > 0,$$

where $s_{j_{\lambda_n}} = s_0 + j_{\lambda_n}s_1$. As

$$\mathcal{Q}_s(T)y = \left((\lambda_n - s_{j_{\lambda_n}})(\lambda_n - \overline{s_{j_{\lambda_n}}})v_n\right)_{n\in\mathbb{N}}$$

and in turn

$$\mathcal{Q}_s(T)^{-1}y = \left((\lambda_n - s_{j_{\lambda_n}})^{-1}(\lambda_n - \overline{s_{j_{\lambda_n}}})^{-1}v_n\right)_{n\in\mathbb{N}},$$

we have $\|\mathcal{Q}_s(T)^{-1}\| \leq 1/\delta_s^2 < +\infty$ such that $s \in \rho_S(T)$. Thus, the relation $\subset$ in (3.26) also holds true.

Now choose a sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lambda_{n,1} \to +\infty$ as $n \to +\infty$ and consider the respective operator T on $\ell^p(\mathbb{N})$. For simplicity, consider for instance $\lambda_n = jn$ with $j \in \mathbb{S}$. By the above considerations, the sequence $s_N = j(N + 1/N)$ with $N = 2, 3, \dots$ does then satisfy $\operatorname{dist}(s_N, \sigma_S(T)) \to 0$ as $N \to +\infty$ and

$$\begin{aligned} \left\|\mathcal{Q}_{s_n}(T)^{-1}\right\| &= \sup_{n\in\mathbb{N}} \frac{1}{|\lambda_n - s_N||\lambda_n - \overline{s_N}|} \\ &= \frac{1}{|\lambda_N - s_N||\lambda_N - \overline{s_N}|} = \frac{1}{2 + \frac{1}{N^2}}. \end{aligned} \tag{3.27}$$

Indeed, if $n < N$, then some simple computations show that the inequality

$$\begin{aligned} \frac{1}{|\lambda_n - s_N||\lambda_n - \overline{s_N}|} &= \frac{1}{N + \frac{1}{N} - n} \frac{1}{n + N + \frac{1}{N}} \\ &< \frac{1}{2 + \frac{1}{N^2}} = \frac{1}{|\lambda_N - s_N||\lambda_N - \overline{s_N}|} \end{aligned}$$

is equivalent to $0 < N^2 - n^2$, which is obviously true. Similarly, in the case $n > N$, the inequality

$$\begin{aligned} \frac{1}{|\lambda_n - s_N||\lambda_n - \overline{s_N}|} &= \frac{1}{n - N - \frac{1}{N}} \frac{1}{n + N + \frac{1}{N}} \\ &< \frac{1}{2 + \frac{1}{N^2}} = \frac{1}{|\lambda_N - s_N||\lambda_N - \overline{s_N}|} \end{aligned}$$

is equivalent to $4 + 1/N^2 < n^2 - N^2$, which holds true since $2 \leq N < n$.

From (3.27), we see that $\|\mathcal{Q}_{s_n}(T)^{-1}\| \leq 2$ although $\operatorname{dist}(s_N, \sigma_S(T)) \to 0$. Consequently, the pseudo-resolvent cannot satisfy an estimate analogue to (3.21).

Also controlling the norm of $T\mathcal{Q}_s(T)^{-1}$ by the norm of $\mathcal{Q}_s(T)^{-1}$ in order to improve (3.22) is not possible: if we consider the operator $T\mathcal{Q}_{s_n}(T)^{-1}$ in the above example, then

$$T\mathcal{Q}_{s_n}(T)^{-1} y = \left(\frac{n}{n - N - \frac{1}{N}} \frac{1}{j\left(n + N + \frac{1}{N}\right)} v_n \right)_{n\in\mathbb{N}}$$

and

$$\|T\mathcal{Q}_{s_n}(T)^{-1}\| \geq \|T\mathcal{Q}_{s_n}(T)^{-1}(e_N)\| = \frac{N^2}{2N + \frac{1}{N}} \to +\infty$$

shows that $\|T\mathcal{Q}_{s_n}(T)^{-1}\|$ tends to infinity although $\|\mathcal{Q}_{s_n}(T)^{-1}\|$ stays bounded.

3.3 Closed operators with commuting components

Closed right linear operators cannot always be decomposed into components as it is the case for bounded operators, cf. (2.38).

However, this is possible if $\mathcal{D}(T)$ is a two-sided subspace of X, that is if it is of the form $\mathcal{D}(T) = X_0 \otimes \mathbb{H}$ for some subspace X_0 of $X_\mathbb{R}$.

If on the other hand $T_0, \dots, T_3$ are operators on $X_\mathbb{R}$, then we can define the operator

$$T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \quad \text{with} \quad \mathcal{D}(T) = \left(\bigcap_{\ell=0}^{4} \mathcal{D}(T_\ell) \right) \otimes \mathbb{H}.$$

Definition 3.3.1. Let X be a two-sided quaternionic Banach space. We define $\mathcal{KC}(X)$ as the set of all operators $T \in \mathcal{K}(X)$ that admit a decomposition of the form $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell$ with closed operators $T_\ell \in \mathcal{K}(X_\mathbb{R})$ such that

(i) $\mathcal{D}(T^2) = \bigcap_{\ell,\kappa=0}^{3} \mathcal{D}(T_\ell T_\kappa) = \bigcap_{\ell=0}^{3} \mathcal{D}(T_\ell^2)$,

(ii) $\mathcal{D}(T_\ell T_\kappa) = \mathcal{D}(T_\kappa T_\ell)$, for $\ell, \kappa \in \{0, \dots, 3\}$,

(iii) $T_\ell T_\kappa y = T_\kappa T_\ell y$, for all $y \in \mathcal{D}(T^2)$ for $\ell, \kappa \in \{0, \dots, 3\}$.

Furthermore, we call a closed operator T a scalar operator if it is of the form $T = T_0$, that is if $T_1 = T_2 = T_3 = 0$ or equivalently if T is the extension of a closed operator on $X_\mathbb{R}$ to X.

Remark 3.3.1. A scalar operator $T \in \mathcal{K}(X)$ commutes with any $a \in \mathbb{H}$.

The S-spectrum $\sigma_S(T)$ of any operator $T \in \mathcal{KC}(X)$ can be characterized in a different way that takes the commutativity of the components into account. The corresponding characterization for bounded operators has been presented in Section 2.3.

Definition 3.3.2. Let X be a two-sided quaternionic Banach space. For a closed operator $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{KC}(X)$ with commuting components, we define $\overline{T} = T - \sum_{\ell=1}^{3} T_\ell e_\ell$ with $\mathcal{D}\left(\overline{T}\right) = \cap_{\ell=0}^{3} \mathcal{D}(T_\ell) = \mathcal{D}(T)$.

McIntosh and Pryde showed in [179, Theorem 3.3] that an operator $T \in \mathcal{B}(X)$ with commuting components is invertible if and only if $T\overline{T} = \overline{T}T = \sum_{\ell=0}^{3} T_\ell^2$ is invertible. This holds true also for an unbounded operator with commuting components as the next lemma shows.

Lemma 3.3.3. *Let $T \in \mathcal{KC}(X)$. Then the following statements are equivalent.*

(i) *The operator T has a bounded inverse.*

(ii) *The operator $\overline{T}$ has a bounded inverse.*

(iii) *The operator $\overline{T}T$ has a bounded inverse.*

Proof. First of all, we observe that, due to $\mathcal{D}(T) = \mathcal{D}\left(\overline{T}\right)$, we have

$$\mathcal{D}\left(\overline{T}T\right) = \{y \in X : Ty \in \mathcal{D}(T)\} = \mathcal{D}\left(T^2\right).$$

Since $\mathcal{D}\left(T^2\right) = \bigcap_{\ell,\kappa=0}^{3} \mathcal{D}\left(T_\ell T_\kappa\right) = \bigcap_{\ell=0}^{3} \mathcal{D}(T_\ell^2)$ and

$$\overline{T}Ty = T_0^2 y + \sum_{\ell=1}^{3} e_\ell T_0 T_\ell y - \sum_{\ell=1}^{3} e_\ell T_\ell T_0 y - \sum_{\ell,\kappa=1}^{3} e_\ell e_\kappa T_\ell T_\kappa y = \sum_{\ell=0}^{3} T_\ell^2 y$$

because $e_\ell e_\kappa = -e_\kappa e_\ell$ and $e_\ell^2 = -1$ for $1 \leq \ell, \kappa \leq 3$ with $\ell \neq \kappa$, we thus have $\overline{T}T = \sum_{\ell=0}^{3} T_\ell^2$. In particular, $\overline{T}T$ is a scalar operator and hence commutes with any quaternion.

If $T\overline{T}$ is invertible, then $\left(T\overline{T}\right)^{-1} = \left(\sum_{\ell=0}^{3} T_\ell^2\right)^{-1}$ commutes with each of the components T_ℓ and it also commutes with the imaginary units e_ℓ. Hence, it commutes with T and so the inverse T^{-1} is given by $T^{-1} = \overline{T}\left(T\overline{T}\right)^{-1}$ because

$$\left(\overline{T}\left(T\overline{T}\right)^{-1}\right) Ty = \overline{T}T\left(\overline{T}T\right)^{-1} y, \quad \forall y \in \mathcal{D}(T)$$

and

$$T\left(\overline{T}\left(T\overline{T}\right)^{-1}\right) y = \left(T\overline{T}\right)\left(T\overline{T}\right)^{-1} y = y, \quad \forall y \in X.$$

Consequently, the invertibility of $T\overline{T}$ implies the invertibility of T.

If on the other hand T is invertible and $T^{-1} = S_0 + \sum_{\kappa=1}^{3} S_\kappa e_\kappa \in \mathcal{B}(X)$, then

$$\begin{aligned}
\mathcal{I}|_{\mathcal{D}(T)} = T^{-1}T &= \left(S_0 + \sum_{\kappa=1}^{3} S_\kappa e_\kappa\right)\left(T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell\right) \\
&= S_0 T_0 - \sum_{\ell=1}^{3} S_\ell T_\ell + (S_2T_3 - S_3T_2)e_1 \\
&\quad + (S_3T_1 - S_1T_3)e_2 + (S_1T_2 - S_2T_1)e_3,
\end{aligned}$$

from which we conclude that

$$\mathcal{I}|_{\mathcal{D}(T)} = S_0T_0 - \sum_{\ell=1}^{3} S_\ell T_\ell \quad \text{and} \quad S_\ell T_\kappa - S_\kappa T_\ell = 0, \quad 1 \leq \ell < \kappa \leq 3.$$

Therefore

$$\begin{aligned}
\overline{S}\,\overline{T} &= \left(S_0 - \sum_{\ell=1}^{3} S_\ell e_\ell\right)\left(T_0 - \sum_{\ell=1}^{3} T_\ell e_\ell\right) \\
&= S_0T_0 - \sum_{\ell=1}^{3} S_\ell T_\ell + (S_2T_3 - S_3T_2)e_1 \\
&\quad + (S_3T_1 - S_1T_3)e_2 + (S_1T_2 - S_2T_1)e_3 = \mathcal{I}|_{\mathcal{D}(T)}.
\end{aligned}$$

Similarly, we see that $TS = \mathcal{I}$ also implies $\overline{T}\,\overline{S} = \mathcal{I}$. Hence, the invertibility of T implies the invertibility of $\overline{T}$ and $\overline{T}^{-1} = \overline{T^{-1}}$. Thus, if T is invertible, we have $\left(T\overline{T}\right)^{-1} = \overline{T}^{-1}T^{-1} \in \mathcal{B}(X)$. Altogether, we find that T is invertible if and only if $T\overline{T} = \overline{T}T$ is invertible. □

Theorem 3.3.4. *Let $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{KC}(X)$ with dense domain. If we set*

$$\mathcal{Q}_{c,s}(T) = s^2\mathcal{I} - 2sT_0 + T\overline{T},$$

then

$$\rho_S(T) = \left\{ s \in \mathbb{H} : \quad \mathcal{Q}_{c,s}(T)^{-1} \in \mathcal{B}(X) \right\} \tag{3.28}$$

and

$$\begin{aligned} S_L^{-1}(s,T) &= (s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T) \\ S_R^{-1}(s,T) &= \mathcal{Q}_{c,s}(T)^{-1}s - \sum_{\ell=0}^{3} T_\ell \mathcal{Q}_{c,s}(T)^{-1}e_\ell. \end{aligned} \tag{3.29}$$

Proof. Since T and $\overline{T}$ commute, we have $\overline{\mathcal{Q}_s(T)} = \mathcal{Q}_s(\overline{T})$ and $\overline{\mathcal{Q}_{c,s}(T)} = \mathcal{Q}_{c,\overline{s}}(T)$. For $y \in \mathcal{D}(T^4) = \mathcal{D}\left(\mathcal{Q}_{c,s}(T)\mathcal{Q}_{c,\overline{s}}(T)\right)$, we thus find

$$\begin{aligned} \mathcal{Q}_{c,s}(T)\overline{\mathcal{Q}_{c,s}(T)}y &= (s^2\mathcal{I} - 2sT_0 + T\overline{T})(\overline{s}^2\mathcal{I} - 2\overline{s}T_0 + T\overline{T})y \\ &= |s|^4\mathcal{I}y - 2s|s|^2T_0y + s^2T\overline{T}y \\ &\quad - 2|s|^2T_0\overline{s}y + 4|s|^2T_0^2y - 2sT_0T\overline{T}y \\ &\quad + \overline{s}^2T\overline{T}y - 2\overline{s}T_0T\overline{T}y + (T\overline{T})^2y \\ &= |s|^4\mathcal{I}y - 2s_0|s|^2Ty - 2s_0|s|^2\overline{T}y + 2\mathrm{Re}(s^2)T\overline{T}y \\ &\quad + 4|s|^2T_0^2y - 2s_0T^2\overline{T}y - 2s_0T\overline{T}^2y + T^2\overline{T}^2y, \end{aligned}$$

where we used in the last identity that $2s_0 = s + \overline{s}$, that $|s|^2 = s\overline{s}$, and that $2T_0y = Ty + \overline{T}y$. As

$$2\mathrm{Re}(s^2)T\overline{T}y = 2s_0^2T\overline{T}y - 2s_1^2T\overline{T}y$$

and

$$4|s|^2T_0^2y = |s|^2(T + \overline{T})^2y = |s|^2T^2y + 2s_0^2T\overline{T}y + s_1^2T\overline{T}y + |s|^2\overline{T}^2y,$$

we further find

$$\begin{aligned} \mathcal{Q}_{c,s}(T)\overline{\mathcal{Q}_{c,s}(T)}y = |s|^2(|s|^2\mathcal{I} - 2s_0T + T^2)y& \\ - 2s_0\overline{T}(|s|^2\mathcal{I} - 2s_0T + T^2)y& \\ + \overline{T}^2(|s|^2\mathcal{I} - 2s_0T + T^2)y &= \mathcal{Q}_s(T)\overline{\mathcal{Q}_s(T)}y. \end{aligned}$$

By the above arguments, we hence have

$$\begin{aligned}\mathcal{Q}_{c,s}(T)^{-1} \in \mathcal{B}(X) &\iff \left(\mathcal{Q}_{c,s}(T)\overline{\mathcal{Q}_{c,s}(T)}\right)^{-1} \in \mathcal{B}(X)\\ &\iff \left(\mathcal{Q}_{s}(T)\overline{\mathcal{Q}_{s}(T)}\right)^{-1} \in \mathcal{B}(X) \iff \mathcal{Q}_{s}(T)^{-1} \in \mathcal{B}(X)\end{aligned}$$

and hence (3.28) holds true.

