Chapter 1

Introduction

In this chapter we summarize the theoretical aspects of the quaternionic spectral theory that will be developed later in this book and we also show some of the possible applications of this theory to fractional diffusion processes.

We denote the skew-field of quaternions by $\mathbb{H}$. An element s of $\mathbb{H}$ is of the form $s = s_0 + s_1e_1 + s_2e_2 + s_3e_3$, $s_\ell \in \mathbb{R}$, $\ell = 0, 1, 2, 3$, where e_1, e_2 and e_3 are the generating imaginary units of $\mathbb{H}$, which satisfy $e_\ell^2 = -1$ and $e_\ell e_\kappa = -e_\kappa e_\ell$ for $\ell, \kappa = 1, 2, 3$ and $\ell \neq \kappa$. The real part s_0 of the quaternion s is also denoted by $\mathrm{Re}(s)$, while its imaginary part is defined as $\mathrm{Im}(s) := s_1e_1 + s_2e_2 + s_3e_3$. We indicate by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, i.e.,

$$\mathbb{S} = \{s = s_1e_1 + s_1e_2 + s_3e_3 : \quad s_1^2 + s_2^2 + s_3^2 = 1\}.$$

Notice that if $j \in \mathbb{S}$, then $j^2 = -1$. For this reason the elements of $\mathbb{S}$ are also called imaginary units. The set $\mathbb{S}$ is a 2-dimensional sphere in $\mathbb{R}^4 \cong \mathbb{H}$. Given a nonreal quaternion $s = \mathrm{Re}(s) + \mathrm{Im}(s)$, we have $s = u + j_s v$ with $u = \mathrm{Re}(s)$, $j_s = \mathrm{Im}(s)/|\mathrm{Im}(s)| \in \mathbb{S}$ and $v = |\mathrm{Im}(s)|$. We can associate to s the 2-dimensional sphere

$$[s] = \{s_0 + j|\mathrm{Im}(s)| : \quad j \in \mathbb{S}\} = \{u + jv : \quad j \in \mathbb{S}\}. \tag{1.1}$$

Before we describe the contents of this book, we recall the problem with the definition of the spectrum of a linear vector operator or, more generally, of a quaternionic linear operator. For more details and for the history of quaternionic spectral theory we refer to the book [57], where we have previously written about the historical development of quaternionic spectral theory and of the related function theories, which can be found in the introduction and in several notes at the end of each chapter. We, however, point out that the main difficulties in developing a mathematically rigorous spectral theory for quaternionic operators was the lack of correct definitions of spectrum and resolvent for such operators. In fact, consider for example, a right linear bounded quaternionic operator $T : X \to X$ acting on a

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two-sided quaternionic Banach space X, that is,

$$T(x\alpha + y\beta) = T(x)\alpha + T(y)\beta,$$

for all $\alpha, \beta \in \mathbb{H}$ and x, $y \in X$. The symbol $\mathcal{B}(X)$ denotes the Banach space of all bounded right linear operators endowed with the natural norm, and we denote by $\mathcal{I}$ the identity operator. The spectrum of an operator should, in an appropriate way, generalize the set of its eigenvalues. But since the quaternionic multiplication is not commutative, even the notion of eigenvalue in this setting is ambiguous. Indeed, one can either consider left or right eigenvalues, which are determined by the equations

$$Tx = sx \quad \text{and} \quad Tx = xs,$$

respectively. In the classical setting, the spectrum $\sigma(A)$ of a complex linear operator A is defined as the set of all $\lambda \in \mathbb{C}$ such that the operator of the eigenvalue equation $\lambda\mathcal{I} - A$ does not have a bounded inverse. If we try to proceed similarly for the left eigenvalue equation, we obtain the left spectrum $\sigma_L(T)$ of T, which is defined as

$$\sigma_L(T) := \{s \in \mathbb{H} : \quad s\mathcal{I} - T \text{ is not invertible in } \mathcal{B}(X)\}, \tag{1.2}$$

where the notation $s\mathcal{I}$ in $\mathcal{B}(X)$ means that $(s\mathcal{I})(x) = sx$. It is associated with the left resolvent operator $(s\mathcal{I} - T)^{-1}$, which is defined on the complement of $\sigma_L(T)$. However, the operator-valued function $s \mapsto (s\mathcal{I} - T)^{-1}$ is not hyperholomorphic in $\mathbb{H} \setminus \sigma_L(T)$ with respect to any known notion of generalized holomorphicity over the quaternions, which limited its usefulness for developing quaternionic spectral theory. Furthermore, the left eigenvalues of an operator (or even a matrix) did not seem to be have any meaningful applications neither in physical applications nor in the mathematical theory.

