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Manipulatives in Mathematics Education

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Keywords

Concrete manipulatives · Virtual manipulatives · Historic-cultural manipulatives · "Artificial" manipulatives

Definition

Mathematical manipulatives are artifacts used in mathematics education: they are handled by students in order to explore, acquire, or investigate mathematical concepts or processes and to perform problem-solving activities drawing on perceptual (visual, tactile, or, more generally, sensory) evidence.

Characteristics

Manipulatives and Mathematics Education

One can distinguish several kinds of manipulatives used in schools and education. Two classifications that emerge from the literature may be suggested, referring to either the quality of interaction user-manipulative or the origin of the manipulative: concrete versus virtual manipulatives and historic-cultural versus "artificial" manipulatives.

Concrete manipulatives are physical artifacts that can be concretely handled by students and offer a large and deep set of sensory experience.

Virtual manipulatives are digital artifacts that resemble physical objects and can be manipulated, usually with a mouse, in a similar way as their authentic, concrete counterparts.

Historic-cultural manipulatives are concrete artifacts that have been created in the longstanding history of mathematics to either explore or solve specific problems, both from inside and from outside mathematics.

"*Artificial*" *manipulatives* are artifacts that have been designed by educators with specific educational aims.

The following table lists some examples according to the combination of the two classifications above.

	Concrete	Virtual
Historic-cultural	Different kinds of abaci; Napier's bones; measuring tools such as graded rulers and protractors; polyhedrons;	Suanpan the Chinese abacus, virtual copies of mathematical machines

(*continued*)

S. Lerman (ed.), *Encyclopedia of Mathematics Education*,
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	Concrete	Virtual
	mathematical machines; topological puzzles; geometrical puzzles; dices and knucklebones; ancient board games	
“Artificial”	Froebel’s gifts, Montessori’s materials, Cuisenaire rods, Dienes’s materials, multibase blocks, fraction strips and circles, bee-bot	Library of virtual manipulatives

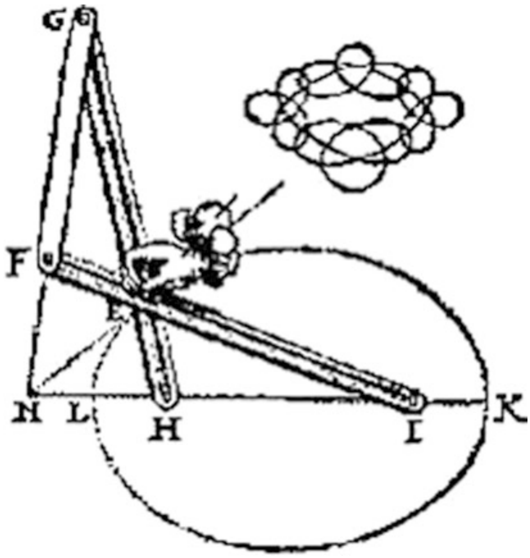
Historic-cultural manipulatives refer to mathematical meanings, as they have paved the way towards today’s mathematics (some examples are discussed in a further section). Artificial manipulatives are the outcomes of an opposite path: an ingenuous educator invented, for specific educational purposes, a new way to embody an established mathematical concept into an object or a game. At the beginning this choice might be considered artificial (and this is the reason of using this term in the classification above). A famous example is given by Dienes who explains the root of multibase blocks and the teachers’ resistance to this introduction, perceived as completely artificial. One might object that the difference between the historic-cultural and artificial ones is fuzzy. Is one allowed to consider Froebel’s gifts artificial and the Slavonic abacus historic-cultural? Not exactly, if one considers that both artifacts date back to the same period and have been designed for educational purposes. The Slavonic abacus was carried to France around 1820 from Russia by Poncelet who transformed the Russian abacus for educational purposes. Froebel gifts were designed around 1840 for activity in the kindergarten. In the proposed classification, the Slavonic abacus is considered a historic-cultural one, because of the strict relationship with other kinds of abaci, while Froebel gifts are considered the ancestors of other artificial manipulatives produced later by educators like Montessori, Cuisenaire, and Dienes (Fig. 1).

Both are examples of the inclination to give value in Europe to active involvement of mathematics students during the nineteenth century (see Bartolini Bussi et al. 2010) and represent the background where the International Commission on Mathematical Instruction (ICMI) started to work with a big emphasis on active methods and laboratory activities.

The distinction between concrete and virtual manipulatives deserves some observation. A whole library of virtual manipulatives is available on the web. In this library, there are digital “objects” (mostly in the form of Java applets) representing many artificial manipulatives and allowing to act on them in a way similar to the action on their concrete counterparts. There are also websites, where digital copies of historic-

Manipulatives in Mathematics Education, Fig. 1 Schoty and Froebel gifts cultural manipulatives are available. In all cases the user-manipulative interaction is limited to mouse piloting and looking at effects. Systematic research on virtual manipulatives and on comparison between concrete and virtual manipulatives is still at the very beginning. Virtual manipulatives are easily available (wherever a computer laboratory is located), are time and space saving, and are motivating, because of the appeal they exert on students accustomed to digital devices. However, if compared with concrete manipulatives, virtual manipulatives seem to highlight mainly visual experience, skipping reference to tactile and other sensory experience. The new touch-screen technology with the possibility of touching simultaneously different points on the screen seems to open new possibilities (see, for instance, Maktrace, by Anna Baccaglini-Frank (Baccaglini-Frank et al. 2012) and TouchCounts by Nathalie Sinclair).

A few studies have been carried out about the comparison between concrete and virtual manipulatives. For instance, Hunt et al. (2011) report the findings of a 3-year study with prospective middle-grade mathematics teachers enrolled in Clayton State University. Perceived advantages and disadvantages of concrete versus virtual manipulatives are compared after a full course where both kinds of manipulatives for Number Concepts had been used. Concrete manipulatives appeared to be more effective for building preservice teachers' and students' conceptual understanding. The virtual manipulatives were used to reinforce those concepts. The usefulness of using both concrete and virtual manipulatives is emphasized by Maschietto and Bartolini Bussi (2011). Both a concrete and a virtual copy of the same manipulative (i.e., the van Schooten ellipsograph by antiparallelogram – Fig. 2) are analyzed, comparing classroom tasks and tasks for teachers about the textual description with "realistic" drawings.



Manipulatives in Mathematics Education, Fig. 2 van Schooten ellipsograph

Critical Issues

The first critical issue concerns the students' autonomy in using manipulatives. In the western tradition, since the time of Montessori, the use of manipulatives was mainly aimed at spontaneous activity within a well-prepared environment: adults organize the environment where students (usually aged between 3 and 10–12) are free to select activity. This trend has to be historically contextualized as a reaction against the lecture-based school, criticized also by Dewey (1907). Yet there are studies (e.g., Uttal et al. 1997; McNeil and Jarvin 2007) which have a more critical approach to manipulatives. The effectiveness of manipulatives over more traditional methods is analyzed, claiming that the sharp distinction between concrete and symbolic forms of mathematical expression is not useful. There is no guarantee that students will establish the necessary connections between manipulatives and more traditional mathematical expressions. In particular this issue calls into play the importance of instruction (or teaching) about manipulatives and the connection between manipulatives and symbols.

The second critical issue concerns the students' age. Most research about manipulatives has been carried out at preschool and primary school level, highlighting the usefulness of manipulatives at a certain age only (e.g., Kamii et al. 2001). In most guides for teachers, the use of manipulatives is especially aimed at either primary school students or students with special needs. Curtain-Phillips complains about the scarce use of manipulatives in

US high schools, quoting, as an exception, Marilyn Burns who used manipulative materials at all levels for 30 years. Moreover, she quotes the attention of the National Council of Teachers of Mathematics (NCTM) that has encouraged the use of manipulatives at all grade levels, in every decade, since 1940. She asks an interesting question: why are high school teachers reluctant to use this type of resources? One reason might be the nearly unique emphasis on artificial manipulatives that have been created with the declared aim to embody an abstract mathematical concept into a concrete (or virtual) object. If this is the shared approach, the effect is that they are used with either young children or students with special needs, who are expected to need more time for concrete-enactive exploration. Nührenbörger and Steinbring (2008) contrast this position emphasizing that manipulatives are symbolic representations in which mathematical relationships, structures, and patterns are contained and can be actively interpreted, exchanged within the discursive context, and checked with regard to plausibility (see also Uttal et al. 1997). The "theoretical ambiguity" of manipulatives is to be considered a central theme in mathematics lessons. This very ambiguity makes manipulatives suitable to all school levels, up to university, as a context where fundamental processes, as defining, conjecturing, arguing, and proving, are fostered. This requires a very strong and deep analysis of manipulatives, from theoretical and epistemological points of view, and a study of the consequence of this analysis in teachers' design of tasks and interventions in the mathematics classroom. To cope with this problem, in our research team (Bartolini Bussi and Mariotti 2008), we have developed the framework of semiotic mediation after a Vygotskian approach. In the following section, we outline this framework together with some examples, mainly taken from the historic tradition.

A Comprehensive Theoretical Approach to Manipulatives: Semiotic Mediation After a Vygotskian Approach

Vygotsky studied the role of artifacts (including language) in the cognitive development and suggested a list of possible examples: "various systems for counting; mnemonic techniques; algebraic symbol systems; works of art; writing; schemes, diagrams, maps, and mechanical drawings; all sorts of conventional signs, etc." (Vygotsky 1981, p. 137). Manipulatives might be included in this list. The introduction of an artifact in a classroom does not automatically determine the way it is used and conceived of by the students and may create the condition for generating the production of different voices. In short, the manipulatives are polysemic, and they may create the condition for generating the production of different voices (*polyphony*). This position is consistent with Nührenbörger and Steinbring's theoretical ambiguity mentioned above (2008). The teacher mediates mathematical meanings, using the artifact as a tool of semiotic mediation. Without teacher's intervention, there might be a fracture between concrete learners' activity on the manipulative and the mathematical culture, hence no learner's construction of mathematical meanings. In this framework the theoretical construct of the *semiotic potential of an artifact* is central: i.e., the double semiotic link which may occur between an artifact and the personal meanings emerging from its use to accomplish a task and at the same time the mathematical meanings evoked by its use and recognizable as mathematics by an expert (Bartolini Bussi and Mariotti 2008).

Some Examples of Manipulatives and Tasks

This section presents the semiotic potential of some manipulatives, known as Mathematical Machines. *A geometrical machine is a tool that forces a point to follow a trajectory or to be transformed according to a given law. An arithmetical machine is a tool that allows the user to perform at least one of the following actions: counting, reckoning, and representing numbers.* They are concretely handled and explored by students at very different school levels, including university. In most cases also virtual copies exist as either available resources (see the right frame at www.macchinematematiche.org) or outcomes of suitable tasks for students themselves (Bartolini Bussi and Mariotti 2008). The historic-cultural feature of these manipulatives allows to create a classroom context where history of mathematics is effectively used to foster students' construction of mathematical meanings (Maschietto and Bartolini Bussi 2011). Each example contains a short description of the manipulative, an exemplary task and the mathematical meaning, as intended by the teacher.

Counting Stick

Counting sticks, dating back to ancient China, are thin bamboo or plastic sticks. The sticks are counted, grouped, and bundled (and tied with ribbons or rubber bands) into tens for counting up to hundred; ten bundles are grouped and bundled into hundreds and so on.

Figure 3a, b is taken from a Chinese textbook: the oral numerals beyond ten are introduced by grouping and tying ten sticks (left, 1st grade) and a "difficult" subtraction is realized by untying and ungrouping a bundle (right, 1st grade).

Tasks: To guess numerals between 10 and 20 in the first case and to calculate 36–8 in the second case.

In this case the triangle of semiotic potential hints at:

Mathematical knowledge: Grouping/regrouping.

There is a perfect correspondence between the two opposite actions: tying/untying and grouping/ungrouping. The former refers to the concrete action with sticks and bundles; the latter refers to a mathematical action with units and tens. It is likely that primary students' descriptions refer to the concrete action (in one class, 1st graders invented the Italian neologism "elasticare," i.e., "rubbering"). It is not difficult for the teacher to guide the transition from the wording of the concrete action towards the wording of a mathematical action. In this way also the need (as perceived by teachers) to use "borrowing" from tens to units is overcome (see Ma 1999, p. 1ff. for a discussion of this issue).

Pascaline

The pascaline is a mechanical calculator (see Fig. 4) (Bartolini Bussi and Boni 2009).

The name of the instrument hints at the design of a mechanical calculator by Blaise Pascal (for details, see Bartolini Bussi et al. 2010). An exemplary task is the following: **Task**: Represent the number 23 and explain how you made it. Different pieces of **mathematical knowledge** may be involved to answer the task, for instance:

- The generation of whichever natural number by iteration of the function "+1" (one step ahead for the right bottom wheel)
- The decomposition of a 2-digit number (23) into 2 tens and 3 units

The first mathematical action may be carried out on the pascaline by iterating 23 times the function "+1"; the second mathematical action may be carried out by iterating the function "+1" 3 times on the right bottom wheel and 2 times on the central bottom wheel.

Manipulatives in Mathematics Education, Fig. 3 Shuxue ISBN 7-107-14-632-7

Pair of Compasses and Other Curve Drawers

The compass (pair of compasses) is the oldest geometrical machine; it is a technical drawing instrument that can be used for inscribing circles or arcs. It is used also as a tool to measure distances, in particular on maps. The compass objectifies, by means of its structure and its functional use, the defining elements of the circle (center and radius) and reflects a clear definition of the circle as a closed curve such that all its points are equidistant from an inside common point (Bartolini Bussi et al. 2007).

Tasks: How is the pair of compasses made? What does it draw? Why does it do that?

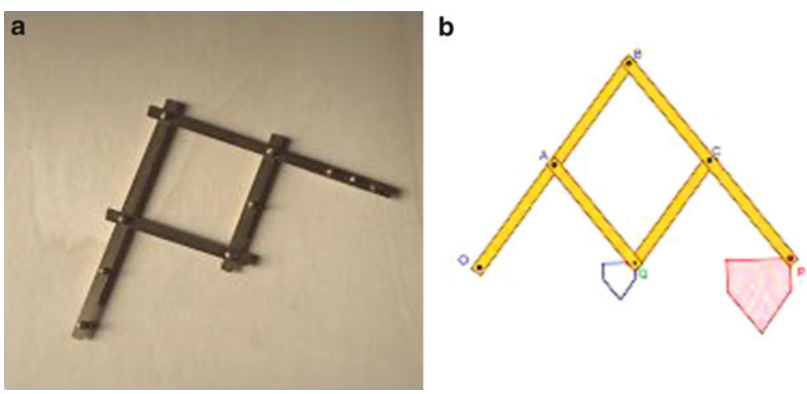
Mathematical knowledge: From primary school the compass can be used and analyzed in order to learn concepts and to understand how it embodies some mathematical laws (Chassapis 1999). The same can be done in the upper grades (up to teacher education programs, Martignone 2011), after the exploration of the compass structure and movements, student can become theoretically aware about how the mathematical law is developed by compass and then they can use this instrument to solve problems and to produce proofs in Euclidean geometry.

Manipulatives in Mathematics Education, Fig. 4 Pascaline "zero + 1"

Even if the compass is the most famous curve drawer, over the centuries many different types of curve drawers have been designed and used as tools for studying mathematics and for solving problems (see http://www.museo.unimo.it/labmat/usa1.htm). The oldest linkages date back to the Alexandrian and Arabic cultures, but it is in seventeenth century, thanks to the work of Descartes (1637), that these machines obtained a wide theoretical importance and played a fundamental role in creating new symbolic languages (see http://kmoddl.library.cornell.edu/linkages/).

