Chapter 2

Preliminary results

This chapter contains two main topics: the theory of slice hyperholomorphic functions and the S-functional calculus for bounded quaternionic operators. We limit ourselves to recalling the results we need in this book. For the proofs of the main theorems, we refer the reader to [57], which currently contains the completest version of quaternionic spectral theory.

2.1 Slice hyperholomorphic functions

There are three possible ways to define slice hyperholomorphic functions, using the definition in [141], using the global operator of slice hyperholomorphic functions introduced in [62] or using the definition that comes from Fueter–Sce–Qian mapping theorem. This last definition is the most appropriate for operator theory so in this chapter we summarize the properties of slice hyperholomorphic functions that we will use in the sequel. The proofs can be found in Chapter 2 of the monograph [57].

We denote by $\mathbb{H}$ the algebra of quaternions. An element q of $\mathbb{H}$ is of the form

$$q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3, \quad q_\ell \in \mathbb{R},\ \ell = 0, 1, 2, 3,$$

where e_1, e_2 and e_3 are the generating imaginary units of $\mathbb{H}$. They satisfy the relations

$$e_1^2 = e_2^2 = e_3^2 = -1 \tag{2.1}$$

and

$$e_1 e_2 = -e_2 e_1 = e_3, \quad e_2 e_3 = -e_3 e_2 = e_1, \quad e_3 e_1 = -e_1 e_3 = e_2. \tag{2.2}$$

The real part, the imaginary part and the modulus of a quaternion $q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$ are defined as $\mathrm{Re}(q) = q_0$, $\mathrm{Im}(q) = q_1 e_1 + q_2 e_2 + q_3 e_3$ and $|q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$. The conjugate of the quaternion q is

$$\bar{q} = \mathrm{Re}(q) - \mathrm{Im}(q) = q_0 - q_1 e_1 - q_2 e_2 - q_3 e_3$$

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and it satisfies

$$|q|^2 = q\bar{q} = \bar{q}q.$$

The inverse of any nonzero element q is hence given by

$$q^{-1} = \frac{\bar{q}}{|q|^2}.$$

We denote by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, i.e.,

$$\mathbb{S} = \{q = q_1e_1 + q_2e_2 + q_3e_3 : q_1^2 + q_2^2 + q_3^2 = 1\}.$$

Notice that if $j \in \mathbb{S}$, then $j^2 = -1$. For this reason the elements of $\mathbb{S}$ are also called imaginary units. The set $\mathbb{S}$ is a 2-dimensional sphere in $\mathbb{R}^4 \cong \mathbb{H}$. Given a nonreal quaternion $q = q_0 + \mathrm{Im}(q)$, we have $q = u + jv$ with $u = \mathrm{Re}(q)$, $v = |\mathrm{Im}(q)|$ and $j = \mathrm{Im}(q)/|\mathrm{Im}(q)| \in \mathbb{S}$. (We will sometimes use the notation $j_q = j = \mathrm{Im}(q)/|\mathrm{Im}(q)|$ when it is necessary to stress the relation between j and q.) We can associate to q the 2-dimensional sphere

$$[q] = \{q_0 + j|\mathrm{Im}(q)| : \quad j \in \mathbb{S}\} = \{u + jv : \quad j \in \mathbb{S}\}.$$

This sphere is centered at the real point $q_0 = \mathrm{Re}(q)$ and has radius $|\mathrm{Im}(q)|$. Furthermore, a quaternion $\tilde{q}$ belongs to $[q]$ if and only if there exists $h \in \mathbb{H} \setminus \{0\}$ such that $\tilde{q} = h^{-1}qh$.

If $j \in \mathbb{S}$, then the set

$$\mathbb{C}_j = \{u + jv : \quad u, v \in \mathbb{R}\}$$

is an isomorphic copy of the complex numbers. If moreover $i \in \mathbb{S}$ with $j \perp i$, then j, i and $k := ji$ is a generating basis of $\mathbb{H}$, i.e., this basis also satisfies the relations (2.1) and (2.2). Hence, any quaternion $q \in \mathbb{H}$ can be written as

$$q = z_1 + z_2 i = z_1 + i\bar{z}_2$$

with unique $z_1, z_2 \in \mathbb{C}_j$ and so

$$\mathbb{H} = \mathbb{C}_j + i\mathbb{C}_j \quad \text{and} \quad \mathbb{H} = \mathbb{C}_j + \mathbb{C}_j i. \tag{2.3}$$

Moreover, we observe that

$$\mathbb{H} = \bigcup_{j \in \mathbb{S}} \mathbb{C}_j.$$

Finally, we introduce the notation $\overline{\mathbb{H}} := \mathbb{H} \cup \{\infty\}$.

Definition 2.1.1. Let $U \subseteq \mathbb{H}$.

(i) We say that U is *axially symmetric* if $[q] \subset U$ for any $q \in U$.

(ii) We say that U is a *slice domain* if $U \cap \mathbb{R} \neq \emptyset$ and if $U \cap \mathbb{C}_j$ is a domain in $\mathbb{C}_j$ for any $j \in \mathbb{S}$.

Definition 2.1.2 (Slice hyperholomorphic functions)**.** Let $U \subseteq \mathbb{H}$ be an axially symmetric open set and let $\mathcal{U} = \{(u,v) \in \mathbb{R}^2 : u + \mathbb{S}v \subset U\}$. A function $f : U \to \mathbb{H}$ is called left slice function, if it is of the form

$$f(q) = f_0(u,v) + jf_1(u,v), \quad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \mathcal{U} \to \mathbb{H}$ that satisfy the compatibility condition

$$f_0(u,-v) = f_0(u,v), \quad f_1(u,-v) = -f_1(u,v). \tag{2.4}$$

If in addition f_0 and f_1 satisfy the Cauchy–Riemann equations

$$\frac{\partial}{\partial u} f_0(u,v) - \frac{\partial}{\partial v} f_1(u,v) = 0, \tag{2.5}$$

$$\frac{\partial}{\partial v} f_0(u,v) + \frac{\partial}{\partial u} f_1(u,v) = 0, \tag{2.6}$$

then f is called left slice hyperholomorphic. A function $f : U \to \mathbb{H}$ is called right slice function if it is of the form

$$f(q) = f_0(u,v) + f_1(u,v)j, \quad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \mathcal{U} \to \mathbb{H}$ that satisfy (2.4). If in addition f_0 and f_1 satisfy the Cauchy–Riemann equations, then f is called right slice hyperholomorphic.

If f is a left (or right) slice function such that f_0 and f_1 are real-valued, then f is called intrinsic.

We denote the sets of left and right slice hyperholomorphic functions on U by $\mathcal{SH}_L(U)$ and $\mathcal{SH}_R(U)$, respectively. The set of intrinsic slice hyperholomorphic functions on U will be denoted by $\mathcal{N}(U)$.

Remark 2.1.1. Any quaternion q can be represented as an element of a complex plane $\mathbb{C}_j$ using at least two different imaginary units $j \in \mathbb{S}$. We have $q = u + jv = u + (-j)(-v)$ and $-j$ also belongs to $\mathbb{S}$. If q is real, then we can use any imaginary unit $j \in \mathbb{S}$ to consider q as an element of $\mathbb{C}_j$. The compatibility condition (2.4) assures that the choice of this imaginary unit is irrelevant. In particular, it forces $f_1(u,v)$ to equal 0 if $v = 0$, that is, if $q \in \mathbb{R}$.

The multiplication and the composition with intrinsic functions preserve slice hyperholomorphicity. This is not true for arbitrary slice hyperholomorphic functions.

Theorem 2.1.3. *With the notation above we have the properties:*

(i) *If $f \in \mathcal{N}(U)$ and $g \in \mathcal{SH}_L(U)$, then $fg \in \mathcal{SH}_L(U)$. If $f \in \mathcal{SH}_R(U)$ and $g \in \mathcal{N}(U)$, then $fg \in \mathcal{SH}_R(U)$.*

(ii) *If $g \in \mathcal{N}(U)$ and $f \in \mathcal{SH}_L(g(U))$, then $f \circ g \in \mathcal{SH}_L(U)$. If $g \in \mathcal{N}(U)$ and $f \in \mathcal{SH}_R(g(U))$, then $f \circ g \in \mathcal{SH}_R(U)$.*

Lemma 2.1.4. *Let $U \subset \mathbb{H}$ be axially symmetric and let f be a left (or right) slice function on U. The following statements are equivalent.*

(i) *The function f is intrinsic.*

(ii) *We have $f(U \cap \mathbb{C}_j) \subset \mathbb{C}_j$ for any $j \in \mathbb{S}$.*

(iii) *We have $f(\overline{q}) = \overline{f(q)}$ for all $q \in U$.*

If we restrict a slice hyperholomorphic function to one of the complex planes $\mathbb{C}_j$, then we obtain a function that is holomorphic in the usual sense.

Lemma 2.1.5 (The Splitting Lemma)**.** *Let $U \subset \mathbb{H}$ be an axially symmetric open set and let $j, i \in \mathbb{S}$ with $i \perp j$. If $f \in \mathcal{SH}_L(U)$, then the restriction $f_j = f|_{U\cap\mathbb{C}_j}$ satisfies*

$$\frac{1}{2}\left(\frac{\partial}{\partial u} f_j(z) + j\frac{\partial}{\partial v} f_j(z)\right) = 0 \tag{2.7}$$

for all $z = u + jv \in U \cap \mathbb{C}_j$. Hence

$$f_j(z) = F_1(z) + F_2(z)i$$

with holomorphic functions $F_1, F_2 : U \cap \mathbb{C}_j \to \mathbb{C}_j$.

If $f \in \mathcal{SH}_R(U)$, then the restriction $f_j = f|_{U\cap\mathbb{C}_j}$ satisfies

$$\frac{1}{2}\left(\frac{\partial}{\partial u} f_j(z) + \frac{\partial}{\partial v} f_j(z)j\right) = 0, \tag{2.8}$$

for all $z = u + jv \in U \cap \mathbb{C}_j$. Hence,

$$f_j(z) = F_1(z) + iF_2(z)$$

with holomorphic functions $F_1, F_2 : U \cap \mathbb{C}_j \to \mathbb{C}_j$.

