Chapter 12

Appendix: Principles of functional Analysis

The principles of functional analysis do not depend on the quaternionic structure, so with minor changes these can be proved also in quaternionic functional analysis. For the convenience of the reader, we collect such results in this appendix. Some of the results were already proved in [110], so we quote those here.

Theorem 12.0.1 (The open mapping theorem)**.** *Let X and W be two right quaternionic Banach spaces, and let T be a right linear continuous quaternionic operator from X onto W. Then the image of every open set is open.*

Proof. Let X and W be two right quaternionic Banach spaces and let $T : X \to W$ be a right linear continuous map such that $TX = W$. It is enough to prove the statement for a neighborhood of 0, more precisely for balls. We denote by $B_X(r)$ the ball in X of radius $r > 0$ and centered at the origin. We prove that the closure $\overline{TB_X(r)}$ of the image of any ball $B_X(r)$ centered at 0 in X contains a neighborhood of 0 in W. Moreover, since $TB_X(r) = rTB_X(1)$, we only need to show that $TB_X(r)$ is a neighborhood of the origin for some positive r.

We will make use of the notation $B_X(r) - B_X(r)$ to denote the set of elements of the form $u - v$ where $u, v \in B_X(r)$.

Observe that the function $u - v$ is continuous in u and v. Also, notice that there exists an open ball $B_X(r')$, for suitable $r' > 0$, such that $B_X(r') - B_X(r') \subseteq B_X(r)$.

For every $v \in X$, we have that $v/n \to 0$ as $n \to \infty$ so $v \in nB_X(r')$ for a suitable $n \in \mathbb{N}$. So

$$X = \bigcup_{n=1}^{\infty} nB_X(r') \quad \text{and} \quad W = TX = \bigcup_{n=1}^{\infty} nTB_X(r').$$

By the Baire category theorem, one of the sets $\overline{nTB_X(r')}$ contains a nonempty

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open set. The map $w \mapsto nw$ is a homeomorphism in W and $\overline{TB_X(r')}$ contains a nonempty open set denoted by $\mathcal{B}$, so

$$\overline{TB_X(r)} \supseteq \overline{TB_X(r') - TB_X(r')} \supseteq \overline{TB_X(r')} - \overline{TB_X(r')} \supseteq \mathcal{B} - \mathcal{B}.$$

The map $w \mapsto u - w$ is a homeomorphism; this implies that the set $u - B_X(r)$ is open. Since the set $\mathcal{B} - \mathcal{B} = \bigcup_{u\in\mathcal{B}}(u - \mathcal{B})$ is the union of open sets, it is open and it contains the origin and so it is a neighborhood of the origin. Thus, we have proved that the closure of the image of the neighborhood of the origin contains a neighborhood of the origin.

For any $\varepsilon > 0$, consider the two spheres $B_X(\varepsilon)$ and $B_W(\varepsilon)$ centered at the origin of X and W, respectively.

Choose an arbitrary positive real number ε_0 and let $\varepsilon_\ell > 0$ be a sequence such that $\sum_{\ell\in\mathbb{N}} \varepsilon_\ell < \varepsilon_0$.

Then, according to what we have proved above, there exists a sequence $\{\theta_\ell\}_{\ell\in\mathbb{N}\cup\{0\}}$ with $\theta_\ell > 0$ and $\theta_\ell \to 0$ such that

$$\overline{TB_X(\varepsilon_\ell)} \supset B_W(\theta_\ell), \quad \ell \in \mathbb{N}\cup\{0\}. \tag{12.1}$$

Now take $w \in B_W(\theta_0)$. We show that there exists $v \in B_X(2\varepsilon_{\theta_0})$ such that $Tv = w$.