Pantographs

Over the century the pantographs were described in different types of documents, such as mathematical texts and technical treatises for architects and painters. In particular, in nineteenth century, when the theory of geometrical transformations became fundamental in mathematics, they were designed and studied by many scientists. A famous linkage is the Scheiner's pantograph: a parallelogram linkage, one of whose joints has its movement duplicated by an attached bar. This has been used for centuries to copy and/or enlarge drawings. Since the end of the sixteenth century, this type of machines was used by painters even if it was improved and described by Scheiner in 1631 (Fig. 5a, b).



Manipulatives in Mathematics Education, Fig. 5 (**a**–**b**) Scheiner's pantograph http://www.macchinematematiche.org/index.php?option=com_content&view=article&id=112&Itemid=195

Tasks: Students can study how the machine is made, how the different components move, what are the constraints, and the variables modeling the structure by means of Euclidean geometry.

Mathematical knowledge: The Scheiner's pantograph can be used for introducing the concept of dilation (homothety) and/or for developing argumentation processes about why the machine does a dilatation.

Finally, it should be emphasized that these ancient technologies, whose use and study date back to past centuries, have modern developments, for example, modeling the robot arms. Also in mathematics, the study of linkages has been recently revived. In the twentieth century, ideas growing from Kempe's work were further generalized by Denis Jordan, Michael Kapovich, Henry King, John Millson, Warren Smith, Marcel Steiner, and others (Demaine and O'Rourke 2007).

Open Questions

There is no best educational choice between different kinds of manipulatives. Rather the choice depends on different factors (what is available, what fits better the students' culture and expectations, and so on) and, above all, on teachers' system of beliefs and view on mathematics. There is never a "natural" access to the embodied mathematics, as no artifact is transparent in its embodied mathematical meaning (Ball 1992; Meira 1998): a suitable context and set of tasks are always required. There are many reasons to support the use of manipulatives in the mathematics classrooms, but the short review of literature above shows that there is still a place for developing studies about:

- Manipulatives: to analyze limits and potentialities of different kinds of manipulatives (concrete vs. virtual; historic-cultural vs. artificial) from an epistemological, cognitive, and didactical perspective
- Classroom practice: to design, test, and analyze tasks about manipulatives at different school levels and in different cultural traditions
- Teacher education and development: to design, test, and analyze tasks for teachers about the use of manipulatives in the mathematics classroom

Cross-References

► Activity Theory in Mathematics Education
► Semiotics in Mathematics Education
► Teaching Practices in Digital Environments

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Mathematical Ability

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Keywords
Development of mathematical ability · Evaluating mathematical ability · High ability · Individual differences · Low ability · Mathematical reasoning · Spectrum of mathematical abilities

Definitions

Mathematical ability is a human construct, which may be defined cognitively or pragmatically, depending on the purpose of definitions. Cognitive definitions are used when relating to this construct from a theoretical perspective; mathematical ability can then be defined as the ability to obtain, process, and retain mathematical information (Krutetskii 1976; Vilkomir and O'Donoghue 2009) or as the capacity to learn and master new mathematical ideas and skills (Koshy et al. 2009). Pragmatic definitions are usually used when looking at this construct from a perspective of evaluation (e.g., when the focus is on identifying learners' potential or assessing learning outcomes). From this perspective, it can be defined as the ability to perform mathematical tasks and to effectively solve given mathematical problems. Such definitions are general in nature and are commonly unpacked into several components, which are not necessarily exclusive to one definition or another. Thus, we speak of an assemblage of mathematical *abilities* rather than a single ability. One of the most acknowledged and widely accepted theories in this respect is that of Krutetskii (1976), who suggested that mathematical ability is comprised of the following abilities: use formal language and operate within formal structures of connections, generalize, think in a logic-sequential manner, perform shortcuts ("curtailments") while solving problems, switch thinking directions, move flexibly between mental processes, and recall previously acquired concepts and generalizations.

Characteristics

The Evolvement of Mathematical Abilities

Mathematical abilities develop in correspondence with the development of rational and logical thinking. According to Piaget's theory of cognitive development (Piaget and Inhelder 1958), logical thinking skills are limited in the first two developmental stages of normative childhood, the sensorimotor stage and the preoperational stage. This means that although young children, who have acquired the use of language (around the age of 2–3), are able to link numbers to objects and may have some understanding of the concepts of numbers and counting, they still cannot comprehend logical notions such as reversible actions or transitivity until they reach the concrete-operational stage, around the age of 7–8. At this stage, a child can comprehend, for example, that the distance from point A to point B is the same as the distance from point B to point A and that if $x \leq y$ and $y \leq z$, then $x \leq z$. During the concrete-operational stage (ages 7–8 to 11–12), a considerable growth in mathematical abilities is enabled due to the acquisition of two additional logical operations: seriation, defined as the ability to order objects according to increasing or decreasing values, and classification, which is grouping objects by a common characteristic (Ojose 2008). Yet, the abstract thinking necessary for grasping and constructing mathematical ideas evolves during the formal-operational stage, around the ages of 11–12 to 14–15. At this stage, according to Piagetian theory, adolescents are able to reason using symbols, make inductive and deductive inferences, form hypotheses, and generalize and evaluate logical arguments.

Piaget's theory was criticized, among other things, for underestimating the abilities of young children while overestimating the abilities of adolescents (Ojose 2008). However, Piaget himself emphasized that the stages in his theory do not necessarily occur in the ages specified. That is, some children will advance more quickly and reach a certain cognitive stage at a relatively early age; others may not arrive at this stage until much later in their lives. The speed of development and the degree to which the last formal-operational stage is realized depend on various personal and environmental attributes. This view corresponds with Vygotsky's theory (Vygotsky 1978) which emphasizes the crucial role that social interactions and adult guidance, available in children's environment, play in their cognitive development. Thus, as a result of variations in individuals' circumstances and available mathematical experiences, we find that the spectrum of mathematical abilities in a specific age group is of a wide magnitude.

Characterizing Different Students on the Spectrum of Mathematical Abilities

Researchers have endeavored to characterize students located close to both ends of the mathematical ability spectrum: on the one hand mathematically gifted and highly able students and on the other hand students who are lacking in their mathematical abilities, compared with their peers.

The aforementioned classical work of Krutetskii (1976) concentrated on the higher end of mathematical abilities. Krutetskii used a wide-ranging set of mathematical problems and an in-depth analysis of children's answers, in an attempt to pinpoint the components of mathematical ability in general and higher ability in particular. Based on his investigations, Krutetskii referred to four groups of children: extremely able, able, average, and low. He inferred that extremely able children are characterized by what he termed as a "mathematical cast of mind." This term designates the tendency to perceive the surrounding environment through lenses of mathematical and logical relationships, to be highly interested in solving challenging mathematical problems, and to keep high levels of concentration during mathematical activities. Interpreting Krutetskii's theory, Vilkomir and O'Donoghue (2009) suggest that a mathematical cast of mind stimulates all other components of mathematical ability to be developed to the highest level, if the student is provided with the necessary environment and instruction.

At the other end of the spectrum, we find learners with low mathematical abilities. Although these learners typically perform poorly in school mathematics, the inverse is not necessarily true. In other words, the presumption that poor mathematical performance of students is indicative of their low mathematical abilities is problematic; a range of social, behavioral, and cultural circumstances can result in low achievements in school mathematics (Secada 1992). In addition, students may develop a negative mathematical self-schema that reduces their motivation to succeed in mathematics, regardless of their overall abilities (Karsenty 2004). Nevertheless, characteristics of low mathematical abilities are available in the literature. Overcoming the abovementioned pitfall may be achieved through careful consideration of a child performance in a supportive environment, under a personal guidance of a trusted adult. Thus, we find that the main features of low mathematical abilities are difficulties in establishing connections between mathematical elements of a problem; inability to generalize mathematical material according to essential attributes, even with help and after a number of practice exercises; lack of capability to deduce one thing from another and find the common principle of series of numbers even with assistance; avoidance from using symbolic notations; and short-lived memory for mathematical procedures (Karsenty et al. 2007; Vilkomir and O'Donoghue 2009). In extreme cases of low mathematical abilities, the term mathematical disability (MD) is used. Research on MD is commonly conducted on subjects with notable deficiencies in basic arithmetic skills and includes explorations of the disability known as dyscalculia. MD is not an uncommon disorder (estimations range between 3% and 8% of the school-age population) and is mainly attributed to cognitive, neuropsychological, and genetic origins (Geary 1993).

Mathematical Abilities and General Intelligence

Despite the popular view that links mathematical ability with intelligence, the relation between these two constructs remains elusive. The original intelligence test developed by Binet and Simon in the early 1900s emphasized mostly verbal reasoning and did not include a mathematical component, except for simple counting. The later version, known as the Stanford-Binet test, which was composed by Terman in 1916 (and is still used today, after several revisions along the years), includes a quantitative reasoning part. Terman assumed that mathematical abilities play some role in determining general intelligence, yet he did not conduct empirical studies to support this argument. Later theories of intelligence also suggested that there is a quantitative element in models describing intelligence. For instance, Thurstone (1935) stated that number facility is one of the seven components of which human intelligence is comprised; Wechsler (1939) included mental arithmetic problems in his widely used IQ tests. There is some evidence that fluid intelligence, defined as general reasoning and problem-solving abilities independent from specific knowledge and culture, is positively correlated with the ability to solve realistic mathematical word problems (Xin and Zhang 2009). However, since mathematical ability stretches far beyond number sense and successful encountering of arithmetic or word problems, we cannot construe on the basis of existing data that intelligence and mathematical ability are mutually related.

Multidimensional theories of intelligence offer a different view on this issue. Gardner, in his seminal work first presented in his book "Frames of Mind" in 1983, suggested that there are several distinct intelligences, one of which is the logical-mathematical intelligence. Gardner argued that traditional models of intelligence, such as Terman's, combine together human capacities that do not necessarily correlate with one another. Thus, a person with high mathematical abilities, as described, for instance, by Krutetskii, will be defined by Gardner's Multiple Intelligences theory as having high logical-mathematical intelligence; this definition does not necessarily imply that this person's score in a conventional IQ test will be superior.

Measuring and Evaluating Students' Mathematical Ability

Following the above, it became clear to researchers that a standard IQ test is not an appropriate tool for evaluating the mathematical ability of students, especially for the purpose of identifying extremely able ones (Carter and Kontos 1982). Instead, one of the most prevalent means for this purpose is known as *aptitude tests*. Aptitude tests are aimed at measuring a specific ability or talent and are often used to predict the likelihood of success in certain areas or occupations (e.g., foreign language learning, military service, or, in this case, mathematics). Among the many existing aptitude tests, a widely known one is the SAT (an acronym which originally stood for Scholastic Aptitude Test), designed by the

College Board in USA for predicting academic success. The SAT includes three parts, one of which is the SAT-M, referring to mathematics. Julian Stanley, founder of SMPY (the Study of Mathematically Precocious Youth) at Johns Hopkins University, found that SAT-M is an efficient means for identifying mathematically gifted students at junior high school age (Stanley et al. 1974). However, the use of aptitude tests like SAT-M for the purpose of measuring mathematical ability was criticized by several scholars as inadequate. For instance, Lester and Schroeder (1983) claimed that multiple-choice, standardized tests, such as SAT-M, provide no information about students' ability to solve nonroutine mathematical problems, and moreover, they cannot reveal the nature and quality of students' mathematical reasoning. These tests focus on a narrow interpretation of mathematical ability, ignoring important problem-solving behaviors that are indicative of this ability. Krutetskii (1976) attacked the credibility of psychometric items for measuring mathematical ability, claiming that (a) a single assessment event is highly affected by the subject's anxiety or fatigue, (b) training and exercise influence the rate of success, and (c) psychometric means concentrate on quantitative rather than qualitative aspects of mathematical ability, i.e., they focus on final outcomes instead of thinking processes, thus missing the central meaning of this construct. Despite criticisms, the current predominant method for assessing students' mathematical ability is still different versions of multiple-choice aptitude tests, most likely due to considerations of time and budget resources. Nevertheless, efforts are being conducted to develop low-cost assessment tools that follow the qualitative approach characteristic of the work of Krutetskii and others (e.g., Vilkomir and O'Donoghue 2009).

Cross-References

- ► Competency Frameworks in Mathematics Education
- ► External Assessment in Mathematics Education
- ► Giftedness and High Ability in Mathematics
- ► Learning Difficulties, Special Needs, and Mathematics Learning
- ► Zone of Proximal Development in Mathematics Education

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Mathematical Approaches

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Keywords
Contributing discipline · Didactical analysis · Didactic transposition · Mathematical analysis · Mathematical approach · Mathematics education community · Mathematical structure

Introduction

Research in mathematics education is interdisciplinary. According to Higginson (1980), mathematics, philosophy, psychology, and sociology are *contributing disciplines* to mathematics education (similar to what Michael Otte called *Bezugsdisziplinen*; Otte et al. 1974, p. 20). Linguistics and semiotics could be added. Framing of research, by means of theories or methods from these, amounts to different *approaches*, mathematics itself being one obvious choice. According to one view, mathematics education as a research field belongs to mathematics: at the second International Congress on Mathematical Education (ICME) in Exeter, Zofia Krygowska suggested that mathematics education should be classified as "a part of mathematics with a status similar to that of analysis or topology" (Howson 1973, p. 48). Another view sees mathematics education as an autonomous science (*didactics of mathematics* as Hans Georg Steiner in 1968 called the new discipline he wanted to establish; see Furinghetti et al. 2008, p. 132), strongly linked to mathematics, as expressed at ICME1 in Lyon 1969: "The theory of mathematical education is becoming a science in its own right, with its own problems both of mathematical and pedagogical content. The new science should be given a place in the mathematical departments of Universities or Research Institutes, with appropriate qualifications available" (quoted in Furinghetti et al. 2008, p. 132). However, in many countries, mathematics education research has an institutional placement mainly in educational departments.

Definition

Mathematical approaches in mathematics education take the characteristics and inner structures of mathematics as a discipline (i.e., the logic of the subject) as its main reference point in curriculum and research studies. These characteristics, however, might be questioned. Studies include philosophical, historical, and didactical analyses of mathematical content and of how it is selected, adapted, or transformed in the process of recontextualization by requirements due to educational constraints, as well as the consequences entailed by these transformations on didactic decisions and processes.

Developments

The field of mathematics education research historically emerged from the scientific disciplines of mathematics and of psychology (Kilpatrick 1992). On an international level, through the activities promoted by ICMI (International Commission on Mathematical Instruction) during the first half of the twentieth century, with their focus on comparing issues of mathematical content in curricula from different parts of the world, with little consideration of research on teaching and learning (Kilpatrick 1992), the approach to secondary and tertiary mathematics education was predominantly mathematical. During the same period, however, in primary mathematics education, the approaches were commonly psychologically or philosophically oriented. Independently, the use of concrete materials in schools is widely developed (Furinghetti et al. 2013). While this situation led to a decrease of ICMI's influence on mathematics education, through and after the *New Math* movement in the 1960s, ICMI regained its voice with support of OEEC/OECD, UNESCO, and through the collaboration of mathematicians with mathematics educators, mainly through CIEAEM, concerned with the full complexity of teaching and learning at all school levels (Furinghetti et al. 2008). The mathematical approach underpinning the reform was warranted not only by the aim to update curricula with modern developments in mathematics but also by Piagetian psychology pointing to "similarities" between mental and mathematical structures (Furinghetti et al. 2008). The aim of the New Math to be a *mathematics for all* was counteracted by its emphasis on general mathematical structures and fundamental concepts. This type of mathematical approach was strongly criticized, most notably by Hans Freudenthal who used the term *anti-didactic inversion* for a static axiomatic ready-made version of mathematics presented to students (Freudenthal 1973, p. 12). An influential similar critique was offered by René Thom (1973, p. 202), who suggested that mathematics education should be founded on meaning rather than rigor.