The notion of right eigenvalues on the other hand seemed to be the more natural notion of eigenvalues, since the considered operators were right linear. Even more, the notion of right eigenvalues had an interpretation in quaternionic quantum mechanics [4] and the spectral theorem for quaternionic matrices is based on the right eigenvalues [114]. The right eigenvalue equation is, however, not quaternionic linear. Indeed, if $Tx = xs$ for some $x \neq 0$ and $a \in \mathbb{H}$ with $as \neq sa$, then

$$T(xa) = T(x)a = (xs)a = x(sa) = (xa)(a^{-1}sa) \neq (xa)s. \tag{1.3}$$

Hence, the operator of the right eigenvalue equation $x \mapsto Tx - xs$ is not linear and its inverse can consequently not be used to define a meaningful notion of quaternionic resolvent. We define the right spectrum $\sigma_R(T)$ of T therefore as the set of right eigenvalues

$$\sigma_R(T) := \{s \in \mathbb{H} : \quad Tx = xs, \text{for some } x \in X \setminus \{0\}\}.$$

Even though the set of right eigenvalues was meaningful both in applications and in the mathematical theory on finite dimensional spaces, it was not clear how to generalize it to a proper notion of spectrum of a right linear operator nor with which resolvent operator this spectrum should be associated. Furthermore, right eigenvalues have two problematic properties that are immediately understood from (1.3). First of all, the set of eigenvectors associated with an individual eigenvalue does not constitute a quaternionic linear space. If x is a right eigenvector of T associated with s, then xa is a right eigenvector of T associated with $a^{-1}sa$ instead of s. The second problem is that right eigenvalues do not appear individually but in terms of equivalence classes of the form

$$[s] = \{a^{-1}sa : \quad a \in \mathbb{H} \setminus \{0\}\}. \tag{1.4}$$

This set agrees with the symmetry class of s defined in (1.1). Hence, the right spectrum $\sigma_R(T)$ of T is axially symmetric.

The solution of these problems came in 2006, when I. Sabadini and one of the authors, introduced the S-spectrum and the S-functional calculus for quaternionic linear operators starting from considerations on slice hyperholomorphic functions, see the introduction of the book [57]. This notion is not intuitive because the S-spectrum of T is defined for those quaternions s such that the second order operator $T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I}$ is not invertible, where $\mathrm{Re}(s)$ is the real part of the quaternion s and $|s|^2$ is its squared norm. There exists also a commutative version of the S-spectrum and it is very useful in applications. We will denote by $\mathcal{BC}(X)$ the subclass of $\mathcal{B}(X)$ that consists of those quaternionic operators T that can be written as $T = T_0 + e_1T_1 + e_2T_2 + e_3T_3$ where the operators T_ℓ, $\ell = 0, 1, 2, 3$, commute mutually, and we set $\overline{T} = T_0 - e_1T_1 - e_2T_2 - e_3T_3$. In this case the S-spectrum has an equivalent definition that takes into account the commutativity of T_ℓ, for $\ell = 0, 1, 2, 3$. In the literature the commutative definition of the S-spectrum is often called the F-spectrum because it is used for the F-functional calculus, see [57]. Let $T \in \mathcal{BC}(X)$, we define the commutative version of the S-spectrum (or F-spectrum $\sigma_F(T)$) of T as those $s \in \mathbb{H}$ such that the operator $s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T}$ is not invertible. The S-resolvent set $\rho_S(T)$ is defined as $\rho_S(T) = \mathbb{H} \setminus \sigma_S(T)$.

Since this book is the natural continuation of [57] where the quaternionic spectral theory based on the S-spectrum is systematically studied, we summarize in two sections the theoretical aspects and the applications that we develop in this book.

1.1 Theoretical aspects

The notion of S-spectrum turned out to be the correct notion of spectrum for a quaternionic linear operator T, see also the books [57, 93], and it was discovered from considerations on slice hyperholomorphic functions. Moreover, the right eigenvalues $\sigma_R(T)$ are equal to the S-eigenvalues of T. We limit the discussion to the case of quaternionic operators but the following definition of S-spectrum can be adapted to the case of n-tuples of non commuting operators. We define

$$\mathcal{Q}_s(T) := T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I}.$$

If T is a linear quaternionic operator then the S-resolvent set is defined as

$$\rho_S(T) = \{s \in \mathbb{H} : \quad \mathcal{Q}_s(T)^{-1} \in \mathcal{B}(X)\},$$

where $\mathcal{Q}_s(T)^{-1}$ is called the pseudo-resolvent operator of T at s, while the S-spectrum is defined as:

$$\sigma_S(T) := \mathbb{H} \setminus \rho_S(T).$$

Due to the non commutativity of the quaternions, there are two resolvent operators associated with a quaternionic linear operator T: when T is bounded, the left S-resolvent operator is defined as

$$S_L^{-1}(s,T) := -\mathcal{Q}_s(T)^{-1}(T - \overline{s}\mathcal{I}), \quad s \in \rho_S(T) \tag{1.5}$$

and the right S-resolvent operator is

$$S_R^{-1}(s,T) := -(T - \overline{s}\mathcal{I})\mathcal{Q}_s(T)^{-1}, \quad s \in \rho_S(T). \tag{1.6}$$

The first main difference with respect to complex operator theory is the fact that the S-resolvent equation involves both the S-resolvent operators

$$\begin{aligned} S_R^{-1}(s,T)S_L^{-1}(p,T) = {} & [[S_R^{-1}(s,T) - S_L^{-1}(p,T)]p \\ & - \overline{s}[S_R^{-1}(s,T) - S_L^{-1}(p,T)]](p^2 - 2s_0p + |s|^2)^{-1}, \end{aligned}$$

for s, $p \in \rho_S(T)$ with $s \notin [p]$. A second major difference is the fact that the operator that defines the S-spectrum is the pseudo-resolvent operator and not the S-resolvent operator, but that the pseudo-resolvent operator $\mathcal{Q}_s(T)^{-1}$ is not slice hyperholomorphic. Only the S-resolvent operators are operator-valued slice hyperholomorphic functions.

The S-functional calculus (also called quaternionic functional calculus) is the quaternionic version of the Riesz-Dunford functional calculus. It is based on the S-spectrum and on the Cauchy formula of slice hyperholomorphic functions. In the next chapter we therefore summarize the main facts on slice hyperholomorphic functions. For more details see [57, 93].