From (12.1), for $\ell = 0$ there exists a $v_0 \in B_X(\varepsilon_0)$ such that

$$\|w - Tv_0\| < \theta_1.$$

Since $w - Tv_0 \in B_W(\theta_1)$ again from (12.1) with $\ell = 1$, there is $v_1 \in B_X(\varepsilon_1)$ with

$$\|w - Tv_0 - Tv_1\| < \theta_2.$$

So we construct a sequence $\{v_n\}_{n\in\cup\{0\}}$ such that $v_n \in B_X(\varepsilon_n)$ and

$$\|w - T\sum_{\ell=0}^{n} v_\ell\| < \theta_{n+1}, \quad n \in \mathbb{N}\cup\{0\}. \tag{12.2}$$

Let us denote $p_m = \sum_{\ell=0}^{m} v_\ell$. So for $m > n$,

$$\|p_m - p_n\| = \|v_{n+1} + \ldots + v_m\| < \varepsilon_{n+1} + \ldots + \varepsilon_m,$$

which shows that p_m is a Cauchy sequence and that the series $\sum_{\ell=0}^{\infty} v_\ell$ converges at a point v with

$$\|v\| \le \sum_{\ell=0}^{\infty} \varepsilon_\ell = 2\varepsilon_0.$$

Now recall that T is continuous and from (12.2) we have $w = Tv$.

This means that an arbitrary sphere $B_X(2\varepsilon_0)$, about the origin in X, maps onto the set $TB_X(2\varepsilon_0)$ which contains the sphere $B_W(\theta_0)$ about the origin in W. So if $\mathcal{X}$ is a neighborhood of the origin in X, then $T\mathcal{X}$ contains a neighborhood of the origin of W. Since T is linear then the above procedure works for every neighborhood of every point. $\square$

Theorem 12.0.2 (The Banach continuous inverse theorem)**.** *Let X and W be two right quaternionic Banach spaces and let T be a right linear continuous quaternionic operator that is one-to-one from X onto W. Then T has a right linear continuous inverse.*

Proof. Let X and W be two right quaternionic Banach spaces and T be a right linear continuous and one-to-one operator such that $TX = W$. By Theorem 12.0.1 T maps open sets onto open sets, so if we write T as $(T^{-1})^{-1}$, it is immediate that T^{-1} is continuous. Now take w_1, $w_2 \in W$ and v_1, $v_2 \in X$ such that $Tv_1 = w_1$, $Tv_2 = w_2$ and $p \in \mathbb{H}$. Then,

$$T(v_1 + v_1) = Tv_1 + Tv_2 = w_1 + w_2, \quad T(v_1 p) = T(v_1)p = w_1 p$$

so that

$$T^{-1}(w_1 + w_2) = v_1 + v_2$$

and

$$T^{-1}(w_1 p) = v_1 p,$$

so T^{-1} is right linear quaternionic operator. □

Definition 12.0.3. Let X and W be two right quaternionic Banach spaces. Suppose that T is a right linear quaternionic operator whose domain $\mathcal{D}(T)$ is a (right) linear manifold contained in X and whose range belongs to W. The graph of T consists of all point (v, Tv), with $v \in \mathcal{D}(T)$, in the product space $X \times W$.

Definition 12.0.4. We say that T is a closed operator if its graph is closed in $X \times W$.

Remark 12.0.1. Equivalently, we can say that T is closed if $v_n \in \mathcal{D}(T)$, $v_n \to v$, and $Tv_n \to y$ imply that $v \in \mathcal{D}(T)$ and $Tv = y$.

Theorem 12.0.5 (The closed graph theorem)**.** *Let X and W be two right quaternionic Banach spaces. Let $T : X \to W$ be a right linear closed quaternionic operator. Then T is continuous.*

Proof. Since X and W are two right quaternionic Banach spaces, we have that $X \times W$ with the norm $\|(v, w)\|_{X\times W} = \|v\|_X + \|w\|_W$ is a right quaternionic Banach space. The graph of T denoted by

$$\mathcal{G}(T) = \{(v, Tv) : \quad v \in \mathcal{D}(T)\}$$

is a closed linear manifold in the product space $X \times W$ so it is a right quaternionic Banach space. The projection

$$P_X : \mathcal{G}(T) \mapsto X, \quad P_X(v, Tv) = v$$

is one-to-one and onto, linear and continuous, so by Theorem 12.0.2, its inverse P_X^{-1} is continuous. Now consider the projection