The eventual failure of the New Math pointed to the need of establishing mathematics education as a discipline "in its own rights" and a wider scope for the work of ICMI. In retrospective, the first ICME congress in 1969 can be said to mark the creation of an autonomous mathematics education community (during a period when several institutions and journals specialized in mathematics education were founded; see, e.g., Furinghetti et al. 2013) and a loosening of the strong link to the community of mathematicians with implications for the "status" of mathematical approaches. With this wider scope, besides mathematical and psychological approaches, a variety of approaches for the study of phenomena within the field was needed, especially with reference to social dimensions.

This development highlights different interpretations of *mathematical approach*. While the New Math was the outcome of a deliberate and research-based program prepared in collaboration, the type of "mathematical approaches" of later movements in the USA, such as *Back to Basics* in the 1970s and even more so the *Math Wars* in the 1990s, is better described as ideologically based reactions to what was seen by some individuals and interest groups as *fuzzy* mathematics. The return to the skill-oriented curriculum advocated failed to take into account not only reported high dropout rates and research showing how it disadvantages underprivileged social groups but also research that highlights the complexity in teaching and learning processes (Goldin 2003; Schoenfeld 2004). In the more research-oriented mathematical approaches that developed in Europe during the same period, it was shown how both the character and learning of mathematics at school are institutionally conditioned.

Characteristics

The following quote gives an argument for taking a mathematical approach to research: "The mathematical science in its real development is therefore the central focus of the mathematics educators, because the separation of creative activity and learning – taking into account the fundamental difference between research and learning – is unfruitful and does not allow to adequately capture the learning nor to properly guide the learning process" (auth. transl., Jahnke et al. 1974, p. 5). To develop mathematical knowledge, the learner must engage in creative mathematical activities. Another rationale for a focus on mathematics itself in didactical research draws on the observation that mathematics "lives" differently in different institutions and is transformed (recontextualized) when moved. In a mathematics classroom, different ideologies influence what kind of mathematical knowledge is proposed as legitimate, requiring from both, the teacher and the researcher, an awareness of the structure of the knowledge produced. The often cited claim by René Thom (1973, p. 204) that "whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics," also applies to research in mathematics education. This can be seen as an argument for the necessity of keeping an awareness of how mathematics is viewed in all approaches to research in mathematical education.

In the following, some examples of theorizing in the field of mathematics education that employ a mathematical approach will be discussed, with a focus on the role given to mathematics as a main point of reference.

Stoffdidaktik

The *Stoffdidaktik* (subject matter didactics, content-oriented analysis) tradition in German-speaking countries, originally with main focus on secondary school mathematics and teacher education, has its modern roots in the efforts by Felix Klein, during the first decades of the twentieth century, to structure *elementary mathematics from an advanced standpoint* and to include lectures on the didactics of mathematics in the education of future teachers. While his classic book (Klein 1908) served the aim to teach (future) teachers to think mathematically, the aim of the lectures was to teach (future) teachers to think didactically (Vollrath 1994). A major aim of Stoffdidaktik is "to make mathematics accessible and understandable to the learner based on an analysis of the subject matter with mathematical means" (Hußmann et al. 2016, p. 2). An historical account can be found in Hefendehl-Hebeker (2016).

According to Tietze (1994), "Stoffdidaktik mainly deals with the subject matter under the aspects of mathematical analysis and of transforming mathematical theories into school mathematics" (p. 42). This approach in mathematics education operates through an explicit didactic transposition of (academic) mathematics for the purpose of making it accessible to students at specific educational levels. Some key principles used in this process, constituted by a mathematical analysis and selection of the content to be taught, are *elementarizing*, *exactifying*, *simplifying*, and *visualizing* (Tietze 1994). Students' problems to cope with, for example, definitions in mathematics, are in this approach seen as based in the complex logical structure of the definitions, which then must be analyzed by way of these principles in order to prepare their teaching.

An example of such analyses is Padberg's (1995) work on fractions, a textbook for teacher education outlining four central aspects (*Größenkonzept*, *Äquivalenzklassenkonzept*, *Gleichungskonzept*, *Operatorkonzept*) and two basic ideas (*Grundvorstellungen*; see below), elaborating on accessible metaphorical descriptions of the concepts but also including a chapter on the mathematical foundation of fractions, presenting an axiomatic characterization of the topic aimed to provide background knowledge for the teacher. Such *mathematical background theories* in mathematics education have commonly been introduced and used within *Stoffdidaktik*. For geometry, Vollrath (1988, pp. 121–127) identifies five (historical) phases of background theories: Euclid's elements (from early times, perfected by Hilbert), transformation geometry (from the early 1800s; e.g., Möbius, later Klein), different axiomatic theories as competing background theories (from early 1900s), an axiomatic theory developed by didacticians from practice of teaching (from 1960s, to decrease the gap from theoretical mathematics to teaching practice; e.g., Steiner 1966), and "The totality of geometric knowledge, including the ideas, connections, applications, and evaluations." As an early example of this kind of mathematical approach, Steiner (1969) outlines *a mathematical analysis* of the relation of rational numbers to measurement and interpretation as operators, with the aim to characterize possibilities for teaching. He calls his procedure a *didactical analysis* (p. 371).

A specific focus for the transposition work is on so-called *fundamental ideas* (*Fundamentale Ideen*; see, e.g., Schweiger 1992). According to Schwill (1993), for an idea to be fundamental, it must appear within different topics of mathematics (*Horizontalkriterium*) and at different levels of the curriculum (*Vertikalkriterium*), be recovered in the historical development of mathematics (*Zeitkriterium*), and be anchored in everyday life activities (*Sinnkriterium*). Using the term *universal ideas*, Schreiber (1983) in a similar vein presents the requirements of *comprehensiveness*, *profusion*, and *meaningfulness*. As an example, Riemann integration is not a fundamental idea but a specific application of the fundamental idea of exhaustion. Other examples are *reversibility* and *symmetry*. Historically, already Whitehead (1913) suggested that school mathematics should emphasize main universally significant general ideas rather than drown in details that may not lead to access to big ideas or provide necessary connections to everyday knowledge. In line with this and with explicit reference to Jerome Bruner's principle that teaching should be oriented toward the structure of science, much work in *Stoffdidaktik* consist of analyses of fundamental ideas in different areas of mathematics. For the teaching of fundamental ideas, Schwill (1993) suggests Bruner's *spiral principle* to be used, in terms of *extendibility*, *prefiguration of notions*, and *anticipated learning*. It still remains unclear; however, at what level of abstraction, fundamental ideas are located (see Vohns 2016, for a critical discussion).

As basis for teaching a mathematical concept, meta-knowledge about the concept is seen as necessary and has to be addressed in teacher education. A theory of concept teaching (e.g., Vollrath 1984) needs to build on the evaluation of mathematical concepts and their hierarchical structure, their historical development, and the *principle of complementarity* (Otte and Steinbring 1977) that concepts should offer both knowledge and use.

Research methods of early work within *Stoffdidaktik* were mainly the same as those of mathematics (Griesel 1974). In Griesel (1969), for example, an axiomatically based mathematical theory for a system of quantities is outlined. It has been pointed out by Griesel, however, that without also empirically investigating the outcomes from such analyses in teaching and learning, the analytical work would not be justified. *Stoffdidaktik* later widened to consider not only academic mathematics along with its epistemology and history but also factors relating to the learner of mathematics. In this context the notion of *Grundvorstellungen* became widely used (e.g., vom Hofe 1995), that is, the basic meanings and representations students should develop about mathematical concepts and their use within and outside mathematics. Conceptualized both as mental objects and as a prescriptive didactical constructs for prototypical metaphorical situations, the epistemological status of *Grundvorstellungen* remains debated (see, e.g., Vohns 2016).

Outside German-speaking countries, mathematics-oriented didactical research has dominated mathematics education, for instance, in the Baltic countries (Lepik 2009). One example of a non-European work employing the approach is Carraher (1993), where a ratio and operator model of rational numbers is developed. There are also regional and international periodic journals for teachers, mathematicians, and mathematics educators that publish mathematical and didactical analyses of elementary topics for school and undergraduate mathematics.

An Epistemological Program

Mathematics also serves as a basic reference point for the "French school" in mathematics education research referred to as an *epistemological program* (Gascon 2003), including the theory of didactical situations (TDS) developed by Guy Brousseau and the anthropological theory of the didactic (ATD) developed by Yves Chevallard. What constitutes mathematical knowledge is here seen as relative to the institution where it is practiced and thus, in research, needs to be questioned regarding its structure and content as practiced. In studies of the diffusion of mathematical knowledge within an institution, it is therefore necessary for the researcher to construct a *reference epistemological model* of the corresponding body of mathematical knowledge (Bosch and Gascon 2006), in order to avoid a bias of the institution studied.

Brousseau (1997) proposes *didactical situations* as epistemological models of mathematical knowledge, both for setting up the target knowledge and for developing it in classroom activity. For the researcher, such models are employed mainly for the analysis of didactical phenomena emerging in the process of instruction. They are also used for *didactical engineering* (e.g., Artigue 1994), where they are analyzed in terms of possible constraints of epistemological, cognitive, or didactical nature (Artigue 1994, p. 32). By investigating the historical development of the mathematical knowledge at issue, as well as its current use, the epistemological constraints can be analyzed. In particular, the functionality of the knowledge to be taught is seen as a key component of a *fundamental didactical situation*, constituting a milieu that promotes the student's use of the knowledge. An idea is here to "restore" the epistemological conditions that were at hand where the knowledge originated but have disappeared in curriculum processes such as decontextualization and sequentialization of knowledge.

In ATD, mathematics is seen as a human activity within institutions (as social organizations), with collective practices that form how the participants think and define their goals. It includes a focus on how mathematical knowledge, having a preexistence outside the educational institution, is *transposed* by institutional constraints when moved into it. The structure of the mathematical knowledge and work is modeled by *praxeologies* (or mathematical organizations) that provide a holistic description of the relations between different aspects of the institutional mathematical practice, in terms of types of tasks and techniques for dealing with these tasks, and those technologies and overall theoretical structures that justify the practice. In didactical research, the characteristics of praxeologies are analyzed in terms of aspects, such as connectedness and levels of generality, and issues linked to the *didactic transposition*, in order to identify possible constraints that are being imposed on students' knowledge development. According to ATD, "phenomena of didactic transposition are at the very core of any didactic problem" (Bosch and Gascon 2006, p. 58). To develop a target mathematical praxeology for classroom teaching, a didactical praxeology needs to be set up. Here one finds a strong emphasis on the functionality of the mathematical knowledge studied (its *raison d'être*), to avoid a *monumentalistic* noncritical selection of traditional school mathematics topics, often described as alien to the reality of the students (Bosch and Gascon 2006).

Realistic Mathematics Education

Realistic mathematics education (RME) views mathematics as an emerging activity: "The learner should reinvent mathematising rather than mathematics, abstracting rather than abstractions, schematising rather than schemes, algorithmising rather than algorithms, verbalising rather than language" (Freudenthal 1991, p. X). While keeping mathematics as a main reference point, researchers within RME take on didactical, phenomenological, epistemological, and historical-cultural analyses as bases for curricular design (see ► "Didactical Phenomenology (Freudenthal)"). Activities of *horizontal mathematization* aim to link mathematical concepts and methods to real situations, while *vertical mathematization* takes place entirely within mathematics. An example of work within RME employing a strong mathematical approach is found in Freudenthal (1983), with its elaborated analyses of mathematical concepts and methods and efforts to root the meanings of those mathematical structures in everyday experiences and language.

Mathematical Knowledge for Teaching

Empirical quantitative research on the amount of mathematical studies needed for a successful or effective teaching of mathematics at different school levels has not been able to settle the issue. Rather, the character of teachers' knowledge and the overall approach to teaching seem to matter more (Ma 1999; Boaler 2002; Hill et al. 2005). With reference to the distinction between subject matter knowledge and *pedagogical content knowledge* (PCK), during the last decades, descriptions and measurements of what has been named *mathematical knowledge for teaching* (MKT; e.g., Hill et al. 2005, p. 373) for use in preservice and in-service teacher education have been developed. This mathematically based approach to mathematics education sets out to characterize the mathematical knowledge that teachers need to effectively teach mathematics and to investigate relations between teaching and learning. MKT stays close to the PCK construct while applying and further detailing the latter in order to grasp the specificities of school mathematics. The approach has much in common with the didactical analyses of mathematical content developed much earlier within *Stoffdidaktik*, though with more focus on primary mathematics. However, as the approach is less systematic and without reference to different possible mathematical background theories, the level of analysis remains unclear. The scope of the empirical research includes efforts to both develop and measure MKT for groups of teachers and its relation to student achievement (e.g., Hill et al. 2005).

Some Further Aspects of Mathematical Approaches

In university mathematics, educational issues identified in beginning courses (such as calculus and linear algebra), especially in the context of the transition from secondary school to university, have commonly been addressed by a mathematical approach by ways of analyses of mathematical structures and processes in the courses. However, in line with the widened scope of mathematics education research since the time of New Math, de Guzman et al. (1998) suggest epistemological and cognitive, sociological and cultural, as well as didactical approaches to study the transition problem. Beside cognitivistic (still constituting the dominating approach), sociological, and discursive approaches, today more recent mathematical approaches (such as the epistemological program) are common for investigating university mathematics education (see, e.g., Artigue et al. 2007).

The importance and relevance of the history of mathematics for mathematics education has long been emphasized in the mathematics education community (e.g., the report from the ICME working group on history in Athen and Kunle 1976, pp. 303–307). In this context, both the didactical analyses of the historical material and the ways of using these in teaching practice often employ a mathematical approach. The claim of a parallel between the historical development and individual learning of mathematical concepts (the *phylogeny-ontogeny parallel*) has been one of the arguments for this approach, while others relate the use of history to motivational and cultural-historical issues or introduce historical outlines as a tool for teaching mathematics (Athen and Kunle 1976).

The examples of theoretical perspectives presented above employ different mathematical approaches to mathematics education as an overarching approach in the research. However, also within other approaches (psychological, social, etc.), mathematical aspects often come into focus. As an example, the APOS framework (e.g., Cottrill et al. 1996) takes a psychological approach to model and study the development of students' conceptual knowledge. However, as a basis for the construction of a *genetic decomposition* of the taught mathematical concept, a mathematical analysis of its structure and historical development is undertaken.

There are several influential mathematics educators whose work cannot be subsumed under the theoretical perspectives considered above, but who have sought to understand and improve mathematics instruction by means of analyzing mathematical processes and structures, often with a focus on developing teaching aids and didactical suggestions. Emma Castelnuovo, Zoltan Dienes, Caleb Gattegno, and George Polya, among others, could be mentioned here.

Unresolved Issues

The community of mathematics education tends to become disintegrated by its diversity of theoretical approaches used in research with a knowledge structure fragmented into what Jablonka and Bergsten (2010) call *branches*. If mathematics education research strives to enhance the understanding of mathematics teaching and learning, including its social, political, and economic conditions and consequences, only a productive interaction of research approaches is likely to move the field forward. Unresolved issues are often due to institutionalized separation of researchers taking distinct approaches, as, for example, epitomized in bemoaning a loss of the focus on mathematics, which need to be resolved through theory (for theory networking, see Prediger et al. 2008). This would, for example, include integrating approaches that focus on mathematical knowledge structures with discursive and sociological approaches. Further, for producing unbiased policy advice, it is necessary to integrate research outcomes on students' and teachers' engagement with mathematics, including cognitive, emotional, language-related, and social dimensions of teaching and learning in classrooms. Such work has been attempted in a range of initiatives and working groups, as, for example, at the conferences of ERME (Prediger et al. 2008). In discussions of goals of mathematics education, mathematical approaches combined with sociological theorizing become pertinent to analyses of the use and exchange values of (school) mathematics for students.