The splitting lemma states that the restriction of any left slice hyperholomorphic function to a complex plane $\mathbb{C}_j$ is left holomorphic, i.e., it is a holomorphic function with values in the left vector space $\mathbb{H} = \mathbb{C}_j + \mathbb{C}_j i$ over $\mathbb{C}_j$. The restriction of a right slice hyperholomorphic function to a complex plane $\mathbb{C}_j$ is right holomorphic, i.e., it is a holomorphic function with values in the right vector space $\mathbb{H} = \mathbb{C}_j + i\mathbb{C}_j$ over $\mathbb{C}_j$.

Theorem 2.1.6 (Identity Principle)**.** *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain, let $f, g : U \to \mathbb{H}$ be left (or right) slice hyperholomorphic functions and set $\mathcal{Z} = \{q \in U : f(q) = g(q)\}$. If there exists $j \in \mathbb{S}$ such that $\mathcal{Z} \cap \mathbb{C}_j$ has an accumulation point in $U \cap \mathbb{C}_j$, then $f = g$.*

The most important property of slice functions (and in particular of slice hyperholomorphic functions) is the Structure Formula, which is often also called Representation Formula.

Theorem 2.1.7 (The Structure (or Representation) Formula)**.** *Let $U \subset \mathbb{H}$ be axially symmetric and let $i \in \mathbb{S}$. A function $f : U \to \mathbb{H}$ is a left slice function on U if and only if for any $q = u + jv \in U$ we have*

$$f(q) = \frac{1}{2}\Big[f(\overline{z}) + f(z)\Big] + \frac{1}{2}ji\Big[f(\overline{z}) - f(z)\Big] \tag{2.9}$$

with $z = u + iv$. A function $f : U \to \mathbb{H}$ is a right slice function on U if and only if for any $q = u + jv \in U$ we have

$$f(q) = \frac{1}{2}\Big[f(\overline{z}) + f(z)\Big] + \frac{1}{2}\Big[f(\overline{z}) - f(z)\Big]ij \tag{2.10}$$

with $z = u + iv$.

Remark 2.1.2. It is sometimes useful to rewrite (2.9) as

$$f(q) = \frac{1}{2}(1 - ij)f(z) + \frac{1}{2}(1 + ij)f(\overline{z})$$

and (2.10) as

$$f(q) = f(z)(1 - ij)\frac{1}{2} + f(\overline{z})(1 + ij)\frac{1}{2}.$$

The representation formula can be written in a different form that shows that $f(s)$ is determined by the values of f at two arbitrary points in the sphere $[s]$, not necessarily by a point and its conjugate.

Corollary 2.1.8. *Let $U \subset \mathbb{H}$ be axially symmetric, let $[s] = u + \mathbb{S}v \subset U$ and let i, j and k be three different imaginary units in $\mathbb{S}$. If f is a left slice function on U, then*

$$\begin{aligned} f(u + jv) = &\left((i - k)^{-1}i + j(k - i)^{-1}\right) f(u + iv) \\ &+ \left((k - i)^{-1}k + j(k - i)^{-1}\right) f(u + kv). \end{aligned} \tag{2.11}$$

Similarly, if f is a right slice function on U, then

$$\begin{aligned} f(u + jv) = &\, f(u + iv) \left(i(i - k)^{-1} + (k - i)^{-1}j\right) \\ &+ f(u + kv) \left(k(k - i)^{-1} + (k - i)^{-1}j\right). \end{aligned} \tag{2.12}$$

As a consequence of the Structure Formula, every holomorphic function that is defined on a suitable open set in $\mathbb{C}_j$ has a slice hyperholomorphic extension.

Lemma 2.1.9. *Let $O \subset \mathbb{C}_j$ be an open set which is symmetric with respect to the real axis. We call the set $[O] = \bigcup_{z\in O}[z]$ the axially symmetric hull of O.*

(i) *Any function $f : O \to \mathbb{H}$ has a unique extension $\mathrm{ext}_L(f)$ to a left slice function on $[O]$ and a unique extension $\mathrm{ext}_R(f)$ to a right slice function on $[O]$.*

(ii) *If* $f : O \to \mathbb{H}$ *is left holomorphic, i.e., it satisfies* (2.7)*, then* $\mathrm{ext}_L(f)$ *is left slice hyperholomorphic.*

(iii) *If* f *is right holomorphic, i.e., it satisfies* (2.8)*, then* $\mathrm{ext}_R(f)$ *is right slice hyperholomorphic.*

Sometimes in the following we will use the notation f_j instead of $f|_{U\cap\mathbb{C}_j}$ for the restriction f to $U\cap\mathbb{C}_j$ without mentioning it explicitly because it is clear from the context. Slice hyperholomorphic functions admit a special kind of derivative, that yields again a slice hyperholomorphic function.

Definition 2.1.10. Let $f : U \subseteq \mathbb{H} \to \mathbb{H}$ and let $q = u + jv \in U$. If q is not real, then we say that f admits left slice derivative in q if

$$\partial_S f(q) := \lim_{p\to q,\, p\in\mathbb{C}_j} (p-q)^{-1}(f_j(p) - f_j(q)) \tag{2.13}$$

exists and is finite. If q is real, then we say that f admits left slice derivative in q if (2.13) exists for any $j \in \mathbb{S}$.

Similarly, we say that f admits right slice derivative at a nonreal point $q = u + jv \in U$ if

$$\partial_S f(q) := \lim_{p\to q,\, p\in\mathbb{C}_j} (f_j(p) - f_j(q))(p-q)^{-1} \tag{2.14}$$

exists and is finite, and we say that f admits right slice derivative at a real point $q \in U$ if (2.14) exists and is finite, for any $j \in \mathbb{S}$.

Observe that $\partial_S f(q)$ is uniquely defined and independent of the choice of $j \in \mathbb{S}$ even if q is real. If f admits slice derivative, then f_j is $\mathbb{C}_j$-complex left (resp. right) differentiable and we find

$$\partial_S f(q) = f_j'(q) = \frac{\partial}{\partial u} f_j(q) = \frac{\partial}{\partial u} f(q), \quad q = u + jv. \tag{2.15}$$

Proposition 2.1.11. *Let* $U \subseteq \mathbb{H}$ *be an axially symmetric open set and let* $f : U \to \mathbb{H}$ *be a real differentiable function.*

(i) *If* f *is left (or right) slice hyperholomorphic, it admits left (resp. right) slice derivative and* $\partial_S f$ *is again left (resp. right) slice hyperholomorphic on* U*.*

(ii) *If* f *is a left (or right) slice function that admits a left (resp. right) slice derivative, then* f *is a left (resp. right) slice hyperholomorphic.*

(iii) *If* U *is a slice domain, then any function that admits left (resp. right) slice derivative is left (resp. right) slice hyperholomorphic.*

Important examples of slice hyperholomorphic functions are power series in the quaternionic variable: power series of the form $\sum_{n=0}^{+\infty} q^n a_n$ with $a_n \in \mathbb{H}$ are left slice hyperholomorphic and power series of the form $\sum_{n=0}^{+\infty} a_n q^n$ are right slice hyperholomorphic. Such a power series is intrinsic if and only if the coefficients a_n are real.

Conversely, any slice hyperholomorphic function can be expanded into a power series, at any real point, due to the splitting lemma.

Theorem 2.1.12. *Suppose that $a \in \mathbb{R}$ and $r > 0$. Let $B_r(a) = \{q \in \mathbb{H} : |q-a| < r\}$. If $f \in \mathcal{SH}_L(B_r(a))$, then*

$$f(q) = \sum_{n=0}^{+\infty} (q-a)^n \frac{1}{n!}, \partial_S{}^n f(a), \quad \forall q = u + jv \in B_r(a). \tag{2.16}$$

If on the other hand $f \in \mathcal{SH}_R(B_r(a))$, then

$$f(q) = \sum_{n=0}^{+\infty} \frac{1}{n!} \left(\partial_S{}^n f(a)\right) (q-a)^n, \quad \forall q = u + jv \in B_r(a).$$

Example 2.1.13. The exponential function is defined by its power series expansion

$$\exp(q) := e^q := \sum_{n=0}^{+\infty} \frac{1}{n!} q^n, \quad q \in \mathbb{H}.$$

Since the power series expansion has real coefficients, the exponential function is both left and right slice hyperholomorphic and even intrinsic. If p and q commute, then it satisfies the usual identity $\exp(p+q) = \exp(p)\exp(q)$. This is, however, not true for arbitrary p and q.

Example 2.1.14. Any $q = u + jv \in \mathbb{H} \setminus \{0\}$ can be written as $q = |q| \exp(j\theta)$ with a unique angle $\theta \in [0, \pi]$. We define the argument of q as

$$\arg(q) := \theta = \arccos(u/|q|).$$

Observe that, $\arg(q)$ is well defined even though j is not unique if q real: if $q > 0$ then $q = |q| \exp(j0)$ for any $j \in \mathbb{S}$ and hence $\arg(q) = 0$ is independent of the choice of $j \in \mathbb{S}$ and if $q < 0$ then $q = |q| \exp(j\pi)$ for any $j \in \mathbb{S}$ and hence $\arg(q) = \pi$ is independent of the choice of $j \in \mathbb{S}$, too.

The logarithm of a quaternion $q = u + jv \in \mathbb{H} \setminus (-\infty, 0]$ is defined as

$$\log(q) := \ln(|q|) + j \arg(q). \tag{2.17}$$

It is an intrinsic slice hyperholomorphic function on $\mathbb{H} \setminus (-\infty, 0]$ and satisfies

$$e^{\log(q)} = q \quad \text{for } q \in \mathbb{H},$$

and

$$\log e^q = q, \quad \text{for } q = u + jv \in \mathbb{H} \text{ with } |\mathrm{Im}(q)| = v < \pi.$$

Contrary to the complex setting it is not possible to choose different intrinsic hyperholomorphic branches of the logarithm.