If the operator T is bounded, then its S-spectrum $\sigma_S(T)$ is a non-empty compact subset of $\mathbb{H}$ that is bounded by the norm of T. We denote by $\mathcal{SH}_L(\sigma_S(T))$ the set of left slice hyperholomorphic functions $f : U \to \mathbb{H}$ where U is a suitable bounded open set that contains $\sigma_S(T)$. Analogously we define $\mathcal{SH}_R(\sigma_S(T))$ for right slice hyperholomorphic functions. The two formulations of the quaternionic functional calculus for left- and right slice hyperholomorphic functions are then given by

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s), \quad f \in \mathcal{SH}_L(\sigma_S(T)), \tag{1.7}$$

and

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,T), \quad f \in \mathcal{SH}_R(\sigma_S(T)), \tag{1.8}$$

where $ds_j = -dsj$, for $j \in \mathbb{S}$. The S-functional calculus is well defined since the integrals depend neither on the open set U with $\sigma_S(T) \subset U$ nor on the imaginary unit $j \in \mathbb{S}$. It is important to note that the definition of the quaternionic functional calculus does not require the linear operator T to be written in terms of components $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell$ with bounded linear operators T_ℓ, $\ell = 0, \dots, 3$, on a real Banach space. Nor does it require that the components T_ℓ commute mutually as it was the case in earlier developed functional calculi for quaternionic linear operators that were based on other function theories. If the components T_ℓ of the operator $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell$ commute mutually, we set for $s \in \mathbb{H}$

$$\mathcal{Q}_{c,s}(T) := s^2\mathcal{I} - 2sT_0 + T\overline{T}$$

and we find that the operator $\mathcal{Q}_{c,s}(T)$ is invertible if and only if $\mathcal{Q}_s(T)$ is invertible and so the S-resolvent set of T can also be characterized as

$$\rho_S(T) = \left\{ s \in \mathbb{H} : \quad \mathcal{Q}_{c,s}(T)^{-1} \in \mathcal{B}(X) \right\}. \tag{1.9}$$

The operator $\mathcal{Q}_{c,s}(T)^{-1}$ is in this case called the commutative pseudo-resolvent operator. Moreover, for $s \in \rho_S(T)$, the commutative S-resolvent operators are

$$S_L^{-1}(s,T) = (s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T)^{-1} \tag{1.10}$$

$$S_R^{-1}(s,T) = \mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I} - \overline{T}). \tag{1.11}$$

The main topics treated in the next chapters are the possible extensions and generalizations of the S-functional calculus to unbounded operators while particular attention is dedicated to sectorial operators.

Direct approach to the S-functional calculus. We develop the S-functional calculus for closed quaternionic linear operators. The S-functional calculus for closed operators has already been considered in the books [57, 93], where the unbounded operator and the function were suitably transformed so that the S-functional calculus for bounded operators could be applied. This strategy is standard in the complex case, but in the quaternionic case it has the disadvantage that it requires that $\rho_S(T) \cap \mathbb{R} \neq \emptyset$. Already the most important quaternionic linear operator, the gradient operator, does not satisfy this condition.

We therefore define the S-functional calculus for closed operators in Chapter 3 directly via a slice hyperholomorphic Cauchy integral formula. If T is a closed operator with nonempty S-resolvent set and f is a function that is left slice hyperholomorphic on a suitable set U with $\sigma_S(T) \subset U$ that contains a neighbourhood of ∞, then we can use the Cauchy formula

$$f(x) = f(\infty) + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s,x)\, ds_j\, f(s), \quad x \subset U.$$

Formally replacing x by T we define

$$f(T) := f(\infty)\mathcal{I} + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s), \quad \sigma_S(T) \subset U.$$

This functional calculus is well-defined and its properties agree with those of the Riesz-Dunford-functional calculus for closed complex linear operators. We investigate these properties in detail. In particular, we discuss the product rule and show that this functional calculus is compatible with intrinsic polynomials of T although these polynomials do not belong to the class of admissible functions because they are not slice hyperholomorphic at infinity. Furthermore, we discuss the relation between the S-functional calculus for left and the S-functional calculus for right slice hyperholomorphic functions, we prove the spectral mapping theorem and we show that the functional calculus is capable of generating Riesz-projectors onto invariant subspaces.

Generation of groups and semigroups and the Phillips functional calculus. If T is a bounded right linear operator on a quaternionic Banach space X then, for s_0 sufficiently large, the left S-resolvent operator can be written as the Laplace transform of e^{tT},

$$S_L^{-1}(s,T) = \int_0^\infty e^{tT} e^{-ts}\, dt,$$

while the right S-resolvent operator can be written as:

$$S_R^{-1}(s,T) = \int_0^\infty e^{-ts} e^{tT}\, dt.$$

The above relations hold true also for a class of unbounded linear operators. We investigate the generation of groups and of semigroups. Moreover, we consider the following perturbation problem. Suppose that the closed right linear quaternionic operator T is the infinitesimal generator of the semigroup $\mathcal{U}_T(t)$. Determine a class of closed right linear quaternionic operators P such that $T+P$ is the generator of a quaternionic semigroup $\mathcal{U}_{T+P}(t)$.