Cross-References

- ▶ Critical Thinking in Mathematics Education
- ▶ Didactic Situations in Mathematics Education
- ▶ Didactic Transposition in Mathematics Education
- ▶ Didactical Phenomenology (Freudenthal)
- ▶ Pedagogical Content Knowledge Within "Mathematical Knowledge for Teaching"
- ▶ Psychological Approaches in Mathematics Education
- ▶ Recontextualization in Mathematics Education
- ▶ Stoffdidaktik in Mathematics Education
- ▶ Subject Matter Knowledge Within "Mathematical Knowledge for Teaching"

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Mathematical Cognition: In Secondary Years [13–18] Part 1

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Cognition · Mathematical thinking · Algebra · Calculus · Generalization

Definition/Introduction

The term cognition is synonymous with "knowing" or "thinking" or the process of knowing or thinking. Hence, mathematical cognition is simply defined as "mathematical thinking or knowing" or the "process of mathematical thinking." In this entry, we examine mathematical cognition as it pertains to the knowing of algebra and calculus, which has been widely studied in the past four decades. This body of work falls under three categories: (1) students' understanding of, and facility with, threshold concepts (Meyer and Land 2005) in algebra and calculus, (2) environments that enhance learners' cognition surrounding those concepts, and (3) learners' global meta-level mathematical activities (Kieran 2007) including problem solving, justifying, and proving as well as describing and justifying properties and relationships of mathematical objects. In this entry, we will focus on the first two categories of research and describe key findings from the body of work reported in this area.

Meyer and Land (2003, as cited in Firth and Lloyd 2016) coined the construct of *threshold concepts* and described them as concepts that serve as the building blocks of a discipline. Elaborating on this characterization, Frith and Lloyd (2016) offered:

> A threshold concept can be conceived of as a gateway, "opening up a new and previously inaccessible way of thinking about something" (Meyer and Land 2003, 1). These are concepts that are not only troublesome to students, but that are transformative – once fully understood, the result is a transformed perception of the concept (and the subject matter and perhaps even the self) and a shift in the use of language associated with it; irreversible – in that the new perspective is not easily undone; and integrative – it enables a view of linkages to other concepts in the discipline. (p.7)

A survey of existing literature on adolescents' mathematical cognition is most intensely studied due to their importance in success in higher mathematics to include ratio and proportions, slope, rate of change, covariation, functions, and functional reasoning. Research has established that these same concepts are considered challenging to develop, difficult to teach, and mathematically complex. The following sections will offer a review of current research on these topics and their associated issues.

Slope and Proportional Reasoning

Proportionality is a multiplicative relationship between two variables whose ratio is constant (Kline 1972). This relationship can be represented as a linear function whose graph passes through the origin. Slope characterizes a line that represents a proportional relationship, also referenced as steepness. The connection between proportional reasoning and slope has certainly been the subject of much scholarly inquiry in the literature. Research into the development of proportional reasoning of children and adolescents has been in existence since Piaget's theory established proportional reasoning as a hallmark of the formal operations stage of development of thinking (Frith and Lloyd 2016, p. 1; Sriraman and Lee 2017). The relationship between research on proportional reasoning and understanding of the concept of slope is, however, a recent development. Cheng et al. (2013) pointed at the strong link between slope and proportionality attributing students' difficulty with slopes to a fragile understanding of proportional relationships. In their study of approximately 413 middle school students' facility with steepness and proportional reasoning, the researchers reported a direct relationship between performance on tasks involving slopes and those involving proportions.

Analysis of data from international studies on student performance on algebra and algebraic contexts which rely on understanding slopes and proportional reasoning reveals global challenges associated with students' success (Gonzales et al. 2008). In a large study of factors leading to mathematics achievement among students in the US and UK, researchers found that an understanding of ratios and proportions was predictive of mathematics achievement especially in algebra (Siegler et al. 2012).

In unpacking the source of students' difficulties with proportional reasoning, scholars have identified the following: reliance on additive instead of multiplicative reasoning (Cheng, Star, Chapin, Cheng et al. 2013; Tague 2015), viewing proportional tasks as occasions to apply rules for computing (Stump 2011), and lack of familiarity with contexts in which problems are situated along with whether the context builds on discrete vs. continuous data (Tague 2015). Cheng et al. (2013) punctuated the absence of explicit instruction on connections between slope and proportions or space for empirical investigation of these connections.

Stump (2011) investigated 22 high school pre-calculus students' conceptions of slope in physical contexts (situations involving measuring slope as steepness) as well as functional reasoning situations (slope as ratio). The participants had previously conducted a physics experiment where they calculated the relationship between pedal revolutions of a bicycle and the distance the bicycle traveled. Stump (2011) interviewed each of the 22 students on 6 tasks: steepness of ski ramps in two contexts, steepness of percent grade on a highway sign, follow-up questions about a graph relating the revolutions of the bicycle pedal to distance, cost of tickets to a dance show, rate of growth of a girl over several years, and lastly a description of slope. The results indicated that many of the participants used angles to think of slope instead of or in addition to ratios. Moreover, the participants had difficulty describing what a ratio meant in terms of a physical rate indicating that connections among representations of rate, slope, steepness, and ratio are particularly difficult to cultivate. Others have reported similar results (Weber and Moore 2017; Thompson and Thompson 1996; Johnson 2012; 2015 a, b; Tague 2015).

Tague (2015) in her study of 877 students enrolled in grades 6 through 10 identified the use of addition on proportional reasoning tasks as the prominent approach used across grade levels. She hypothesized that one of the reasons for why, when given two proportions a/b and c/d, they may add them together could be an *epistemological obstacle with linearity* (Modestou and Gagatsis 2007) emerging from poor approach to teaching proportionality where students' tendency to generalize additive reasoning to proportions remains unchallenged. Tague's results contrast those offered by Fernandez et al. (2012). Fernandez et al. (2012) investigated the development of proportional and additive methods along primary and secondary school learners by analyzing the use of additive methods in proportional word problems and the use of proportional methods in additive word problems. Relying on a test consisting of additive and proportional missing-value word problems, data was collected from 755 primary and secondary school students (from fourth to tenth grade). Results indicated that the use of additive methods in proportional situations increased during primary school and decreased during secondary school, whereas the use of proportional methods in additive situations increased along primary and secondary school. The authors argued that the presence or absence of integer ratios strongly affected students' choices; however the nature of quantities only has a small influence on the use of proportional methods. Despite some differences in findings, there exists general agreement among researchers that additional, long-term research with a focus on investigating epistemological and curricular factors that contribute to the students' preference for additive reasoning is certainly needed. In doing so, need also exists for studies that explore the impact of various curricular and instructional interventions that can facilitate a shift from additive to multiplicative reasoning around proportional reasoning and steepness (Frith and Lloyd 2016; Roorda et al. 2015).

Rate of Change

The concept of rate of change has been studied as a part of calculus (Tague 2015), as covariation (Carlson et al. 2002; Carlson and Moore 2015), as limit (Tall 1986), as a ratio (Thompson and Thompson 1994, 1996; Confrey and Smith 1994), through dynamic simulations (Roschelle et al. 2000; Johnson 2010), and in modeling contexts (Ärlebäck et al. 2013, Ärlebäck and Doerr 2017). The extensive body of work on this topic identifies the concept as one of the most difficult for adolescents to learn, for teachers to teach, and for researchers to study (Tague 2015).

The structure of K-12 curriculum dictates that calculus is the first place where students are formally introduced to the concept of rate of change. English (2008) challenged this curricular approach and noted that in a world where complex systems exist, it is inappropriate to deprive the K-12 curriculum of modeling and thus of rate of change concept, which is the tool to capture real life, complex systems. Rochelle et al. (2007) also expressed the need for the rate of change to be treated as a unifying theme across K-12 curriculum so to provide opportunities for students to gain access to a wider array of mathematical analysis.

Confrey and Smith (1994, 1995) provided some of the earliest insights into school learners' conceptions of rate of change. They argued two approaches to introducing functions in mathematics: a *correspondence* approach and a *covariational* approach. The correspondence approach occurs when students are introduced to functions as a one-to-one relationship with the vertical line test. Alternatively, the covariational approach develops as students examine and create tables where the x-value (independent variable) determines the y-value (the independent variable). The authors argued that children are led to an understanding of functions through exploring the concept of rate and later they tend to use three approaches: additive rate of change, multiplicative rate of change, and "proportional new to old" rate of change (Confrey and Smith 1994, p. 141). Through these initial conceptions of rate of change, Confrey and Smith advocated that in order for a robust understanding of the concept of rate to occur, a multiplicative unit should be reinforced in curriculum and instruction (Confrey and Smith 1994, 1995). This view was challenged by Thompson in the context of functions that involve exponential growth.

Saldanha and Thompson (1998) investigated the type of conceptual operations that an 8th grade student used to reason about continuous covariation of quantities. The authors reported, "that understanding graphs as representing a continuum of states of covarying quantities is nontrivial and should not be taken for granted" (Saldanha and Thompson 1998, p. 7). The body of work offers that the use of dynamic environments that capitalize on multiple representations, linking graphs with tabular and symbolic notating system, assists in building a deeper understanding of functional relationships that build on proportional reasoning and quantification of rate of change across contexts, including discrete and continuous variations.

Tague (2015) examined students' conceptions of rate of change across middle school, high school, undergraduate calculus, and into undergraduate differential equations. Her work, arguably, unique in its scope and range, offered an epistemological, curricular, and conceptual analysis of links across the topics across the grade levels in order to design tasks used with the entire population. Her goal was to provide an overtime growth of understanding of the interconnected concepts spanning additive and multiplicative modes of reasoning including rate, rate of change, proportionality, and functional reasoning. Relying on both data from nearly 900 students in written form and interview data from a selected sample from each participating grade level, she proposed that a solid understanding of rate of change requires the piecing together of multiple mathematical representations and concepts in subtle ways that develop over the course of an individual's mathematical experiences. If we are to understand students' obstacles in understanding rate of change in algebra and upper level mathematics, we need to examine how and if students use rate of change in concepts that might be related to their future understanding of rate of change (Tague 2015, p. 320). She associated students' difficulties in forming a coherent understanding of the concept of rate, at its various degrees of sophistication demanded by curriculum, to the absence of an emphasis on building learners' representational fluency in a conceptually sound and developmentally appropriate manner.

Covariance

Much of the literature on middle and high school students' conceptions of rate of change aims to address how to steer students toward covariational thinking (Tague 2015). Saldanha and Thompson (1998) defined covariational thinking as the ability to imagine two different quantities changing simultaneously. We note that scholars have identified covariational thinking in different ways depending on the type of mathematical contexts and representational environments they used in their studies. These various approaches have led to institution of some global understanding of school learners' covariational thinking though little constancy exists among the findings of reported work, hence, limiting the ability to a coherent theory on learners' cognition about this topic. Most of the existing research surrounding school learners' covariational reasoning covers topics in advanced mathematics and statistics with a majority of this body of work highlighting the benefits of covariational reasoning on students' development of the concept of functions and other related algebraic topics (Confrey and Smith 1994, 1995; Saldanha and Thompson 1998; Warren 2005a, b). Others have considered ways in which covariational reasoning facilitates learning of calculus topics (Carlson et al. 2002; Carlson et al. 2001; Oehrtman et al. 2008), trigonometry (Paoletti and Moore 2017), and statistics (Zieffler and Garfield 2009).

Johnson (2012) studied four secondary students' understanding of covariation prior to their exposure to a formal mathematical definition. Specifically she aimed "to characterize a way of reasoning about covarying quantities involved in rate of change that could potentially serve as a cognitive root for calculus" (p. 314). Johnson (2012) aimed to identify where the reasoning was covariational (Carlson et al. 2002), transformational (Simon 1996), and proportional (Lamon 2007) and found while her subjects were able to describe changes verbally, they were not always successful in translating their verbal reasoning to written symbolic structures including ratio, limit, and function. These findings were fairly consistent across the different contexts she used in her research challenging scholars who had previously capitalized the importance of context on learners' approach to the use of rate of change and covariation.

Coulombe (1997) examined students' conceptions of covariation with a focus on linear functions over the course of an algebra I course. The sample consisted of 121 8th and 9th grade students who completed an assessment of covariation of variables. Follow-up interviews were conducted with several students on covariation of distance, time, and speed. Four themes were present in the analysis: (1) dependency, (2) multiple patterns of covariation, (3) linear patterns of covariation, and (4) generalizability (Coulombe 1997, p. i). Dependency indicated that the participant understood the effect the independent variable has on the dependent variable. Multiple patterns were defined as when there are piecewise-defined functions used in real-world situations. The linear theme was chosen because it was the underlying covariation that was being studied. Lastly generalizability occurred when the participant demonstrated the ability to generalize rules or patterns. The data suggested that 8th and 9th graders relied on intuitive representational schemes when describing covariation. However, those intuitive understandings were unstable. That is, children drew on different aspects of covariation or rate when presented with differing contexts.

Moritz (2004) examined 3rd, 5th, 7th, and 9th graders' performance in three areas dealing with covariation, using open-ended questions: translating a verbal statement into a graph, translating a scatter plot into a verbal statement, reading values, and interpolating. Students' responses were coded according to whether they had used appropriate covariation. The researcher reported that 3rd and 5th grade students were typically successful in translating a verbal statement into a graph; however, as the tasks became more complex, their performance declined. Seventh and ninth grades tended to manage covariations more effectively. The author concluded that covariational reasoning might be age dependent since students' responses became more robust according to grade level. Such developmental growth of cognition was supported by other researchers (Billings et al. 2007, Kaput 2008). For instance, relying on data obtained from analysis of pictorial growth pattern tasks, Billings et al. (2008) reported that students' correspondence reasoning followed covariational reasoning. Payne (2012) however raised the issues that the complexity of the functional reasoning tasks dictates whether students use correspondence or variational reasoning. According to her, the quality of task determines the type of reasoning students may offer.

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Functions

A function is a unique correspondence between two sets such that each element in the first set corresponds to exactly one element in the second set (Vinner and Dreyfus 1989).[1] Warren et al. (2006) describe a function "as a relationship between a first variable quantity and a second variable quantity or in terms of the change from the second to the first" (pp. 208–209).

Functions, functional reasoning, and functional thinking constitute key topics in secondary school mathematics, leading to higher-level mathematics courses beginning with calculus. Algebra is the study of structures, symbols, functions, and relations. There is consensus that key elements associated with reasoning and sensemaking with functions include the following: Different representations of a function – tables, graphs or diagrams, symbolic expressions, and verbal descriptions – exhibit different properties and using a variety of representations can help make functions more understandable to a wider range of students than with symbolic representations alone. An understanding of functions and facility with functional relationships underlies development of concepts in calculus including limit, continuity, derivative, and integral. Due to its importance, a number of research studies in the past two decades have examined high school and undergraduate students' facility with this concepts and ways that they conceptualize it. This body of work has documented that students at all levels struggle with the function concept and indeed many learners hold misconceptions about it (Carlson 1998; Eisenberg 1991; Leinhardt et al. 1990; Vinner and Dreyfus 1989). Analysis of substantial samples of students' performance on large national and international assessments such as the National Assessment of Educational Progress (Perez 2013) and Third International Mathematics and Science Study (Gonzales et al. 2008) on middle and high school students' facility with functions and functional reasoning also indicates that learners, globally, maintain a fragile understanding of functions (Payne 2012).