If $y \in \mathcal{D}(T^2) = \mathcal{D}(\mathcal{Q}_{c,s}(T))$ with $\mathcal{Q}_{c,s}(T) \in \mathcal{D}(T)$, we have

$$\begin{aligned}&(s\mathcal{I} - T)\mathcal{Q}_{c,s}(T)y\\ &= (\overline{s}\mathcal{I} - T)\left(s^2\mathcal{I} - 2sT_0 + T\overline{T}\right)y\\ &= |s|^2 s\mathcal{I}y - Ts^2y - 2|s|^2T_0y + 2TT_0sy + \overline{s}T\overline{T}y - T^2\overline{T}y\\ &= |s|^2 s\mathcal{I}y - Ts^2y - |s|^2Ty - |s|^2\overline{T}y + T^2sy + T\overline{T}sy + \overline{s}T\overline{T}y - T^2\overline{T}y\\ &= |s|^2\left(s\mathcal{I} - \overline{T}\right)y - 2s_0T\left(s\mathcal{I} - \overline{T}\right)y + T^2\left(s\mathcal{I} - \overline{T}\right)y\\ &= \left(T^2 - 2s_0T + |s|^2\mathcal{I}\right)\left(s\mathcal{I} - \overline{T}\right)y = \mathcal{Q}_s(T)\left(s\mathcal{I} - \overline{T}\right)y.\end{aligned}$$

For any $x \in \mathcal{D}(T)$, we can set $y = \mathcal{Q}_{c,s}(T)^{-1}x \in \mathcal{D}(T^2)$. If we apply the operator $\mathcal{Q}_s(T)^{-1}$ to the above identity from the right, we then obtain

$$S_L^{-1}(s,T)x = \mathcal{Q}_s(T)^{-1}(s\mathcal{I} - T)x = \left(s\mathcal{I} - \overline{T}\right)\mathcal{Q}_{c,s}(T)^{-1}x$$

and a density argument shows that (3.29) holds true for the left S-resolvent operator. Similar computations show also the identity for the right S-resolvent equation. □

3.4 The S-functional calculus and its properties

We want to define the S-functional calculus for an arbitrary operator in $\mathcal{K}(X)$ with nonempty S-resolvent set via the slice hyperholomorphic Cauchy integral. The domain of integration is thereby the boundary of a suitable slice Cauchy domain U in one of the complex planes $\mathbb{C}_j$, for $j \in \mathbb{S}$. In order for the S-functional calculus to be well-defined, we have to show that these integrals are independent of the choice of the slice Cauchy domain U and of the complex plane $\mathbb{C}_j$. We follow the strategy known from the bounded case.

Theorem 3.4.1. *Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \neq \emptyset$. If $f \in \mathcal{SH}_L(\sigma_S(T) \cup \{\infty\})$, then there exists an unbounded slice Cauchy domain U with $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$. The integral*

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s) \tag{3.30}$$

defines an operator in $\mathcal{B}(X)$ and this operator is the same for any choice of the imaginary unit $j \in \mathbb{S}$ and for any choice of the slice Cauchy domain U that satisfies the above conditions.

Similarly, if $f \in \mathcal{SH}_R(\sigma_S(T) \cup \{\infty\})$, then there exists an unbounded slice Cauchy domain U such that $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$. Again, the integral

$$\frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,T)$$

defines an operator in $\mathcal{B}(X)$ and this operator is the same for any choice of the imaginary unit $j \in \mathbb{S}$ and for any choice of the slice Cauchy domain U that satisfies the above conditions.

Proof. Let $f \in \mathcal{SH}_L(\sigma_S(T) \cup \{\infty\})$ and $q \in \rho_S(T)$. Since $\rho_S(T)$ is open, there exists a closed ball $\overline{B_\varepsilon(q)} \subset \rho_S(T)$ and since $\rho_S(T)$ is axially symmetric we have

$$\left[\overline{B_\varepsilon(q)}\right] = \{s = s_0 + j_s s_1 \in \mathbb{H} : \quad (s_0 - q_0)^2 + (s_1 - q_1)^2 \leq \varepsilon\} \subset \rho_S(T).$$

The existence of the slice Cauchy domain U follows from Theorem 2.1.31 applied with $C = \sigma_S(T)$ and $O = \mathcal{D}(f) \cap \left(\mathbb{H} \setminus \overline{B_\varepsilon(q)}\right)$.

The boundary of U in $\mathbb{C}_j$ consists of a finite set of closed piecewise differentiable Jordan curves and so it is compact. Hence, (3.30) is the integral of a bounded integrand over a compact domain. Thus, it converges in $\mathcal{B}(X)$ and defines an operator in $\mathcal{B}(X)$.

We now show the independence of the slice Cauchy domain. Consider first the case of another unbounded slice Cauchy domain U' such that $\sigma_S(T) \subset U'$ and $\overline{U'} \subset \mathcal{D}(f)$. Let us for the moment furthermore assume that $\overline{U'} \subset U$. Then the set $W = U \setminus \overline{U'}$ is a bounded slice Cauchy domain and

$$\partial(W \cap \mathbb{C}_j) = \partial(U \cap \mathbb{C}_j) \cup \Big(-\partial(U' \cap \mathbb{C}_j)\Big),$$

where $-\partial(U' \cap \mathbb{C}_j)$ denotes the inversely orientated boundary of U' in $\mathbb{C}_j$. Moreover, the function $s \mapsto S_L^{-1}(s,T)$ is right and the function $s \mapsto f(s)$ is left slice hyperholomorphic on $\overline{W}$. Thus, Theorem 2.1.20 implies

$$\begin{aligned} 0 &= \frac{1}{2\pi} \int_{\partial(W\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s) \\ &= \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s) - \frac{1}{2\pi} \int_{\partial(U'\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s). \end{aligned}$$

If $\overline{U'}$ is not contained in U, then $U \cap U'$ is an axially symmetric open set that contains $\sigma_S(T)$ such that $\partial(U \cap U')$ is nonempty and bounded. Theorem 2.1.31 implies the existence of a third slice Cauchy domain W such that $\sigma_S(T) \subset W$ and $\overline{W} \subset U \cap U'$. By the above arguments, the choice of any of them yields the same operator in (3.30).

Finally, we consider another imaginary unit $i \in \mathbb{S}$ and choose another unbounded slice Cauchy domain W with $\sigma_S(T) \subset W$ and $\overline{W} \subset U$. By the above arguments and the Cauchy formulae, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s) = \frac{1}{2\pi}\int_{\partial(W\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s)\\
&= \frac{1}{(2\pi)^2}\int_{\partial(W\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\left(f(\infty)+\int_{\partial(U\cap\mathbb{C}_i)} S_L^{-1}(q,s)\,dq_i\,f(q)\right)\\
&= \frac{1}{(2\pi)^2}\int_{\partial(W\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(\infty)\\
&\quad - \frac{1}{(2\pi)^2}\int_{\partial(U\cap\mathbb{C}_i)}\int_{\partial(W^{\mathsf{c}}\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,S_L^{-1}(q,s)\,dq_i\,f(q),
\end{aligned}$$

where Fubini's theorem allows us to exchange the order of integration in the last equation because we integrate a bounded function over a finite domain. The set W^{c} is a bounded slice Cauchy domain and the left S-resolvent is right slice hyperholomorphic in s on $\overline{W^{\mathsf{c}}}$. Theorem 2.1.20 implies

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{\partial(W\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(\infty)\\
&= -\frac{1}{(2\pi)^2}\int_{\partial(W^{\mathsf{c}}\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(\infty) = 0.
\end{aligned}$$

Since any $q\in\partial(U\cap\mathbb{C}_j)$ belongs to W^{c} by our choices of U and W and since $S_L^{-1}(q,s) = -S_R^{-1}(s,q)$, we deduce from the Cauchy formulae

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s)\\
&= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_i)}\left(\frac{1}{2\pi}\int_{\partial(W^{\mathsf{c}}\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,S_R^{-1}(s,q)\right)dq_i\,f(q)\\
&= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_i)} S_L^{-1}(q,T)\,dq_i\,f(q).
\end{aligned}$$

□

Definition 3.4.2. Let $T\in\mathcal{K}(X)$ with $\rho_S(T)\neq\emptyset$. For any $f\in\mathcal{SH}_L(\sigma_S(T)\cup\{\infty\})$, we define

$$f(T) := f(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s), \tag{3.31}$$

and for $f\in\mathcal{SH}_R(\sigma_S(T)\cup\{\infty\})$, we define

$$f(T) := f(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\,S_R^{-1}(s,T), \tag{3.32}$$

where $j\in\mathbb{S}$ is arbitrary and U is any slice Cauchy domain as in Theorem 3.4.1.

Remark 3.4.1. If $\rho_S(T)\cap\mathbb{R}\neq\emptyset$, then our approach is consistent with the approach that defines the S-functional calculus of an unbounded operator by suitably transforming both the function and the operator and then applying the S-functional calculus for bounded operators. Precisely, one chooses $\alpha\in\rho_S(T)\cap\mathbb{R}$ and sets $\Phi_\alpha(s)=(s-\alpha)^{-1}$. Then $A:=(T-\alpha\mathcal{I})^{-1}=S_R^{-1}(\alpha,T)$ is a bounded operator and formally corresponds to $\Phi_\alpha(T)$. Furthermore, a function f belongs to $\mathcal{SH}_L(\sigma_S(T)\cup\{\infty\})$ or $\mathcal{SH}_R(\sigma_S(T)\cup\{\infty\})$ if and only if $f\circ\Phi_\alpha^{-1}$ belongs to $\mathcal{SH}_L(\sigma_S(A))$ (resp. $\mathcal{SH}_R(\sigma_S(A))$). One then defines

$$f(T):=f\circ\Phi_\alpha^{-1}(A).$$

This approach was presented in [57, 93]. In the complex setting, it is equivalent to the direct approach via a Cauchy integral, which was developed above. In the quaternionic setting it, however, requires that $\rho_S(T)\cap\mathbb{R}\neq\emptyset$, which is not always true.

The S-functional calculus for closed operators is furthermore consistent with the S-functional calculus for bounded operators. Since we do not require connectedness of $\mathcal{D}(f)$ in Definition 3.4.2, we might extend $f\in\mathcal{SH}_L(\sigma_S(T))$ for bounded T to a function in $\mathcal{SH}_L(\sigma_S(T)\cup\{\infty\})$, for instance by setting $f(s)=c$ with $c\in\mathbb{H}$ on $\mathbb{H}\setminus B_r(0)$. We can then use the unbounded slice Cauchy domain $(\mathbb{H}\setminus B_r(0))\cup U$ in (3.31). Since the left S-resolvent is then right slice hyperholomorphic on $\mathbb{H}\setminus B_r(0)$ and $f(s)$ is left slice hyperholomorphic on this set, we obtain

$$\begin{aligned}f(T)&=f(\infty)\mathcal{I}+\frac{1}{2\pi}\int_{-\partial(B_r(0)\cap\mathbb{C}_j)}f(s)\,ds_j\,S_L^{-1}(s,T)\\&\qquad+\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)}f(s)\,ds_j\,S_L^{-1}(s,T)\\&=\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)}f(s)\,ds_j\,S_L^{-1}(s,T)\end{aligned}$$

because Theorem 2.1.20 implies that the sum of $f(\infty)\mathcal{I}$ and the integral over the boundary of $B_r(0)$ vanishes.

Example 3.4.3. Let $T\in\mathcal{K}(X)$ with $\rho_S(T)\neq\emptyset$. Consider the left slice hyperholomorphic function $f(s)=a$ for some $a\in\mathbb{H}$ and choose an arbitrary unbounded slice Cauchy domain U with $\sigma_S(T)\subset U$ and an imaginary unit $j\in\mathbb{S}$. Then

$$f(T)=f(\infty)\mathcal{I}+\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)}S_L^{-1}(s,T)\,ds_j\,f(s)=a\mathcal{I},\tag{3.33}$$

because $f(\infty)=a$ and the integral vanishes by Theorem 2.1.20 as the left S-resolvent is right slice hyperholomorphic in s on a superset of $\overline{\mathbb{H}\setminus U}$ and vanishes at infinity. An analogue argument shows that also $f(T)=\mathcal{I}a$ if f is considered right slice hyperholomorphic.

The following algebraic properties of the S-functional calculus follow immediately from the left and right linearity of the integral.

Corollary 3.4.4. *Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \neq \emptyset$.*

(i) *If $f, g \in \mathcal{SH}_L(\sigma_S(T) \cup \{\infty\})$ and $a \in \mathbb{H}$, then*

$$(f+g)(T) = f(T) + g(T) \quad \text{and} \quad (fa)(T) = f(T)a.$$

(ii) *If $f, g \in \mathcal{SH}_R(\sigma_S(T) \cup \{\infty\})$ and $a \in \mathbb{H}$, then*

$$(f+g)(T) = f(T) + g(T) \quad \text{and} \quad (af)(T) = af(T).$$

Theorem 3.4.1 ensures that the S-functional calculus for left slice hyperholomorphic functions and the S-functional calculus for right slice hyperholomorphic functions are well-defined in the sense that they are independent of the choices of the imaginary unit $j \in \mathbb{S}$ and the slice Cauchy domain U. Another important question is whether they are consistent. We show now that this is the case, if the function f is intrinsic.

Lemma 3.4.5. *Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \neq \emptyset$ and let $f \in \mathcal{N}(\sigma_S(T) \cup \{\infty\})$. Furthermore, consider a slice Cauchy domain U such that $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$ and some imaginary unit $j \in \mathbb{S}$. If $\gamma_1, \ldots, \gamma_N$ is the part of $\partial(U \cap \mathbb{C}_j)$ that lies in $\mathbb{C}_j^+$ as in Definition 2.1.34, then*

$$\begin{aligned}\int_{\partial(U\cap\mathbb{C}_j)} & f(s)\, ds_j\, S_R^{-1}(s,T)\\ &= \sum_{\ell=1}^{N} \int_0^1 2\mathrm{Re}\left(f(\gamma_\ell(t))(-j)\gamma_\ell'(t)\overline{\gamma_\ell(t)}\right) \mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\, dt \\ &\quad - \sum_{\ell=1}^{N} \int_0^1 2\mathrm{Re}\Big(f(\gamma_\ell(t))(-j)\gamma_\ell'(t)\Big) T \mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\, dt.\end{aligned} \tag{3.34}$$

Proof. We have

$$\begin{aligned}\int_{\partial(U\cap\mathbb{C}_j)} & f(s)\, ds_j\, S_R^{-1}(s,T)\\ &= \sum_{\ell=1}^{N} \int_{\gamma_\ell} f(s)\, ds_j\, S_R^{-1}(s,T) + \sum_{\ell=1}^{N} \int_{-\overline{\gamma_\ell}} f(s)\, ds_j\, S_R^{-1}(s,T)\\ &= \sum_{\ell=1}^{N} \int_0^1 f(\gamma_\ell(t))(-j)\gamma_\ell'(t) \left(\overline{\gamma_\ell(t)} - T\right) \mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\, dt \\ &\quad + \sum_{\ell=1}^{N} \int_0^1 f\left(\overline{\gamma_\ell(1-t)}\right) j \overline{\gamma_\ell'(1-t)} (\gamma_\ell(1-t) - T) \mathcal{Q}_{\overline{\gamma_\ell(1-t)}}(T)^{-1}\, dt.\end{aligned}$$

Since $f(\overline{s}) = \overline{f(s)}$ as f is intrinsic and $\mathcal{Q}_{\overline{s}}(T)^{-1} = \mathcal{Q}_s(T)^{-1}$ for $s \in \rho_S(T)$, we get, after a change of variables in the integrals of the second sum,

$$\begin{aligned}
&\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\, S_R^{-1}(s,T)\\
&= \sum_{\ell=1}^{N}\int_0^1 f(\gamma_\ell(t))(-j)\gamma_\ell'(t)\left(\overline{\gamma_\ell(t)} - T\right)\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\,dt\\
&\quad + \sum_{\ell=1}^{N}\int_0^1 \overline{f(\gamma_\ell(t))(-j)\gamma_\ell'(t)}(\gamma_\ell(t) - T)\mathcal{Q}_{\gamma(t)}(T)^{-1}\,dt\\
&= \sum_{\ell=1}^{N}\int_0^1 2\mathrm{Re}\left(f(\gamma_\ell(t))(-j)\gamma_\ell'(t)\overline{\gamma_\ell(t)}\right)\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\,dt\\
&\quad - \sum_{\ell=1}^{N}\int_0^1 2\mathrm{Re}\Big(f(\gamma_\ell(t))(-j)\gamma_\ell'(t)\Big)T\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\,dt. \qquad \square
\end{aligned}$$

Theorem 3.4.6. *Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \neq \emptyset$. If $f \in \mathcal{N}(\sigma_S(T)\cup\{\infty\})$, then*

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\,S_R^{-1}(s,T),$$

for any $j \in \mathbb{S}$ and any slice Cauchy domain as in Theorem 3.4.1.