In the case the operator T is the generator of a strongly continuous group of quaternionic linear operators, then one can define a slice hyperholomorphic functional calculus via the quaternionic Laplace–Stieltjes transform. The so-called Phillips functional calculus applies to a larger class of functions with respect to the

S-functional calculus because it does not require slice hyperholomorphicity at infinity. If T is the infinitesimal generator of a strongly continuous group $\{\mathcal{U}_T(t)\}_{t\in\mathbb{R}}$ with growth bound $\omega > 0$, that is,

$$\sigma_S(T) \subset \{s \in \mathbb{H} : \quad -\omega \leq \operatorname{Re}(s) \leq \omega\} \tag{1.12}$$

and

$$\|\mathcal{U}_T(t)\| \leq Me^{-|t|\omega}, \tag{1.13}$$

for some constant $M > 0$, then we consider the subset $\mathbf{S}(T)$ of all quaternion-valued measures on $\mathbb{R}$ given by

$$\mathbf{S}(T) := \left\{\mu \in \mathcal{M}(\mathbb{R}, \mathbb{H}) : \quad \int_{\mathbb{R}} e^{-|t|(\omega+\varepsilon)}\, d|\mu|(t) < +\infty\right\}, \tag{1.14}$$

where $\varepsilon > 0$ might depend on the measure μ. The quaternionic Laplace–Stieltjes transform

$$f(q) := \int_{\mathbb{R}} d\mu(t)\, e^{-tq}, \quad -(\omega+\varepsilon) < \operatorname{Re}(q) < \omega + \varepsilon$$

is then a right slice hyperholomorphic function and we can define

$$f(T) := \int_{\mathbb{R}} d\mu(t)\, \mathcal{U}_T(-t).$$

We discuss the quaternionic Laplace–Stieltjes transform and show that this functional calculus is well-defined. We study its algebraic properties and show its compatibility with the S-functional calculus defined in Chapter 3. Finally, we conclude by showing how to invert the operator $f(T)$ for intrinsic f using an inverting sequence of polynomials.

The general version of the H^∞-functional calculus. This is the natural functional calculus for sectorial quaternionic operators and in this book we introduce it in its full generality. Any quaternion can be written as $s = |s|e^{j_s \arg(s)}$ with a unique angle $\arg(s) \in [0, \pi]$. A quaternionic right linear operator is called sectorial if its S-spectrum is contained in the closure of a symmetric sector around the positive real axis of the form

$$\Sigma_\omega = \{s \in \mathbb{H} : \quad \arg(s) \in [0, \omega)\}$$

with $\omega \in (0, \pi)$ and for any $\varphi \in (\omega, \pi)$ there exists a constant $C > 0$ such that

$$\left\|S_L^{-1}(s,T)\right\| \leq \frac{C}{|s|} \quad \text{and} \quad \left\|S_R^{-1}(s,T)\right\| \leq \frac{C}{|s|},$$

for all $s \in \mathbb{H} \setminus \Sigma_\varphi$. If f is left slice hyperholomorphic on a sector Σ_φ for some $\varphi \in (\omega, \pi)$ and has polynomial limit 0 both at 0 and at infinity, then we can choose $\varphi' \in (\omega, \varphi)$ and define $f(T)$ by a Cauchy integral as

$$f(T) := \frac{1}{2\pi} \int_{\partial(\Sigma_{\varphi'} \cap \mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s). \tag{1.15}$$

If f is left slice hyperholomorphic on Σ_φ and has finite polynomial limits at 0 and infinity, then it is of the form

$$f(q) = \tilde{f}(q) + a + (1+q)^{-1}b \tag{1.16}$$

with $a, b \in \mathbb{H}$ and $\tilde{f}$ admissible for (1.15). Since $-1 \in \rho_S(T)$, the operator

$$-S_L^{-1}(s,T) = (\mathcal{I} + T)^{-1}$$

exists, and we can define for such functions

$$f(T) := \tilde{f}(T) + \mathcal{I}a + (\mathcal{I}+T)^{-1}b, \tag{1.17}$$

where $\tilde{f}(T)$ is intended in the sense of (1.15). We denote the class of functions of the form (1.16) by $\mathcal{E}_L(\Sigma_\varphi)$ and the class of intrinsic functions of the form (1.16) by $\mathcal{E}(\Sigma_\varphi)$. Finally, the class of admissible functions can be extended even further, which yields the H^∞-functional calculus. A regulariser for a left slice meromorphic function f on Σ_φ is a function $e \in \mathcal{E}(\Sigma_\varphi)$ such that $ef \in \mathcal{E}_L(\Sigma_\varphi)$ and such that $e(T)$ is injective. If such a regulariser exists for f, then we define

$$f(T) := e(T)^{-1}(ef)(T),$$

where $e(T)$ and $(ef)(T)$ are intended in the sense of (1.17). This operator is not necessarily bounded, because $e(T)^{-1}$ can be unbounded.

We define this functional calculus precisely and discuss its properties. We focus in particular on the composition rule and the spectral mapping theorem. As we will see there are several technical difficulties that have to be overcome when generalising them from the complex to the quaternionic setting.