Dreyfus and Eisenberg (1983) found in their interviews that some students perceived a relation to be a function only when it could be represented by a single formula. This finding was further supported by Vinner and Dreyfus (1989) who reported that students rejected certain graphs of functions because of their perceptions of continuity and viewed algebraic data and graphical data as separate and that graphical representation with no formula was not perceived as meaningful. A focus on computational aspect of relations rather than its conceptual links dominated student thinking. Graham and Ferrini-Mundy (1989) substantiated this data drawing from the results of their own on research. The authors reported that when given a function, the students assumed they were expected to substitute a value in it. Their participants viewed the function as a static quantity.

This body of work offers that functions are often viewed by students as either an action, a process, an object, or a schema. Combining these perspectives on functions, Asiala et al. (1996) proposed APOS (Action-Process-Object-Schema) Theory offering that students with an action view see functions as merely a means for performing a particular action, such as computation. Those with a process view see a function as a collection of actions all at once and can comprehend the connections between those actions and what they can produce together. The object view of functions and the function schema are even more sophisticated. Several researchers (Carlson et al. 2010; Dubinskey and Harel 1992; Oehrtman et al. 2008) have claimed that students need at least a process view in order to develop a strong understanding of functions and have used APOS Theory to help explain students' impoverished function sense. Consensus exists the productive and effective approach to teaching functions requires a balance of viewing functions as each of the proposed models contextually and in tandem (Oehrtman et al. 2008; Tague 2015).

Other scholars have attributed poor facility with functional thinking to an overemphasis on procedures and rules without understanding (Oehrtman et al. 2008; Blanton 2008; Payne 2012) and

[1]We acknowledge that this is a modern definition of function. For a detailed analysis of historical development of definition of function, see Selden and Selden (1992).

curricular and instruction failure to capitalize on children's intuitive understanding of functions prior to its formal introduction (Eisenmann 2009, Kaput 2008, Blanton and Kaput 2004. Blanton and Kaput (2004) argued that students as early as third grade are capable of engaging in functional thinking and even representing functional relationships symbolically. Stressing the overtime results of instructional practices focused on building students' functional thinking using pattern generalizing tasks, the authors proposed that students in their study were able to express mathematical relationships using tables, graphs, pictures, words, and symbols in increasingly sophisticated ways. Student development was closely linked to the teacher's deliberate attempts at scaffolding learners' thinking while strategically introducing representations and vocabulary.

A large body of research argue that the type of tasks that provide opportunities for students to engage in functional reasoning (Carlson and Moore 2015) by looking for patterns (Kaput 2008), examining multiple representations (Johnson 2015a, b), reasoning about the relationships between quantities, and making generalizations (Blanton and Kaput 2008; Warren 2005b) facilitates learners' ability to deal with formal representations of function. Evidence exists that by refraining from stressing algorithmic fluency and symbolic manipulations (Payne 2012) and instead capitalizing on students' intuitive and informal knowledge to build their functional thinking, teachers secure greater chance of helping adolescents growth of cognition (Johnson 2015a, b).

In summary, findings offer that students:

- Perceive a function as a single formula.
- Experience difficulty reconciling the view of functions as process and objects.
- Fail to see connection between algebraic and graphical representations of a function.
- In order to develop functional reasoning, educators should begin the process early by engaging students in tasks that allow them to reason about the relationships between quantities and make generalizations (Blanton and Kaput 2005; Confrey and Smith 1994; Warren 2005b).
- Scholars have argued that:
  - Students are capable of engaging in functional reasoning as early as elementary school (e.g., Blanton and Kaput 2004; Warren 2005a).
  - Some children develop an intuitive understanding of functions before any formal introduction (Eisenmann 2009).
  - A large number of students, even in higher-level mathematics courses, experience difficultly understanding algebraic functions (Warren et al. 2006).

Generalizing

Panorkou et al. (2013) noted that patterns form the foundation for student's later understanding of proportional relationships and slope. Fonger (2014) proposed that in order for students to understand linear equations and functions, they first need to understand patterns. Patterns might connect because in developing an understanding of the growth or lack of growth of a pattern, students might later connect this reasoning to examination of growth of a function or the growth of distance between objects.

In mathematics education literature, the study of patterns has generally embedded in students attempts toward generalizing mathematical relationships and properties of objects. Indeed, Kaput (2008) defined algebraic and functional reasoning to consist of building, describing, and reasoning with and about functions by making generalizations about how data are related and later using symbols to act on these generalizations.

Despite frequent references to "generalizing" when elaborating on learners' algebraic and calculus-based cognition, differences exist in how "generalization" and "generalizing" have been defined in the literature. Polya (1957) defines generalization as "passing from the consideration of one object to the consideration of a set containing that object; or passing from the consideration of a restricted set to that of a more comprehensive set containing the restricted one" (p. 108). Krutetskiĭ (1976, p. 236) argued that the ability to generalize mathematical idea can be considered from two aspects “as a person’ ability to see something general and known to him in what is particular and concrete” (subsume a particular case under a general concept) and “the ability to see something general and still unknown to him in what is isolated and particular” (deduce general from particular cases). Dörfler (1991) defined generalization as “a social-cognitive process which leads to something general (or more general) and whose product consequently refers to an actual or potential manifold (collection, set, variety) in a certain way” (p. 63). Kaput (1999) described generalization as “extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases, or lifting the reasoning or communication to a level where the focus is no longer on the cases or situations themselves but rather on the patterns, procedures, structures, and the relations across and among them (which, in turn, become new, higher level objects of reasoning or communication).” (p. 137). From an actor-oriented view, Ellis (2007) described generalization as cognitive processes revealed in one of the three activities: (a) identifying commonality across cases, (b) extending one’s reasoning beyond the range in which it is oriented, or (c) deriving broader results from particular cases (Ellis 2007, p. 197).

These various descriptions highlight several aspects of mathematical generalization. First, they imply multiple characteristics of generalization. The first characteristic associated with generalization is abstraction as generalizing always involves extracting properties that are common or invariant among a given class of objects. The second characteristic associated with generalization is “extending” since generalization often involves going beyond the boundaries of a given class of objects. The third characteristic is that generalizing involves both seeing a generality through the particular and seeing the particular in the general.

Types of Mathematical Generalization

Existing studies have identified different categories of mathematical generalizations. Table 1 is a summary of generalization categories identified in mathematics education literature. As illustrated in the table, empirical generalization and theoretical generalization are two major categories of generalization. A theoretical generalization can be produced by a focus on the invariants of the action itself, invariants of the conditions of the action, or the result of the action. Even though inductive generalization often starts from empirical cases, mathematical generalization does not always rely on empirical cases as it can also be produced by dropping, ignoring, relaxing, or combining the conditions of given mathematical statement. The basic process of empirical generalization is to detect a common quality or property among two or more objects or situations based on perceptions and then to record these qualities as being common and general to these objects or situations. In contrast, theoretical generalization is constructed through abstraction of the essential invariants of a system of actions. Therefore, the abstracted properties are relations among objects rather than objects themselves. A major challenge in mathematics education is to develop students’ abilities to generalize based on mathematical structures rather than perception or the evidence offered by the regularities found in a few examples (Davydov 1990; Sriraman 2004).

Ellis (2007) developed a student-centered generalization taxonomy to describe the different types of generalizations that students create when reasoning abstractly. The taxonomy distinguishes between students’ generalizing actions and the product of generalizing (i.e., reflection generalizations). Generalizing actions include *relating*, *searching*, and *extending*. When *relating*, students form an association between two or more problems, situations, ideas, and mathematical objects by recalling a prior situation, inventing a new one, or focusing on similar properties or forms of mathematical objects. When *searching*, students engage in a repeated mathematical action, such as calculating a ratio, to find an invariance relationship, procedure, or result. *Extending* involves expansion of pattern, relationship, or rule into a more general structure. Reflection generalizations include *identifications or statements*, *definitions of classes objects*, and *influence*. When a student identifies a

Mathematical Cognition: In Secondary Years [13–18] Part 1, Table 1 Categories of generalization identified in literature

Author(s)	Criteria	Categories
Dörfler (1991)	Object of abstracting (common properties vs. invariants of actions)	Empirical generalization Theoretical generalization Generalization of the invariants of actions Generalization of the conditions for actions Generalization of results of actions.
Harel and Tall (1991)	Status of the cognitive schema of the individual	Expansive generalization Reconstructive generalization Disjunctive generalization
Harel (2001)	Ways of student thinking in relation to tasks that involve mathematical induction	Result pattern generalization Process pattern generalization
Yerushalmy (Yerushalmy 1993)	Sources of generalization (empirical examples vs. ideas)	Generalization from examples Generalization of ideas
Holland et al. (Holland et al. 1986)	Specific method used to produce a generalization	Condition-simplifying generalization instance-based generalization
Krygowska (1979, in Ciosek 2012)	Specific method used to produce a generalization	Generalization through induction Generalization through generalizing the reasoning Generalization through unifying specific cases Generalization through perceiving recurrence
Radford (Radford 2008)	Strategies used to identify and describe patterns	Naïve induction Arithmetic generalization Algebraic generalization

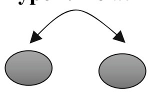
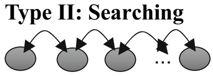
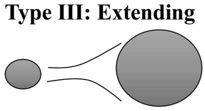
generalization and then articulates it, he or she may refer to a general pattern, property, rule, strategy, or a common element across cases or situations. The final product of a generalization can also be a *definition* of a class of object all satisfying a given relationship, pattern, or other phenomena. *Influence* refers to implementation of a previously developed generalization into a new context or problem. A student may implement a prior idea or strategy or may modify a prior idea as he or she approaches a new problem. This taxonomy provides a useful tool to describe students' generalizing behaviors and the generalizations they produced. However, it cannot tell us whether a generalization is mathematically sound or at which level a student is generalizing (Table 2).

Strategies in Pattern Generalizing

Stacey (1989) focused her exploration on pictorial linear pattern tasks. Students aged between 9 and 13 were asked to determine the number of matches needed to make a ladder with certain number of rungs or the number of lights in a Christmas tree of a given size. In her study, Stacey (1989) distinguished between "near generalization" tasks, which can be solved by step-by-step counting or drawing, and "far generalization" tasks, which go beyond reasonable practical limits of such a step-by-step approach. Four main generalization strategies are identified in her study: *counting method*, counting from drawing or successive adding; *difference method*, multiplying the term number by the common difference; *whole object method*, using the smaller figure as a unit and scaling this unit by a factor to find the larger figure; and linear method, implicitly or explicitly using the linear model $f(n) = an + b$. Stacey (1989) found that the constant difference property of consecutive term was widely recognized and most students in her study can move from $f(n)$ to $f(n + 1)$. And students were not consistent in their strategy use and tended to impose simple rule to the pattern without checking its validity.

Bishop (2000) interviewed 23 students in 8th and 9th grade as they solved four linear geometric

Mathematical Cognition: In Secondary Years [13–18] Part 1, Table 2 Generalization taxonomy (Ellis 2007)

Generalizing actions		
Type I: Relating 	1. *Relating situations*: The formation of an association between two or more problems or situations	*Connecting back*: The formation of a connection between a current situation and a previously encountered situation
		Creating new: The invention of a new situation viewed as similar to an existing situation
	2. *Relating objects*: The formation of an association of similarity between two or more present objects	*Property*: The association of objects by focusing on a property similar to both
		Form: The association of objects by focusing on their similar form
Type II: Searching 	1. *Searching for the same relationship:* The performance of a repeated action in order to detect a stable relationship between two or more objects	
	2. *Search for the same procedure*: The repeated performance of a procedure in order to test whether it remains valid for all cases	
	3. *Searching for the same pattern*: The repeated action to check whether a detected pattern remains stable across all cases	
	4. *Search for the same solution or result*. The performance of a repeated action in order to determine if the outcome of the action is identical every time	
Type III: Extending 	1. *Expanding the range of applicability*: The application of a phenomenon to a larger range of cases than that from which it originated	
	2. *Removing particulars*: The removal of some contextual details in order to develop a global case	
	3. *Operating*: The act of operating upon an object in order to generate new cases	
	4. *Continuing*: The act of repeating an existing pattern in order to generate new cases	
Reflection generalizations		
Type IV: Identification or statement	1. *Continuing phenomenon*: The identification of a dynamic property extending beyond a specific instance	
	2. *Sameness*: The statement of commonality or similarity	*Common property*: The identification of the property common to objects or situations
		Objects or representations: The identification of objects as similar or identical
		Situations: The identification of situations as similar or identical
	3. *General principle*: a statement of a general phenomenon	*Rule*: The description of a general formula or fact
		Pattern: The identification of a general pattern
		Strategy or procedure: The description of a method extending beyond a specific case
		Global rule: The statement of the meaning of an object or idea
Type V: Definition	1. *Class of objects*: The definition of a class of objects all satisfying a given relationship, pattern, or other phenomena	
Type VI: **Influence**	1. *Prior idea or strategy*: The implementation of a previously developed generalization	
	2. *Modified idea or strategy*: The adaptation of an existing generalization to apply to a new problem or situation	

pattern problems. In each of the patterning problem, four tasks were presented to the students in sequence. First, after the first four figures were presented, students were first asked to find the perimeter or area of certain figure numbers (the figure numbers in each problem were chosen to avoid obvious multiples of 2, 3, and 4); second, students were asked to verbally express the relationship they observed in the first task; third students were asked to assess the algebraic rule provided by the researchers; and lastly students were asked to find the figure number given the number with a given perimeter or area. Bishop (2000) identified several strategies students used to continue the patterns: *model* the required figure with pattern blocks, *multiply* the figure number by the constant difference, *apply proportional reasoning*, *skip counting* by or *adding* on the constant differences, and *use a linear expression*. And strategies for assessing the algebraic expression provided by others include *substitute values* into the expression, *compare the expression* with his/her own expression, and *relate the expression to the shapes*. And strategies for solving the inverse task include *model*, *guess and check*, *divide*, *apply proportional reasoning*, *skip count*, *solve equation*, *work backward*, and *analyze structure*. And Bishop (2000) noticed that students are not consistent on their strategy use: not only did individual students frequently use different strategies for different pattern problems on each task, but they also used nonparallel strategies on different tasks for each pattern problem. Nevertheless, after careful analysis of students' problem-solving strategies on the four tasks of the three pattern problems, Bishop (2000) found that students thinking about linear geometric pattern problem tends to fall into four categories: *concrete modeling and counting*, *inappropriate use of proportion*, *focus on recursive relationships*, and *analysis of the functional relation between a perimeter or area and the shape number*. What Bishop (2000) called the model, multiply, proportional, and use expression strategies on Task 1 correspond to Stacey's (1989) counting, difference, whole object, and linear methods, respectively. And what Bishop called the skip count/add strategy and use expression strategies correspond to the recursive and functional strategies identified by Swafford and Langrall (2000).

Lannin (2003) described six strategies students use to develop generalizations in pattern problem: *counting*, constructing a model to represent the situation and counting the desired attribute; *recursion*, building on a previous term or terms in the sequence to construct the next term; *whole object*, using a portion as a unit to construct a larger unit using multiples of the unit; *contextual*, constructing a rule on the basis of a relationship that is determined from the problem situation; *guess and check*, guessing a rule without regard to why the rule may work; and *rate-adjust*, using the constant rate of change as a multiplying factor and then an adjustment is made by adding or subtracting a constant to attain a particular value of the dependent variable. Lannin (2003) pointed out that students often use multiple strategies when they attempt to generalize a situation. After describing those strategies, Lannin (2003) also discussed the types of justification students provided. The first type of justification is *proof by example*, a common strategy that occurs through K-12 (Hoyles 1997). And the second type of justification is *linking the rule to the problem context*. Explanations provided for the recursive and contextual strategies might fall into this type. And another type of justification is *using proof by induction*, which is sometimes offered by students who use the rate-adjust strategy.