Proof. Fix U and $j \in \mathbb{S}$, let $\gamma_1, \dots \gamma_N$ be the part of $\partial(U\cap\mathbb{C}_j)$ that lies in $\mathbb{C}_j^+$ and write the integral involving the right S-resolvent as an integral over these paths as in (3.34). Any operator commutes with real numbers and $f(\gamma_\ell(t))$, $\gamma_\ell'(t)$ and $\overline{\gamma_\ell(t)}$ commute mutually since they all belong to the same complex plane $\mathbb{C}_j$. Hence,

$$\begin{aligned}
&\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\, S_R^{-1}(s,T)\\
&= \sum_{\ell=1}^{N}\int_0^1 \mathcal{Q}_{\gamma(t)}(T)^{-1}2\mathrm{Re}\left(\overline{\gamma_\ell(t)}\gamma_\ell'(t)(-j)f(\gamma_\ell(t))\right)\,dt\\
&\quad - \sum_{\ell=1}^{N}\int_0^1 T\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}2\mathrm{Re}\Big(\gamma_\ell'(t)(-j)f(\gamma_\ell(t))\Big)\,dt\\
&= \sum_{\ell=1}^{N}\int_0^1 \left(T\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1} - \mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\overline{\gamma_\ell(t)}\right)\gamma_\ell'(t)(-j)f(\gamma_\ell(t))\,dt\\
&\quad + \sum_{\ell=1}^{N}\int_0^1 \left(T\mathcal{Q}_{\overline{\gamma_\ell(t)}}(T)^{-1} - \mathcal{Q}_{\overline{\gamma_\ell(t)}}(T)^{-1}\gamma_\ell(t)\right)\overline{\gamma_\ell'(t)}j\overline{f(\gamma_\ell(t))}\,dt
\end{aligned}$$

$$= \sum_{\ell=1}^{N} \int_{\gamma_\ell} S_L^{-1}(s,T)\, ds_j\, f(s) + \sum_{\ell=1}^{N} \int_{-\overline{\gamma_\ell}} S_L^{-1}(s,T)\, ds_j\, f(s)$$
$$= \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s). \qquad \square$$

Since $f(\infty) = \lim_{s\to\infty} f(s) \in \mathbb{R}$ as $f(s) \in \mathbb{R}$ for $s \in \mathbb{R}$ if f is intrinsic, we can rephrase the above result as,

Corollary 3.4.7. *Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \neq \emptyset$. The S-functional calculus for left slice hyperholomorphic functions and the S-functional calculus for right slice hyperholomorphic functions agree for intrinsic functions: if $f \in \mathcal{N}(\sigma_S(T) \cup \{\infty\})$, then (3.31) and (3.32) give the same operator.*

Remark 3.4.2. For intrinsic functions, slice hyperholomorphic Cauchy integrals of the form (3.31) and (3.32) are always equivalent. We have shown this only for the S-functional calculus, but with the same technique one can show this equivalence also for the H^∞-functional calculus or for fractional powers of quaternionic linear operators. Since the technique for showing this equivalence is the same in any situation, we will use it without proving it explicitly at every occurrence.

We have shown that the two versions of the S-functional calculus are consistent for intrinsic functions. However, there exist functions that are both left and right slice hyperholomorphic, but not intrinsic. We want to clarify the relation between the versions of the S-functional calculus for such functions and we start by characterising functions of this type.

Recall that a function f on U is called locally constant if every point $q \in U$ has a neighborhood $B_q \subset U$ such that f is constant on U. A locally constant function f is constant on every connected subset of its domain. Thus, since every sphere $[q]$ is connected, the function f is constant on every sphere if its domain U is axially symmetric, i.e., it is of the form $f(q) = c(u,v)$ for $q = u + jv$, where c is locally constant on an appropriate subset of $\mathbb{R}^2$. Therefore, f can be considered a left and a right slice function and it is even left and right slice hyperholomorphic because the partial derivatives of a locally constant function vanish.

Lemma 3.4.8. *A function f is both left and right slice hyperholomorphic if and only if $f = c + \tilde{f}$, where c is a locally constant slice function and $\tilde{f}$ is intrinsic slice hyperholomorphic.*

Proof. Obviously any function that admits a decomposition of this type is both left and right slice hyperholomorphic. Assume on the other hand that f is left and right slice hyperholomorphic such that for $q = u + jv$

$$f(q) = f_0(u,v) + jf_1(u,v)$$

and

$$f(q) = \hat{f}_0(u,v) + \hat{f}_1(u,v)j.$$

The compatibility condition (2.4) implies

$$f_0(u,v) = \frac{1}{2}\left(f(q) + f(\overline{q})\right) = \hat{f}_0(u,v),$$

from which we deduce $jf_1(u,v) = f(q_j) - f_0(u,v) = \hat{f}_1(u,v)j$ with $q_j = u + jv$ for any $j \in \mathbb{S}$. Hence, we have

$$jf_1(u,v)j^{-1} = \hat{f}_1(u,v).$$

If we choose j such that $f_1(u,v) \in \mathbb{C}_j$, then j and $f_1(u,v)$ commute and we obtain $f_1(u,v) = \hat{f}_1(u,v)$. We further conclude that $f_1(u,v)$ commutes with every $j \in \mathbb{S}$ because

$$jf_1(u,v) = \hat{f}_1(u,v)j = f_1(u,v)j.$$

This implies that $f_1(u,v)$ is real.

Since f_1 takes real values, its partial derivatives $\frac{\partial}{\partial u} f_1(u,v)$ and $\frac{\partial}{\partial v} f_1(u,v)$ are real-valued too. Thus, since f_0 and f_1 satisfy the Cauchy–Riemann equations (2.5), the partial derivatives of f_0 also take real-values.

Now define $\tilde{f}_0(u,v) = \mathrm{Re}(f_0(u,v))$ and $\tilde{f}_1(u,v) = f_1(u,v)$ and set $\tilde{f}(q) = \tilde{f}_0(u,v) + j\tilde{f}_1(u,v)$ and $c(q) = f(q) - \tilde{f}(q) = \mathrm{Im}(f_0(u,v))$ for $q = u + jv$. Obviously, $\tilde{f}_0$ and $\tilde{f}_1$ satisfy the compatibility condition (2.4). Moreover, the partial derivatives of $\tilde{f}_0$ and $\tilde{f}_1$ coincide with the partial derivatives of f_0 (resp. f_1). For $f_1 = \tilde{f}_1$ this is obvious and for $\tilde{f}_0$ this follows from

$$\frac{\partial}{\partial \nu}\tilde{f}_0(u,v) = \frac{\partial}{\partial \nu}\mathrm{Re}(f_0(u,v)) = \mathrm{Re}\left(\frac{\partial}{\partial \nu} f_0(u,v)\right) = \frac{\partial}{\partial \nu} f_0(u,v)$$

for $\nu \in \{u,v\}$ since $\frac{\partial}{\partial \nu} f_0(u,v)$ is real-valued by the above arguments. We conclude that $\tilde{f}_0$ and $\tilde{f}_1$ satisfy the Cauchy–Riemann equations (2.5) because f_0 and f_1 satisfy them. Therefore, $\tilde{f}$ is a left slice hyperholomorphic function with real-valued components, thus intrinsic.

It remains to show that c is locally constant. Since $c(q) = c(u + jv) = \mathrm{Im}(f_0(u,v))$ depends only on u and v but not on the imaginary unit j, it is constant on every sphere $[q] \subset U$. Moreover, as the sum of two left slice hyperholomorphic functions, it is left slice hyperholomorphic and thus, its restriction c_j to any complex plane $\mathbb{C}_j$ is a $\mathbb{H}$-valued left holomorphic function. But

$$c_j'(q) = \frac{\partial}{\partial q_0} c_j(q) = \frac{\partial}{\partial q_0} f(q) - \frac{\partial}{\partial q_0}\tilde{f}(q) = 0, \quad q \in U \cap \mathbb{C}_j$$

and hence c is locally constant on $U \cap \mathbb{C}_j$. If $q = u + jv \in U$, we can therefore find a neighborhood B_j of q in $U \cap \mathbb{C}_j$ such that c_j is constant on B_j. Since c is constant on every sphere, it is even constant on the axially symmetric hull $B = [B_j]$ of B_j, which is a neighborhood of q in U. □

Corollary 3.4.9. *Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \neq \emptyset$ and let f be both left and right slice hyperholomorphic on $\sigma_S(T)$ and at infinity. If $\mathcal{D}(f)$ is connected, then (3.31) and (3.32) give the same operator.*

Proof. By applying Lemma 3.4.8 we obtain a decomposition $f = c + \tilde{f}$ of f into the sum of a locally constant function c and an intrinsic function $\tilde{f}$. Since $\mathcal{D}(f)$ is connected, c is a constant function. Thus, Corollary 3.4.7 and Example 3.4.3 imply

$$\begin{aligned}
&f(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,T)\\
&= c\left(\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} ds_j\, S_R^{-1}(s,T)\right)\\
&\quad + \tilde{f}(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} \tilde{f}(s)\, ds_j\, S_R^{-1}(s,T)\\
&= c\mathcal{I} + \tilde{f}(T) = \mathcal{I}c + \tilde{f}(T)\\
&= \left(\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\right) c\\
&\quad + \tilde{f}(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, \tilde{f}(s)\\
&= f(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s),
\end{aligned}$$

where U and $j \in \mathbb{S}$ are chosen as in Definition 3.4.2. □

Remark 3.4.3. As we have shown, the two versions of the S-functional calculus are consistent for intrinsic slice hyperholomorphic functions and for functions defined on connected sets. However, in general, this is not true. If $\mathcal{D}(f)$ is not connected, then c is only locally constant, i.e., it is of the form $c(s) = \sum_\ell \chi_{\Delta_\ell}(s)c_\ell$ with $c_\ell \in \mathbb{H}$, where the Δ_ℓ are disjoint axially symmetric sets. The function $\chi_{\Delta_\ell}(s)$ is the characteristic function of Δ_ℓ, which is obviously intrinsic. The functional calculi for left and right slice hyperholomorphic functions yield then $c(T) = \sum_\ell \chi_{\Delta_\ell}(T)c_\ell$ and $c(T) = \sum_\ell c_\ell\chi_{\Delta_\ell}(T)$, respectively. These two operators coincide only if the operators $\chi_{\Delta_\ell}(T)$ commute with the scalars c_ℓ. As we will see in Section 3.7, the operators $\chi_{\Delta_\ell}(T)$ are projections onto invariant subspaces of the operator T. Since the operator T is right linear, its invariant subspaces are right subspaces of X. But if a projection $\chi_{\Delta_\ell}(T)$ commutes with any scalar, then

$$ay = a\chi_{\Delta_\ell}(T)y = \chi_\Delta(T)ay \in \chi_{\Delta_\ell}(T)X,$$

for any $y \in \chi_{\Delta_\ell}(T)X$ and any $a \in \mathbb{H}$. Thus, $\chi_{\Delta_\ell}(T)X$ is also a left-sided and therefore, even a two-sided subspace of X. In general, this is not true: the invariant subspaces obtained from spectral projections are only right-sided. Hence, the projections $\chi_{\Delta_\ell}(T)$ do not necessarily commute with any scalar and it might happen that

$$\sum_\ell \chi_{\Delta_\ell}(T)c_\ell \neq \sum_\ell c_\ell \chi_{\Delta_\ell}(T),$$

i.e., the two functional calculi give different operators for the same function. An explicit example for this situation is given in Example 3.7.9.

Finally, we show that the S-functional calculus admits, for intrinsic functions, a representation that only depends on the right linear structure of the space. In particular, this representation also shows the compatibility of the S-functional calculus and its classical counterpart form the theory of complex linear operators, the Riesz–Dunford functional calculus for holomorphic functions.

Definition 3.4.10. Let $T \in \mathcal{K}(X)$. We define the X-valued function

$$\mathcal{R}_s(T;y) = \mathcal{Q}_s(T)^{-1}y\overline{s} - T\mathcal{Q}_s(T)^{-1}y \quad \forall y \in X,\ s \in \rho_S(T).$$

Remark 3.4.4. By Theorem 3.1.8, the mapping $y \mapsto \mathcal{R}_s(T;y)$ coincides with the resolvent of T at s applied to y if T is considered a $\mathbb{C}_{j_s}$-linear operator on $X_{j_s} = X$.

Theorem 3.4.11. *Let $T \in \mathcal{K}(X)$ be a closed operator on a two-sided quaternionic Banach space X with $\rho_S(T) \neq \emptyset$ and let $f \in \mathcal{N}(\sigma_{SX}(T)))$. For any $j \in \mathbb{S}$ and any unbounded slice Cauchy domain U with $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(F)$, the operator $f(T)$ obtained via the S-functional calculus satisfies*

$$f(T)y = yf(\infty) + \int_{\partial(U\cap\mathbb{C}_j)} \mathcal{R}_z(T;y)f(z)\,dz\frac{-j}{2\pi} \quad \forall y \in X. \tag{3.35}$$

Proof. Let U be a slice Cauchy domain such that $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$. We then have for any $j \in \mathbb{S}$ and any $y \in X$ that

$$f(T)y = f(\infty)y + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\, S_R^{-1}(s,T)y. \tag{3.36}$$

If $\gamma_\ell : [0,1] \to \mathbb{C}_j^+$, $\ell = 1,\dots N$, is the part of $\partial(U\cap\mathbb{C}_j)$ that lies in $\mathbb{C}_j^+$ as in Definition 2.1.34, then we have by Lemma 3.4.5 that

$$\begin{aligned}
&\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\, S_R^{-1}(s,T)y\\
&= \sum_{\ell=1}^N \int_0^1 2\mathrm{Re}\left(f(\gamma_\ell(t))(-j)\gamma_\ell'(t)\overline{\gamma_\ell(t)}y\right)\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}y\,dt\\
&\quad - \sum_{\ell=1}^N \int_0^1 2\mathrm{Re}\Big(f(\gamma_\ell(t))(-j)\gamma_\ell'(t)\Big)T\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}y\,dt.
\end{aligned} \tag{3.37}$$

Since $\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}y$ and $TQ_{\gamma_\ell(t)}(T)^{-1}y$ commute with real numbers, we furthermore have

$$\begin{aligned}
&\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\,S_R^{-1}(s,T)y\\
&=\sum_{\ell=1}^{N}\int_0^1 \mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}y\,2\mathrm{Re}\left(f(\gamma_\ell(t))(-j)\gamma_\ell'(t)\overline{\gamma_\ell(t)}\right)\,dt\\
&\quad-\sum_{\ell=1}^{N}\int_0^1 T\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}y\,2\mathrm{Re}\Big(f(\gamma_\ell(t))(-j)\gamma_\ell'(t)\Big)\,dt\\
&=\sum_{\ell=1}^{N}\int_0^1 \left(\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}y\overline{\gamma_\ell(t)}-T\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}y\right)f(\gamma_\ell(t))\gamma_\ell'(t)\,dt(-j)\\
&\quad-\sum_{\ell=1}^{N}\int_0^1 \left(\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}y\gamma_\ell(t)-T\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}y\right)\overline{f(\gamma_\ell(t))\gamma_\ell'(t)}\,dt(-j).
\end{aligned}$$

Recalling that $f(\overline{x})=\overline{f(x)}$ because f is intrinsic, that $\mathcal{Q}_{\overline{s}}(T)^{-1}=\mathcal{Q}_s(T)^{-1}$ for any $s\in\rho_S(T)$ and that $(-\overline{\gamma_\ell})(t)=-\overline{\gamma_\ell'(1-t)}$, we thus find

$$\begin{aligned}
&\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\,S_R^{-1}(s,T)y\\
&=\sum_{\ell=1}^{N}\int_{\gamma_\ell}\left(\mathcal{Q}_z(T)^{-1}y\overline{z}-T\mathcal{Q}_z(T)^{-1}y\right)f(z)\,dz(-j)\\
&\quad+\sum_{\ell=1}^{N}\int_{-\overline{\gamma_\ell}}\left(\mathcal{Q}_z(T)^{-1}y\overline{z}-T\mathcal{Q}_z(T)^{-1}y\right)f(z)\,dt(-j)\\
&=\int_{\partial(U\cap\mathbb{C}_j)}\left(\mathcal{Q}_z(T)^{-1}y\overline{z}-T\mathcal{Q}_z(T)^{-1}y\right)f(z)\,dz(-j)\\
&=\int_{\partial(U\cap\mathbb{C}_j)}\mathcal{R}_z(T;y)f(z)\,dz(-j).
\end{aligned}$$

Finally, observe that $f(\infty)=\lim_{s\to\infty}f(s)\in\mathbb{R}$ because, as an intrinsic function, f takes only real values on the real line. Since any vector commutes with real numbers, we can hence rewrite (3.36) as

$$f(T)y=yf(\infty)+\int_{\partial(U\cap\mathbb{C}_j)}\mathcal{R}_z(T;y)f(z)\,dz\frac{(-j)}{2\pi}. \qquad \square$$

Remark 3.4.5. We point out that (3.35) contains neither the multiplication of vectors with non-real scalars from the left nor the multiplication of any operator with a non-real scalar. Hence, this expression is independent from the left multiplication defined on the X, cf. Remark 2.2.7. Instead, it shows that the operator $f(T)$

can be expressed in terms of only the right linear structure on the space X if f is intrinsic.