$$P_W : \mathcal{G}(T) \mapsto W, \quad P_W(v, Tv) = Tv,$$

since $T = P_W P_X^{-1}$, so we get the statement. □

Theorem 12.0.6 (The Hahn–Banach theorem)**.** *Let X_0 be a right subspace of a right quaternionic Banach space X on $\mathbb{H}$. Suppose that p is a norm on X and let ϕ be a linear and continuous functional on X_0 such that*

$$|\langle \phi, v\rangle| \leq p(v), \quad \forall v \in X_0. \tag{12.3}$$

Then it is possible to extend ϕ to a linear and continuous functional Φ on X satisfying the estimate (12.3) for all $v \in X$.

Proof. Note that, for any quaternion q, we have $q = q_0 + q_1 i + q_2 j + q_3 k = z_1(q) + z_2(q)j$, where $z_1, z_2 \in \mathbb{C}_i = \mathbb{R} + \mathbb{R}i$ and $qj = -z_2(q) + z_1(q)j$, so $q = z_1(q) - z_1(qj)j$. The functional ϕ can be written as $\phi = \phi_0 + \phi_1 i + \phi_2 j + \phi_3 k = \psi_1(\phi) + \psi_2(\phi)j$, with $\psi_1(\phi) = \phi_0 + \phi_1 i$ and $\psi_2(\phi) = \phi_2 + \phi_3 i$ which are complex functionals. It is immediate that

$$\langle \phi, v\rangle = \langle \psi_1, v\rangle - \langle \psi_1, vj\rangle j, \quad \forall v \in X_0,$$

where ψ_1 is a $\mathbb{C}$–linear functional. So we can apply the complex version of the Hahn–Banach theorem to deduce the existence of a functional $\tilde{\psi}_1$ that extends ψ_1 to the whole of X. The functional Ψ, given by

$$\langle \Psi, v\rangle = \langle \tilde{\psi}_1, v\rangle - \langle \tilde{\psi}_1, vj\rangle j,$$

is defined on X and it is the extension that satisfies estimate (12.3) for all $v \in X$. □

The following result is an immediate consequence of the quaternionic version of the Hahn–Banach theorem.

Corollary 12.0.7. *Let X be a right quaternionic Banach space and let $v \in X$. If $\langle \phi, v\rangle = 0$ for every linear and continuous functional ϕ in X^*, then $v = 0$.*

We can reformulate this with the following corollary.

Corollary 12.0.8. *The dual space of a quaternionic right Banach space separates points.*

In the following paragraphs, we have restated the quaternionic version of the results that we have previously used in this book. The proofs in the complex case, found in [110], are very similar.

Theorem 12.0.9 (Uniform boundedness principle)**.** *Let X and W be two right quaternionic Banach spaces and let $\{T_\alpha\}_{\alpha \in A}$ be bounded linear maps from X to W. Suppose that*

$$\sup_{\alpha \in A} \|T_\alpha v\| < \infty, \quad v \in X.$$

Then,

$$\sup_{\alpha \in A} \|T_\alpha\| < \infty.$$

Proof. For the proof see Theorem 11 in [110, p. 52]. □

Also, the following extension theorem is in [110].

Theorem 12.0.10 (Extension by continuity)**.** *Let X and W be two-sided quaternionic Banach spaces. Let $F : \mathcal{D} \subset X \to W$ be a uniformly continuous operator and suppose that $\mathcal{D}$ is dense in X. Then F has a unique continuous extension $\widetilde{F} : X \to W$ which is uniformly continuous.*

Lemma 12.0.11 (Corollary of Ascoli–Arzelá theorem)**.** *Let $\mathcal{G}_1$ be a compact subset of a topological group $\mathcal{G}$ and let $\mathcal{K}$ be a bounded subset of the space of continuous functions $\mathcal{C}(\mathcal{G}_1)$. Then $\mathcal{K}$ is conditionally compact if and only if for every $\varepsilon > 0$ there is a neighborhood $\mathcal{U}$ of the identity in G such that $|f(t) - f(s)| < \varepsilon$ for every $f \in K$ and every pair s, t in S with $t \in \mathcal{U}$.*

Proof. It is Corollary 9 [110, p. 267] and its proof can be obtained with the same arguments. □

Definition 12.0.12. We say that a quaternionic topological vector space $\mathcal{X}$ has the fixed point property if for every continuous mapping $T : \mathcal{X} \to \mathcal{X}$, there exists $u \in \mathcal{X}$ such that $u = T(u)$.