El Mouhayar and Jurdak (2015) surveyed 1232 students from grade 4 to grade 11 to investigate the variation of strategy use (counting from a drawing, recursive, chunking, functional, and whole object) in pattern generalization across different generalization types (immediate generalization, near generalization, and far generalization) and across grade level. Students in the study were provided four figural pattern tasks, two linear pattern tasks, and two quadratic pattern tasks. Result from the survey showed that the frequency of strategy use differed according to the generalization type and that recursive strategy was most frequently used in each of the generalization tasks. And as the demand of the task changed from constructing a step-by-step solution to finding a general formula, the use of recursive strategy increased, whereas the use of functional strategy decreased. Findings also revealed that the use of recursive strategy increased across grade 4 to grade 8 and then it decreased across grade 8 to grade 11, whereas the functional strategy increased across grade 4 to grade 11.

Cognitive Difficulties in Generalizing Mathematical Ideas

Despite the fact that students can use various strategies to generalize mathematical patterns, they also experience difficulties in the process of generalizing. The first difficulty concerns grasping the expected mathematical structure (Sriraman 2004). Many researchers have noted that children were not reluctant to generalize, rather they constructed generalizations too readily with an eye on simplicity rather than accuracy. Stacey (1989) wrote that "the greatest puzzle is to explain why so many children are apparently content to use generalizations which can very easily be shown to be false, even using only the data visible on the page" (p. 161). The frequent use of difference of consecutive terms and recursion in pattern generalization suggests that children tend to grasp the local regularities instead of the expected global structure. Children's lack of success in grasping the expected mathematical structure is not only due to their immaturity in mathematics but also the nature of mathematical objects. Duval (2006) argued that mathematical objects are not objects that can be directly perceived or observed with instrument and the access of mathematical objects is bound to the use of a system of representations which themselves might be open for multiple interpretations. Therefore, pedagogical interventions are needed for students to recognize the expected mathematical structure (Jurow 2004).The second difficulty concerns expressing the perceived generality, and the third difficulty concerns formalizing the articulated generality (Sriraman 2004). Some studies have shown that the passage from pre-symbolic to symbolic generalizations requires a specific kind of rupture with the ostensive gestures and contextually based key linguistic terms underpinning pre-symbolic generalizations (Radford 2008).

There is evidence that students make overgeneralizations in the process of generalizing. Overgeneralization occurs when individuals make a general claim based on insufficient evidence or apply a generalization beyond its range of applicability. It is an easy thinking error as it is a simple way to organize and make sense of things. Overgeneralizations may take in different forms, such as applying a generalization beyond the cases in which it is truly valid and imposing a pattern by selectively focusing on specific cases. For instance, some studies reported that chunking and whole object are frequently used strategies in pattern generalization (Stacey 1989; Bishop 2000). The inappropriate uses of proportion in the two strategies are overgeneralizations of proportion. Many other examples of overgeneralization have also been documented in other domains of mathematics. For instance, multiplication always makes a number bigger and division always makes a number smaller that are overgeneralizations of properties of natural number to rational number. And the idea that the rule for the product of radicals $\sqrt{a}\sqrt{b} = \sqrt{a \cdot b}$ is also applicable to adding or subtracting radicals, i.e., $\sqrt{a} + \sqrt{b} = \sqrt{a + b}$ is another overgeneralization. And generalizing based on patterns in data is an important mathematical practice. However, overgeneralizing might occur when students selectively focus on a few cases and make general claim based on results from these special cases. For instance, after trying a few examples, a student might claim that the sum of two composite numbers is a composite number. Inadequate generalizations come from different sources, such as intuition, met-before, external similarity, and so on.

Justification of Mathematical Generalization

Studies have shown that students prefer to use empirical evidences to justify their own generalizations. For instance, when examining students' generalization and justification in pattern activities, Lannin (2005) characterized students' justification into five levels: *no justification*, responses do not address justification; *appear to external authority*, reference is made to the correctness stated by others or reference material; *empirical evidence*, justification is provided through the correctness of particular cases; *generic example*, deductive justification is expressed in a particular case; and *deductive justification*, validity is given though deductive argument that is independent of particular cases. Twenty-five sixth graders were studied to understand how students justify their generalization when engaging in pattern activities. Results from data analysis showed that students in the study tended to use empirical evidence and generic examples to justify their generalizations. The use of empirical justification was generally due to a lack of connections to a geometric scheme that established a connection between the rule and the context. Even though some arguments students provided reflect the general relation, Lannin admitted that it is unclear whether students in the study understood the difference between empirical argument and generic example.

Studies have also shown that when a generalization and various forms of justifying it are presented by peers, students consider empirical arguments as more convincing. For instance, Healy and Hoyles (2000) presented to 2459 middle students two mathematical conjectures and a range of arguments supporting their validity. Students were asked to select among these arguments nearest to their own approaches. They found that arguments presented in words were popular as students' choices of their own approaches to a proof; students were reasonably successful at evaluating these types of arguments and were likely to see them as explanatory. In contrast, the participants found that arguments containing symbols were hard to follow and that they offered little in terms of communicating and explaining the mathematics involved. The results also showed that empirical argument dominated students' own justification of the general statements, although most students were aware of their limitations. Similarly, Liu (2013) designed four mathematical statements in four different mathematical contexts, each of which is justified by arguments with different representations (visual, narrative, numerical, symbolic) and resources (authority, example, imaginary, fact, assumption, opinion). Students in the study were asked to decide which argument type they found more convincing, exploratory, and appealing. Results from the survey of over 500 middle school students and the follow-up interviews revealed that the use of examples was the most referenced type of evidence to support the validity of an argument. Most of Liu's participants didn't consider the general validity of a conjecture as a requirement for convincing argument.

Summary

- Majority of the work on middle and high school students' mathematical cognition unfolds obstacles to learning of key mathematical ideas, as opposed to offering an epistemological account of their growth of understanding pertaining to these topics.
- Studies of environmental influences on attitudes and performance of students highlight the direct impact of instructional practices and pedagogical tools including technology and curriculum types on students' acquisition of mathematical concepts.
- Although some evidence exists on the developmental nature of growth of cognition regarding covariational and correspondence reasoning, additional research is needed providing greater detail regarding this development.
- Tasks and contexts that encourage the generalization of rules support the development of functional reasoning.
- Absence of consistent tools for measuring learners' development, unifying definitions and long-term studies on cognitive development of children has been identified as a major barrier in creating a coherent body of research that might advance mathematics education.

Cross-References

▶ Abstraction in Context
▶ Algebra Teaching and Learning
▶ Calculus Teaching and Learning

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Mathematical Cognition: In Secondary Years [13–18] Part 2

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Definition/Introduction

As defined in part 1, the term cognition is synonymous with "knowing" or "thinking" or the process of knowing or thinking. Hence, mathematical cognition is simply defined as "mathematical thinking or knowing" or the "process of mathematical thinking." In this entry we examine mathematical cognition as it pertains to the knowing of geometry and proofs, highlighting the major theoretical views that account for adolescent mathematical cognition. We note that a majority of research reports surrounding mathematical thinking and cognition of students in this age group concerns conditions under which student learning has been explored, among many include technology, curriculum, and classroom interactions. Furthermore, a large portion of studies that shed light on mathematical cognition in areas relevant to secondary school mathematics have relied on data collected from undergraduate student populations.

Geometry

Battista (2007) characterized geometry as "a complex interconnected network of concepts, ways of reasoning, and representation systems that is used to conceptualize and analyze physical and imaged spatial environments" (p. 843). Geometry entails, among many, interconnected skills, visualization and construction of images of geometric concepts, realizing and appropriating relationships between concepts, making and justifying generalizations, and proving. It also plays a central role in connecting various mathematical subjects including discrete and continuous mathematics, functions, limits, and trigonometry (Goldenberg et al. 1998). Despite its importance, scholars' worldwide have consistently documented school learners' difficulties with the subject (Berenger 2017). Among many influential factors most widely cited include the language of geometry, weak visualization skills, ineffective instruction, and poor reasoning skills (Lew et al. 2012). Some have attributed learners' poor performance in geometry to its absence from school curriculum (Thompson et al. 2012; Berenger 2017). Transition from elementary to secondary geometry has also been identified as rough, highlighting little gains in geometric understanding of students as they progress across grade levels and complete courses in secondary schools (Usiskin 1982).

Theoretical Models Guiding Research on Geometry Learning

While the extensive body of research on geometry learning has relied on a variety of theoretical models, the most influential includes Piaget's stages of cognitive development which describes the process of the formation of spatial representations central to geometric reasoning.

Piaget believed that geometric thought developed in stages according to experiential order, starting with topological relations (such as connectedness and continuity), followed by projective (rectilinearity) and then Euclidean relations (Jones 2002, p. 130). Piaget proposed that geometric thinking is developed with the age of the child (Huitt and Hummel 2003; Mason 1998) and that mental structures developed through the child's own activity and interactions within the environment (Clements and Battista 1992; Sriraman and Lee 2017). Piaget viewed knowledge as made up of logical structures resulting from actions and contributing to a total mental structure. Reliance on Piagetian approach to the study of geometry cognition has been most prominent at the elementary grade levels as it stresses maturation and age-appropriate experiences in building up of mental representation (Clements 1999; Battista 2007).

The second model, the *van Hiele Model of Geometric Thought*, is the one most widely used as a framework for tracing adolescences' geometric cognition. Van Heile's model (1992) while offering a level-based development of geometry thinking, associates growth across levels to experience rather than age. Van Hiele levels include "visual," "descriptive/analytical," "informal deductive," "formal deductive," and "rigor" (Burger and Shaughnessy 1986). At the visual level (Level 1), learners could identify, name, and compare geometric figures, such as triangles, rectangles, angles, parallel lines, etc., according to how they look. At the descriptive/analytical level (Level 2), learners can recognize components and properties of a figure; however, they cannot reason upon those properties. They are able to describe figures in terms of their parts and relationships among these parts, to summarize the properties of a class of figures, and to use properties to solve basic identification problems, but they cannot yet conduct deduction. At the informal deductive level (Level 3), learners are able to connect figures with their properties. They can justify figures by their properties as well as articulate the properties of a given figure. The learners can understand and use precise definitions. They are capable of using "if-then" thinking, but they cannot consciously use mathematically correct language, nor can they realize the deductive property of their reasoning. Their reasoning is based on intuition instead of a mathematical foundation. At the deductive level (Level 4), learners can reason about geometric objects using their defined properties in a deductive manner. They could consciously construct the types of proofs that one would find in a typical high school geometry course. They are aware of what counts as a legitimate proof in mathematics. At the highest level, rigor (Level 5), learners can compare different axiomatic systems. Learners fully understand the structure of a system as well as its applications and limitations. They can analyze and compare these systems. According to van Hiele, progress through the levels is dependent on experience and that instruction central to facilitating learners' cognitive development.[1]

Usiskin (1982) initiated a watershed effort to examine the validity of van Hiele's theory. He developed and used a test of geometric reasoning eliciting student knowledge along the five van Hiele's levels. Relying on a pre-post procedure, he traced the growth of geometric reasoning of approximately 2500 high school students over the course of 3 years. Results reported that van Hiele levels are an adequate classification of the student's current foundation in geometry and adequate predictors of later geometric achievement. His study confirmed that the use of van Hiele theory could explain why many students struggle with learning geometry. Usiskin questioned the quality of mathematics instruction students received in a year-long course in geometry since a majority of the learners left their courses with little to no improvement in their geometric knowledge. Similar results persist globally. Jones (2002), reporting on the state of geometry teaching and learning in secondary schools in England, proposed, "most lower secondary students perform at levels one or two with almost 40% of students completing secondary school below level 2 (Jones 2002, p. 130). Most recently, Berenger (2017) examined the geometric thinking of students in Years 7 and 8 at two schools in Australia. Classroom-based data was collected to examine how students and teachers communicated their understanding of geometric concepts relating to two-dimensional shapes. The author reported that students operated at level two (analysis) of van Hiele model. The students' progress in geometric reasoning was hindered not only by students' misconceptions but also by teacher's own ill-structured understanding. This finding further substantiates the influence of instruction on advancing geometric reasoning of children.

An impressive volume of research on conditions and environments that facilitate adolescents' geometric knowledge growth points at the utility of dynamic geometry software (Mason 2007, 2009; Laborde 2002; de Villiers 2003; Sinclair and Yurita 2008), instructional practices responsive to the van Hiele's developmental stages (Kuzniak and Rauscher 2011; Swafford et al. 1997), the use of exploratory tasks (Fujita and Jones 2003), an emphasis on conceptual understanding of geometric ideas (Jones and Herbst 2012; Henningsen and Stein 1997; Sinclair et al. 2016; Schoenfeld 1988; Reid 2011; Recio and Godino 2001), and collaborative discourse (Kramarski and Mevarech 2003; Kunimune et al. 2010; Kuchemann and Hoyles 2009; Pierce 2014; Walmsley and Muniz 2003).

[1]The van Hiele model has been modified and extended by scholars to meet particular research interest. For example, Clements and Battista (1992) added a level 0, "pre-recognition," where children were not able to visually identify the difference between shapes, depicting their cognition in geometry at the very beginning stage. Pegg and Davey (1998) integrated the van Hiele model with another learning theory, the SOLO taxonomy (Biggs and Collis 1982a, b), to describe how learning develops within and through the levels.

Proofs

The study of learners' proof schemes has a long history and is currently a mainstream in didactics of mathematics (Liu 2013). While a majority of the courses offered in geometry at the high school level tend to build students' understanding of axiomatic structure of the discipline and deductive reasoning toward building proofs, school learners, even in presence of proof-based courses in geometry, exhibit difficulty in meeting these curricular goals (Battista 2007). Literature on secondary students' performance on tasks and contexts that demand deductively structured proofs indicates that students fail to see a need for proofs and are unable to distinguish between verifying, explaining, and proving (Jones 2002). Senk (1982, 1985), in a large-scale study of high school students' proving performance, reported that only about 30% of students completing a full-year course in geometry showed mastery of proof writing at a level that indicated above average (75%) mastery.

Healy and Hoyles (2000) categorized students' view of proof and its purposes in a large-scale empirical study of children aged 14–15. They found that 28% of the students didn't show any understanding of the purpose of proof. In addition, only 1% of them acknowledged that proof might help discover new theories or systemize ideas. The most recognized functions of proof were verification and explanation.[2] Furthermore, Healy and Hoyles posited that students' understanding of the purposes of proof had a significant influence on their ability to identify and construct a proof. Liu (2013) analyzed survey results from 476 eighth grade students who were enrolled in 5 different public schools to determine whether the students' preference for a particular kind of argument was consistent across different subject areas. In contrast to the findings of previous research which illustrated that students excluded algebraic arguments when they were asked to select an argument that they found convincing and explanatory (Healy and Hoyles 2000), Liu reported that his study participants' preference for argument type was highly inconsistent across content areas (Freudenthal 1971), and hence an overarching preference of proof type is unlikely to be achieved at early cognitive stages. A half of the participants in Liu's study seemed to realize that testing special cases was not sufficient for claiming proof of the statements. However, most were unaware of the advantage of symbolic expressions which could represent general cases. Liu (2013) concluded: learners' understanding of proof develops locally and doesn't automatically transfer to other fields. Students may appreciate deductive reasoning in one area but still find visual illustration and use of examples convincing in other contexts. Since proof ability essentially concerns the relationships among concepts and properties, it is crucial for students to develop a conceptual understanding of mathematical topics. When reasoning is addressed in different content areas, there is greater potential for development of a coherent perception of mathematical structure among learners (p. 247).