Remark 3.4.6. Theorem 3.4.11 shows that complex and quaternionic operator theory are consistent. Indeed, we can also obtain $f(T)$ by the following procedure: we choose $j \in \mathbb{S}$ and consider the complex numbers as embedded into the quaternions by identifying them with the plane $\mathbb{C}_j$ determined by j. The quaternionic Banach space X is then also a complex Banach space over $\mathbb{C}_j$ and we denote the space X considered as a complex Banach space over $\mathbb{C}_j$ by X_j. Any operator $T \in \mathcal{K}(X)$ is then also a complex linear operator on X_j. We have $\sigma_{\mathbb{C}_j}(T) = \sigma_S(T) \cap \mathbb{C}_j$ and $\mathcal{R}_s(T; y)$ is for $s \in \rho_{\mathbb{C}_j} = \rho_S(T) \cap \mathbb{C}_j$ exactly the resolvent of T as a complex linear operator on X_j, cf. Theorem 3.1.8. If $f \in \mathcal{N}(\sigma_S(T))$, then $f_j = f|_{\mathbb{C}_j}$ is a holomorphic function on $\sigma_{\mathbb{C}_j}(T)$ and the right-hand side of (3.35) is hence the formula that determines $f_j(T)$ in terms of the Riesz-Dunford functional calculus for T on X_j. (A similar relation also holds for other functional calculi such as the Phillips functional calculus or the continuous functional calculus.) The converse is however not true: if f_j is an arbitrary holomorphic function on a neighborhood of $\sigma_{\mathbb{C}_j}(T)$ in $\mathbb{C}_j$, then $f_j(T)$ obtained by the Riesz–Dunford functional calculus does not coincide with the operator $f(T)$ obtained by applying the S-functional calculus to $f = \operatorname{ext}_L(f_j)$. This is only true if f is an intrinsic function. Indeed, $f_j(T)$ is otherwise only $\mathbb{C}_j$-linear, but not necessarily quaternionic linear.

3.5 The product rule and polynomials in T

One of the most important properties of the S-functional calculus is the product rule.

Theorem 3.5.1 (Product Rule)**.** *Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \neq \emptyset$. If $f \in \mathcal{N}(\sigma_S(T) \cup \{\infty\})$ and $g \in \mathcal{SH}_L(\sigma_S(T) \cup \{\infty\})$, then*

$$(fg)(T) = f(T)g(T). \tag{3.38}$$

Similarly, if $f \in \mathcal{SH}_R(\sigma_S(T) \cup \{\infty\})$ and $g \in \mathcal{N}(\sigma_S(T) \cup \{\infty\})$, then the product rule (3.38) also holds true.

Proof. Let $f \in \mathcal{N}(\sigma_S(\sigma_S(T) \cup \{\infty\})$ and let $g \in \mathcal{SH}_L(\sigma_S(T) \cup \{\infty\})$. By Theorem 3.4.1, there exist unbounded slice Cauchy domains U_p and U_s such that $\sigma_S(T) \subset U_p$ and $\overline{U_p} \subset U_s$ and $\overline{U_s} \subset \mathcal{D}(f) \cap \mathcal{D}(g)$. The subscripts s and p indicate the respective variable of integration in the following computation. Moreover, we use the notation $[\partial O]_j := \partial(O \cap \mathbb{C}_j)$ for an axially symmetric set O in order to obtain compacter formulas.

Recall that the operator $f(T)$ can, by Theorem 3.4.6, also be represented using the right S-resolvent operator and hence

$$f(T)g(T) = \left(f(\infty)\mathcal{I} + \frac{1}{2\pi} \int_{[\partial U_s]_j} f(s)\, ds_j\, S_R^{-1}(s,T) \right) \cdot \left(g(\infty)\mathcal{I} + \frac{1}{2\pi} \int_{[\partial U_p]_j} S_L^{-1}(p,T)\, dp_j\, g(p) \right).$$

For the product of the integrals, the S-resolvent equation gives us that

$$\begin{aligned}
&\int_{[\partial U_s]_j} f(s)\, ds_j\, S_R^{-1}(s,T) \int_{[\partial U_p]_j} S_L^{-1}(p,T)\, dp_j\, g(p)\\
&= \int_{[\partial U_s]_j} \int_{[\partial U_p]_j} f(s)\, ds_j\, S_R^{-1}(s,T) S_L^{-1}(p,T)\, dp_j\, g(p)\\
&= \int_{[\partial U_s]_j} \int_{[\partial U_p]_j} f(s)\, ds_j\, S_R^{-1}(s,T) p(p^2 - 2s_0 p + |s|^2)^{-1}\, dp_j\, g(p)\\
&\quad - \int_{[\partial U_s]_j} \int_{[\partial U_p]_j} f(s)\, ds_j\, S_L^{-1}(p,T) p(p^2 - 2s_0 p + |s|^2)^{-1}\, dp_j\, g(p)\\
&\quad - \int_{[\partial U_s]_j} \int_{[\partial U_p]_j} f(s)\, ds_j\, \overline{s} S_R^{-1}(s,T) (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_j\, g(p)\\
&\quad + \int_{[\partial U_s]_j} \int_{[\partial U_p]_j} f(s)\, ds_j\, \overline{s} S_L^{-1}(p,T) (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_j\, g(p).
\end{aligned}$$

For the sake of readability, let us denote these last four integrals by $I_1, \dots I_4$.

If $r > 0$ is large enough, then $\mathbb{H} \setminus U_s$ is entirely contained in $B_r(0)$. In particular, $W := B_r(0) \cap U_p$ is then a bounded slice Cauchy domain with boundary

$$\partial(W \cap \mathbb{C}_j) = \partial(U_p \cap \mathbb{C}_j) \cup \partial(B_r(0) \cap \mathbb{C}_j).$$

From Lemma 2.2.24, we deduce

$$\begin{aligned}
I_1 &= \int_{[\partial U_s]_j} f(s)\, ds_j\, S_R^{-1}(s,T) \int_{[\partial U_p]_j} p(p^2 - 2s_0 p + |s|^2)^{-1}\, dp_j\, g(p)\\
&= \int_{[\partial U_s]_j} f(s)\, ds_j\, S_R^{-1}(s,T) \int_{[\partial W]_j} p(p^2 - 2s_0 p + |s|^2)^{-1}\, dp_j\, g(p)\\
&\quad - \int_{[\partial U_s]_j} f(s)\, ds_j\, S_R^{-1}(s,T) \int_{[\partial B_r(0)]_j} p(p^2 - 2s_0 p + |s|^2)^{-1}\, dp_j\, g(p)\\
&= - \int_{[\partial U_s]_j} f(s)\, ds_j\, S_R^{-1}(s,T) \int_{[\partial B_r(0)]_j} p(p^2 - 2s_0 p + |s|^2)^{-1}\, dp_j\, g(p),
\end{aligned}$$

where the last equality follows from the Cauchy integral theorem since, by our choice of U_s and U_p, the function $p \mapsto p(p^2 - 2s_0 p + |s|^2)^{-1}$ is left slice hyperholomorphic and the function $p \mapsto g(p)$ is right slice hyperholomorphic on $\overline{W}$. If we let r tend to $+\infty$ and apply Lebesgue's theorem in order to exchange limit and integration, the inner integral tends to $2\pi g(\infty)$ and hence

$$I_1 = -2\pi \left(\int_{[\partial U_s]_j} f(s)\, ds_j\, S_R^{-1}(s,T) \right) g(\infty).$$

We also have

$$-I_2 + I_4 = \int_{[\partial U_s]_j} \int_{[\partial U_p]_j} f(s)\, ds_j\, \left(\overline{s} S_L^{-1}(p,T) - p S_L^{-1}(p,T) \right) \cdot (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_j\, g(p)$$

and applying Fubini's theorem allows us to change the order of integration. If we now set $W = B_r(0) \cap U_s$ with r sufficiently large, we obtain, as before, a bounded slice Cauchy domain with $\partial(W \cap \mathbb{C}_j) = \partial(U_s \cap \mathbb{C}_j) \cup \partial(B_r(0) \cap \mathbb{C}_j)$. Applying Lemma 2.2.24 with $B = S_L^{-1}(p,T)$, we find

$$\begin{aligned}
-I_2 + I_4 = & \int_{[\partial U_p]_j} \int_{[\partial W]_j} f(s)\, ds_j\, \left(\overline{s} S_L^{-1}(p,T) - p S_L^{-1}(p,T) \right) \cdot \\
& \qquad \cdot (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_j\, g(p) \\
& - \int_{[\partial U_p]_j} \int_{[\partial B_r(0)]_j} f(s)\, ds_j\, \left(\overline{s} S_L^{-1}(p,T) - p S_L^{-1}(p,T) \right) \cdot \\
& \qquad \cdot (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_j\, g(p) \\
= & \ 2\pi \int_{[\partial U_p]_j} S_L^{-1}(p,T) f(p)\, dp_j\, g(p) \\
& - \int_{[\partial U_p]_j} \int_{[\partial B_r(0)]_j} f(s)\, ds_j\, \overline{s} S_L^{-1}(p,T) (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_j\, g(p) \\
& - \int_{[\partial U_p]_j} \int_{[\partial B_r(0)]_j} f(s)\, ds_j\, p S_L^{-1}(p,T) (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_j\, g(p).
\end{aligned}$$

Observe that the third integral tends to zero as $r \to +\infty$. For the second one, by applying Lebesgue's theorem, we obtain

$$\begin{aligned}
& \int_{[\partial U_p]_j} \int_{[\partial B_r(0)]_j} f(s)\, ds_j\, \overline{s} S_L^{-1}(p,T) (p^2 - 2s_0 p + |s|^2)^{-1}\, dp_j\, g(p) \\
& = \int_{[\partial U_p]_j} \left(\int_0^{2\pi} f(re^{i\phi}) r^2 S_L^{-1}(p,T) (p^2 - 2r\cos(\phi) p + r^2)^{-1}\, d\phi \right) dp_j\, g(p) \\
& \overset{r \to +\infty}{\longrightarrow} 2\pi f(\infty) \int_{[\partial U_p]_j} S_L^{-1}(p,T)\, dp_j\, g(p).
\end{aligned}$$

Since f is intrinsic, $f(p)$ commutes with dp_j, and hence

$$-I_2 + I_4 = 2\pi \int_{[\partial U_p]_j} S_L^{-1}(p,T)\, dp_j\, f(p)g(p) \\ - 2\pi f(\infty) \int_{[\partial U_p]_j} S_L^{-1}(p,T)\, dp_j\, g(p).$$

Finally, we consider the integral I_3. If we set again $W = B_r(0) \cap U_p$ with r sufficiently large, then

$$-I_3 = - \int_{[\partial U_s]_j} \int_{[\partial W]_j} f(s)\, ds_j\, \overline{s} S_R^{-1}(s,T)(p^2 - 2s_0 p + |s|^2)^{-1}\, dp_j\, g(p) \\ + \int_{[\partial U_s]_j} \int_{[\partial B_r(0)]_j} f(s)\, ds_j\, \overline{s} S_R^{-1}(s,T)(p^2 - 2s_0 p + |s|^2)^{-1}\, dp_j\, g(p).$$

By our choice of U_s and U_p, the functions $p \mapsto (p^2 - 2s_0 p + |s|^2)^{-1}$ and $p \mapsto g(p)$ are left (resp. right) slice hyperholomorphic on $\overline{W}$. Hence, Cauchy's integral theorem implies that the first integral equals zero. Letting r tend to infinity, we can apply Lebesgue's theorem in order to exchange limit and integration and we see that

$$-I_3 = \int_{[\partial U_s]_j} \int_{[\partial B_r(0)]_j} f(s)\, ds_j\, \overline{s} S_R^{-1}(s,T)(p^2 - 2s_0 p + |s|^2)^{-1}\, dp_j\, g(p) \to 0.$$

Altogether, we obtain

$$\frac{1}{(2\pi)^2} \int_{[\partial U_s]_j} f(s)\, ds_j\, S_R^{-1}(s,T) \int_{[\partial U_p]_j} S_L^{-1}(p,T)\, dp_j\, g(p) \\ = -\frac{1}{2\pi} \left(\int_{[\partial U_s]_j} f(s)\, ds_j\, S_R^{-1}(s,T) \right) g(\infty) + \frac{1}{2\pi} \int_{[\partial U_p]_j} S_L^{-1}(p,T)\, dp_j\, f(p)g(p) \\ - f(\infty) \frac{1}{2\pi} \int_{[\partial U_p]_j} S_L^{-1}(p,T)\, dp_j\, g(p).$$

We thus have

$$f(T)g(T) = f(\infty)g(\infty)\mathcal{I} + f(\infty) \frac{1}{2\pi} \int_{[\partial U_p]_j} S_L^{-1}(p,T)\, dp_j\, g(p) \\ + \left(\frac{1}{2\pi} \int_{[\partial U_s]_j} f(s)\, ds_j\, S_R^{-1}(s,T) \right) g(\infty) \\ + \frac{1}{(2\pi)^2} \int_{[\partial U_s]_j} f(s)\, ds_j\, S_R^{-1}(s,T) \int_{[\partial U_p]_j} S_L^{-1}(p,T)\, dp_j\, g(p) \\ = f(\infty)g(\infty)\mathcal{I} + \frac{1}{2\pi} \int_{[\partial U_p]_j} S_L^{-1}(p,T)\, dp_j\, f(p)g(p) = (fg)(T). \qquad \square$$

If the operator T is bounded, then slice hyperholomorphic polynomials of T belong to the class of functions that are admissible within S-functional calculus. In the unbounded case, this is not true, but the S-functional calculus is in some sense still compatible, at least with intrinsic polynomials. For such polynomial $P(s) = \sum_{k=0}^{n} a_k s^k$ with $a_k \in \mathbb{R}$, the operator $P(T)$ is as usual defined as the operator

$$P(T)y := \sum_{k=0}^{n} a_k T^k y, \quad y \in \mathcal{D}(T^n).$$

Lemma 3.5.2. *Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \neq \emptyset$, let $f \in \mathcal{N}(\sigma_S(T) \cup \{\infty\})$ and let P be an intrinsic polynomial of degree $n \in \mathbb{N}_0$. If $y \in \mathcal{D}(T^n)$, then $f(T)y \in \mathcal{D}(T^n)$ and*

$$f(T)P(T)y = P(T)f(T)y.$$

Proof. We consider first the special case $P(s) = s$. Let U be an unbounded slice Cauchy domain with $\sigma_S(T) \subset U$, let $j \in \mathbb{S}$ and let $\{\gamma_1, \ldots, \gamma_n\}$ be the part of $\partial(U \cap \mathbb{C}_j)$ in $\mathbb{C}_j^+$ as in Definition 2.1.34. We apply Lemma 3.4.5 and write

$$\begin{aligned}
&\int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,T)\\
&= \sum_{\ell=1}^{N} \int_0^1 2\mathrm{Re}\left(f(\gamma_\ell(t))(-j)\gamma_\ell'(t)\overline{\gamma_\ell(t)}\right) \mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\, dt\\
&\quad - \sum_{\ell=1}^{N} \int_0^1 2\mathrm{Re}\Big(f(\gamma_\ell(t))(-j)\gamma_\ell'(t)\Big) T\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}\, dt.
\end{aligned}$$

Observe that $\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}Ty = T\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}y$ for $y \in \mathcal{D}(T)$ and that T also commutes with real numbers. By applying Hille's theorem for the Bochner integral, Theorem 20 in [110, Chapter III.6], we can move T in front of the integral and find

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,T)Ty\\
&= \sum_{\ell=1}^{n} T\frac{1}{2\pi} \int_0^1 2\mathrm{Re}\left(f(\gamma_\ell(t))(-j)\gamma_\ell'(t)\overline{\gamma_\ell(t)}\right) \mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}y\\
&\quad - \sum_{\ell=1}^{n} T\frac{1}{2\pi} \int_0^1 2\mathrm{Re}\Big(f(\gamma_\ell(t))(-j)\gamma_\ell'(t)\Big) T\mathcal{Q}_{\gamma_\ell(t)}(T)^{-1}y\\
&= T\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,T)y,
\end{aligned}$$

where the last equation follows again from Lemma 3.4.5. Finally, observe that $f(\infty) = \lim_{s\to\infty} f(s)$ is real since $f(s) \in \mathbb{R}$ for any $s \in \mathbb{R}$ because f is intrinsic.