Remark 2.1.3. Observe that there exist other definitions of the quaternionic logarithm in the literature. In [162], the logarithm of a quaternion is for instance defined as

$$\log_{k,\ell} x := \begin{cases} \ln|x| + j_x\left(\arccos\frac{x_0}{|x|} + 2k\pi\right), & |\underline{x}| \neq 0 \text{ or } |\underline{x}| = 0,\ x_0 > 0, \\ \ln|x| + e_\ell\pi, & |\underline{x}| = 0,\ x_0 < 0, \end{cases}$$

where $k \in \mathbb{Z}$ and e_ℓ is one of the generating units of $\mathbb{H}$. This logarithm is, however, not continuous (and therefore, in particular, not slice hyperholomorphic) on the real line, unless $k = 0$. But for $k = 0$ this definition of the logarithm coincides with the one given above. Even more, the identity principle implies that (2.17) defines the maximal slice hyperholomorphic extension of the natural logarithm on $(0, +\infty)$ to a subset of the quaternions.

Example 2.1.15. For $\alpha \in \mathbb{R}$, we define the fractional power q^α of a quaternion $q = u + jv \in \mathbb{H} \setminus (-\infty, 0]$ as

$$q^\alpha := e^{\alpha \log q} = e^{\alpha(\ln|q| + j\arg(q))}. \tag{2.18}$$

The function $q \mapsto q^\alpha$ is an intrinsic slice hyperholomorphic function on its domain $\mathbb{H} \setminus (-\infty, 0]$ by Theorem 2.1.3 and Lemma 2.1.4 as it is the composition of two intrinsic slice hyperholomorphic functions. However, if we try to define fractional powers of non-real components by the formula (2.18), then we do not obtain a slice hyperholomorphic function: the composition of two such functions is only intrinsic if the inner function is slice hyperholomorphic and this is not the case for $\alpha \notin \mathbb{R}$.

As pointed out above, the product of two slice hyperholomorphic functions is not slice hyperholomorphic unless the factor on the appropriate side is intrinsic. However, there exists a regularised product that preserves slice hyperholomorphicity.

Definition 2.1.16. For $f = f_0 + jf_1, g = g_0 + jg_1 \in \mathcal{SH}_L(U)$, we define their left slice hyperholomorphic product as

$$f *_L g = (f_0g_0 - f_1g_1) + j(f_0g_1 + f_1g_0).$$

For $f = f_0 + f_1j, g = g_0 + g_1j \in \mathcal{SH}_R(U)$, we define their right slice hyperholomorphic product as

$$f *_R g = (f_0g_0 - f_1g_1) + (f_0g_1 + f_1g_0)j.$$

The slice hyperholomorphic product is associative and distributive, but it is in general not commutative. If f is intrinsic, then $f *_L g$ coincides with the pointwise product fg and

$$f *_L g = fg = g *_L f. \tag{2.19}$$

Similarly, if g is intrinsic, then $f *_R g$ coincides with the pointwise product fg and

$$f *_R g = fg = g *_R f. \tag{2.20}$$

Example 2.1.17. If $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{+\infty} q^n b_n$ are two left slice hyperholomorphic power series, then their slice hyperholomorphic product equals the usual product of formal power series with coefficients in a non-commutative ring

$$\left(\sum_{n=0}^{+\infty} q^n a_n\right) *_L \left(\sum_{n=0}^{+\infty} q^n b_n\right) = (f *_L g)(q) = \sum_{n=0}^{+\infty} q^n \sum_{k=0}^{n} a_k b_{n-k}. \tag{2.21}$$

Similarly, we have for right slice hyperholomorphic power series that

$$\left(\sum_{n=0}^{+\infty} a_n q^n\right) *_R \left(\sum_{n=0}^{+\infty} b_n q^n\right) = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) q^n. \tag{2.22}$$

Definition 2.1.18. We define for $f = f_0 + jf_1 \in \mathcal{SH}_L(U)$ its slice hyperholomorphic conjugate $f^c = \overline{f_0} + j\overline{f_1}$ and its symmetrisation $f^s = f *_L f^c = f^c *_L f$. Similarly, we define for $f = f_0 + f_1 j \in \mathcal{SH}_R(U)$ its slice hyperholomorphic conjugate as $f^c = \overline{f_0} + \overline{f_1} j$ and its symmetrisation as $f^s = f *_R f^c = f^c *_R f$.

The symmetrisation of a left slice hyperholomorphic function $f = f_0 + jf_1$ is explicitly given by

$$f^s = |f_0|^2 - |f_1|^2 + j2\mathrm{Re}\left(f_0\overline{f_1}\right).$$

Hence, it is an intrinsic function. It is $f^s(q) = 0$ if and only if $f(\tilde{q}) = 0$ for some $\tilde{q} \in [q]$. Furthermore, if $q = u + jv$, one has

$$f^c(q) = \overline{f_0(u,v)} + j\overline{f_1(u,v)} = \overline{f_0(u,v)} + \overline{f_1(u,v)(-j)} = \overline{f(\overline{q})} \tag{2.23}$$

and an easy computation shows that

$$f *_L g(q) = f(q) g\left(f(q)^{-1} q f(q)\right), \quad \text{if } f(q) \neq 0. \tag{2.24}$$

For $f(q) \neq 0$, one has

$$\begin{aligned} f^s(q) &= f(q) f^c\left(f(q)^{-1} q f(q)\right) \\ &= f(q)\overline{f\left(\overline{f(q)^{-1} q f(q)}\right)} = f(q)\overline{f\left(f(q)^{-1}\overline{q} f(q)\right)}. \end{aligned} \tag{2.25}$$

Similar computations hold true in the right slice hyperholomorphic case. Finally, if f is intrinsic, then $f^c(q) = f(q)$ and $f^s(q) = f(q)^2$. Some consequences of the above definitions are collected in the following corollary.

Corollary 2.1.19. *The following statements hold true.*

(i) *For $f \in \mathcal{SH}_L(U)$ with $f \not\equiv 0$, its slice hyperholomorphic inverse f^{-*_L}, which satisfies $f^{-*_L} *_L f = f *_L f^{-*_L} = 1$, is given by*

$$f^{-*_L} = (f^s)^{-1} *_L f^c = (f^s)^{-1} f^c$$

and it is defined on $U \setminus [\mathcal{Z}_f]$, where $\mathcal{Z}_f = \{s \in U : f(s) = 0\}$.

(ii) *For $f \in \mathcal{SH}_R(U)$ with $f \not\equiv 0$, its slice hyperholomorphic inverse f^{-*_R}, which satisfies $f^{-*_R} *_R f = f *_R f^{-*_R} = 1$, is given by*

$$f^{-*_R} = f^c *_R (f^s)^{-1} = f^c (f^s)^{-1}$$

and it is defined on $U \setminus [\mathcal{Z}_f]$, where $\mathcal{Z}_f = \{s \in U : f(s) = 0\}$.

(iii) *If $f \in \mathcal{N}(U)$ with $f \not\equiv 0$, then $f^{-*_L} = f^{-*_R} = f^{-1}$.*

We observe that the modulus $|f^{-*_L}|$ is in a certain sense comparable to $1/|f|$. Since f^s is intrinsic, we have $|f^s(q)| = |f^s(\tilde{q})|$ for any $\tilde{q} \in [q]$. As $f(q)qf(q)^{-1} \in [q]$, we find, for $f(q) \neq 0$ because of (2.25), that

$$\begin{aligned} |f^s(q)| &= \left| f^s\left(f(q)qf(q)^{-1}\right)\right| \\ &= \left| f\left(f(q)qf(q)^{-1}\right) \overline{f(\overline{q})} \right| \\ &= \left| f\left(f(q)qf(q)^{-1}\right)\right| \left|f(\overline{q})\right|. \end{aligned}$$

Therefore we have, because of (2.23), that

$$\begin{aligned} \left|f^{-*_L}(q)\right| &= \left|f^s(q)^{-1}\right| \left|f^c(q)\right| \\ &= \frac{1}{\left|f\left(f(q)qf(q)^{-1}\right)\right| \left|f(\overline{q})\right|} \left|f(\overline{q})\right| \\ &= \frac{1}{\left|f\left(f(q)\overline{q}f(q)^{-1}\right)\right|} \end{aligned}$$

and so

$$\left|f^{-*_L}(q)\right| = \frac{1}{|f(\tilde{q})|} \quad \text{with } \tilde{q} = f(q)\overline{q}f(q)^{-1} \in [q]. \tag{2.26}$$

An analogous estimate holds for the slice hyperholomorphic inverse of a right slice hyperholomorphic function.

Slice hyperholomorphic functions satisfy a version of Cauchy's integral theorem and a Cauchy formula with a slice hyperholomorphic integral kernel. However, left and right slice hyperholomorphic functions satisfy Cauchy formulas with different kernels. This is contrary to the case of Fueter regular functions, where both left and right Fueter regular functions satisfy a Cauchy formula with the same kernel.

Theorem 2.1.20 (Cauchy's integral theorem)**.** *Let $U \subset \mathbb{H}$ be open, let $j \in \mathbb{S}$ and let $f \in \mathcal{SH}_L(U)$ and $g \in \mathcal{SH}_R(U)$. Moreover, let $D_j \subset U \cap \mathbb{C}_j$ be an open and bounded subset of the complex plane $\mathbb{C}_j$ with $\overline{D_j} \subset U \cap \mathbb{C}_j$ such that ∂D_j is a finite union of piecewise continuously differentiable Jordan curves. Then*

$$\int_{\partial D_j} g(s)\, ds_j\, f(s) = 0,$$

where $ds_j = ds(-j)$.

In order to determine the left and right slice hyperholomorphic Cauchy kernels, we start from an analogy with the classical complex case. We consider the series expansion of the complex Cauchy kernel and determine its closed form under the assumption that s and q are quaternions that do not commute.

Theorem 2.1.21. *Let $q, s \in \mathbb{H}$ with $|q| < |s|$. Then,*

$$\sum_{n=0}^{+\infty} q^n s^{-n-1} = -(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}(q - \overline{s}) \tag{2.27}$$

and

$$\sum_{n=0}^{+\infty} s^{-n-1} q^n = -(q - \overline{s})(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}. \tag{2.28}$$

Definition 2.1.22 (Slice hyperholomorphic Cauchy kernels)**.** We define the left slice hyperholomorphic Cauchy kernel as

$$S_L^{-1}(s,q) := -(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}(q - \overline{s}), \quad q \notin [s]$$

and the right slice hyperholomorphic Cauchy kernel as

$$S_R^{-1}(s,q) := -(q - \overline{s})(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}, \quad q \notin [s].$$

The left and right slice hyperholomorphic Cauchy kernels are proper generalizations of the classical Cauchy kernel.