Studies of school learners' perceptions of and facility with proofs provide consistent results: students fail to consider proofs as important and find them difficult and irrelevant (Senk 1982, 1985; Usiskin 1982; Knuth et al. 2009). Most students view a mathematical proof as a method to check and verify a particular case and tend to judge the validity of a proof by its appearance (Martin and Harel 1989). There is also evidence that even mathematically successful students fail to associate meaning to what is expected and explored in traditional proof-based courses in geometry (Stylianides and Stylianides 2008). Most high school students do not have adequate exposure to the process of proving (Dreyfus 1999; Recio and Godino 2001; Schoenfeld 1988; Jones 2002; Pierce 2014; Reid 2011; Panaoura and Gagatsis 2009; Sinclair et al. 2016). Dreyfus argued that instruction must provide opportunities for students to build on their tacit knowledge when justifying and proving students. Segal (1999) further stressed this point by suggesting that students' perception of validity of a proof is influenced by the norm set by the teacher. His assertion was

[2] In Healy and Hoyles' (2000) classification of students' view of the purposes of proof, the category named "explanation" included both explanation and communication as identified in de Villiers' (1990, de Villiers 2003) model.

substantiated by findings of research conducted by other researchers (McCrone et al. 2002; Thompson et al. 2012; Berenger 2017). This body of work suggests that advancing learners' proving capacity demands greater emphasis on sense making through exploratory work and authentic inquiry (Herbst et al. 2017).

Genres of Investigation

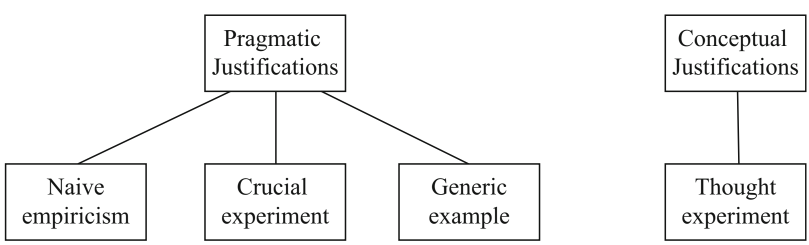
Stylianides and Stylianides (2008) identified three cohorts of scholarly investigations focused on studying proofs in mathematics education research. The first cohort seeks evidence that students possess the ability to use deductive reasoning in constructing arguments and proofs, even at the early elementary grades. The second cohort describes students' common difficulties and mistakes in producing proofs across the grade levels and content areas. The third cohort offers an account of pedagogical factors that could facilitate students' learning about proofs.

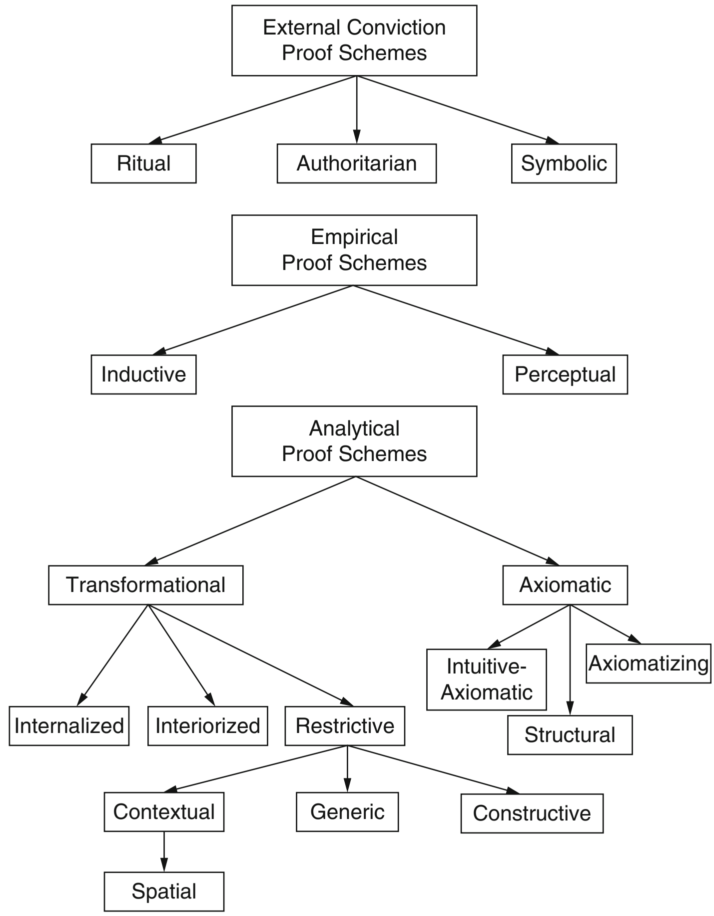
Although these three cohorts of studies, including both empirical reports and theoretical investigations, provide insights into students' analytics as well as challenges experienced in learning proofs do not offer a framework that captures features of students' thinking when performing proof-related tasks. Studies of students' proof schemes tend to close this gap by classifying the different types of proofs that students produce. Following previous scholars' work (Balacheff 1988, 1991), Harel and Sowder (2007) organized the types of proof students may use in various content areas of mathematics and proposed a taxonomy of proof schemes consisting of three main categories, i.e., "external," "empirical," and "analytical," each of which encompasses several subcategories.

Balacheff (1988) coined "pragmatic" and "conceptual" as two prominent modes of justification used by students. *Pragmatic* justifications are based on the use of examples (or on actions), and conceptual justifications are based on abstract formulations of properties and of relationships among properties. He further identified three types of pragmatic justifications to include "naive empiricism," in which a statement to be proved is checked in a few (somewhat randomly chosen) examples; "crucial experiment," in which a statement is checked in a carefully selected example; and "generic example," in which the justification is based on operations or transformations on an example which is selected as a characteristic representative of a class. "Thought experiment" is identified as conceptual justification, in which actions are internalized and dissociated from the specific examples and the justification is based on the use of and the transformation of formalized symbolic expressions (see Fig. 1). Balacheff (1988) concluded that while students experience difficulty producing proofs, they do however show awareness of the necessity to prove and to use logical reasoning.

Harel and Sowder (1998) proposed a taxonomy of proof schemes consisting of three main categories, i.e., "external," "empirical," and "analytical," each of which encompasses several subcategories (See Fig. 2). In particular, *external* conviction proof schemes include instances where students determine the validity of an argument by referring to external sources, such as the appearance of the argument instead of its content (e.g., they tend to judge upon the kind of symbols used in the argument instead of the embedded concepts and connection of those symbols) or words in a textbook or told by a teacher. *Empirical* proof schemes, inductive or perceptual, include

Mathematical Cognition: In Secondary Years [13–18] Part 2, Fig. 1 Balacheff's (1988) classification of students' proving schemes





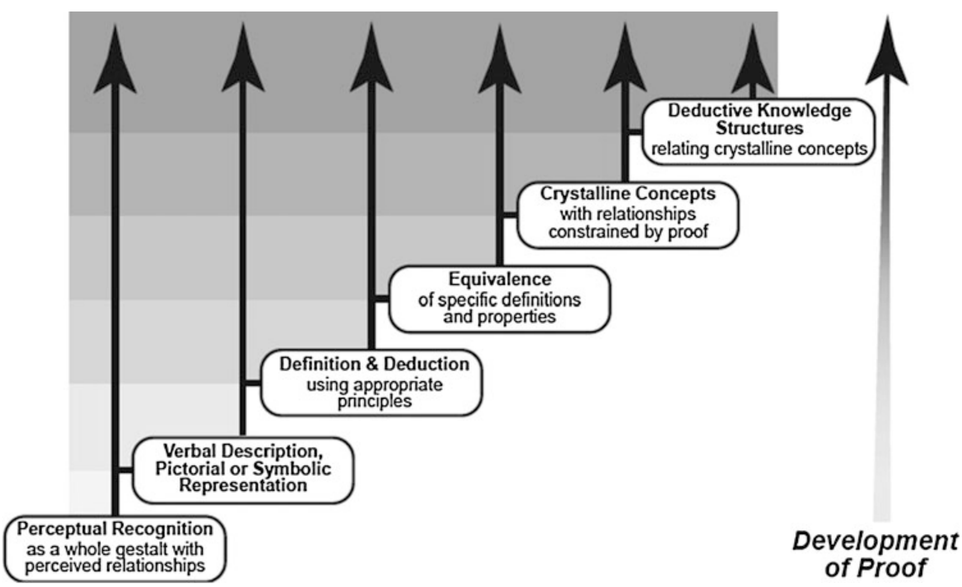
Mathematical Cognition: In Secondary Years [13–18] Part 2, Fig. 2 Proof schemes and sub schemes (Sowder and Harel 1998)

instances when a student relies on examples or mental images to verify the validity of an argument; the prior draws heavily on examination of cases for convincing oneself, while the latter is grounded in more intuitively coordinated mental procedures without realizing the impact of specific transformations. Lastly, *analytical* proof schemes rely on either transformational structures (operations on objects) or axiomatic modes of reasoning which include resting upon defined and undefined terms, postulates, or previously proven conjectures.

Harel and Sowder (1998, 2007) observed that students could simultaneously hold different proof schemes when working on different problems. Their model detects such a difference but does not explain why such inconsistency might exist. The cognitive development models can capture students' progress in producing logical reasoning in a certain mathematical field but fail to describe why and how such a development may emerge across content area differences. Despite this, Harel and Sowder's typology is arguably the most widely used scheme in research surrounding proving style of secondary school pupils. The universal appeal of the model has allowed for generation of a substantial body of work that highlights adolescence's tendencies when constructing proofs, stressing obstacles students experience when expected to producing analytical proofs (Berenger 2017). Only recently, some researchers have begun to examine the impact of content and context on why students may persist on producing certain types of proofs (Liu 2013; Berenger 2017), highlighting that a focus on typology of students' proving scheme is limited in accounting for what kind of mathematical arguments students find appealing, convincing, or explanatory since even arguments that are classified as the same type can be judged quite differently among people and across the content areas. Liu (2013) called the need for conceptualizing a more precise proof classification framework that responds to these environmental and epistemological conditions.

There is also consensus that in order for the instruction to enable students to understand and appreciate proof as a reliable way of reasoning (de Villiers 2003; Fawcett 1938/1995; Reid 2011), learning about ways to help students realize proof as a reasoning methodology is equally important as teaching the skills of producing specific proofs. To address this, Tall et al. (2012) proposed a two-dimensional model to depict the development of factors that are involved in the maturation of one's proof ability (see Fig. 3). This framework captures six key components (i.e., *perceptual recognition*, *verbal description and pictorial or symbolic representation*, *definition and deduction*, *equivalence*, *crystalline concepts*, and *deductive knowledge structur*e) and their relationships in the broad maturation of proof structure. Different from the van Hiele model, Tall et al. (2012) suggest that the perceptual understanding doesn't develop only at earlier stages. Instead it continues to be refined when the understanding of the concept and deductive process is advanced. This idea is consistent with the perspective of *constructivism*, in the sense that the mathematical system possesses a dynamic structure so that a shift in understanding of a factor may impact other components (Lakatos 1976; Tall 2005).

Nevertheless, Tall et al. (2012) don't suggest that all the components in the structure develop simultaneously. Instead, certain types of understanding serve as a prerequisite for others to occur. This feature is denoted by the "height" of each component. *Crystalline concept* introduced in this framework plays a crucial role in the development of proof structure. According to the authors, it is a concept with a pack of associated knowledge attached to it. In order to construct deductive reasoning, involved concepts must not be perceived as isolated objects, and only when the roads are built can a pass be drawn. This approach to growth of proving cognition offers a fruitful link between instruction and development of reasoning skills. To date, this theoretical model remains open to elaboration, and only a handful of studies (Liu 2013) have attempt to examine its



Mathematical Cognition: In Secondary Years [13–18] Part 2, Fig. 3 The broad maturation of proof structure (Tall et al. 2012) utility for studying proving cognition of school learners.

Cross-References

▶ Deductive Reasoning in Mathematics Education
▶ Shape and Space: Geometry Teaching and Learning

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Mathematical Cognition: In the Elementary Years [6–12]

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Introduction

Mathematical cognition in the elementary years is a vast subject of study with entire handbooks devoted to understanding its different aspects, viz., computational views, dyscalculia, neuroscientific views, psychological views, and sociocultural views (Ashcraft, 1995; Campbell, 2005; Gallistel and Gelman, 2005; Radford, 2014). In this entry we view mathematical cognition as relating to the epistemology of mathematics and analyze cognition as an imprint of mathematical structures naturally occurring and perceived in the world. In particular, we synthesize Piagetian and non-Piagetian views on the development of mathematical cognition in children (ages 5–12) across two major areas of mathematics extensively studied by pupils in their elementary school years: geometry and enumeration and whole-number arithmetic.

Cognition in Elementary Years: Geometric Thinking

Piagetian Views

Piaget's goal was to study children to answer basic philosophical questions about the nature and origins of knowledge. His focus from philosophy was the child's understanding of space, time, and causality, of number and quantity, and of classes and relations of invariance and change. Piaget's perspectives from biology were organization, development, and adaptation interlacing four periods of cognitive development, namely, sensorimotor, pre-operational, concrete operational, and formal operational.

Piaget repeatedly defined operations as internalized actions and often went so far as to say they derive directly from the subject's physical actions as enacted in sensorimotor behavior. The emphasis upon physical action created a terminology that is perhaps more metaphorical than exact when it comes to describing the activity of operational thought. For example, with the operation defined as an "action" enacted in thought, the *grouping* structure of operations (where each operation is always implicitly bound up with a whole system or structure of interrelated operations), together with the emphasis upon the operation as a *reversible* action, leads one to imagine the mind as comprising a system of extremely rapid and "machine-like" executions of these actions quite literally. However, this kind of terminology should be treated more as a metaphor or analogy, which, as such, leads to its own problems. Piaget's treatment of concrete operations, as actions, also leads to the problem of defining the qualitative difference between the two types of reversibility. On the one hand, we have operations where the reversibility is characterized by direct negation: for example, the action to place an object is negated by the action to remove the object. And on the other hand, we have operations where the reversibility is characterized by reciprocity: the relation of "bigger than" is not put in opposition to the relation of "not bigger than" but to the relation of "smaller than." However, this last statement in itself shows us the difficulty of realizing operations of reciprocity as deriving from physical actions enacted by the subject: what is the *action* of "bigger than?" Rather, the relation of "bigger than" is just that – a relation – and not necessarily the direct result of a physical action.

If Piaget's apparent reduction of operations to physical actions is difficult to sustain, it would be wrong to conclude that he did indeed reduce operations to actions. In fact, one can argue that Piaget made a number of attempts to provide different qualitative descriptions of operational thought. In *The Origins of Intelligence in Children* (Piaget 1952a), in the table in the introduction, we see the function of assimilation, not accommodation, as leading to the operations. For another, in *Possibility and Necessity*, Piaget (1987) elaborated upon the functions of differentiation and integration as the mechanism for abstracting from the concrete to the abstract. And, in the various essays on equilibration, Piaget attempted to describe operational structures not so much in terms of systems of actions but rather as systems of dynamic equilibrium. Of course, all these different perspectives amount to the same thing, not to a change of view. Piaget was attempting to describe better and better what he saw as the (rather difficult) truth and was not simply changing his mind.

Moreover, with the difficulty of maintaining Piaget's apparent reduction of operations to physical actions, it would be wrong to go to the other extreme of relying only upon notions of passive *perceptions* of physical relations. Perception, when conceived as the rather inert function of mere assimilation, does not sufficiently allow for the fact that cognitive structures develop. More significantly, pre-operational and operational structures do not just change but undergo profound qualitative advances in their capabilities. To get from one level to the next, the subject has to be active within the milieu of those relations, the internalization of which precisely defines the next level over the previous. The definition of an operation as an action retains the sense of active and constructive participation by the subject, whereas some notion of simply "seeing" (i.e., perceiving) these relations fails to address these necessary attributes.