Hence, we find

$$\begin{aligned} f(T)Ty &= f(\infty)Ty + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\,S_R^{-1}(s,T)Ty \\ &= Tf(\infty)y + T\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\,S_R^{-1}(s,T)y = Tf(T)y. \end{aligned}$$

In particular, this implies $f(T)y \in \mathcal{D}(T)$.

We show the general statement by induction with respect to the degree n of the polynomial. If $n = 0$ then the statement follows immediately from Example 3.4.3. Now assume that it is true for $n-1$ and consider $P(s) = a_k s^n + P_{n-1}(s)$, where $a_n \in \mathbb{R}$ and $P_{n-1}(s)$ is an intrinsic polynomial of degree lower or equal to $n-1$. For $y \in \mathcal{D}(T^n)$ the above argumentation implies then $f(T)T^{n-1}y \in \mathcal{D}(T)$ and

$$\begin{aligned} f(T)P(T)y &= f(T)a_nT^ny + f(T)P_{n-1}(T)y \\ &= a_nTf(T)T^{n-1}y + f(T)P_{n-1}(T)y. \end{aligned}$$

From the induction hypothesis, we further deduce $f(T)T^{n-1}y = T^{n-1}f(T)y$ and $f(T)P_{n-1}(T)y = P_{n-1}(T)f(T)y$ and hence

$$f(T)P(T)y = a_nT^nf(T)y + P_{n-1}(T)f(T)y = P(T)f(T)y.$$

In particular, we see that $f(T)y$ belongs to $\mathcal{D}(T^n)$ and we obtain that the statement is true. □

Remark 3.5.1. We only considered intrinsic polynomials in Lemma 3.5.2 because only multiplying with such functions yields again a slice hyperholomorphic function. However, even the definition of $P(T)$ is not straightforward if P does not have real coefficients. Indeed, if T is unbounded and $P(s) = \sum_{k=0}^n s^k a_k$ with $a_k \notin \mathbb{R}$, then setting $P(T)v = \sum_{k=0}^n T^k a_k v$ might not be meaningful for all $v \in \mathcal{D}(T^n)$. Unless $\mathcal{D}(T^n)$ is a two-sided subspace of X, it is not clear that $a_k v \in \mathcal{D}(T^k)$ even if $v \in \mathcal{D}(T)$.

As in the complex case, we say that f has a zero of order n at ∞ if the first $n-1$ coefficients in the Taylor series expansion of $s \mapsto f(s^{-1})$ at 0 vanish and the n-th coefficient does not. Equivalently, f has a zero of order n if $\lim_{s\to\infty} f(s)s^n$ is bounded and nonzero. We say that f has a zero of infinite order at infinity, if it vanishes on a neighborhood of ∞.

Lemma 3.5.3. *Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \neq \emptyset$ and assume that $f \in \mathcal{N}(\sigma_S(T)\cup\{\infty\})$ has a zero of order $n \in \mathbb{N}_0 \cup \{+\infty\}$ at infinity.*

(i) *For any intrinsic polynomial P of degree lower than or equal to n, we have $P(T)f(T) = (Pf)(T)$.*

(ii) *If $y \in \mathcal{D}(T^m)$ for some $m \in \mathbb{N}_0 \cup \{\infty\}$, then $f(T)y \in \mathcal{D}(T^{m+n})$.*

Proof. Assume first that f has a zero of order greater than or equal to one at infinity and consider $P(s) = s$. Then $Pf \in \mathcal{N}(\sigma_S(T) \cup \{\infty\})$ and for $y \in X$

$$(Pf)(T)y = \lim_{s\to\infty} sf(s)y + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, sf(s)y,$$

with an appropriate slice Cauchy domain U and any imaginary unit $j \in \mathbb{S}$. Since s and ds_j commute, we deduce from the left S-resolvent equation that

$$\begin{aligned}&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, sf(s)y \\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} TS_L^{-1}(s,T)\, ds_j\, f(s)y + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} ds_j\, f(s)y\end{aligned}$$

Any sufficiently large Ball $\overline{B_r(0)}$ contains ∂U. The function $f(s)y$ is then right slice hyperholomorphic on $\overline{B_r(0) \cap U}$ and Cauchy's integral theorem implies

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} ds_j\, f(s)y &= \lim_{r\to+\infty} -\frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_j)} ds_j\, f(s)y \\ &= \lim_{r\to+\infty} -\frac{1}{2\pi}\int_0^{2\pi} re^{j\varphi} f(re^{j\varphi})y\, d\varphi = -\lim_{s\to+\infty} sf(s)y.\end{aligned}$$

Thus, after applying Hille's theorem for the Bochner integral, Theorem 20 in [110, Chapter III.6], in order to write the operator T in front of the integral, we obtain

$$(Pf)(T)y = T\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s)y = P(T)f(T)y.$$

In particular, we see that $f(T)y \in \mathcal{D}(T)$.

We show (i) for monomials by induction and assume that it is true for $P(s) = s^{n-1}$ if f has a zero of order greater than or equal to $n-1$ at infinity. If the order of f at infinity is even greater than or equal to n, then $g(s) = s^{n-1}f(s)$ has a zero of order at least 1 at infinity and, from the above argumentation and the induction hypothesis, we conclude for $P(s) = s^n$

$$(Pf)(T)y = Tg(T)y = TT^{n-1}f(T)y = T^n f(T)y,$$

which implies also $f(T)y \in \mathcal{D}(T^n)$. For arbitrary intrinsic polynomials the statement finally follows from the linearity of the S-functional calculus.

In order to show (ii) assume first $y \in \mathcal{D}(T^m)$ for $m \in \mathbb{N}$. If f has a zero of order $n \in \mathbb{N}$ at infinity, then (i) with $P(s) = s^n$ and Lemma 3.5.2 imply

$$(Pf)(T)T^m y = T^n f(T)T^m y = T^n T^m f(T)y = T^{m+n} f(T)y$$

and hence $f(T)y \in \mathcal{D}(T^{m+n})$. Finally, if $m = +\infty$ then $y \in \mathcal{D}(T^k)$ and hence $f(T)y \in \mathcal{D}(T^{k+n})$ for any $k \in \mathbb{N}$. Thus, $y \in \mathcal{D}(T^\infty)$. □

Corollary 3.5.4. *Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \neq \emptyset$. For any intrinsic polynomial P, the operator $P(T)$ is closed.*

Proof. We choose $s \in \rho_S(T)$ and $n \in \mathbb{N}$ such that $m \leq 2n$, where m is the degree of P. Then $f(p) = P(p)\mathcal{Q}_s(p)^{-n}$ belongs to $\mathcal{N}(\sigma_S(T) \cup \{\infty\})$ and has a zero of order $2n - m$ at infinity. Applying Lemma 3.5.3, we see that

$$P(T)y = P(T)\mathcal{Q}_s(T)^n \mathcal{Q}_s(T)^{-n}y = \mathcal{Q}_s(T)^n P(T)\mathcal{Q}_s(T)^{-n}y = \mathcal{Q}_s(T)^n f(T)y$$

for $y \in \mathcal{D}(T^m)$. Since its inverse is bounded, the operator $\mathcal{Q}_s(T)^n$ is closed and in turn $P(T)$ is closed as it is the composition of a closed and a bounded operator. □

Corollary 3.5.5. *Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \neq \emptyset$. If $f \in \mathcal{N}(\sigma_S(T) \cup \{\infty\})$ does not have any zeros on $\sigma_S(T)$ and a zero of even order n at infinity, then* $\operatorname{ran}(f(T)) = \mathcal{D}(T^n)$ *and $f(T)$ is invertible in the sense of closed operators. If $\rho_S(T) \cap \mathbb{R} \neq \emptyset$, this holds true for any order $n \in \mathbb{N}$.*

Proof. Let $p \in \rho_S(T)$ and set $k = n/2$. The function $h(s) = f(s)\mathcal{Q}_p(s)^k$ with $\mathcal{Q}_p(s) = s^2 - 2p_0 s + |p|^2$ belongs to $\mathcal{N}(\sigma_S(T) \cup \{\infty\})$ and does not have any zeros in $\sigma_S(T)$. Furthermore, $h(\infty) = \lim_{s\to\infty} h(s)$ is finite and nonzero. Hence, the function $s \mapsto h(s)^{-1}$ belongs to $\mathcal{N}(\sigma_S(T) \cup \{\infty\})$ and we deduce from Theorem 3.5.1 that $h(T)$ is invertible in $\mathcal{B}(X)$ with $h(T)^{-1} = h^{-1}(T)$. Theorem 3.5.1 moreover implies $f(T) = \mathcal{Q}_p(T)^{-k}h(T)$. Now observe that $h(T)$ maps X bijectively onto X and that $\mathcal{Q}_p(T)^{-k}$ maps X onto $\mathcal{D}(T^{2k}) = \mathcal{D}(T^n)$. Thus $\operatorname{ran}(f(T)) = \mathcal{D}(T^n)$.

Finally, $f(T)^{-1} := h^{-1}(T)\mathcal{Q}_p(T)^k$ is a closed operator because h is bijective and continuous and $\mathcal{Q}_p(T)^k$ is closed by Corollary 3.5.4. So it satisfies $f(T)^{-1}f(T)y = y$ for $y \in X$ and $f(T)f(T)^{-1}y = y$ for $y \in \mathcal{D}(T^n)$. Thus, it is the inverse of $f(T)$.

In the case there exists a point $a \in \rho_S(T) \cap \mathbb{R}$, similar arguments hold with $P(s) = (s-a)^n$ instead of $\mathcal{Q}_p(s)^k$. In particular, this allows us to include functions with a zero of odd order at infinity too. □

We conclude this section by determining the slice derivatives of the left and right S-resolvents of T as an application of the above theorems.

Definition 3.5.6. Let $T \in \mathcal{B}(X)$ and let $s \in \rho_S(T)$. For $n \geq 0$, we define

$$S_L^{-n}(s,T) := \sum_{k=0}^{n} (-1)^k \binom{n}{k} T^k \mathcal{Q}_s(T)^{-n} \overline{s}^{n-k}$$

and, similarly, we define

$$S_R^{-n}(s,T) := \sum_{k=0}^{n} (-1)^k \binom{n}{k} \overline{s}^{n-k} T^k \mathcal{Q}_s(T)^{-n}.$$

Remark 3.5.2. Since the function $Q_s(q)^{-n}$ is intrinsic, the above definitions are due to the product rule compatibility with the S-functional calculus, that is,

$$\left[S_L^{-n}(s,\cdot)\right](T) = S_L^{-n}(s,T) \quad \text{and} \quad \left[S_R^{-n}(s,\cdot)\right](T) = S_R^{-n}(s,T).$$

Proposition 3.5.7. *Let $T \in \mathcal{B}(X)$ and let $s \in \rho_S(T)$. Then*

$$\partial_S{}^m S_L^{-1}(s,T) = (-1)^m m!\, S_L^{-(m+1)}(s,T) \tag{3.39}$$

and

$$\partial_S{}^m S_R^{-1}(s,T) = (-1)^m m!\, S_R^{-(m+1)}(s,T), \tag{3.40}$$

for any $m \geq 0$.

Proof. Recall that the slice derivative coincides with the partial derivative with respect to the real part of s. We show only (3.39), since (3.40) follows by analogous computations.

We prove the statement by induction. For $m = 0$, the identity (3.39) is obvious. We assume that $\partial_S{}^{m-1} S_L^{-1}(s,T) = (-1)^{m-1}(m-1)!\, S_L^{-m}(s,T)$ and we compute $\partial_S{}^m S_L^{-1}(s,T)$. We represent $S_L^{-m}(s,T)$ using the S-functional calculus. If we choose the path of integration $\partial(U \cap \mathbb{C}_j)$ in the complex plane $\mathbb{C}_j$ that contains s, then $S_L^{-m}(s,p) = (s-p)^{-m}$ for any $p \in \partial(U \cap \mathbb{C}_j)$ so that

$$\begin{aligned}
\partial_S S_L^{-m}(s,T) &= \partial_S \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(p,T)\, dp_j\, S_L^{-m}(s,p) \\
&= \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(p,T)\, dp_j\, \frac{\partial}{\partial s_0}(s-p)^{-m} \\
&= -m \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(p,T)\, dp_j\, (s-p)^{-(m+1)} \\
&= -m\, S_L^{-(m+1)}(s,T),
\end{aligned}$$

and in turn,

$$\begin{aligned}
\partial_S{}^m S_L^{-1}(s,T) &= \partial_S \left(\partial_S{}^{m-1} S_L^{-1}(s,T)\right) \\
&= (-1)^{m-1}(m-1)! \partial_S S_L^{-m}(s,T) = (-1)^m m!\, S_L^{-(m+1)}(s,T). \qquad \square
\end{aligned}$$

3.6 The spectral mapping theorem

We recall that the extended S-spectrum $\sigma_{SX}(T)$ equals $\sigma_S(T)$ if T is bounded and it equals $\sigma_S(T) \cup \{\infty\}$ if T is unbounded.

Theorem 3.6.1 (Spectral Mapping Theorem)**.** *Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \neq \emptyset$. For any function $f \in \mathcal{N}(\sigma_S(T) \cup \{\infty\})$, we have $\sigma_S(f(T)) = f(\sigma_{SX}(T))$.*

Proof. Let us first show the relation $\sigma_S(f(T)) \supset f(\sigma_{SX}(T))$. For $p \in \sigma_S(T)$ consider the function

$$g(s) := (f(s)^2 - 2\mathrm{Re}(f(p))f(s) - |f(p)|^2)(s^2 - 2\mathrm{Re}(p)s + |p|^2)^{-1},$$

which is defined on $\mathcal{D}(f) \setminus [p]$. If we set $p_{j_s} = p_0 + j_s p_1$, then p_{j_s} and s commute. Since f is intrinsic, it maps $\mathbb{C}_j$ into $\mathbb{C}_j$ and hence $f(p_{j_s})$ and $f(s)$ commute, too. Thus

$$g(s) = \frac{(f(s) - f(p_{j_s}))(f(s) - \overline{f(p_{j_s})})}{(s - p_{j_s})(s - \overline{p_{j_s}})}$$

and we can extend g to all of $\mathcal{D}(f)$ by setting

$$g(s) = \begin{cases} \partial_S f(s) \left(f(p)\, p^{-1} \right), & s \in [p] \text{ if } p \notin \mathbb{R}, \\ (\partial_S f(s))^2, & s = p, \text{ if } p \in \mathbb{R}. \end{cases} \tag{3.41}$$

Now observe that

$$(s^2 - 2\mathrm{Re}(p)s + |p|^2)g(s) = f(s)^2 + 2\mathrm{Re}(f(p))f(s) + |f(p)|^2$$

and that g has a zero of order greater or equal to 2 at infinity. Hence, we can apply the S-functional calculus to deduce from Lemma 3.5.3, Theorem 3.5.1 and Example 3.4.3 that

$$(T^2 - 2\mathrm{Re}(p)T + |p|^2\mathcal{I})g(T)y = (f(T)^2 + 2\mathrm{Re}(f(p))f(T) + |f(p)|\mathcal{I})y,$$

for any $y \in X$ and

$$g(T)(T^2 - 2\mathrm{Re}(p)T + |p|^2\mathcal{I})y = (f(T)^2 + 2\mathrm{Re}(f(p))f(T) + |f(p)|\mathcal{I})y,$$

for $y \in \mathcal{D}(T^2)$. If $f(p) \in \rho_S(T)$, then

$$\mathcal{Q}_{f(p)}(f(T)) = f(T)^2 - 2\mathrm{Re}(f(p))f(T) + |f(p)|\mathcal{I}$$

is invertible and

$$\mathcal{Q}_{f(p)}(f(T))^{-1}g(T) = g(T)\mathcal{Q}_{f(p)}(f(T))^{-1}$$

is the inverse of the operator $\mathcal{Q}_p(T) = T^2 - 2\mathrm{Re}(p)T + |p|^2\mathcal{I}$. Hence, $f(p) \notin \sigma_S(f(T))$ implies $p \notin \sigma_S(T)$ and as a consequence $p \in \sigma_S(T)$ implies $f(p) \in \sigma_S(T)$, that is $f(\sigma_S(T)) \subset \sigma_S(f(T))$.