Lemma 2.1.23. *If s and q commute, then the left and the right slice hyperholomorphic Cauchy kernel reduce to the complex Cauchy kernel, i.e.,*

$$S_L^{-1}(s,q) = (s - q)^{-1} = S_R^{-1}(s,q) \quad \text{if } sq = qs.$$

Remark 2.1.4. The left and right slice hyperholomorphic Cauchy kernels are the left and right slice hyperholomorphic inverses of the function $q \mapsto s - q$, which is another analogy to the classical case.

As the next proposition shows, the slice hyperholomorphic Cauchy kernels $S_L^{-1}(s,q)$ and $S_R^{-1}(s,q)$ can be written in two different ways, which justifies Definition 2.1.25.

Proposition 2.1.24. *If $q, s \in \mathbb{H}$ with $q \notin [s]$, then*

$$-(q^2 - 2q\mathrm{Re}(s) + |s|^2)^{-1}(q - \overline{s}) = (s - \overline{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1} \tag{2.29}$$

and

$$(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}(s - \overline{q}) = -(q - \overline{s})(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}. \tag{2.30}$$

Definition 2.1.25. Let $q, s \in \mathbb{H}$ with $q \notin [s]$.

- We say that $S_L^{-1}(s,q)$ is written in the form I if

$$S_L^{-1}(s,q) := -(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}(q - \overline{s}).$$

- We say that $S_L^{-1}(s,q)$ is written in the form II if

$$S_L^{-1}(s,q) := (s - \bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}.$$

- We say that $S_R^{-1}(s,q)$ is written in the form I if

$$S_R^{-1}(s,q) := -(q - \overline{s})(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}.$$

- We say that $S_R^{-1}(s,q)$ is written in the form II if

$$S_R^{-1}(s,q) := (s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}(s - \bar{q}).$$

Corollary 2.1.26. *For $q, s \in \mathbb{H}$ with $s \notin [q]$, we have*

$$S_L^{-1}(s,q) = -S_R^{-1}(q,s).$$

It is essential for the theory that the left and right slice hyperholomorphic Cauchy kernels are slice hyperholomorphic in both variables. This is what the next lemma shows.

Lemma 2.1.27. *Let $q, s \in \mathbb{H}$ with $q \notin [s]$. The left slice hyperholomorphic Cauchy kernel $S_L^{-1}(s,q)$ is left slice hyperholomorphic in q and right slice hyperholomorphic in s. The right slice hyperholomorphic Cauchy kernel $S_R^{-1}(s,q)$ is left slice hyperholomorphic in s and right slice hyperholomorphic in q.*

As pointed out in Remark 2.1.4, the left and the right Cauchy kernel are the slice hyperholomorphic inverses of the mapping $q \mapsto s - q$. In analogy with the classical case, their slice derivatives are multiples of the n-th slice hyperholomorphic inverses of this function.

Definition 2.1.28. For $s, q \in \mathbb{H}$ with $s \notin [q]$ and $n \in \mathbb{N}$, we define

$$S_L^{-n}(s,q) := (s-q)^{-*_L n} = (q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-n} \sum_{k=0}^{n} \binom{n}{k} (-q)^k \overline{s}^{n-k},$$

$$S_R^{-n}(s,q) := (s-q)^{-*_R n} = \sum_{k=0}^{n} \binom{n}{k} \overline{s}^{n-k} (-q)^k (q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-n}.$$

Lemma 2.1.29. *For $s, q \in \mathbb{H}$ with $s \notin [q]$ and $n \in \mathbb{N}$, the slice derivatives in the variable q of the right and left slice hyperholomorphic Cauchy kernels are*

$$\partial_S{}^n S_L^{-1}(s,q) = (-1)^n \, n! \, S_L^{-(n+1)}(s,q)$$

and

$$\partial_S{}^n S_R^{-1}(s,q) = (-1)^n \, n! \, S_R^{-(n+1)}(s,q).$$

The domains of integration that appear in the slice hyperholomorphic Cauchy formulas as well as in the S-functional calculus are slice Cauchy domains. Before we state the Cauchy formulas, we recall some properties of the slice Cauchy domains, the proofs of which can be found in [130].

Definition 2.1.30 (Slice Cauchy domain)**.** An axially symmetric open set $U \subset \mathbb{H}$ is called a slice Cauchy domain, if $U \cap \mathbb{C}_j$ is a Cauchy domain in $\mathbb{C}_j$ for any $j \in \mathbb{S}$. More precisely, U is a slice Cauchy domain if, for any $j \in \mathbb{S}$, the boundary $\partial(U \cap \mathbb{C}_j)$ of $U \cap \mathbb{C}_j$ is the union of a finite number of non-intersecting piecewise continuously differentiable Jordan curves in $\mathbb{C}_j$.

Remark 2.1.5. Any slice Cauchy domain has only finitely many components (i.e., maximal connected subsets). Moreover, at most one of them is unbounded and if there exists an unbounded component, then it contains a neighborhood of ∞ in $\mathbb{H}$.

Theorem 2.1.31. *Let C be a closed and let O be an open axially symmetric subset of $\mathbb{H}$ such that $C \subset O$ and such that ∂O is nonempty and bounded. Then there exists a slice Cauchy domain U such that $C \subset U$ and $\overline{U} \subset O$. If O is unbounded, then U can be chosen to be unbounded, too.*

The boundary of a slice Cauchy domain in a complex plane $\mathbb{C}_j$ is of course symmetric with respect to the real axis. Hence, it can be fully described by the part that lies in the closed upper half plane

$$\mathbb{C}_j^+ := \{z_0 + jz_1 : \quad z_0 \in \mathbb{R},\ z_1 \geq 0\}.$$

We specify this idea in the following statements.

Definition 2.1.32. For a path $\gamma : [0,1] \to \mathbb{C}_j$, we define the paths $(-\gamma)(t) := \gamma(1-t)$ and $\overline{\gamma}(t) := \overline{\gamma(t)}$.

Lemma 2.1.33. *Let γ be a Jordan curve in $\mathbb{C}_j$ whose image is symmetric with respect to the real axis. Then $\gamma_+ := \gamma \cap \mathbb{C}_j^+$ consists of a single curve and $\gamma = \gamma_+ \cup \gamma_-$ with $\gamma_- := -\overline{\gamma_+}$.*

Let now U be a slice Cauchy domain and consider any $j \in \mathbb{S}$. The boundary $\partial(U \cap \mathbb{C}_j)$ of U in $\mathbb{C}_j$ consists of a finite union of piecewise continuously differentiable Jordan curves and is symmetric with respect to the real axis. Hence, whenever a curve γ belongs to $\partial(U \cap \mathbb{C}_j)$, the curve $-\overline{\gamma}$ belongs to $\partial(U \cap \mathbb{C}_j)$ too. We can therefore decompose $\partial(U \cap \mathbb{C}_j)$ as follows:

- First define $\gamma_{+,1}, \dots, \gamma_{+,\kappa}$ as those Jordan curves that belong to $\partial(U \cap \mathbb{C}_j)$ and lie entirely in the open upper complex halfplane $\mathbb{C}_j^+$.

  The curves $-\overline{\gamma_{+,1}}, \dots, -\overline{\gamma_{+,\kappa}}$ are then exactly those Jordan curves that belong to $\partial(U \cap \mathbb{C}_j)$ and lie entirely in the lower complex halfplane $\mathbb{C}_j^-$.

- In a second step, consider the curves $\gamma_{\kappa+1}, \dots, \gamma_N$ that belong to $\partial(U \cap \mathbb{C}_j)$ and take values both in $\mathbb{C}_j^+$ and $\mathbb{C}_j^-$. Define $\gamma_{+,\ell}$ for $\ell = \kappa+1, \dots, N$ as the part of γ_ℓ that lies in $\mathbb{C}_j^+$ and $\gamma_{-,\ell} = -\overline{\gamma_{+,\ell}}$ as the part of γ_ℓ that lies in $\mathbb{C}_j^-$, cf. Lemma 2.1.33.

Overall, we obtain the following decomposition of $\partial(U \cap \mathbb{C}_j)$:

$$\partial(U \cap \mathbb{C}_j) = \bigcup_{1 \leq \ell \leq N} \gamma_{+,\ell} \cup -\overline{\gamma_{+,\ell}}.$$

Definition 2.1.34. We call the set $\{\gamma_{1,+}, \ldots, \gamma_{N,+}\}$ the part of $\partial(U \cap \mathbb{C}_j)$ that lies in $\mathbb{C}_j^+$.

Finally, we are now able to formulate the Cauchy formulas for slice hyperholomorphic functions. These formulas are also the starting point for the definition of the S-functional calculus.

Theorem 2.1.35 (The Cauchy formulas)**.** *Let $U \subset \mathbb{H}$ be a bounded slice Cauchy domain, let $j \in \mathbb{S}$ and set $ds_j = ds(-j)$. If f is a (left) slice hyperholomorphic function on a set that contains $\overline{U}$, then*

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s,q)\, ds_j\, f(s), \quad \text{for any } q \in U. \tag{2.31}$$

If f is a right slice hyperholomorphic function on a set that contains $\overline{U}$, then

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,q), \quad \text{for any } q \in U. \tag{2.32}$$

These integrals depend neither on U nor on the imaginary unit $j \in \mathbb{S}$.

Theorem 2.1.36 (Cauchy formulas on unbounded slice Cauchy domains)**.** *Let $U \subset \mathbb{H}$ be an unbounded slice Cauchy domain and let $j \in \mathbb{S}$ and set $ds_j = ds(-j)$. If $f \in \mathcal{SH}_L(\overline{U})$ and $f(\infty) := \lim_{|q| \to \infty} f(q)$ exists, then*

$$f(q) = f(\infty) + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s,q)\, ds_j\, f(s), \quad \text{for any } q \in U.$$

If $f \in \mathcal{SH}_R(\overline{U})$ and $f(\infty) := \lim_{|q| \to \infty} f(q)$ exists, then

$$f(q) = f(\infty) + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,q), \quad \text{for any } q \in U.$$

Finally, just as holomorphic functions, slice hyperholomorphic functions can be approximated by rational functions.