Fractional powers of quaternionic linear operators. We first define fractional powers of sectorial operators with bounded inverse directly by the slice hyperholomorphic Cauchy integral formula

$$T^{-\alpha} := \frac{1}{2\pi}\int_\Gamma s^{-\alpha}\, ds_j\, S_R^{-1}(s,T),$$

where Γ is a path that goes from $-\infty e^{j\theta}$ to $\infty e^{-j\theta}$ in the set $\mathbb{C}_j \setminus (\Sigma_\varphi \cup B_\varepsilon(0))$ for sufficiently small $\varepsilon > 0$, sufficiently large $\theta \in (0,\pi)$ and arbitrary $j \in \mathbb{S}$ and avoids the negative real axis. We then discuss the properties of these fractional powers. In particular, we prove several integral representations and the semigroup property. We point out that in the quaternionic setting there exist integral representations that do not exist in the complex setting, for example when $\sigma_S(T) \subset \{s \in \mathbb{H} : \mathrm{Re}(s) > 0\}$ for $\alpha \in (0,1)$, we have

$$T^{-\alpha} = \frac{1}{\pi}\int_0^{+\infty} \tau^{-\alpha}\left(\cos\left(\frac{\alpha\pi}{2}\right)T + \sin\left(\frac{\alpha\pi}{2}\right)\tau\mathcal{I}\right)(T^2+\tau^2\mathcal{I})^{-1}\, d\tau.$$

The inverse of the quadratic operator $T^2+\tau^2\mathcal{I}$, that appears in the above formula, comes from the S-resolvent operator and it has no analogue in the complex setting because of noncommutativity. As a second approach, we define the fractional powers of positive exponent via the H^∞-functional calculus. In a third approach we introduce the fractional powers indirectly using an approach of Kato. We first define for $\alpha \in (0,1)$ the operator-valued function

$$F_\alpha(p,T) := \frac{\sin(\alpha\pi)}{\pi} \int_0^{+\infty} t^\alpha \left(p^2 - 2pt^\alpha \cos(\alpha\pi) + t^{2\alpha}\right)^{-1} S_R^{-1}(-t,T)\, dt,$$

which corresponds to an integral representation of $S_R^{-1}(p, T^\alpha)$ of the form (1.15), in which we let φ' tend to π. Then we show that there actually exists a unique closed operator B_α such that $F_\alpha(p,T) = S_R^{-1}(s, B_\alpha)$ and we define $T^\alpha := B_\alpha$.

1.2 Applications to fractional diffusion processes

One of the most important facts about quaternionic spectral theory is that it contains, as a particular case, the spectral theory of vector operators like the gradient operator and its generalizations with non constant coefficients. We explain in the following the ideas behind the definition of new fractional diffusion operators that generalize the Fourier law to non-local diffusion processes.

Our strategy for fractional diffusion problems does not apply only to the Fourier law with constant coefficients, but it works for general non constant coefficients Fourier law and it generates the associated non local diffusion operator. Moreover, we can apply our techniques for bounded domains as well as for unbounded domains.

To explain our approach we consider the case of fractional evolution on $\mathbb{R}^3$. We denote by $v : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}$ the temperature and by q the heat flow. We set the thermal diffusivity equal to 1 and set $x = (x_1, x_2, x_3)$. The heat equation is then deduced from the two laws

$$q(x,t) = -\nabla v(x,t) \quad \text{(Fourier's law)}, \tag{1.18}$$

$$\partial_t v(x,t) + \operatorname{div} q(x,t) = 0 \quad \text{(Conservation of Energy)}. \tag{1.19}$$

Replacing the heat flow in the law of conservation of energy using Fourier's law, we get the classical heat equation:

$$\partial_t v(x,t) - \Delta v(x,t) = 0, \quad (x,t) \in \mathbb{R}^3 \times (0,\infty). \tag{1.20}$$

The fractional heat equation is an alternative model, that takes non-local interactions into account. It is obtained by replacing the negative Laplacian in the heat equation by its fractional power so that one obtains the equation

$$\partial_t v(x,t) + (-\Delta)^\alpha v(x,t) = 0, \quad (x,t) \in \mathbb{R}^3 \times (0,\infty), \quad \alpha \in (0,1), \tag{1.21}$$

where the fractional Laplacian is given by

$$(-\Delta)^\alpha v(x) = c(\alpha) P.V. \int_{\mathbb{R}^3} \frac{v(x) - v(y)}{|x-y|^{3+2\alpha}} dy,$$

and the integral is defined in the sense of the principal value, $c(\alpha)$ is a known constant, and $v : \mathbb{R}^3 \to \mathbb{R}$ must belong to a suitable function space.

The fractional powers of the gradient operator. The new approach to fractional diffusion presented in this book consists in replacing the gradient in (1.18) by its fractional power before combining it with (1.19) instead of replacing the negative Laplacian in (1.20) by its fractional power. This is done by interpreting the gradient as a quaternionic linear operator, which allows us to define its fractional powers using the techniques presented in this book.

The following two observations are of crucial importance for defining the new procedure for fractional diffusion processes.

(I) The S-spectrum of the gradient operator ∇ on $L^2(\mathbb{R}^3, \mathbb{H})$ is $\sigma_S(\nabla) = \mathbb{R}$. Since the map $s \mapsto s^\alpha$ with $\alpha \in (0,1)$ is not defined on $(-\infty, 0)$, we have to consider the projections of the fractional power ∇^α to the subspace associated with the subset $[0, +\infty)$ of the S-spectrum of ∇. Only for these spectral values, the function $s \mapsto s^\alpha$ is well defined and slice hyperholomorphic. We denote these projections by $P_\alpha(\nabla)$. Precisely, what we call the fractional power $P_\alpha(\nabla)$ of the gradient, is given by the quaternionic Balakrishnan formula (deduced from the H^∞-functional calculus)

$$P_\alpha(\nabla)v = \frac{1}{2\pi} \int_{-j\mathbb{R}} S_L^{-1}(s, \nabla)\, ds_j\, s^{\alpha-1} \nabla v,$$

for $v : \mathbb{R}^3 \to \mathbb{R}$ in $\mathcal{D}(\nabla)$. The path of integration is chosen to take into account just the part of the S-spectrum with $\mathrm{Re}(\sigma_S(\nabla)) \geq 0$.