Finally, observe that $f(\infty) = \lim_{p\to\infty} f(p)$ is real because f is intrinsic and thus takes real values on the real line. If T is unbounded and $f(\infty) \neq f(p)$ for any point $p \in \sigma_S(T)$ (otherwise we already have $f(\infty) \in f(\sigma_S(T)) \subset \sigma_S(f(T))$), then the function $h(s) = (f(s) - f(\infty))^2$ belongs to $\mathcal{N}(\sigma_S(T) \cup \{\infty\})$ and has a zero of even order n at infinity but no zero in $\sigma_S(T)$. By Corollary 3.5.5, the range of $h(T) = \mathcal{Q}_{f(\infty)}(f(T))$ is $\mathcal{D}(T^n)$. Thus, it does not admit a bounded inverse and we obtain $f(\infty) \in \sigma_S(f(T))$. Altogether, we have $f(\sigma_{SX}(T)) \subset \sigma_S(f(T))$.

In order to show the relation $\sigma_S(f(T)) \subset f(\sigma_{SX}(T))$, we first consider a point $c \in \sigma_S(f(T))$ such that $c \neq f(\infty)$. We want to show $c \in f(\sigma_S(T))$ and assume the converse, i.e., $f(s) - c$ has no zeros on $\sigma_S(T)$.

If c is real, then the function $h(s) = f(s) - c$ is intrinsic, has no zeros on $\sigma_S(T)$ and $\lim_{s\to\infty} h(s) = f(\infty) - c \neq 0$. Hence, $h^{-1}(s) = (f(s) - c)^{-1}$ belongs to $\mathcal{N}(\sigma_S(T) \cup \{\infty\})$. Applying the S-functional calculus, we deduce from Theorem 3.5.1 that $h^{-1}(T)$ is the inverse of $f(T) - c\mathcal{I}$ and hence $\mathcal{Q}_c(f(T))^{-1} = (h^{-1}(T))^2$, which is a contradiction as $c \in \sigma_S(f(T))$. Thus, $c = f(p)$ for some $p \in \sigma_S(T)$.

If on the other hand $c = c_0 + ic_1$ is not real, then $f - c_j \neq 0$ for any $c_j = c_0 + jc_1 \in [c]$. Indeed, $f(p) = f_0(u, v) + kf_1(u, v) = c_0 + jc_1$ for $p = u + kv$ would imply $k = j$ and $f_0(u, v) = c_0$ and $f_1(u, v) = c_1$ as f_0 and f_1 are real-valued because f is intrinsic. This would in turn imply $f(p_i) = f(u + iv) = f_0(u, v) + if_1(u, v) = c$ for $p = u + iv$, which would contradict our assumption. Therefore, the function

$$h(s) = (f(s)^2 - 2\mathrm{Re}(c)f(s) + |c|^2) = (f(s) - c_{j_s})(f(s) - \overline{c_{j_s}})$$

with $c_{j_s} = c_0 + jc_2$ for $s = u + jv$ does not have any zeros on $\sigma_S(T)$. Moreover, since $f(\infty)$ is real, we have

$$h(\infty) = (f(\infty) - c)\overline{(f(\infty) - c)} = |f(\infty) - c|^2 \neq 0$$

and hence $h^{-1}(s) = (f(s)^2 - 2\mathrm{Re}(c)f(s) + |c|^2)^{-1}$ belongs to $\mathcal{N}(\sigma_S(T) \cup \{\infty\})$. Applying the S-functional calculus, we deduce again from Theorem 3.5.1 that the operator $h^{-1}(T)$ is the inverse of $\mathcal{Q}_c(T)$, which contradicts $c \in \sigma_S(f(T))$. Hence, there must exist some $p \in \sigma_S(T)$ such that $c = f(p)$.

Altogether, we obtain $\sigma_S(f(T)) \setminus \{f(\infty)\}$ is contained in $f(\sigma_S(T))$.

Finally, let us consider the case that the point $c = f(\infty)$ belongs to $\sigma_S(f(T))$. If T is unbounded, then $\infty \in \sigma_{SX}(T)$ and hence $c \in f(\sigma_{SX}(T))$. If on the other hand T is bounded, then there exists a function $g \in \mathcal{N}(\sigma_S(T) \cup \{\infty\})$ that coincides on an axially symmetric neighborhood $\sigma_S(T)$ with f but satisfies $c \neq g(\infty)$. In this case $f(T) = g(T)$, as pointed out in Remark 3.4.1, and we can apply the above argumentation with g instead of f to see that $c \in g(\sigma_S(T)) = f(\sigma_S(T))$. □

Theorem 3.6.2. *If $T \in \mathcal{K}(X)$ with $\sigma_S(T) \neq \emptyset$, then $P(\sigma_S(T)) = \sigma_S(P(T))$ for any intrinsic polynomial P.*

Proof. The arguments are similar to those in the proof of Theorem 3.6.1: in order to show $P(\sigma_S(T)) \subset \sigma_S(P(T))$, we consider the polynomial $\mathcal{Q}_{P(p)}(P(s))$, which is given by $\mathcal{Q}_{P(p)}(P(s)) = P(s)^2 - 2\mathrm{Re}(P(p))P(s) + |P(p)|^2$ for any $p \in \sigma_S(T)$. As p and $\overline{p}$ are both zeros of $\mathcal{Q}_{P(p)}(P(s))$ (resp. as p is a zero of even order of $\mathcal{Q}_{P(p)}(P(s)) = (P(s) - P(p))^2$ if p is real), there exists an intrinsic polynomial $R(s)$ such that

$$\mathcal{Q}_{P(p)}(P(s)) = \mathcal{Q}_p(s)R(s).$$

If $P(p) \notin \sigma_S(P(T))$, then $\mathcal{Q}_{P(p)}(P(T))$ is invertible and Lemma 3.5.3 and Example 3.4.3 imply that $\mathcal{Q}_{P(p)}(P(T))^{-1}R(T)$ is the inverse of $\mathcal{Q}_p(T)$, which is a contradiction because we assumed $p \in \sigma_S(T)$. Therefore $P(p) \in \sigma_S(P(T))$.

Conversely assume that $p \notin P(\sigma_S(T))$. Then the function

$$\mathcal{Q}_p(P(s)) = P(s)^2 - 2\mathrm{Re}(p)P(s) + |p|^2$$

does not take any zero on $\sigma_S(T)$ and we conclude from Corollary 3.5.5 that $\mathcal{Q}_p(P(T))$ has a bounded inverse. Thus $p \notin \sigma_S(P(T))$ and in turn $\sigma_S(P(T)) \subset P(\sigma_S(T))$. $\square$

Theorem 3.6.3 (Composition rule)**.** *Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \neq \emptyset$. If $f \in \mathcal{N}(\sigma_S(T) \cup \{\infty\})$ and $g \in \mathcal{SH}_L(f(\sigma_{SX}(T))$ or $g \in \mathcal{SH}_R(f(\sigma_{SX}(T))$, then*

$$(g \circ f)(T) = g(f(T)).$$

Proof. Because of Remark 3.4.1, we can assume that $f(\infty)$ belongs to $f(\sigma_{SX}(T))$. We apply Theorem 2.1.31 in order to choose an unbounded slice Cauchy domain U_p such that $\sigma_S(f(T)) = f(\sigma_{SX}(T)) \subset U_p$ and $\overline{U_p} \subset \mathcal{D}(g)$ and a second unbounded slice Cauchy domain U_s such that $\sigma_S(T) \subset U_s$ and $\overline{U_s} \subset f^{-1}(U_p) \cap \mathcal{D}(f)$. The subscripts are chosen in order to indicate the respective variable of integration in the following computation.

After choosing an imaginary unit $j \in \mathbb{S}$, we deduce from Cauchy's integral formula, that

$$\begin{aligned}
&(g \circ f)(T) - (g \circ f)(\infty)\mathcal{I} \\
&= \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, (g \circ f)(s) \\
&= \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j \left(\frac{1}{2\pi} \int_{\partial(U_p \cap \mathbb{C}_j)} S_L^{-1}(p, f(s))\, dp_j\, g(p) \right).
\end{aligned}$$

Changing the order of integration by applying Fubini's theorem, we obtain

$$\begin{aligned}
&(g \circ f)(T) - (g \circ f)(\infty)\mathcal{I} \\
&= \frac{1}{2\pi} \int_{\partial(U_p \cap \mathbb{C}_j)} \left(\frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, S_L^{-1}(p, f(s)) \right) dp_j\, g(p) \\
&= \frac{1}{2\pi} \int_{\partial(U_p \cap \mathbb{C}_j)} S_L^{-1}(p, f(T))\, dp_j\, g(p) \\
&\quad - \frac{1}{2\pi} \int_{\partial(U_p \cap \mathbb{C}_j)} S_L^{-1}(p, f(\infty))\, dp_j\, g(p)\mathcal{I} \\
&= g(f(T)) - g(f(\infty))\mathcal{I}
\end{aligned}$$

and hence $(g \circ f)(T) = g(f(T))$. $\square$

3.7 Spectral sets and projections onto invariant subspaces

As in the complex case, the S-functional calculus allows to associate subspaces of X that are invariant under T to certain subsets of $\sigma_S(T)$.

Definition 3.7.1 (Spectral set)**.** A subset σ of $\sigma_{SX}(T)$ is called a spectral set if it is open and closed in $\sigma_{SX}(T)$.

Just as $\sigma_S(T)$ and $\sigma_{SX}(T)$, every spectral set is axially symmetric: if $s \in \sigma$ then the entire sphere $[s]$ is contained in σ. Indeed, the set $\sigma \cap [s]$ is then a nonempty, open and closed subset of $\sigma_{SX}(T) \cap [s] = [s]$. Since $[s]$ is connected this implies $\sigma \cap [s] = [s]$. Moreover, if σ is a spectral set, then $\sigma' = \sigma_{SX}(T) \setminus \sigma$ is a spectral set, too.

If σ is a spectral set of T, then σ and σ' can be separated in $\mathbb{H}_\infty$ by axially symmetric open sets and hence Theorem 2.1.31 implies the existence of two slice Cauchy domains U_σ and $U_{\sigma'}$ containing σ and σ', respectively, such that one of them is unbounded and $\overline{U} \cap \overline{U_{\sigma'}} = \emptyset$. We define

$$\chi_\sigma(x) := \begin{cases} 1 & \text{if } x \in U_\sigma, \\ 0 & \text{if } x \in U_{\sigma'}. \end{cases}$$

The function $\chi_\sigma(x)$ obviously belongs to $\mathcal{N}(\sigma_S(T) \cup \{\infty\})$.

Definition 3.7.2 (Spectral projection)**.** Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \neq \emptyset$ and let $\sigma \subset \sigma_S(T)$ be a spectral set of T. The spectral projection associated with σ is the operator $E_\sigma := \chi_\sigma(T)$ obtained by applying the S-functional calculus to the function χ_σ. Furthermore, we define $X_\sigma := E_\sigma X$ and $T_\sigma = T|_{\mathcal{D}(T_\sigma)}$ with $\mathcal{D}(T_\sigma) = \mathcal{D}(T) \cap X_\sigma$.

Explicit formulas for the operator E_σ are for bounded σ are given by

$$E_\sigma = \frac{1}{2\pi} \int_{\partial(U_\sigma \cap \mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j = \frac{1}{2\pi} \int_{\partial(U_\sigma \cap \mathbb{C}_j)} ds_j\, S_R^{-1}(s,T)$$

and for unbounded σ

$$E_\sigma = \mathcal{I} + \frac{1}{2\pi} \int_{\partial(U_\sigma \cap \mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j = \mathcal{I} + \frac{1}{2\pi} \int_{\partial(U_\sigma \cap \mathbb{C}_j)} ds_j\, S_R^{-1}(s,T),$$

where the imaginary unit $j \in \mathbb{S}$ can be chosen arbitrarily.

Corollary 3.7.3. *Let $T \in \mathcal{K}(X)$ such that $\rho_S(T) \neq \emptyset$ and let σ be a spectral set of T.*

(i) *The operator E_σ is a projection, i.e., $E_\sigma^2 = E_\sigma$.*

(ii) *Set $\sigma' = \sigma_{SX}(T) \setminus \sigma$. Then $E_\sigma + E_{\sigma'} = \mathcal{I}$ and $E_\sigma E_{\sigma'} = E_{\sigma'} E_\sigma = 0$.*

Proof. This follows immediately from the algebraic properties of the S-functional calculus shown in Corollary 3.4.4 and Theorem 3.5.1 as $\chi_\sigma^2 = \chi_\sigma$ and $\chi_\sigma + \chi_{\sigma'} = 1$ and $\chi_\sigma \chi_{\sigma'} = \chi_{\sigma'} \chi_\sigma = 0$. □

The following Lemma 3.7.4 is a special case of [47, Chapter II §1.9, Proposition 14] and Lemma 3.7.5 is an immediate consequence of the fact that any projection with closed range is continuous.

Lemma 3.7.4. *Let A, B, M and N be right linear subspaces of a quaternionic right vector space X_R such that $A \subset M$ and $B \subset M$. If $A \oplus B = M \oplus N$, then $A = M$ and $B = N$.*

Lemma 3.7.5. *Let A, B, M and N be right linear subspaces of a quaternionic Banach vector space X_R such that $A \subset M$, $B \subset N$ and such that M, N and $M \oplus N$ are closed. Then $A \oplus B$ is dense in $M \oplus N$ if and only if A is dense in M and B is dense in N.*

Definition 3.7.6. Let $T : D(T) \to X$. We split the S-spectrum into the three disjoint sets:

(P) The point S-spectrum of T:

$$\sigma_{Sp}(T) = \{s \in \mathbb{H} : \quad \ker(\mathcal{Q}_s(T)) \neq \{0\}\}.$$

(R) The residual S-spectrum of T:

$$\sigma_{Sr}(T) = \left\{ s \in \mathbb{H} : \quad \ker(\mathcal{Q}_s(T)) = \{0\},\ \overline{\operatorname{ran}(\mathcal{Q}_s(T))} \neq X \right\}.$$

(C) The continuous S-spectrum of T:

$$\sigma_{Sc}(T) = \left\{ s \in \mathbb{H} : \quad \ker(\mathcal{Q}_s(T)) = \{0\},\ \overline{\operatorname{ran}(\mathcal{Q}_s(T))} = X,\ \mathcal{Q}_s(T)^{-1} \notin \mathcal{B}(X) \right\}.$$

There are different possible ways to split the S-spectrum. We refer to Section 9.2 in [57] for more details and comments.

Theorem 3.7.7. *Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \neq \emptyset$ and let $E_1, E_2 \in \mathcal{B}(X)$ be projections such that $E_1 + E_2 = \mathcal{I}$ (and hence $E_1E_2 = E_2E_1 = 0$). Denote $X_\ell := E_\ell(X)$ and $\mathcal{D}(T_\ell) := E_\ell(\mathcal{D}(T))$ and assume that $T(\mathcal{D}(T_\ell)) \subset X_\ell$ such that $T_\ell := T|_{\mathcal{D}(T_\ell)}$ is a closed operator on the right Banach space X_ℓ for $\ell = 1, 2$. Then*

(i) $E_\ell T y = T E_\ell y$ *for* $y \in \mathcal{D}(T)$,

(ii) $\mathcal{D}(T_\ell^2) = E_\ell(\mathcal{D}(T^2))$ *for* $\ell = 1, 2$,

(iii) $\operatorname{ran}(\mathcal{Q}_s(T)) = \operatorname{ran}(\mathcal{Q}_s(T_1)) \oplus \operatorname{ran}(\mathcal{Q}_s(T_2))$, *for any* $s \in \mathbb{H}$,

(iv) $\sigma_S(T) = \sigma_S(T_1) \cup \sigma_S(T_2)$ *and*

(v) $\sigma_{Sp}(T) = \sigma_{Sp}(T_1) \cup \sigma_{Sp}(T_2)$.

If moreover $\sigma_S(T_1) \cap \sigma_S(T_2) = \emptyset$, *then*

(vi) $\sigma_{Sc}(T) = \sigma_{Sc}(T_1) \cup \sigma_{Sc}(T_2)$ *and*

(vii) $\sigma_{Sr}(T) = \sigma_{Sr}(T_1) \cup \sigma_{Sr}(T_2)$.