Definition 2.1.37. A function r is called left rational if it is of the form $r(q) = P(q)^{-1}Q(q)$ with polynomials $P \in \mathcal{N}(\mathbb{H})$ and $Q \in \mathcal{SH}_L(\mathbb{H})$.

A function r is called right rational if it is of the form $r(q) = Q(q)P(q)^{-1}$ with polynomials $P \in \mathcal{N}(\mathbb{H})$ and $Q \in \mathcal{SH}_R(\mathbb{H})$.

Finally, a function r is called intrinsic rational if it is of the form $r(q) = P(q)^{-1}Q(q)$ with two polynomials $P, Q \in \mathcal{N}(\mathbb{H})$.

Remark 2.1.6. The requirement that P is intrinsic is necessary because the function P^{-1} is otherwise not slice hyperholomorphic. This is, however, not a serious restriction since any rational left slice hyperholomorphic function as $f(q) = P(q)^{-*_L} *_L Q(p)$ with left slice hyperholomorphic (but not necessarily intrinsic) polynomials can be represented in the above form $f(q) = \tilde{P}(q)^{-1}\tilde{Q}(q)$ with an intrinsic polynomial $\tilde{P}$ and a left slice hyperholomorphic polynomial $\tilde{Q}$. (Precisely, we have $\tilde{P} = P^s$ and $\tilde{Q} = P^c *_L Q$, cf. Corollary 2.1.19.) An analogous result holds for the right slice hyperholomorphic case.

Corollary 2.1.38. *Let $f \in \mathcal{SH}_L(U)$, let $j, i \in \mathbb{S}$ with $i \perp j$ and write $f_j = F_1 + F_2 i$ with holomorphic components $F_1, F_2 : U \cap \mathbb{C}_j \to \mathbb{C}_j$ according to Lemma 2.1.5. Then f is left rational if and only if F_1 and F_2 are rational functions on $\mathbb{C}_j$.*

Similarly, if $f \in \mathcal{SH}_R(U)$ and we write $f_j = F_1 + iF_2$ with holomorphic components F_1 and F_2 according to Lemma 2.1.5, then f is right rational if and only if F_1, F_2 are rational functions on $\mathbb{C}_j$.

Theorem 2.1.39 (Runge's Theorem)**.** *Let $K \subset \mathbb{H}$ be an axially symmetric compact set and let A be an axially symmetric set such that $A \cap C \neq \emptyset$ for any connected component C of $(\mathbb{H} \cup \{\infty\}) \setminus K$.*

If f is left slice hyperholomorphic on an axially symmetric open set U with $K \subset U$, then, for any $\varepsilon > 0$, there exists a left rational function r whose poles lie in A such that

$$\sup\{|f(q) - r(q)| : \quad q \in K\} < \varepsilon. \tag{2.33}$$

Similarly, if f is right slice hyperholomorphic on an axially symmetric open set U with $K \subset U$, then, for any $\varepsilon > 0$, there exists a right rational function r whose poles lie in A such that (2.33) holds.

Finally, if $f \in \mathcal{N}(U)$ for some axially symmetric open set U with $K \subset U$, then, for any $\varepsilon > 0$, there exists a real rational function r whose poles lie in A such that (2.33) holds.

2.2 The S-functional calculus for bounded operators

This section is a preliminary to the arguments dealt with in this book. As pointed out in the introduction, the fundamental difficulty in developing a mathematically rigorous theory of quaternionic linear operators was the identification of suitable notions of quaternionic spectrum and of quaternionic resolvent operators. These problems were solved with the introduction of the S-spectrum and the S-resolvent operators. We summarize these concepts and we define the quaternionic S-functional calculus for bounded operators, for the proofs see [57].

Let us start with a precise definition of the various structures of quaternionic vector, Banach and Hilbert spaces.

Definition 2.2.1. A quaternionic right vector space is an additive group $(X, +)$ endowed with a quaternionic scalar multiplication from the right such that for all

$x, y \in X$ and all $a, b \in \mathbb{H}$

$$(x+y)a = xa + ya, \quad x(a+b) = xa + xb, \quad y(ab) = (ya)b, \quad y1 = y. \tag{2.34}$$

A quaternionic left vector space is an additive group $(X, +)$ endowed with a quaternionic scalar multiplication from the left such that for all $x, y \in X$ and all $a, b \in \mathbb{H}$

$$a(x+y) = ax + ay, \quad (a+b)y = ay + by, \quad (ab)y = a(by), \quad 1y = y. \tag{2.35}$$

Finally, a two-sided quaternionic vector space is an additive group $(X, +)$ endowed with a quaternionic scalar multiplication from the right and a quaternionic scalar multiplication from the left that satisfy (2.34) (resp. (2.35)) such that in addition $ay = ya$ for all $y \in X$ and all $a \in \mathbb{R}$.

Remark 2.2.1. Starting from a real vector space $X_{\mathbb{R}}$, we can easily construct a two-sided quaternionic vector space by setting

$$X_{\mathbb{R}} \otimes \mathbb{H} = \left\{ \sum_{\ell=0}^{3} y_\ell \otimes e_\ell : \quad y_\ell \in X_{\mathbb{R}} \right\},$$

where we denote $e_0 = 1$ for neatness. Together with the componentwise addition $X_{\mathbb{R}} \otimes \mathbb{H}$ forms an additive group. It is a two-sided quaternionic vector space, if we endow it with the right and left scalar multiplications

$$ay = \sum_{\ell,\kappa=0}^{3} (a_\ell y_\kappa) \otimes (e_\ell e_\kappa) \quad \text{and} \quad ya = \sum_{\ell,\kappa=0}^{3} (a_\ell y_\kappa) \otimes (e_\kappa e_\ell)$$

for $a = \sum_{\ell=0}^{3} a_\ell e_\ell \in \mathbb{H}$ and $y = \sum_{\kappa=0}^{3} y_\kappa \otimes e_\kappa \in X_{\mathbb{R}} \otimes \mathbb{H}$. Usually one omits the symbol $\otimes$ and simply writes $y = \sum_{\ell=0}^{3} y_\ell e_\ell$.

Any two-sided quaternionic vector space is essentially of this form. Indeed, we can set

$$X_{\mathbb{R}} = \{y \in X : \quad ay = ya, \ \forall a \in \mathbb{H}\} \tag{2.36}$$

and find that X is isomorphic to $X_{\mathbb{R}} \otimes \mathbb{H}$. If we set $\mathrm{Re}(y) := \frac{1}{4} \sum_{\ell=0}^{3} \overline{e_\ell} y e_\ell$, then $\mathrm{Re}(y) \in X_{\mathbb{R}}$ and $y = \sum_{\ell=0}^{3} \mathrm{Re}(\overline{e_\ell} y) e_\ell$.

Remark 2.2.2. A quaternionic right or left vector space also carries the structure of a real vector space: if we simply restrict the quaternionic scalar multiplication to $\mathbb{R}$, then we obtain a real vector space. Similarly, if we choose some $j \in \mathbb{S}$ and identify $\mathbb{C}_j$ with the field of complex numbers, then X also carries the structure of a complex vector space over $\mathbb{C}_j$. Again we obtain this structure by restricting the quaternionic scalar multiplication to $\mathbb{C}_j$.

If we consider a two-sided quaternionic vector space, then the left and the right scalar multiplication coincide for real numbers so that we can restrict them to $\mathbb{R}$ in order to obtain again a real vector space. This is, however, not true for the multiplication with scalars in one complex plane $\mathbb{C}_j$. In general, $zy \neq yz$ for $z \in \mathbb{C}_j$ and $y \in X$. Hence, we can only restrict either the left or the right multiplication to $\mathbb{C}_j$ in order to consider X as a complex vector space over $\mathbb{C}_j$, but not both simultaneously.

Definition 2.2.2. A norm on a right, left or two-sided quaternionic vector space X is a norm in the sense of real vector spaces (cf. Remark 2.2.2) that is compatible with the quaternionic right, left (resp. two-sided) scalar multiplication. Precisely, this means that $\|ya\| = \|y\||a|$ (or $\|ay\| = |a|\|y\|$ (resp. $\|ay\| = |a|\|y\| = \|ya\|$)) for all $a \in \mathbb{H}$ and all $y \in X$. A quaternionic right, left or two-sided Banach space is a quaternionic right, left or two-sided vector space that is endowed with a norm $\|\cdot\|$ and complete with respect to the topology induced by this norm.

Remark 2.2.3. Similar to Remark 2.2.2, we obtain a real Banach space if we restrict the left or right scalar multiplication on a quaternionic Banach space to $\mathbb{R}$ and we obtain a complex Banach space over $\mathbb{C}_j$ if we restrict the left or right scalar multiplication to $\mathbb{C}_j$ for some $j \in \mathbb{S}$.

Definition 2.2.3. A quaternionic right Hilbert space $\mathcal{H}$ is a quaternionic right vector space equipped with a scalar product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{H}$ so that for all vectors $x, y, w \in \mathcal{H}$ and all scalars $a \in \mathbb{H}$

(i) $\langle x, x \rangle \geq 0$,

(ii) $\langle x, ya + w \rangle = \langle x, y \rangle a + \langle x, w \rangle$,

(iii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$,

and so that $\mathcal{H}$ is complete with respect to the norm $\|y\| = \sqrt{\langle y, y \rangle}$.

Remark 2.2.4. In order to be consistent with our notation, we shall also assume the scalar product of a complex Hilbert space to be sesquilinear in the first and linear in the second variable.

Remark 2.2.5. The fundamental concepts that appear in complex Hilbert spaces such as orthogonality, orthonormal bases, etc. can be defined in a similar way in the quaternionic setting and fundamental results such as the Riesz representation theorem hold also in this noncommutative setting.