(II) The above procedure gives a quaternionic operator

$$P_\alpha(\nabla) = Z_0 + e_1 Z_1 + e_2 Z_2 + e_3 Z_3,$$

where Z_ℓ, $\ell = 0, 1, 2, 3$, are real operators obtained by the functional calculus. Finally, we take the vector part $P_\alpha(\nabla)$

$$\mathrm{Vect}(P_\alpha(\nabla)) = e_1 Z_1 + e_2 Z_2 + e_3 Z_3$$

of the quaternionic operator $P_\alpha(\nabla)$ so that we can apply the divergence operator.

With the above definitions and the surprising expression for the left S-resolvent operator

$$S_L^{-1}(-jt, \nabla) = (-jt + \nabla) \underbrace{(-t^2 + \Delta)^{-1}}_{= R_{-t^2}(-\Delta)},$$

the fractional powers $P_\alpha(\nabla)$ become

$$P_\alpha(\nabla)v = \underbrace{\frac{1}{2}(-\Delta)^{\frac{\alpha}{2}-1}\nabla^2 v}_{:=\mathrm{Scal}P_\alpha(\nabla)v} + \underbrace{\frac{1}{2}(-\Delta)^{\frac{\alpha-1}{2}}\nabla v}_{:=\mathrm{Vec}P_\alpha(\nabla)v}.$$

Now we observe that

$$\mathrm{div}\mathrm{Vec}P_\alpha(\nabla)v = -\frac{1}{2}(-\Delta)^{\frac{\alpha}{2}+1}v.$$

The fractional heat equation for $\alpha \in (1/2, 1)$

$$\partial_t v(t,x) + (-\Delta)^\alpha v(t,x) = 0$$

can hence be written as

$$\partial_t v(t,x) - 2\mathrm{div}\left(\mathrm{Vec}P_\beta(\nabla)v\right) = 0, \quad \beta = 2\alpha - 1,$$

so this approach coincided with the classical one for the gradient operator.

The S-spectrum approach to fractional diffusion processes. We say that T represents a Fourier law with commuting components, when T is a vector operator of the form

$$T = e_1\, a_1(x_1)\partial_{x_1} + e_2\, a_2(x_2)\partial_{x_2} + e_3\, a_3(x_3)\partial_{x_3},$$

where a_1, a_2, $a_3 : \Omega \to \mathbb{R}$ are suitable real-valued functions that depend on the space variables x_1, x_2, x_3, respectively, where $(x_1, x_2, x_3) \in \Omega$ and $\Omega \subseteq \mathbb{R}^3$. In this case the real operators $a_\ell(x_\ell)\partial_{x_\ell}$, for $\ell = 1, 2, 3$, commute among themselves. Then the S-spectrum of T can also be determined using the commutative pseudo-resolvent operator

$$Q_{c,s}(T) := s^2\mathcal{I} - 2sT_0 + T\overline{T} = a_1^2(x_1)\partial_{x_1}^2 + a_2^2(x_2)\partial_{x_2}^2 + a_3^2(x_3)\partial_{x_3}^2 + s^2\mathcal{I}$$

because $\mathcal{Q}_{c,s}(T)$ is invertible if and only if $\mathcal{Q}_s(T)$ is invertible. The operator $\mathcal{Q}_{c,s}(T)$ is a scalar operator if s^2 is a real number. Since T is a vector operator, we have $T_0 = 0$ and $T\overline{T}$ does not contain the imaginary units of the quaternions. Using the non commutative expression of the pseudo-resolvent operator $\mathcal{Q}_s(T)$, we obtain

$$\begin{aligned}\mathcal{Q}_s(T) = &-(a_1(x_1)\partial_{x_1})^2 - (a_2(x_2)\partial_{x_2})^2 - (a_3(x_3)\partial_{x_3})^2 \\ &- 2s_0(e_1\, a_1(x_1)\partial_{x_1} + e_2\, a_2(x_2)\partial_{x_2} + e_3\, a_3(x_3)\partial_{x_3}) + |s|^2\mathcal{I}.\end{aligned}$$

We observe that, according to what we need to show in the commutative case, we have two possibilities to determine the S-spectrum: $Q_{c,s}(T)$ and $\mathcal{Q}_s(T)$.

Now we explicitly describe the procedure of the S-spectrum approach to fractional diffusion processes. Suppose that $\Omega \subseteq \mathbb{R}^3$ is a suitable bounded or unbounded domain and let X be a two-sided Banach space. We consider the initial-boundary value problem for non-homogeneous materials. We denote by T the heat flow $q(x, \partial_x)$ and we restrict ourselves to the case of homogeneous boundary conditions (for $\tau > 0$):

$$
\begin{aligned}
&T(x) = a_1(x_1)\partial_{x_1} e_1 + a_2(x_2)\partial_{x_2} e_2 + a_3(x_3)\partial_{x_3} e_3, \quad x = (x_1, x_2, x_3) \in \Omega, \\
&\partial_t v(x,t) + \operatorname{div} T(x) v(x,t) = 0, \quad (x,t) \in \Omega \times (0, \tau], \\
&v(x,0) = f(x), \quad x \in \Omega, \\
&v(x,t) = 0, \quad x \in \partial\Omega, \ t \in [0, \tau].
\end{aligned}
$$