Lemma 12.0.13. *Let $\mathcal{K}$ be a compact convex subset of a locally convex linear quaternionic space $\mathcal{X}$ and let $T : \mathcal{K} \to \mathcal{K}$ be continuous. If $\mathcal{K}$ contains at least two points, then there exists a proper closed convex subset $\mathcal{K}_1 \subset \mathcal{K}$ such that $T(\mathcal{K}_1) \subseteq \mathcal{K}_1$.*

Theorem 12.0.14 (Schauder–Tychonoff)**.** *A compact convex subset of a locally convex quaternionic linear space has the fixed point property.*

Proof. The proof is based on Zorn lemma and on Lemma 12.0.13, see [110]. □

Definition 12.0.15. Let X_0 be a subset of X and let $\operatorname{span}(X_0)$ be the subspace of X spanned by X_0. We say that X_0 is a fundamental set if $\operatorname{span}(X_0) = X$.

The above definition is useful to state the following result.

Theorem 12.0.16. *Let X be a quaternionic Banach space and let $\mathcal{A}_m$ be a sequence of linear bounded quaternionic operators on X to itself. Then the limit $\mathcal{A}v = \lim_{m\to\infty} \mathcal{A}_m v$ exists for every $v \in X$ if and only if*

(a) *the limit $\mathcal{A}v$ exists for every fundamental set,*

(b) *for each $v \in X$ we have $\sup_{m\in\mathbb{N}} \|\mathcal{A}_m v\| < \infty$.*

When the limit $\mathcal{A}v$ exists for each $v \in X$, the operator $\mathcal{A}$ is bounded and

$$\|\mathcal{A}\| \leq \liminf_{m\to\infty} \|\mathcal{A}_m\| \leq \sup_{m\in\mathbb{N}} \|\mathcal{A}_m\| < \infty.$$

Proof. It mimics the proof of Theorem II.3.6 in [110] for complex Banach spaces. □

Lemma 12.0.17. *Let F and G belong to $L^1(\mathbb{R},\mathbb{H})$ with respect to the Lebesgue measure. Then the convolution*

$$(F * G)(t) := \int_0^t F(t-\tau)G(\tau)d\tau$$

is defined for almost all t, is a function in $L^1(\mathbb{R},\mathbb{H})$, thus

$$\|(F * G)\|_{L^1} \leq \|F\|_{L^1}\|G\|_{L^1}.$$

(a) *If $F \in L^1(\mathbb{R},\mathbb{H})$ and there exists a positive constant M such that $|G(t)| \leq M$, then*

$$\|(F * G)\|_{L^1} \leq M\|F\|_{L^1}.$$

(b) *Let F and G be defined for $t \geq 0$ and let them be Lebesgue integrable over every finite interval. Then $(F * G)(t)$ is Lebesgue integrable over every finite interval.*

Proof. It follows the proof of Lemma 24 in [110, p.634]. □

Theorem 12.0.18. *Let V and W be quaternionic two-sided Banach spaces and let T be a closed linear quaternionic operator on a domain $\mathcal{D}$ and with range W. Let (S,μ) be a measure quaternionic space and let $\mathcal{F}$ be a μ-integrable function with values in $\mathcal{D}$. Suppose that $T\mathcal{F}$ is a μ-integrable function then we have*

(a) $\int_S \mathcal{F}(\tau)\mu(d\tau) \in \mathcal{D}$, *and*

(b) $T\int_S \mathcal{F}(\tau)\mu(d\tau) = \int_S T\mathcal{F}(\tau)\mu(d\tau)$.

Proof. It follows with obvious modifications from the proof of the Theorem 20 in [110, p. 153]. □