Notation 2.2.4. Since we are working with different number systems and vector space structures, we introduce for a set of vectors $\mathbf{B} := (b_\ell)_{\ell \in \Lambda}$ the quaternionic right-linear span of $\mathbf{B}$

$$\operatorname{span}_{\mathbb{H}} \mathbf{B} := \left\{ \sum_{\ell \in I} b_\ell x_\ell : \quad x_\ell \in \mathbb{H},\ I \subset \Lambda \text{ finite} \right\}$$

and the $\mathbb{C}_j$-linear span of $\mathbf{B}$

$$\operatorname{span}_{\mathbb{C}_j} \mathbf{B} := \left\{ \sum_{\ell \in I} b_\ell z_\ell : \quad z_\ell \in \mathbb{C}_j,\ I \subset \Lambda \text{ finite} \right\}.$$

Definition 2.2.5. A mapping $T : X_R \to W_R$ between two quaternionic right Banach spaces X_R and W_R is called right linear if $T(xa + y) = T(x)a + T(y)$ for all $x, y \in X_R$ and $a \in \mathbb{H}$. It is bounded if $\|T\| := \sup_{\|y\|=1} \|Ty\|$ is finite. We denote the set of all bounded right linear operators $T : X_R \to X_R$ by $\mathcal{B}(X_R)$.

Remark 2.2.6. We consider right linear operators by convention. One can also consider left linear operators, which leads to an equivalent theory.

Remark 2.2.7. The set $\mathcal{B}(X_R)$ of all bounded right linear operators on a quaternionic right Banach space X_R is a real Banach space with the pointwise addition $(T + U)(y) := T(y) + U(y)$ and the multiplication $(Ta)(y) := T(ya)$ with scalars $a \in \mathbb{R}$. However, if we define $(Ta)(y) := T(ya)$ for $a \in \mathbb{H} \setminus \mathbb{R}$, then we do not obtain a quaternionic right linear operator as

$$(Ta)(y)b = T(y)ab \neq T(yba) = T(yb)a = (Ta)(yb)$$

if $a, b \in \mathbb{H}$ do not belong to the same complex plane. Hence, $Ta \notin \mathcal{B}(X_R)$ for $a \notin \mathbb{R}$ and so $\mathcal{B}(X_R)$ is not a quaternionic linear space.

The space $\mathcal{B}(X)$ of all bounded right linear operators on a two-sided quaternionic Banach space X is on the other hand again a two-sided quaternionic Banach space with the scalar multiplications

$$(aT)(y) = a(T(y)) \quad \text{and} \quad (Ta)(y) = T(ay). \tag{2.37}$$

For this reason, the theory of quaternionic linear operators is usually developed on two-sided quaternionic Banach spaces and we also work on two-sided spaces in this book. An exception is the article [131], which discusses the minimal structure that is necessary to develop quaternionic operator theory and shows that the essential results can also be obtained on one-sided spaces.

If T is a bounded operator on a two-sided quaternionic Banach space $X = X_{\mathbb{R}} \otimes \mathbb{H}$, then we can write T as

$$T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \tag{2.38}$$

with $\mathbb{R}$-linear components $T_\ell \in \mathcal{B}(X_{\mathbb{R}})$ for $\ell = 0, \dots, 3$. If we set $e_0 = 1$ for neatness, then T acts as

$$Ty = \sum_{\ell,\kappa=0}^{3} T_\ell y_\kappa e_\ell e_\kappa \quad \text{for } y = \sum_{\kappa=0}^{3} y_\kappa e_\kappa \in X = X_{\mathbb{R}} \otimes \mathbb{H}.$$

We thus have

$$\mathcal{B}(X) = \mathcal{B}(X_{\mathbb{R}}) \otimes \mathbb{H}.$$

Definition 2.2.6. Let X be a two-sided quaternionic Banach space. We denote the set of all operators $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{B}(X)$ with commuting components by $\mathcal{BC}(X)$. Furthermore, we call a bounded operator T a scalar operator if it is of the form $T = T_0$, that is if $T_1 = T_2 = T_3 = 0$.

We now want to introduce suitable notions of spectrum and resolvent operator for quaternionic operators. In the classical setting, the spectrum $\sigma(A)$ of a complex operator A is the set of complex numbers λ, for which the resolvent operator $R_\lambda(A) = (\lambda\mathcal{I} - A)^{-1}$ does not exist as a bounded operator. The resolvent on the other hand formally corresponds to the Cauchy kernel $(\lambda - z)^{-1}$, in which the scalar variable z is formally replaced by the operator A. This relation is fundamental for the definition of the Riesz–Dunford functional calculus for holomorphic functions.

In order to extend the concepts of spectrum and resolvent operator in the quaternionic setting, we consider the series expansions (2.27) and (2.28) of the slice hyperholomorphic Cauchy kernels, that is the series

$$\sum_{n=0}^{+\infty} q^n s^{-n-1} \quad \text{and} \quad \sum_{n=0}^{+\infty} s^{-n-1} q^n$$

and we give the following definition.

Definition 2.2.7. Let $T \in \mathcal{B}(X)$ and $s \in \mathbb{H}$. We call the series

$$\sum_{n=0}^{+\infty} T^n s^{-n-1} \quad \text{and} \quad \sum_{n=0}^{+\infty} s^{-n-1} T^n$$

the left and right Cauchy kernel operator series, respectively.

Lemma 2.2.8. *Let $T \in \mathcal{B}(X)$. For $\|T\| < |s|$ the left and the right Cauchy kernel operator series converge in the operator norm.*

Our goal is now to determine the closed form of the Cauchy kernel operator series. We collect in the following theorem some important results.

Theorem 2.2.9. *Let $T \in \mathcal{B}(X)$ and let $s \in \mathbb{H}$ with $\|T\| < |s|$. Then the following results hold.*

(i) *We have*

$$\left(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I}\right)^{-1} = \sum_{n=0}^{+\infty} T^n \sum_{k=0}^{n} \overline{s}^{-k-1} s^{-n+k-1}. \tag{2.39}$$

(ii) *The left Cauchy kernel series equals*

$$\sum_{n=0}^{+\infty} T^n s^{-n-1} = -(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}).$$

(iii) *The right Cauchy kernel series equals*

$$\sum_{n=0}^{+\infty} s^{-n-1} T^n = -(T - \overline{s}\mathcal{I})(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}.$$

The previous result motivates the following definition.

Definition 2.2.10. Let $T \in \mathcal{B}(X)$. For $s \in \mathbb{H}$, we set

$$\mathcal{Q}_s(T) := T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I}.$$

We define the *S-resolvent set* $\rho_S(T)$ of T as

$$\rho_S(T) := \{s \in \mathbb{H} : \quad \mathcal{Q}_s(T) \text{ is invertible in } \mathcal{B}(X)\}$$

and we define the *S-spectrum* $\sigma_S(T)$ of T as

$$\sigma_S(T) := \mathbb{H} \setminus \rho_S(T).$$

For $s \in \rho_S(T)$, the operator $\mathcal{Q}_s(T)^{-1} \in \mathcal{B}(X)$ is called the *pseudo-resolvent operator* of T at s.

As the following results show, the S-spectrum has a structure that is compatible with the structure of slice hyperholomorphic functions and with the symmetry of the set of right eigenvalues of T. Moreover, it generalizes the set of right eigenvalues just as the classical spectrum generalizes the set of eigenvalues of a complex linear operator and has analogous properties.

Theorem 2.2.11. *Let $T \in \mathcal{B}(X)$.*

(i) *The sets $\rho_S(T)$ and $\sigma_S(T)$ are axially symmetric.*

(ii) *The S-spectrum $\sigma_S(T)$ of T is a nonempty, compact set contained in the closed ball $\overline{B_{\|T\|}(0)}$.*

(iii) *Let $T \in \mathcal{B}(X)$. Then* $\ker \mathcal{Q}_s(T) \neq \{0\}$ *if and only if s is a right eigenvalue of T. In particular, any right eigenvalue belongs to $\sigma_S(T)$.*

On the S-resolvent set we can now define the slice hyperholomorphic resolvents. As in the complex case, they correspond to the slice hyperholomorphic Cauchy kernel, in which we formally replace the scalar variable q by the operator T. Since we distinguish between left and right slice hyperholomorphicity, two different resolvent operators are associated with an operator T in the quaternionic setting.

Definition 2.2.12. Let $T \in \mathcal{B}(X)$. For $s \in \rho_S(T)$, we define the *left S-resolvent operator* as

$$S_L^{-1}(s,T) = -\mathcal{Q}_s(T)^{-1}(T - \overline{s}\mathcal{I}),$$

and the *right S-resolvent operator* as

$$S_R^{-1}(s,T) = -(T - \overline{s}\mathcal{I})\mathcal{Q}_s(T)^{-1}.$$

To develop the slice hyperholomorphic functional calculus it is necessary that the S-resolvents are slice hyperholomorphic in the scalar variable s. We observe that the left resolvent operator $(s\mathcal{I} - T)^{-1}$ is not slice hyperholomorphic nor Fueter regular.

Lemma 2.2.13. *Let $T \in \mathcal{B}(X)$.*

(i) *The left S-resolvent $S_L^{-1}(s,T)$ is a $\mathcal{B}(X)$-valued right slice hyperholomorphic function of the variable s on $\rho_S(T)$.*

(ii) *The right S-resolvent $S_R^{-1}(s,T)$ is a $\mathcal{B}(X)$-valued left slice hyperholomorphic function of the variable s on $\rho_S(T)$.*

The S-resolvent operators in general do not commute with T. However, they satisfy the following relations, that are useful for compensating this fact.

Theorem 2.2.14. *Let $T \in \mathcal{B}(X)$ and let $s \in \rho_S(T)$. The left S-resolvent operator satisfies the* left S-resolvent equation

$$S_L^{-1}(s,T)s - TS_L^{-1}(s,T) = \mathcal{I} \tag{2.40}$$

and the right S-resolvent operator satisfies the right S-resolvent equation

$$sS_R^{-1}(s,T) - S_R^{-1}(s,T)T = \mathcal{I}. \tag{2.41}$$

The left and the right S-resolvent equations cannot be considered the generalizations of the classical resolvent equation

$$R_\lambda(A) - R_\mu(A) = (\mu - \lambda)R_\lambda(A)R_\mu(A), \quad \text{for } \lambda, \mu \in \rho(A), \tag{2.42}$$

where $R_\lambda(A) = (\lambda\mathcal{I} - A)^{-1}$ is the resolvent operator of A at $\lambda \in \rho(A)$. This equation provides the possibility to write the product $R_\lambda(A)R_\mu(A)$ in terms of the difference $R_\lambda(A) - R_\mu(A)$. This is not the case for the left and the right S-resolvent equations. The proper generalization of (2.42), which preserves this property, is the S-resolvent equation that we show in the following theorem. It is remarkable that this equation involves both the left and the right S-resolvent operators while a generalization of (2.42), that includes just one of them, has never been found.