Our general procedure consists of the following steps:

(S1) We study the invertibility of the operator

$$Q_{c,s}(T) := s^2\mathcal{I} - 2sT_0 + T\overline{T} = a_1^2(x_1)\partial_{x_1}^2 + a_2^2(x_2)\partial_{x_2}^2 + a_3^2(x_3)\partial_{x_3}^2 + s^2\mathcal{I},$$

where $\overline{T} = -T$, to get the S-resolvent operator. Precisely, let $F : \Omega \to \mathbb{H}$ be a given function with a suitable regularity and denote by $X : \Omega \to \mathbb{H}$ the unknown function of the boundary value problem:

$$
\begin{aligned}
&\Big(a_1^2(x_1)\partial_{x_1}^2 + a_2^2(x_2)\partial_{x_2}^2 + a_3^2(x_3)\partial_{x_3}^2 + s^2\mathcal{I}\Big) X(x) = F(x), \quad x \in \Omega, \\
&X(x) = 0, \quad x \in \partial\Omega.
\end{aligned}
$$

We study under which conditions on the coefficients a_1, a_2, $a_3 : \mathbb{R}^3 \to \mathbb{R}$ the above equation has a unique solution. We can similarly use the non commutative version of the pseudo-resolvent operator $\mathcal{Q}_s(T)$. In the case we deal with an operator T with non-commuting components, then we have to consider $\mathcal{Q}_s(T)$, only.

(S2) From (S1) we get that $s \in \mathbb{H} \setminus \{0\}$ with $\operatorname{Re}(s) = 0$ belongs to $\rho_S(T)$, so we obtain the unique pesudo-resolvent operator $\mathcal{Q}_{c,s}(T)^{-1}$ and we define the S-resolvent operator

$$S_L^{-1}(s,T) = (s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T)^{-1}.$$

Then we prove that, for every $s \in \mathbb{H} \setminus \{0\}$ with $\operatorname{Re}(s) = 0$, the S-resolvent operators satisfy the estimates

$$\left\|S_L^{-1}(s,T)\right\| \leq \frac{\Theta}{|s|} \quad \text{and} \quad \left\|S_R^{-1}(s,T)\right\| \leq \frac{\Theta}{|s|} \tag{1.22}$$

with a constant $\Theta > 0$ that does not depend on s.

(S3) Using the Balakrishnan, formula we define $P_\alpha(T)$ as:

$$P_\alpha(T)v = \frac{1}{2\pi} \int_{-j\mathbb{R}} s^{\alpha-1}\, ds_j\, S_R^{-1}(s,T)Tv, \quad \text{for } \alpha \in (0,1),$$

and $v \in \mathcal{D}(T)$. Analogously, one can use the definition of $P_\alpha(T)$ related to the left S-resolvent operator.

(S4) After we define the fractional powers $P_\alpha(T)$ of the vector operator T, we consider its vector part $\mathrm{Vec}(P_\alpha(T))$ and we obtain the fractional evolution equation:

$$\partial_t v(t,x) - \mathrm{div}(\mathrm{Vec}(P_\alpha(T)v)(t,x) = 0.$$

As an application of our theory we get Theorems 10.3.1 and 10.3.2 that we summarize in the following result.

Let Ω be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary. Let $T = e_1\, a_1(x_1)\partial_{x_1} + e_2\, a_2(x_2)\partial_{x_2} + e_3\, a_3(x_3)\partial_{x_3}$ and assume that the coefficients $a_\ell : \overline{\Omega} \to \mathbb{R}$, for $\ell = 1,2,3$, belong to $\mathcal{C}^1(\overline{\Omega}, \mathbb{R})$ and $a_\ell(x_\ell) \geq m$ in $\overline{\Omega}$ for some $m > 0$. Moreover, assume that

$$\inf_{x\in\Omega} \left|a_\ell(x_\ell)^2\right| - \frac{\sqrt{C_\Omega}}{2} \left\|\partial_{x_\ell} a_\ell(x_\ell)^2\right\|_\infty > 0, \quad \ell = 1,2,3,$$

and

$$\frac{1}{2} - \frac{1}{2}\|\Phi\|_\infty^2 C_\Omega^2 C_a^2 > 0,$$

where C_Ω is the Poincaré constant of Ω and

$$\Phi(x) := \sum_{\ell=1}^{3} e_\ell \partial_{x_\ell} a_\ell(x_\ell) \quad \text{and} \quad C_a := \sup_{\substack{x\in\Omega \\ \ell=1,2,3}} \frac{1}{|a_\ell(x_\ell)|} = \frac{1}{\inf_{\substack{x\in\Omega \\ \ell=1,2,3}} |a_\ell(x_\ell)|}.$$

Then any $s \in \mathbb{H}\setminus\{0\}$ with $\mathrm{Re}(s) = 0$ belongs to $\rho_S(T)$ and the S-resolvents satisfy the estimate

$$\left\|S_L^{-1}(s,T)\right\| \leq \frac{\Theta}{|s|} \quad \text{and} \quad \left\|S_R^{-1}(s,T)\right\| \leq \frac{\Theta}{|s|}, \quad \text{if } \mathrm{Re}(s) = 0, \tag{1.23}$$

with a constant $\Theta > 0$ that does not depend on s. Moreover, for $\alpha \in (0,1)$, and for any $v \in \mathcal{D}(T)$, the integral

$$P_\alpha(T)v := \frac{1}{2\pi} \int_{-j\mathbb{R}} s^{\alpha-1}\, ds_j\, S_R^{-1}(s,T)Tv$$

converges absolutely in $L^2(\Omega, \mathbb{H})$.