Proof. The assertions (i) to (iii) are obvious. Now assume that $s \in \rho_S(T)$. Then $\operatorname{ran}(\mathcal{Q}_s(T)) = X$ and from (iii) we deduce

$$X_1 \oplus X_2 = X = \operatorname{ran}(\mathcal{Q}_s(T)) = \operatorname{ran}(\mathcal{Q}_s(T_1)) \oplus \operatorname{ran}(\mathcal{Q}_s(T_2)).$$

As $\operatorname{ran}(\mathcal{Q}_s(T_\ell)) \subset X_\ell$, Lemma 3.7.4 implies $\operatorname{ran}(\mathcal{Q}_s(T_\ell)) = X_\ell$ and hence $\mathcal{Q}_s(T_\ell)^{-1} = \mathcal{Q}_s(T)^{-1}|_{X_\ell}$ as $\mathcal{Q}_s(T_\ell) = \mathcal{Q}_s(T)|_{\mathcal{D}(T_\ell^2)}$. Indeed, we have

$$\mathcal{Q}_s(T)^{-1}\mathcal{Q}_s(T_\ell)y = \mathcal{Q}_s(T)^{-1}\mathcal{Q}_s(T)y = y \quad \text{for } y \in \mathcal{D}(T_\ell^2)$$

and, since $\mathcal{Q}_s(T)^{-1}y \in \mathcal{D}(T_\ell^2)$ for $y \in X_\ell$, also

$$\mathcal{Q}_s(T_\ell)\mathcal{Q}_s(T)^{-1}y = \mathcal{Q}_s(T)\mathcal{Q}_s(T)^{-1}y = y \quad \text{for } y \in X_\ell.$$

Thus, $s \in \rho_S(T_1) \cap \rho_S(T_2)$. Conversely, if $s \in \rho_S(T_1) \cap \rho_S(T_2)$, then the operator $\mathcal{Q}_s(T_1)^{-1}E_1 + \mathcal{Q}_s(T_2)^{-1}E_2$ is the inverse of $\mathcal{Q}_s(T)$ and hence $s \in \rho_S(T)$. Altogether, $\rho_S(T) = \rho_S(T_1) \cap \rho_S(T_2)$, which is equivalent to $\sigma_S(T) = \sigma_S(T_1) \cup \sigma_S(T_2)$ and hence (iv) holds true.

Obviously, $\sigma_{Sp}(T_\ell) \subset \sigma_{Sp}(T)$ as any S-eigenvector of T_ℓ is also an S-eigenvector of T associated with the same eigensphere. Conversely, if $y \neq 0$ is an S-eigenvector of T associated with the eigensphere $[s] = s_0 + \mathbb{S}s_1$, then set $y_\ell = E_\ell y$ and we observe that

$$0 = \mathcal{Q}_s(T)y = \mathcal{Q}_s(T_1)y_1 + \mathcal{Q}_s(T_2)y_2.$$

As $\mathcal{Q}_s(T_\ell)y_\ell \in X_\ell$ and $X_1 \cap X_2 = \{0\}$, this implies $\mathcal{Q}_s(T_\ell)y_\ell = 0$ for $\ell = 1, 2$. As $y \neq 0$, at least one of the vectors y_ℓ is nonzero and therefore an S-eigenvalue of T_ℓ associated with the eigensphere $[s]$. Thus $[s] \subset \sigma_{Sp}(T_1) \cup \sigma_{Sp}(T_2)$ and in turn $\sigma_{Sp}(T) = \sigma_{Sp}(T_1) \cup \sigma_{Sp}(T_2)$ so that (v) holds true.

We assume now that $\sigma_S(T_1) \cap \sigma_S(T_2) = \emptyset$. Then assertions (iv) and (v) imply that $s \in \sigma_{Sc}(T) \cup \sigma_{Sr}(T)$ if and only if $s \in \sigma_{Sc}(T_\ell) \cup \sigma_{Sr}(T_\ell)$ for either $\ell = 1$ or $\ell = 2$. We assume without loss of generality $s \in \sigma_{Sc}(T_1) \cup \sigma_{Sr}(T_1)$ and thus $s \in \rho_S(T_2)$. As $\operatorname{ran}(\mathcal{Q}_s(T_2)) = X_2$, we deduce from (iii) and Lemma 3.7.5 that $\operatorname{ran}(\mathcal{Q}_s(T))$ is dense in $X = X_1 \oplus X_2$ if and only if $\operatorname{ran}(\mathcal{Q}_s(T_1))$ is dense in X. In other words, $s \in \sigma_{Sc}(T)$ if and only if $s \in \sigma_{Sc}(T_1)$ and in turn $s \in \sigma_{Sr}(T)$ if and only if $s \in \sigma_{Sr}(T_1)$. □

Theorem 3.7.8. *Let* $T \in \mathcal{K}(X)$ *with* $\rho_S(T) \neq \emptyset$ *and let* $\sigma \subset \sigma_S(T)$ *be a spectral set of* T. *Then*

(i) $E_\sigma(\mathcal{D}(T)) \subset \mathcal{D}(T)$,

(ii) $T(\mathcal{D}(T) \cap X_\sigma) \subset X_\sigma$,

(iii) $\sigma = \sigma_{SX}(T_\sigma)$,

(iv) $\sigma \cap \sigma_{Sp}(T) = \sigma_{Sp}(T_\sigma)$,

(v) $\sigma \cap \sigma_{Sc}(T) = \sigma_{Sc}(T_\sigma)$,

(vi) $\sigma \cap \sigma_{Sr}(T) = \sigma_{Sr}(T_\sigma)$.

If the spectral set σ is bounded, then we further have:

(vii) $X_\sigma \subset \mathcal{D}(T^\infty)$ *and*

(viii) T_σ *is a bounded operator on* X_σ.

Proof. Assertion (i) follows from the definition of E_σ and Lemma 3.5.2. In order to prove (ii), we observe that if $y \in \mathcal{D}(T) \cap X_\sigma$, then $E_\sigma y = y$. Hence, we deduce from Lemma 3.5.2 that $E_\sigma T y = T E_\sigma y = Ty$, which implies $Ty \in X_\sigma$.

If σ is bounded, then we can choose U_σ bounded and hence χ_σ has a zero of infinite order at infinity. We conclude from Lemma 3.5.3 that $y = E_\sigma y = \chi_\sigma(T)y \in \mathcal{D}(T^\infty)$ for any $y \in X_\sigma$ and hence (vii) holds true. In particular, $X_\sigma \subset \mathcal{D}(T)$. Therefore, T_σ is a bounded operator on X_σ as it is closed and everywhere defined.

We show now assertion (iii) and consider first a point $s \in \mathbb{H} \setminus \sigma$. We show that s belongs to $\rho_S(T_\sigma)$. For an appropriately chosen slice Cauchy domain U_σ, the function $f(s) := \mathcal{Q}_s(p)^{-1}\chi_{U_\sigma}(s)$ belongs to $\mathcal{N}(\sigma_S(T) \cup \{\infty\})$. By Lemma 3.5.3 and Lemma 3.5.2, we have

$$f(T)\mathcal{Q}_s(T)y = \chi_{U_\sigma}(T)y = E_\sigma y, \quad \text{for } y \in \mathcal{D}(T^2) \cap X_\sigma$$

and

$$\mathcal{Q}_s(T)f(T)y = \chi_{U_\sigma}(T)y = E_\sigma y = y \quad \text{for } y \in X_\sigma.$$

Hence, $\mathcal{Q}_s(T_\sigma) = \mathcal{Q}_s(T)|_{X_\sigma \cap \mathcal{D}(T^2)}$ has the inverse $f(T)|_{X_\sigma} \in \mathcal{B}(X_\sigma)$. Thus, we find $s \in \rho_S(T_\sigma)$ and in turn $\sigma_S(T_\sigma) \subset \sigma \cap \mathbb{H} =: \sigma_1$. The same arguments applied to $T_{\sigma'}$ with $\sigma' = \sigma_{SX}(T) \setminus \sigma$ show that $\sigma_S(T_{\sigma'}) \subset \sigma' \cap \mathbb{H} := \sigma_2$. But by (iv) in Theorem 3.7.7, we have

$$\sigma_S(T_\sigma) \cup \sigma_S(T_\sigma) = \sigma_S(T) = \sigma_1 \cup \sigma_2$$

and hence $\sigma_S(T_\sigma) = \sigma_1 = \sigma \cap \mathbb{H}$ and $\sigma_S(T_{\sigma'}) = \sigma_2 = \sigma' \cap \mathbb{H}$. If σ is bounded, then this is equivalent to (iii) because of (viii). If σ is not bounded, then $\infty \in \sigma$ and T is not bounded on X. However, in this case σ' is bounded and hence $T_\sigma \in \mathcal{B}(X_\sigma)$. But as $T = T_\sigma E_\sigma + T_{\sigma'} E_{\sigma'}$, we conclude that T_σ is unbounded as T is unbounded. Hence, $\infty \in \sigma_{SX}(T_\sigma)$ and (viii) holds true also in this case.

Finally, (iv) to (vi) are direct consequences of (v) to (vii) in Theorem 3.7.7 as we know now that $\sigma_S(T_\sigma)$ and $\sigma_S(T_{\sigma'})$ are disjoint. $\square$

Example 3.7.9. We choose a generating basis j, i and $k := ji$ of $\mathbb{H}$ and consider the quaternionic right-linear operator T on $X = \mathbb{H}^2$ that is defined by its action on the two right linearly independent right eigenvectors $y_1 = (1, j)^T$ and $y_2 = (i, -k)^T$, namely

$$\begin{pmatrix}1\\ j\end{pmatrix} \mapsto \begin{pmatrix}0\\ 0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}i\\ -k\end{pmatrix} \mapsto \begin{pmatrix}-k\\ -i\end{pmatrix} = \begin{pmatrix}i\\ -k\end{pmatrix} j.$$

Its matrix representation is

$$T = \frac{1}{2}\begin{pmatrix}-j & 1\\ -1 & -j\end{pmatrix}.$$

Since, for operators on finite-dimensional spaces, the S-spectrum coincides with the set of right-eigenvalues by Theorem 3.1.6, we have $\sigma_S(T) = \sigma_R(T) = \{0\} \cup \mathbb{S}$. Indeed, we have

$$\begin{aligned}\mathcal{Q}_s(T) &= \frac{1}{2}\begin{pmatrix}-1 & -j\\ j & -1\end{pmatrix} - s_0\begin{pmatrix}-j & 1\\ -1 & -j\end{pmatrix} + |s|^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}-\frac{1}{2} + |s|^2 + s_0 j & -s_0 - \frac{1}{2}j\\ s_0 + \frac{1}{2}j & -\frac{1}{2} + |s|^2 + s_0 j\end{pmatrix}\end{aligned}$$

and hence

$$\mathcal{Q}_s(T)^{-1} = |s|^{-2}(-1 + 2js_0 + |s|^2)^{-1}\begin{pmatrix}-\frac{1}{2} + |s|^2 + js_0 & \frac{1}{2}j + s_0\\ -\frac{1}{2}j - s_0 & -\frac{1}{2} + |s|^2 + js_0\end{pmatrix},$$

which is defined for any $s \notin \{0\} \cup \mathbb{S}$. For any $s \in \rho_S(T)$, the left S-resolvent is therefore given by

$$\begin{aligned}S_L^{-1}(s,T) = &\frac{1}{2}|s|^{-2}(-1 + |s|^2 + 2js_0)^{-1}\\ &\cdot \begin{pmatrix}|s|^2(j + 2\overline{s}) + \overline{s}(-1 + 2js_0) & -|s|^2 + \overline{s}(j + 2s_0)\\ |s|^2 - \overline{s}(j + 2s_0) & |s|^2(j + 2\overline{s}) + \overline{s}(-1 + 2js_0)\end{pmatrix}.\end{aligned}$$

Since $\sigma_S(T) \cap \mathbb{C}_j = \{0, j, -j\}$, we choose $U_{\{0\}} = B_{1/2}(0)$ and set $U_{\mathbb{S}} = B_2(0) \setminus B_{2/3}(0)$. For $s = \frac{1}{2}e^{j\varphi} \in \partial U_{\{0\}}(0) \cap \mathbb{C}_j$, we have

$$\begin{aligned}S_L^{-1}(s,T) = &2e^{-j\varphi}\left(3j + 4\mathrm{Re}\left(e^{j\varphi}\right)\right)^{-1}\\ &\cdot \begin{pmatrix}j + e^{j\varphi} + 2\cos(\varphi) & 2 + je^{j\varphi} + 2j\cos\varphi\\ -2 - je^{j\varphi} + 2j\cos\varphi & j + e^{j\varphi} + 2\cos\varphi\end{pmatrix}\end{aligned}$$

and so

$$
\begin{aligned}
E_{\{0\}} &= \frac{1}{2\pi}\int_{\partial(U_{\{0\}}\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\\
&= \frac{1}{2\pi}\int_0^{2\pi} 2e^{-j\varphi}\left(3j+4\mathrm{Re}\left(e^{j\varphi}\right)\right)^{-1}\cdot\\
&\qquad\cdot\begin{pmatrix} j+e^{j\varphi}+2\cos(\varphi) & 2+je^{j\varphi}-2j\cos\varphi\\ -2-je^{j\varphi}+2j\cos\varphi & j+e^{j\varphi}+2\cos\varphi\end{pmatrix}\frac{1}{2}e^{j\varphi}j(-j)\,d\varphi\\
&= \frac{1}{2}\begin{pmatrix}1 & -j\\ j & 1\end{pmatrix}.
\end{aligned}
$$

A similar computation shows that

$$
E_{\mathbb{S}} = \frac{1}{2\pi}\int_{\partial(U_{\mathbb{S}}\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j = \frac{1}{2}\begin{pmatrix}1 & j\\ -j & 1\end{pmatrix}.
$$

Straightforward calculations show that these matrices actually define projections on $\mathbb{H}^2$ with $E_{\{0\}}+E_{\mathbb{S}}=\mathcal{I}$. Moreover, we have $E_{\{0\}}y_1 = y_1$ and $E_{\mathbb{S}}y_1 = 0$ as well as $E_{\{0\}}y_2 = 0$ and $E_{\mathbb{S}}y_2 = y_2$. Thus, the invariant subspace $E_{\{0\}}X$ associated with the spectral set $\{0\}$ is the right linear span of y_1, which consist of all eigenvectors with respect to the real eigenvalue 0 as $T(y_1a) = T(y_1)a = 0$ for all $a\in\mathbb{H}$. The invariant subspace $E_{\mathbb{S}}$ associated with the spectral set $\mathbb{S}$ consists of the right linear span of y_2. For $a\in\mathbb{H}\setminus\{0\}$, we have $T(y_2a) = T(y_2)a = y_2ja = (y_2a)(a^{-1}ja)$. Thus, as $a^{-1}ja\in\mathbb{S}$, the subspace $E_{\mathbb{S}}$ consists of all right eigenvectors associated with eigenvalues in $\mathbb{S}$. (This is true only because the associated subspace is one-dimensional! Otherwise, the subspace would consist of sums of eigenvectors associated with possibly different eigenvalues in the sphere $\mathbb{S}$. Such vectors are in general not right eigenvectors, but they are S-eigenvectors associated with the eigensphere $\mathbb{S}$.)

Finally, we can construct functions, which are left and right slice hyperholomorphic on $\sigma_S(T)$, but for which the S-functional calculi for left and right slice hyperholomorphic functions yield different operators: consider the function

$$
f(s) = c_1\chi_{U_{\{0\}}}(s) + c_2\chi_{U_{\mathbb{S}}}(s)
$$

such that c_1 or c_2 does not belong to $\mathbb{C}_j$. Choose for instance $c_1 = i$ and $c_2 = 0$ for the sake of simplicity. This function is a locally constant slice function on $U = U_{\{0\}}\cup U_{\mathbb{S}}$ and thus left and right slice hyperholomorphic by Lemma 3.4.8. Then

$$
\begin{aligned}
\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j f(s) &= \left(\frac{1}{2\pi}\int_{\partial(B_{1/2}(0)\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\right) i\\
&= \frac{1}{2}\begin{pmatrix}1 & -j\\ j & 1\end{pmatrix} i = \frac{1}{2}\begin{pmatrix}i & -k\\ k & i\end{pmatrix},
\end{aligned}
$$

but

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\,S_R^{-1}(s,T) = i\left(\frac{1}{2\pi}\int_{\partial(B_{1/2}(0)\cap\mathbb{C}_j)} ds_j\,S_R^{-1}(s,T)\right)$$
$$= \frac{1}{2}i\begin{pmatrix}1 & -j\\ j & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}i & k\\ -k & i\end{pmatrix}.$$

The reason for why we obtain different operators is that the spectral projections $E_{\mathbb{S}}$ and $E_{\{0\}}$ cannot commute with arbitrary scalars because the respective invariant subspaces are not two-sided. Indeed, $-iy_2 = (1,j) = y_1$, which obviously does not belong to $E_{\mathbb{S}}X$.