Theorem 2.2.15 (The S-resolvent equation)**.** *Let $T \in \mathcal{B}(X)$ and let $s, q \in \rho_S(T)$ with $q \notin [s]$. Then the equation*

$$\begin{aligned} S_R^{-1}(s,T)S_L^{-1}(p,T) = \big[&\left(S_R^{-1}(s,T) - S_L^{-1}(q,T)\right) q \\ &-\overline{s}\left(S_R^{-1}(s,T) - S_L^{-1}(q,T)\right)\big] (q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1} \end{aligned} \tag{2.43}$$

holds true. Equivalently, it can also be written as

$$\begin{aligned} S_R^{-1}(s,T)S_L^{-1}(q,T) = &(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1} \\ &\cdot \left[\left(S_L^{-1}(q,T) - S_R^{-1}(s,T)\right)\overline{q} - s\left(S_L^{-1}(q,T) - S_R^{-1}(s,T)\right)\right]. \end{aligned} \tag{2.44}$$

We can now define the S-functional calculus for a bounded quaternionic linear operator T on a two-sided quaternionic Banach space X. The S-functional calculus is the quaternionic version of the Riesz–Dunford functional calculus for complex linear operators. We consider a function f that is slice hyperholomorphic on $\sigma_S(T)$ and we use the slice hyperholomorphic Cauchy formula. In order to define $f(T)$, we formally replace the scalar variable q by the operator T, in the Cauchy kernels $S_L^{-1}(s,q)$ (resp. $S_R^{-1}(s,q)$). We thus obtain the corresponding S-resolvent operators $S_L^{-1}(s,T)$ (resp. $S_R^{-1}(s,T)$). The main references in which the formulations and the properties of S-functional calculus for quaternionic operators has been studied are [11, 82, 83].

Before we define the S-functional calculus, we show that the procedure described above is actually meaningful. In particular, it must be consistent with functions of T that we can define explicitly, that is with polynomials of T.

Theorem 2.2.16. *Let $T \in \mathcal{B}(X)$, let U be a bounded slice Cauchy domain that contains $\sigma_S(T)$, let $j \in \mathbb{S}$ and set $ds_j = ds(-j)$. For any left slice hyperholomorphic polynomial $P(q) = \sum_{\ell=0}^{n} q^\ell a_\ell$ with $a_\ell \in \mathbb{H}$, we set $P(T) = \sum_{\ell=0}^{n} T^\ell a_\ell$. Then*

$$P(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, P(s). \tag{2.45}$$

Similarly, we set $P(T) = \sum_{\ell=0}^{n} a_\ell T^\ell$ for any right slice hyperholomorphic polynomial $P(q) = \sum_{\ell=0}^{n} a_\ell q^\ell$ with $a_\ell \in \mathbb{H}$. Then

$$P(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} P(s)\, ds_j\, S_R^{-1}(s,T). \tag{2.46}$$

In particular, the operators in (2.45) and (2.46) coincide for any intrinsic polynomial $P(q) = \sum_{\ell=0}^{n} q^\ell a_\ell$ with real coefficients $a_\ell \in \mathbb{R}$.

The S-functional calculus applies to functions that are slice hyperholomorphic on the S-spectrum of T. We introduce the following notation for this class of functions.

Definition 2.2.17. Let $T \in \mathcal{B}(X)$. We denote by $\mathcal{SH}_L(\sigma_S(T))$, $\mathcal{SH}_R(\sigma_S(T))$ and $\mathcal{N}(\sigma_S(T))$ the sets of all left, right and intrinsic slice hyperholomorphic functions f with $\sigma_S(T) \subset \mathcal{D}(f)$, where $\mathcal{D}(f)$ is the domain of the function f.

Remark 2.2.8. If $f \in \mathcal{SH}_L(\sigma_S(T))$ or $f \in \mathcal{SH}_R(\sigma_S(T))$, then the set $\mathcal{D}(f)$ is an axially symmetric open set that contains the compact axially symmetric set $\sigma_S(T)$. By Theorem 2.1.31 there exists a bounded slice Cauchy domain U such that $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$.

Definition 2.2.18 (S-functional calculus)**.** Let $T \in \mathcal{B}(X)$. For any $f \in \mathcal{SH}_L(\sigma_S(T))$, we define

$$f(T) := \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s), \tag{2.47}$$

where j is an arbitrary imaginary unit in $\mathbb{S}$, $ds_j = ds(-j)$ and U is an arbitrary slice Cauchy domain U as in Remark 2.2.8. For any $f \in \mathcal{SH}_R(\sigma_S(T))$, we define

$$f(T) := \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,T), \tag{2.48}$$

where j is again an arbitrary imaginary unit in $\mathbb{S}$, $ds_j = ds(-j)$ and U is an arbitrary slice Cauchy domain as in Remark 2.2.8.

Theorem 2.2.16 shows that the S-functional calculus is meaningful because it is consistent with polynomials of T. As the next crucial result shows, it is moreover well-defined.

Theorem 2.2.19. *Let $T \in \mathcal{B}(X)$. For any $f \in \mathcal{SH}_L(\sigma_S(T))$, the integral in (2.47) that defines the operator $f(T)$ is independent of the choice of the slice Cauchy domain U and of the imaginary unit $j \in \mathbb{S}$. Similarly, for any $f \in \mathcal{SH}_R(\sigma_S(T))$, the integral in (2.48) that defines the operator $f(T)$ is also independent of the choice of U and $j \in \mathbb{S}$.*

Theorem 2.2.19 shows that the S-functional calculus is well defined for any left or right slice hyperholomorphic function and Theorem 2.2.16 shows that it is consistent with slice hyperholomorphic polynomials. Even more, it is compatible with any rational slice hyperholomorphic function and with limits of uniformly convergent sequences of functions.

Lemma 2.2.20. *Let $T \in \mathcal{B}(X)$. If P is an intrinsic polynomial such that $P^{-1} \in \mathcal{N}(\sigma_S(T))$, then $P^{-1}(T) = P(T)^{-1}$. Moreover, if $r(q) = P(q)^{-1}Q(q)$ is an intrinsic rational function and $P^{-1} \in \mathcal{N}(\sigma_S(T))$, then (2.47) and (2.48) give the same operator $r(T) = P(T)^{-1}Q(T)$.*

Theorem 2.2.21. *Let $T \in \mathcal{B}(X)$. Let $f_n, f \in \mathcal{SH}_L(\sigma_S(T))$ or let $f_n, f \in \mathcal{SH}_R(\sigma_S(T))$ for $n \in \mathbb{N}$. If there exists a bounded slice Cauchy domain U with $\sigma_S(T) \subset U$ such that $f_n \to f$ uniformly on $\overline{U}$, then $f_n(T)$ converges to $f(T)$ in the norm topology of $\mathcal{B}(X)$.*

Lemma 2.2.20 implies in particular that the S-functional calculus for left slice hyperholomorphic functions and the S-functional calculus for right slice hyperholomorphic functions are consistent for intrinsic rational functions. Since, by Theorem 2.1.39, any intrinsic slice hyperholomorphic function can be uniformly approximated by intrinsic rational functions, Theorem 2.2.21 implies that both versions of the S-functional calculus are consistent for arbitrary intrinsic slice hyperholomorphic functions.

Theorem 2.2.22. *Let $T \in \mathcal{B}(X)$. If $f \in \mathcal{N}(\sigma_S(T))$, then both versions of S-functional calculus give the same operator $f(T)$. Precisely, we have*

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,T).$$

Remark 2.2.9. We point out that Theorem 2.2.22 is in general not true for arbitrary intrinsic slice hyperholomorphic functions, cf. Example 3.7.9 and the discussion in Section 3.8.

An immediate consequence of Definition 2.2.18 is that the S-functional calculus for left slice hyperholomorphic functions is quaternionic right linear and that the S-functional calculus for right slice hyperholomorphic functions is quaternionic left linear.

Lemma 2.2.23. *Let $T \in \mathcal{B}(X)$.*

(i) *If $f, g \in \mathcal{SH}_L(\sigma_S(T))$ and $a \in \mathbb{H}$, then*

$$(f+g)(T) = f(T) + g(T) \quad \textit{and} \quad (fa)(T) = f(T)a.$$

(ii) *If $f, g \in \mathcal{SH}_R(\sigma_S(T))$ and $a \in \mathbb{H}$, then*

$$(f+g)(T) = f(T) + g(T) \quad \textit{and} \quad (af)(T) = af(T).$$

Since the product of two slice hyperholomorphic functions is not necessarily slice hyperholomorphic, we cannot expect to obtain a product rule for arbitrary slice hyperholomorphic functions. However, if $f \in \mathcal{N}(\sigma_S(T))$ and $g \in \mathcal{SH}_L(\sigma_S(T))$ then $fg \in \mathcal{SH}_L(\sigma_S(T))$ and if $f \in \mathcal{SH}_R(\sigma_S(T))$ and $g \in \mathcal{N}(\sigma_S(T)$ then $fg \in \mathcal{SH}_R(\sigma_S(T))$. In order to show that the S-functional calculus is at least in these cases compatible with the multiplication of functions, we have used the following lemma.

Lemma 2.2.24. *Let $B \in \mathcal{B}(X)$. For any $q, s \in \mathbb{H}$ with $q \notin [s]$, we have*

$$(\overline{s}B - Bq)(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1} = (s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}(sB - B\overline{q}). \tag{2.49}$$

If, moreover, f is an intrinsic slice hyperholomorphic function and U is a bounded slice Cauchy domain with $\overline{U} \subset \mathcal{D}(f)$, then

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, (\overline{s}B - Bq)(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1} = Bf(q),$$

for any $q \in U$ and any $j \in \mathbb{S}$.