The above result in particular holds for v real-valued.

This approach has several advantages:

(I) It modifies the Fourier law but keeps the law of conservation of energy.

(II) It is applicable to a large class of operators that includes the gradient but also operators with non-constant coefficients.

(III) Fractional powers of the operator T provide a more realistic model for nonhomogeneous materials.

(IV) The fact that we keep the evolution equation in divergence form allows an immediate definition of the weak solution of the fractional evolution problem.

1.3 On quaternionic spectral theories

For the convenience of the reader, we recall in this section some considerations that were already discussed in [57] in order to put the spectral theory on the S-spectrum into perspective. The quaternionic spectral theories arise from the Fueter–Sce–Qian mapping theorem that has been widely treated in [57].

In classical complex operator theory, the Cauchy formula of holomorphic functions is a fundamental tool for defining functions of operators. Moreover, the Cauchy–Riemann operator factorizes the Laplace operator, so holomorphic functions also play a crucial role in harmonic analysis and in boundary value problems. In higher dimensions, for quaternion-valued functions or more in general for Clifford-algebra-valued functions, there appear two different notions of hyperholomorphicity. The first one is called slice hyperholomorphicity and the second one is known under different names, depending on the dimension of the algebra and the range of the functions: Cauchy–Fueter regularity for quaternion-valued functions and monogenicity for Clifford algebra-valued functions. The Fueter–Sce–Qian mapping theorem reveals a fundamental relation between the different notions of hyperholomorphicity and it can be illustrated by the following two maps

$$F_1 : \mathrm{Hol}(\Omega) \mapsto \mathcal{N}(U) \quad \text{and} \quad F_2 : \mathcal{N}(U) \mapsto \mathcal{AM}(U).$$

The map F_1 transforms holomorphic functions in $\mathrm{Hol}(\Omega)$, where Ω is a suitable open set Ω in $\mathbb{C}$, into intrinsic slice hyperholomorphic functions in $\mathcal{N}(U)$ defined on the open set U in $\mathbb{H}$. Applying the second transformation F_2 to intrinsic slice hyperholomorphic functions, we get axially Fueter regular (resp. axially monogenic) functions. Roughly speaking the map F_1 is defined as follows:

1. We consider a holomorphic function $f(z)$ that depends on a complex variable $z = u + \iota v$ in an open set of the upper complex halfplane. (In order to distinguish the imaginary unit of $\mathbb{C}$ from the quaternionic imaginary units, we denote it by ι). We write

$$f(z) = f_0(u, v) + \iota f_1(u, v),$$

where f_0 and f_1 are $\mathbb{R}$-valued functions that satisfy the Cauchy–Riemann system.

2. For suitable quaternions q, we replace the complex imaginary unit ι in $f(z) = f_0(u, v) + \iota f_1(u, v)$ by the quaternionic imaginary unit $\frac{\mathrm{Im}(q)}{|\mathrm{Im}(q)|}$ and we set $u = \mathrm{Re}(q) = q_0$ and $v = |\mathrm{Im}(q)|$. We then define

$$f(q) = f_0(q_0, |\mathrm{Im}(q)|) + \frac{\mathrm{Im}(q)}{|\mathrm{Im}(q)|} f_1(q_0, |\mathrm{Im}(q)|).$$

The function $f(q)$ turns out to be slice hyperholomorphic by construction.

When considering quaternion-valued functions, the map F_2 is the Laplace operator, i.e., $F_2 = \Delta$. When we work with Clifford algebra-valued functions, then $F_2 = \Delta_{n+1}^{(n-1)/2}$, where n is the number of generating units of the Clifford algebra and Δ_{n+1} is the Laplace operator in dimension $n+1$. The Fueter–Sce–Qian mapping theorem can be adapted to the more general case in which $\mathcal{N}(U)$ is replaced by slice hyperholomorphic functions and the axially regular (or axially monogenic) functions $\mathcal{AM}(U)$ are replaced by monogenic functions. The generalization of holomorphicity to quaternion- or Clifford algebra-valued functions produces two different notions of hyper-holomorphicity that are useful for different purposes. Precisely, we have that:

(I) The Cauchy formula of slice hyperholomorphic functions leads to the definition of the S-spectrum and the S-functional calculus for quaternionic linear operators. Moreover, the spectral theorem for quaternionic linear operators is based on the S-spectrum. The aim of this book and of the monograph [57] is to give a systematic treatment of this theory and of its applications.

(II) The Cauchy formula associated with Cauchy–Fueter regularity (resp. monogenicity) leads to the notion of monogenic spectrum and produces the Cauchy–Fueter functional calculus for quaternion-valued functions and the monogenic functional calculus for Clifford algebra-valued functions. This theory has applications in harmonic analysis in higher dimensions and in boundary value problems. For an overview on the monogenic functional calculus and its applications see [171] and for applications to boundary values problems see [163] and the references contained in those books.

In this book and in the monograph [57] we treat the quaternionic spectral theory on the S-spectrum so, very often, we will refer to it as quaternionic spectral theory because no confusion arises with respect to the monogenic spectral theory.