3.8 The special roles of intrinsic functions and the left multiplication

As we saw in this chapter, the role of intrinsic slice hyperholomorphic functions stands out in quaternionic operator theory. Important results such as the product rule, the spectral mapping theorem and the composition rule only hold for these functions. This is not surprising since, on the level of functions, slice hyperholomorphicity is only compatible with multiplication and composition of intrinsic functions, not of arbitrary slice hyperholomorphic functions. There exists, however, a deeper, more fundamental reason for this special role of intrinsic functions that we want to explain in the following.

A functional calculus for an operator T is a mathematical method that allows to define an operator $f(T)$ such that $f(T)$ generalizes the mapping behavior of T for each f in a certain class of functions on the spectrum of T (for instance the class of holomorphic, continuous, measurable or slice hyperholomorphic functions on the spectrum of T). This is useful for generating new operators, and it is also useful for understanding the operator T itself. The way $f(T)$ changes as f varies in the corresponding class of functions gives information about T and allows to identify, for instance, eigenspaces, invariant subspaces, or, if T is a normal operator on a Hilbert space, even its spectral resolution. This is, however, only possible if the mapping behavior of T and $f(T)$ are related in a suitable way. Intuitively, the operator $f(T)$ should be obtained by letting f act on spectral values of T. In particular, if $v \in X \setminus \{0\}$ is an eigenvector of T associated with s, i.e.,

$$Tv = vs, \tag{3.42}$$

then v should be an eigenvector of $f(T)$ associated with $f(s)$, i.e.,

$$f(T)v = vf(s). \tag{3.43}$$

One of the fundamental peculiarities of operator theory in the quaternionic setting is the axial symmetry of the set of eigenvalues and the S-spectrum of an operator. In particular, if (3.42) holds and $h \in \mathbb{H} \setminus \{0\}$, then

$$T(vh) = (Tv)h = vsh = (vh)(h^{-1}sh). \tag{3.44}$$

Consequently, if (3.42) implies (3.43), then vh is an eigenvector of $f(T)$ associated with $f(h^{-1}sh)$, that is

$$f(T)(vh) = (vh)f(h^{-1}sh). \tag{3.45}$$

On the other hand, (3.43) implies

$$f(T)(vh) = (f(T)v)h = vf(s)h = (vh)(h^{-1}f(s)h). \tag{3.46}$$

Combining (3.45) and (3.46), we find that f must satisfy

$$f(h^{-1}sh) = h^{-1}f(s)h \quad \forall h \in \mathbb{H} \setminus \{0\}. \tag{3.47}$$

Now assume that $s = u + jv \in \mathbb{H}$ and choose $h = j$. Since s and j commute, we conclude from (3.47) that

$$jf(s) = jf(j^{-1}sj) = jj^{-1}f(s)j = f(s)j.$$

A quaternion commutes with j if and only it belongs to $\mathbb{C}_j$. Hence, $f(s) \in \mathbb{C}_j$ and so $f(s)$ belongs to the same complex plane as s. Let $\alpha, \beta \in \mathbb{R}$ such that $f(s) = \alpha + j\beta$. If $\tilde{s} = u + iv \in [s]$ with $i \in \mathbb{S}$ arbitrary, then there exists $h \in \mathbb{H} \setminus \{0\}$ such that $\tilde{s} = h^{-1}sh$. Furthermore, we conclude from

$$u + iv = \tilde{s} = h^{-1}sh = u + h^{-1}jhv$$

that $i = h^{-1}jh$. The identity (3.47) then implies

$$f(u + iv) = f(\tilde{s}) = f(h^{-1}sh) = h^{-1}f(s)h = \alpha + (h^{-1}jh)\beta = \alpha + i\beta.$$

Setting $f_0(u, v) := \alpha \in \mathbb{R}$ and $f_1(u, v) := \beta \in \mathbb{R}$, we find

$$f(u + iv) = f_0(u, v) + if_1(u, v), \quad \forall i \in \mathbb{S}.$$

Thus, f is an intrinsic slice function. In the quaternionic setting, any proper functional calculus must therefore necessarily apply to a class of intrinsic slice functions—otherwise it does not follow the most fundamental intuition of such calculus, namely that (3.42) implies (3.43), and the mapping behavior of $f(T)$ is not related with the mapping behavior of T.

This explains why the S-functional calculus shows undesirable properties when non-intrinsic functions are considered, such as the voidness of the product rule and the spectral mapping theorem or such as the inconsistencies between the S-functional calculi for left- and right slice hyperholomorphic functions. (These phenomena are not restricted to the S-functional calculus but appear, due to the reasons explained above, in any quaternionic functional calculus, for instance, in the continuous functional calculus for normal quaternionic operators [57, 148].)

We make another important observation: intrinsic slice hyperholomorphic functions of an operator can be expressed in terms of only the right linear structure on the space, cf. Theorem 3.4.11. Hence, they do not depend on the left multiplication. A right linear operator is, via the linearity condition, only related with the right multiplication, not with the left multiplication on the space. It is therefore plausible that, as a general principle, only the right linear structure should be important for the spectral properties of such operator. Indeed, we assume the existence of a left multiplication on X only because the space $\mathcal{B}(X)$ of right linear operators on X is otherwise only a real, not a quaternionic Banach space.

We can show the independence of intrinsic slice hyperholomorphic functions of an operator of the left multiplication with a different argument, which applies in other situations, too. If f is an intrinsic slice hyperholomorphic function on $\sigma_{SX}(T)$, then f can be approximated uniformly on $\sigma_{SX}(T)$ by intrinsic rational functions R_n due to Runge's theorem. Intrinsic rational functions are rational functions with real coefficients. Hence, they are precisely those rational functions of T that can be defined even if $\mathcal{B}(X)$ is considered only as a real Banach space, that is, if only the right linear structure on X is considered. For any R_n, the operator $R_n(T)$ therefore does not depend on the left multiplication. Instead, $R_n(T)$ is fully determined by the right linear structure on X. Furthermore, the operator norm $\|T\| = \sup_{\|v\|=1} \|Tv\|$, and in turn also the topology on $\mathcal{B}(X)$, is independent of whether we consider X as a quaternionic two-sided Banach space or a quaternionic right Banach space. We have

$$f = \lim_{n\to+\infty} R_n$$

uniformly on $\sigma_S(T)$. As the S-functional calculus is compatible with uniform limits, we find

$$f(T) = \lim_{n\to+\infty} R_n(T).$$

Since the operators $R_n(T)$ and the topology on $\mathcal{B}(X)$ are determined by the right linear structure on X and do not depend on the left multiplication, this is also true for the operator $f(T)$.

Similarly, the continuous functional calculus for a bounded normal quaternionic operator T is defined by approximating a continuous intrinsic slice function f on $\sigma_S(T)$ uniformly by intrinsic polynomials in s and $\overline{s}$, that is by polynomials of the form

$$P_n(s) = \sum_{0\leq\ell,k\leq n} a_{\ell,k} s^\ell \overline{s}^k \quad \text{with } a_{\ell,k} \in \mathbb{R}.$$

The operator

$$P_n(T) := \sum_{0\leq\ell,k\leq n} a_{\ell,k} T^\ell (T^*)^k,$$

where T^* denotes the adjoint of T, is then again fully determined by the right linear structure on the space since it contains only real coefficients. Consequently, also the operator $f(T) = \lim_{n\to+\infty} P_n(T)$ depends only on the right linear structure and not on any left multiplication [57].

Other important functional calculi such as the H^∞-functional calculus or the measurable functional calculus are extensions of these two calculi. Hence, they inherit the independence from the left linear structure on X (as long as only intrinsic slice functions are considered).

The fact that functional calculi for quaternionic right linear operators are determined by the right linear structure on the space brings up the question of clarifying the role that the left multiplication plays in this theory. In particular, we have to ask whether it has any influence on the spectral properties of an operator or not. The spectral properties of a quaternionic operator T must be independent of the concrete model of this operator that is considered a change of basis for instance, must not effect these properties. More general, let X be a two-sided quaternionic Banach space and let $T \in \mathcal{K}(X)$. If Y is another two-sided quaternionic Banach space and $U : X \to Y$ is a norm-preserving and bijective right-linear mapping, then

$$S := UTU^{-1}$$

is a model for T in Y. The spectral properties of S should correspond to the spectral properties of T and, indeed, we have

$$\mathcal{Q}_s(S) = \mathcal{Q}_s\left(UTU^{-1}\right) = U\mathcal{Q}_s(T)U^{-1}, \quad \forall s \in \mathbb{H}.$$

Hence, we find

$$\rho_S(T) = \rho_S(S) \quad \text{and} \quad \sigma_S(T) = \sigma_S(S),$$

and

$$\mathcal{Q}_s(S)^{-1} = U\mathcal{Q}_s(T)^{-1}U^{-1}, \quad \forall s \in \rho_S(T).$$

If $P(s) = \sum_{k=0}^{n} a_k s^k$ with $a_n \in \mathbb{R}$ is an intrinsic polynomial, then

$$P(S) = \sum_{k=0}^{n} a_k S^k = U \sum_{k=0}^{n} a_k T^k U^{-1} = UP(T)U^{-1}.$$

For any intrinsic rational function $R(s) = P(s)Q(s)^{-1}$ with intrinsic polynomials P and Q such that the zeros of Q (resp. the poles of R) lie in $\rho_S(T) = \rho_S(S)$, we therefore find that

$$R(S) = P(S)Q(S)^{-1} = UP(T)Q(T)^{-1}U^{-1} = UR(T)U^{-1}.$$

If $f \in \mathcal{N}(\sigma_{SX}(T)) = \mathcal{N}(\sigma_{SX}(S))$, then Runge's theorem implies the existence of a sequence of intrinsic rational functions $R_n, n \in \mathbb{N}$, the poles of which lie in $\rho_S(T) = \rho_S(S)$ such that $f(s) = \lim_{n\to+\infty} R_n(s)$ uniformly on $\sigma_{SX}(T)$. Since the S-functional calculus is compatible with uniform limits on $\sigma_{SX}(T)$, we obtain that

$$f(S) = \lim_{n\to+\infty} R_n(S) = \lim_{n\to+\infty} R_n(S) = \lim_{n\to+\infty} UR_n(T)U^{-1} = Uf(T)U^{-1}.$$

Similarly, it also follows that $f(S) = Uf(T)U^{-1}$ for any continuous intrinsic slice function f on $\sigma_S(T)$ if T is a normal operator on a quaternionic Hilbert space and U is a unitary right linear bijection. This correspondence is inherited by the extensions of these functional calculi such as the H^∞-functional calculus or the measurable functional calculus for normal operators. (For the S-functional calculus the identity $f(S) = Uf(T)U^{-1}$ can also be deduced directly from the integral representation (3.35). However, for the intrinsic functional calculus such integral representation does not exist and one has to follow the strategy described above.)

Objects and techniques that depend on the left multiplication or that apply to functions other than intrinsic functions are, on the other hand, not invariant under the transformation U. Consider for instance the constant function $f(s) = a$ with $a \in \mathbb{H} \setminus \mathbb{R}$. This function is both left and right slice hyperholomorphic on $\sigma_{SX}(T) = \sigma_{SX}(S)$, but not intrinsic. If we apply the S-functional calculus, we find $f(S) = a\mathcal{I}_Y$ and $f(T) = a\mathcal{I}_X$. However, unless $aU = Ua$, we have

$$f(S) = a\mathcal{I}_Y \neq U(a\mathcal{I}_X)U^{-1} = Uf(T)U^{-1}.$$

Actually, even for the S-resolvents, in general, we have

$$S_L^{-1}(s,S) \neq US_L^{-1}(s,T)U^{-1} \quad \text{and} \quad S_R^{-1}(s,S) \neq US_R^{-1}(s,T)U^{-1}.$$

Indeed, unless $U\overline{s} = \overline{s}U$, it is

$$\begin{aligned} S_L^{-1}(s,S) &= \mathcal{Q}_s(S)^{-1}\overline{s} - S\mathcal{Q}_s(S)^{-1} \\ &= U\mathcal{Q}_s(T)^{-1}U^{-1}\overline{s} - UT\mathcal{Q}_s(T)^{-1}U^{-1} \\ &\neq U\left(\mathcal{Q}_s(T)^{-1}\overline{s} - T\mathcal{Q}_s(T)^{-1}\right)U^{-1} = US_L^{-1}(s,T)U^{-1}. \end{aligned}$$

Due to the symmetry of the path of integration in the S-functional calculus, the S-resolvents are always simultaneously evaluated at s and $\overline{s}$ and it is this fact that ensures the independence of the S-functional calculus for intrinsic functions from the left multiplication.

One could argue that, since we are working on two-sided quaternionic Banach spaces, only transformations U that are compatible with the entire structure of X, that is with both the left and the right multiplication, should be considered in the arguments above. Hence, one should assume that U is both left and right linear. Such a transformation would satisfy $aU = Ua$ for all $a \in \mathbb{H}$ and the problems described above would not occur. The transformations of this type can be characterized easily: if

$$X_{\mathbb{R}} = \{v \in X : \quad av = va, \ \forall a \in \mathbb{H}\} \quad \text{and} \quad Y_{\mathbb{R}} = \{v \in X : \quad av = va, \ \forall a \in \mathbb{H}\}$$

such that

$$X = X_{\mathbb{R}} \otimes \mathbb{H} \quad \text{and} \quad Y = Y_{\mathbb{R}} \otimes \mathbb{H},$$

then an operator $U : X \to Y$ is both left and right linear if and only if it is the quaternionic right linear extension of an $\mathbb{R}$-linear operator $U_{\mathbb{R}} : X_{\mathbb{R}} \to Y_{\mathbb{R}}$. (In the terminology introduced in Definitions 2.2.6 and 3.3.1, this is equivalent to U being a scalar operator.)

However, restricting ourselves to such transformations is not feasible. We can consider for example $X = \mathbb{H}^n$ with its natural left and right multiplication and endowed with an arbitrary norm. This yields a two-sided quaternionic Banach space of dimension n. A right linear operator T on $\mathbb{H}^n$ can be represented by an $n \times n$-matrix with quaternionic entries and an operator is both left and right linear if it is represented by a matrix with only real coefficients. This matrix can be put in Jordan normal form, i.e., there exists an invertible matrix U such that $T = USU^{-1}$, where S is a block diagonal matrix with the diagonal that consists of Jordan blocks [190]. The matrix U however does not necessarily only have real entries. It is in general a matrix with quaternionic entries and hence it represents an operator that is only right, but not necessarily left linear. If we require that the spectral properties are only invariant under transformations that are both left and right linear, this would imply that the spectral properties of T are not necessarily invariant under the transformation U. Hence, T and its model in Jordan normal form S might have different spectral properties, which is absurd.

Spectral properties of an operator must therefore be invariant under norm-preserving and bijective right linear transformations. Since the left multiplication is not invariant under such transformations, the spectral properties of an operator cannot depend on it. *We conclude that right linear quaternionic operators have to be understood in terms of the right linear structure only. The left multiplication is a useful auxiliary tool, but spectral properties of the operator cannot depend on it.* However, the left multiplication is necessary in order to consider $\mathcal{B}(X)$ as a quaternionic linear space. Without it, it is not possible to apply quaternionic techniques to elements of $\mathcal{B}(X)$ and to give intuitive integral representations for the S-functional calculus in $\mathcal{B}(X)$. Furthermore, without assuming the existence of a left multiplication, it would not have been possible to develop the fundamental concepts of quaternionic operator theory. In particular, the S-spectrum could not have been found as its definition was understood by finding the closed form of the Cauchy kernel operator series $\sum_{n=0}^{+\infty} T^n s^{-(n+1)}$, cf. Theorem 2.2.9. Giving meaning to this series requires $\mathcal{B}(X)$ to be a quaternionic linear space.

Finally, in certain situations, the left multiplication of X is particularly useful for simplifying computations, since this left multiplication might allow us to write T in terms of components as $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell$, cf. Definition 2.2.6 and Definition 3.3.1. If there exists a model of the operator T in a space with a left multiplication, such that T has commuting components, then this model can be used to significantly simplify computations for investigating T, cf. Theorem 3.3.4.