From the above lemma and some computations we get the product rule.

Theorem 2.2.25 (Product rule)**.** *Let $T \in \mathcal{B}(X)$ and let $f \in \mathcal{N}(\sigma_S(T))$ and $g \in \mathcal{SH}_L(\sigma_S(T))$ or let $f \in \mathcal{SH}_R(\sigma_S(T))$ and $g \in \mathcal{N}(\sigma_S(T))$. Then*

$$(fg)(T) = f(T)g(T).$$

An immediate consequence is the following corollary.

Corollary 2.2.26. *Let $T \in \mathcal{B}(X)$ and let $f \in \mathcal{N}(\sigma_S(T))$. If $f^{-1} \in \mathcal{N}(\sigma_S(T))$, then $f(T)$ is invertible and $f(T)^{-1} = f^{-1}(T)$.*

Finally, the S-functional calculus has the capability to define the quaternionic Riesz projectors and allows in turn to identify invariant subspaces of T that are associated with sets of spectral values.

Theorem 2.2.27 (Riesz's projectors)**.** *Let $T \in \mathcal{B}(X)$ and assume that $\sigma_S(T) = \sigma_1 \cup \sigma_2$ with*

$$\operatorname{dist}(\sigma_1, \sigma_2) > 0.$$

We choose an open axially symmetric set O with $\sigma_1 \subset O$ and $\overline{O} \cap \sigma_2 = \emptyset$ and define $\chi_{\sigma_1}(s) = 1$ for $s \in O$ and $\chi_{\sigma_2}(s) = 0$ for $s \notin O$. Then $\chi_{\sigma_1} \in \mathcal{N}(\sigma_S(T))$ and

$$P_{\sigma_1} := \chi_{\sigma_1}(T) = \frac{1}{2\pi} \int_{\partial(O \cap \mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j$$

is a continuous projection that commutes with T. Hence, $P_{\sigma_1}X$ is a right linear subspace of X that is invariant under T.

Similar to the product rule, the spectral mapping theorem does not hold for arbitrary slice hyperholomorphic functions. This is not surprising: it is clear that it can only hold true for slice hyperholomorphic functions that preserve the fundamental geometry of the S-spectrum, namely its axially symmetry. Again, the class of intrinsic slice hyperholomorphic functions stands out here; in fact this class of functions maps axially symmetric sets to axially symmetric sets.

Theorem 2.2.28 (The Spectral Mapping Theorem)**.** *Let $T \in \mathcal{B}(X)$ and let $f \in \mathcal{N}(\sigma_S(T))$. Then*

$$\sigma_S(f(T)) = f(\sigma_S(T)) = \{f(s) : \quad s \in \sigma_S(T)\}.$$

The Spectral Mapping Theorem allows us to generalize the Gelfand formula for the spectral radius to quaternionic linear operators.

Definition 2.2.29. Let $T \in \mathcal{B}(X)$. Then the S-spectral radius of T is defined to be the nonnegative real number

$$r_S(T) := \sup\{|s| : \quad s \in \sigma_S(T)\}.$$

Theorem 2.2.30. *For $T \in \mathcal{B}(X)$, we have*

$$r_S(T) = \lim_{n \to +\infty} \|T^n\|^{\frac{1}{n}}.$$

Finally, the Spectral Mapping Theorem also allows us to generalize the composition rule.

Theorem 2.2.31 (Composition rule)**.** *Let $T \in \mathcal{B}(X)$ and let $f \in \mathcal{N}(\sigma_S(T))$. If $g \in \mathcal{SH}_L(\sigma_S(f(T))$, then $g \circ f \in \mathcal{SH}_L(\sigma_S(T))$ and if $g \in \mathcal{SH}_R(f(\sigma_S(T)))$, then $g \circ f \in \mathcal{SH}_R(\sigma_S(T))$. In both cases,*

$$g(f(T)) = (g \circ f)(T).$$

We recall Theorem 4.7 in [83], or also see Theorem 4.14.14 in [93].

Theorem 2.2.32 (Perturbation of the S-functional calculus)**.** *Let $T, Z \in \mathcal{B}(X)$, $f \in \mathcal{SH}_L(\sigma_S(T))$ and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that, for $\|Z - T\| < \delta$, we have $f \in \mathcal{SH}_L(\sigma_S(Z))$and*

$$\|f(Z) - f(T)\| < \varepsilon,$$

where

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s, T) \; ds_j \; f(s)$$

and $U \subset \mathbb{H}$ is a Cauchy domain, $ds_j = ds/j$ for $j \in \mathbb{S}$.

2.3 Bounded operators with commuting components

If the components of T commute, then the S-spectrum can be characterized by a different operator, which is often easier to handle in the applications. The S-resolvent operators, in this case, can be expressed in a form that corresponds to replacing the scalar variable q by the operator T in the slice hyperholomorphic Cauchy kernels when they are written in form II, see Proposition 2.1.24.

We recall that any two-sided quaternionic vector space X is essentially of the form $X = X_{\mathbb{R}} \otimes \mathbb{H}$, where $X_{\mathbb{R}}$ is the real vector space consisting of those vectors that commute with all quaternions. If $x = \sum_{\ell=0}^{3} x_\ell e_\ell$ with $x_\ell \in X_{\mathbb{R}}$, where we set $e_0 = 1$ for neatness, then we can write any operator $T \in \mathcal{B}(X)$ as $T = \sum_{\ell=0}^{3} T_\ell e_\ell$ with components $T_\ell \in \mathcal{B}(X_{\mathbb{R}})$, where this operator acts as

$$Tx = \left(\sum_{\ell=0}^{3} T_\ell e_\ell \right) \left(\sum_{\kappa=0}^{3} x_\kappa e_\kappa \right) = \sum_{\ell,\kappa=0}^{3} T_\ell(x_\kappa) e_\ell e_\kappa.$$

We find $\mathcal{B}(X) = \mathcal{B}(X_{\mathbb{R}}) \otimes \mathbb{H}$ and hence we call any operator in $\mathcal{B}(X_{\mathbb{R}})$ a scalar operator on X. Furthermore, we denote the space of all $T = \sum_{\ell=0}^{3} T_\ell e_\ell \in \mathcal{B}(X)$ with mutually commuting components $T_\ell \in \mathcal{B}(X_{\mathbb{R}}), \ell = 0, \ldots, 3$, by $\mathcal{BC}(X)$, cf. Definition 2.2.6.

Definition 2.3.1. For $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{BC}(X)$, we set

$$\overline{T} := T_0 - \sum_{\ell=1}^{3} T_\ell e_\ell.$$

The following statement shows that for an operator $T \in \mathcal{BC}(X)$ the analogues of the scalar identities $s+\overline{s} = 2\mathrm{Re}(s)$ and $s\overline{s} = \overline{s}s = |s|^2$ hold true. This motivates the idea that we can write the S-resolvent operators, for such operators, also by formally replacing q by T in the slice hyperholomorphic Cauchy kernels, when they are written in form II.

Lemma 2.3.2. *Let $T = T_0 + \sum_{\ell=1}^3 T_\ell e_\ell \in \mathcal{BC}(X)$. Then $2T_0 = T + \overline{T}$ and $T\overline{T} = \overline{T}T = \sum_{\ell=0}^3 T_\ell^2$.*

Lemma 2.3.3. *If $T = T_0 + \sum_{\ell=1}^3 T_\ell e_\ell \in \mathcal{BC}(X)$, then the following statements are equivalent.*

(i) *The operator T is invertible.*

(ii) *The operator $\overline{T}$ is invertible.*

(iii) *The operator $T\overline{T}$ is invertible.*

In this case we have

$$\overline{T}^{-1} = \overline{T^{-1}} \quad \text{and} \quad T^{-1} = (T\overline{T})^{-1}\overline{T}. \tag{2.50}$$

Definition 2.3.4. Let $T = T_0 + \sum_{\ell=1}^3 T_\ell e_\ell \in \mathcal{BC}(X)$. For $s \in \mathbb{H}$, we define the operator

$$\mathcal{Q}_{c,s}(T) := s^2\mathcal{I} - 2sT_0 + T\overline{T}.$$

Theorem 2.3.5. *Let $T = T_0 + \sum_{\ell=1}^3 T_\ell e_\ell \in \mathcal{BC}(X)$. Then $\mathcal{Q}_{c,s}(T)$ is invertible if and only if $\mathcal{Q}_s(T)^{-1}$ is invertible and so*

$$\rho_S(T) = \left\{ s \in \mathbb{H} : \quad \mathcal{Q}_{c,s}(T)^{-1} \in \mathcal{B}(X) \right\}. \tag{2.51}$$

Moreover, for $s \in \rho_S(T)$, we have

$$S_L^{-1}(s,T) = (s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T)^{-1}, \tag{2.52}$$

$$S_R^{-1}(s,T) = \mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I} - \overline{T}). \tag{2.53}$$

Definition 2.3.6 (SC-resolvent operators)**.** Let $T \in \mathcal{BC}(X)$. For $s \in \rho_S(T)$, we define the left and right SC-resolvent operators of T as

$$S_{c,L}^{-1}(s,T) = (s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T)^{-1}$$

$$S_{c,R}^{-1}(s,T) = \mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I} - \overline{T}).$$

So for $T \in \mathcal{BC}(X)$ we have the equivalent definitions of the S-functional calculus for operators with commuting components. For $f \in \mathcal{SH}_L(\sigma_S(T))$, we have

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_{c,L}^{-1}(s,T)\, ds_j\, f(s)$$

and for $f \in \mathcal{SH}_R(\sigma_S(T))$ we have

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, S_{c,R}^{-1}(s,T),$$

for any imaginary unit $j \in \mathbb{S}$ and any bounded slice Cauchy domain U with $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$.

The S-functional calculus for operators with commuting components defined by the above integrals, that involve the SC-resolvents, is often also referred to as the SC-functional calculus. Similarly, the S-spectrum is sometimes called F-spectrum, when it is characterized by the operator $\mathcal{Q}_{c,s}(T)^{-1}$, because it was used in the definition of the F-functional calculus.