So, concluding with the claim that to define operations as internalized *actions* leads to difficulties we could do without (and more such examples will be described below), we now have to suggest a more useful terminology that does not suffer the deficiencies Piaget's own terminology sought to overcome. Piaget believed that the development of reasoning occurs in stages. Three separate stages (and two substages) of reasoning are described and can be characterized by the amount and structure of related propositions children are able to use in justifying the truth of ideas (Inhelder and Piaget 1958; Piaget 1987). For example, the level of reasoning may be assessed by the student's ability to recall, organize, and decide which information is necessary and sufficient to establish a proposition. The nature of change from one level to the next is characterized by states of equilibrium within the organizational structures of knowledge (e.g., possibility, reversibility, conversation, transformations). In one experiment, children were asked to uncover a large irregular closed figure. The problem was for a child to determine which figure among a given set was under the rectangular cover by uncovering the fewest number of clues as possible to be sure of their answer. The figures used were examples of the 12 figures the children were given as possibilities for the covered figure. The findings of this study were as follows:

Level I (ages 5–7): children were unable to use formal reasoning. They were unable to retain, relate, and coordinate clues or relevant information. The students tended to focus on the last and most meaningful (to them) clue and disregard the importance of other clues or relationships between clues.

Level IIA (ages 7–9): reasoning becomes anticipatory in nature. Piaget believed that this level coincides with the onset of concrete operations. In the hidden-figure experiment, children were able to classify figures as possible or not possible. However, they were unable to explicitly explain their methods of classification. Children at this level have a vague global intuition of concepts but are unable to make explicit their understanding.

Level IIB (ages 7–9): children begin to be able to describe explicitly their intuitions in solving a problem. In the hidden-figure experiment, children would be able to explain that a shape is not a possibility because it lacks certain characteristics which separate it from the hidden figure.

The key difference between level II and level I is that the second level is anticipatory and comprehensive. There is significant switch in the direction of thinking from the actual situation to the potential.

Level III (ages 9–12): children begin to make general hypothesis to explain why things must occur. Piaget believed that this coincides with the onset of formal operations.

Level IIIA: children are still bound to the concrete situation in which problems arise. Children can argue that enough information about a situation has been determined to guarantee an outcome is certain. It is not necessary for these children to uncover more clues to be sure of the identity of a

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shape, once a necessary and sufficient set has been uncovered. However it is difficult for children at this sublevel to explain why the set is necessary and sufficient to solve the problem absolutely.

Level IIIB: children are after the general (abstract) properties and the relationships that affect how they operate. In the hidden-figure experiment, these children would not only be able to explain why a set of clues is necessary and sufficient but also if other necessary and sufficient sets of clues exist.

Another feature of level III reasoning is that possibility is not bound to be an extension of empirical situations. Children at this level can formulate hypothetical situations and draw out necessary consequences without ever observing these consequences.

Piaget's theory suggests that there is a "structural mechanism" which enables students to compare various combinations of facts and decide which facts constitute necessary and sufficient conditions to ascertain truth. He believed that this structural mechanism is functional in children only after they are able to transform propositions about reality (or to abstract reality) so that the relevant variables can be isolated and relations deduced. For this to occur, children must be at the stage of formal operations (Inhelder and Piaget 1958).

To demonstrate the different levels of reasoning, Piaget performed a series of experimental studies where he investigated several phenomena such as the equality of angles of incidence and reflection and the operations of reciprocal implication, the law of floating bodies and the elimination of contradiction, the oscillation of a pendulum, and the operations of exclusion, among others. All of these studies identified significant changes in children's ability to reason (Inhelder and Piaget 1958).

Piaget described reasoning as a separate structural mechanism which controls the use of knowledge and is capable of creating new knowledge *without additional external influence*. Thus, in theory, reasoning serves the purposes of establishing truth of existing knowledge and constructing new knowledge. The development of reasoning is therefore dependent on factors that cause changes in this structural mechanism. Piaget believed that as the brain develops physically, this mechanism changes and has the potential to change the level of thinking and reasoning.

Non-Piagetian Views: Van Hiele Levels of Geometric Thinking

In 1957, Dutch educators, a husband and wife, Pierre and Dina van Hiele completed "companion" doctoral dissertations, in which Pierre described a system of levels of thinking in geometry and Dina focused on the teaching structures and experiments that can help improve students' learning and progressions within these levels. Their model involved five levels of geometric thought (for more information on the levels, see Fuys et al. 1988; Wu and Ma 2005): visualization, analysis, abstraction, deduction, and rigor. The first level is characterized by students recognizing figures in their global appearance, i.e., they see geometric figures as visual gestalts. For example, students may distinguish between triangles and quadrilaterals, but not able to distinguish between a rhombus and a parallelogram. At this level students recognize figures visually, by appearance, often comparing them to a known prototype. The properties of figures are not yet understood, and the decisions are made based on visual observations and perceptions rather than reasoning. At the second level, students are able to analyze or list properties of geometric figures; the properties of geometric figures become vehicles for identification and description. The third-level students begin to relate and integrate properties into necessary and sufficient sets for geometric shapes. Students at this level understand that some figures can be defined in terms of others. For example, a square is a rectangle with consecutive sides equal. At the fourth level, students develop sequences of statements to deduce one statement from another. Formal deductive proof appears for the first time at this level. Finally, the fifth level is where students are able to analyze and compare different deductive systems, by establishing and comparing mathematical systems and argumentations. Students at this level understand the utility of widely used geometric proofs, such as indirect proofs and proofs by contrapositive, and understand (or cognitively ready to understand) non-Euclidean geometry.

There has been recent evidence that supports a level of thinking that appears before van Hiele's first level (Battista and Clements 1988, 1989; Usiskin 1982, 1987). Students at this level labeled "precognition" by Battista and Clements focus on only part of the visual characteristics. Other studies attempt to describe how students reason at different van Hiele levels depending on the given task. Mayberry (1981) found that students do not necessarily think at the same level in each topic area.

The most notable characteristics of the van Hiele levels of geometric thinking are that they are hierarchical and sequential, the levels are discrete rather than continuous, and the structure of geometric knowledge is unique for each level. Van Hiele (1986) posited that children must pass through each level of geometric thinking in the development process. No student can skip a level or be thinking at a higher level and digress to a lower level of thinking. Children first encounter ideas implicitly, and when these ideas are understood explicitly, through the understanding of language and interconnecting new ideas with existing ideas, children progress to the next level of thinking.

It should be noted that the initial work of van Hiele was influenced by the psychology known to them, particularly Gestalt psychology. Dina van Hiele mentioned the work of Kohler and Duncker. Van Hiele (1959) also recognizes the contribution of Piaget to this work. He agrees with Piaget's observations on concept formation and believes that Piaget's stages of intellectual development provide a valuable contribution to developmental psychology. But van Hiele argues that the sensorimotor, concrete operational, and formal operational stages do not develop uniformly across school subjects, and they are not linked as much to biological age as Piaget and his followers imply. While Piaget interprets protocols in which students do not solve or misunderstand a problem to mean that students at that particular age are incapable of solving the problem, van Hiele sees such protocols as an indictment of contemporary school practices. No doubt van Hiele believes that had Piaget's subjects been instructed according to the van Hiele model, they would do more than Piaget reports. Interestingly enough in later years, Piaget (1972) admitted that interest, culture, and experience do play a role in determining changes in cognitive development.

Cognition in Elementary Years: Enumeration and (Whole) Number Operations

Piagetian Views

We draw on Piaget's work with children related to correspondence, quantities, and equivalence, in which he investigated classes and relations, as well as numbers as cognitive domains, to provide evidence for a hypothesis that the construction of number is closely related to child's development of logic; see *The Child's Conception of Number* (1952). Piaget asserted, "the construction of number goes hand-in-hand with the development of logic, and that pre-numerical period corresponds to the pre-logical level" (Piaget 1952b, p. viii).

Number is organized in a close connection "with the gradual elaboration of systems of inclusions (hierarchy of logical classes) and systems of asymmetrical relations (qualitative seriations), and sequence of numbers thus resulting from an operational synthesis of classification and seriation" (Piaget 1952, p. viii). As a result, *logical* and *arithmetical* operations (with numbers) are psychologically natural systems, the second resulting from generalizations and fusions of the first, under complementary headings of inclusion of classes and seriation of relations. When the child applies this operational system to sets, the emergence of "inclusion and seriation of the elements into a single operational totality takes place, and this totality constitutes the sequence of whole numbers, which are indissociably cardinal and ordinal" (Piaget 1952, p. viii). Below, we explore this notion at greater details and provide specific examples from Piaget's work, based upon which he made his assumptions and which help us make further connections between his work and the work of others (discussed in the next section), related to child's cognition within mathematical domains of enumeration and whole-number operations.

First, we draw attention to the series of experiments conducted with children related to *cardinal and ordinal one-to-one correspondence*. Specifically, Piaget (1952) asked his participants to describe quantitative relationships between a row of five glasses and a row of six bottles positioned closer together (than glasses). At Stage I (age 4–5), children indicated that a row of five glasses contained more elements than a row of six bottles. Furthermore, some children, when the glasses were moved closer together, indicated that the quantities were (now) either "less" or "equivalent" to the row of bottles, indicating lack of one-to-one correspondence and the perceptions that the notion of equivalence between two sets is not lasting (i.e., equivalence depends on other factors). At Stage II (age 5–6), children were able to recognize one-to-one correspondence (assigning bottles to glasses, one by one); however, at this stage, they continued to reason about quantities based on their global appearance. For example, after the glasses were pulled into one group together, children were asked "Where are there more?" to which they responded, "There are more where its bigger," therefore suggesting that, given one-to-one correspondence, the equivalence remains not lasting. At Stage III (age 6–7), all children in the experiments demonstrated understanding of both one-to-one correspondence and lasting equivalence. These stages were evident and consistent across all *cardinal and ordinal one-to-one correspondence* experiments conducted with children (see experiments with flowers and vases, eggs and egg cups, and one-to-one exchange of pennies for objects).

Second, we draw attention to an additional example, from a series of experiments in *additive composition of classes within the relation between class and number* domain, in which children were asked to divide quantity (18) into two equal parts (division of whole numbers without remainders). At Stage I (age 5–6), children employed various strategies, all of which indicated that, at this stage, children have neither yet developed counting skills and cardinality nor one-to-one correspondence. For example, some put a hand over the pile and made a rough division into two parts; others took the counters one by one and separated them into two piles. However, children were unable to determine if the two piles were of equal quantity and made inferences based on the density of the piles/heaps rather than quantity. At Stage II (age 6–7), children separated quantities into two groups and formed subsets or familiar shapes with the counters (in each group) to help compare them. For example, one student arranged the counters (in each group) in a row of pairs and quickly realized that he made a mistake (10 and 8). He spaced out the pairs of the row of 8 so that it was the same length as the row of 10, but seeing the difference in density, he took 1 counter from the 10 and added it to the 8 to make the groups equal. Piaget (1952) classified this as cardinality and one-to-one correspondence, including the evident skills for children to be able to compare and equalize two unequal sets of quantities, yet, still, without lasting equivalence or conservations of the whole (Piaget 1952, p. 196). At Stage III (age 7–8), all children were able to take 1 or more counters at a time, put them into two sets of 9, and were confident that the 2 sets were equal. A common response from children, at this stage, was "they are equal, because I put the same amount in each group" regardless of the density of the groups.

Piaget indicated that *cardinal and ordinal one-to-one correspondence* experiments conducted with children demonstrate their progress toward enumerations, whereas experiments related to *additive composition of classes within the relation between class and number* provide evidence about progress toward addition. The author argued that *enumeration* and *addition* are mutually dependent; however, they are not the same. For example, if a child counts ninepins in a set, one after another, each time saying "one, one, one . . .," it is not addition, since there is no clear awareness of the sum. There is merely awareness of a succession of events, and "naturally, the idea of quantity is present in it, but this quantification is not yet numerical, since 'one' and 'another' are neither units of number nor elements of classes" (Piaget 1952, p. 199). The author further argues that the reason why primitive enumeration (e.g., one, another) does not give rise to addition is because it does not lead to a stable totality. Similarly, the reason why primitive addition does not give rise to numerical sets that are categories (i.e., *colligation*) is because additive numeration is lacking.

While it is clear that neither process is sufficient by itself, the early signs of enumeration and addition (according to Piaget) become visible when, in comparing configurations, the child is able to recognize the resemblance between the details and the quantity as a whole. It is precisely this resemblance that helps the child to develop *one-to-one correspondence*. Furthermore, if a child is able to provide an intuitive colligation, while the enumeration of the elements takes the form of seriations, based on their positions or other (perceived) qualities, then this intuitive synthesis (of enumeration becoming seriation and addition becoming intuitive composition) indicates a definite progress toward *additive composition*. However, at this level, *operational addition* yet does not exist. It is at the final stage of development (Stage III) where the synthesis between enumeration and colligation becomes lasting: both become operational and independent of the perceived figures or qualities, and the child is able to count the elements of a set and understand "that the position of each term in the series is defined in relation to the set of seriated elements, the set constituting an invariant whole" (Piaget 1952, p. 200). Thus, the child shows evidence for development of serial addition, addition of classes, and of numerical addition.

Non-Piagetian Views: Cognitively Guided Instruction

Although Piaget was neither directly concerned with children's learning of mathematics nor mathematical instruction, many studies have drawn on his work to seek to establish relationships between child development and mathematics teaching practices (e.g., Carpenter et al. 1988; Peterson et al. 1991; Putnam et al. 1990). The broader aim was to develop a comprehensive framework for examining and advancing child cognition, particularly because Piagetian experiments provided guidance for mathematics achievement but did very little to enhance children's development of mathematical concepts (Young-Loveridge 1987; Aubrey 1993). This work was particularly needed due to studies repeatedly documenting that young children are capable of engaging in mathematical activity and abstract thought reaching far beyond concrete experiences (see Sarama et al. 2017). For example, Aubrey (1993) reported that conceptions such as sorting, matching, classifying, joining and separating of sets, counting and ordering, recognizing and writing numbers 0–10, and demonstrating mathematic relationships through the use of concrete object, including topics of measurement, geometry, and pictorial representations, were all found to be part of early childhood (ages 4–5) development and cognition. The authors noted, "Whilst they may not possess the formal conventions for representing it, children clearly enter school having acquired already much of this mathematical content" (Aubrey 1993, p. 32). As a result, projects like *Cognitively Guided Instruction* (CGI) have been launched and sponsored, focusing on children's cognition across the topics of enumeration and whole-number operations, with a parallel goal of supporting teachers' instructional practices and professional development (Carpenter et al. 1989, 2000; Fennema et al. 1996).

In contrast to Piaget, the CGI project focused on children's developmental stages of cognition, comprehension, and language development across different types of mathematical (story) problems within each operation: addition, subtraction, multiplication, and division (see Carpenter et al. 1999). In fact, when designing addition tasks, three mathematical structures were identified, where one of the addends or the sum was unknown. For example, the *join/addition problems* involved *result unknown* (e.g., 3 + 7 =?), *change unknown* (e.g., 3 +? = 10), and *initial unknown* (e.g.,? + 3 = 10). The *initial unknown* addition task involved specific mathematical language and structure: *Sally had some rocks. John gave her three more rocks. Now she has ten rocks. How many rocks did Sally have to start with?*

For subtraction, the same three structures were used, where the minuend, subtrahend, or difference was unknown. The project considered other addition and subtraction problem types, including *part-part-whole* problems, where a part or a whole was unknown (e.g., *Sally has ten rocks, of which three are red and the rest are blue. How many blue rocks does Sally have?*), and *compares* problems, where the difference, quantity, or referent was unknown

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(e.g., *Sally has ten rocks. She has three more rocks than John. How many rocks does John have?*). Note that children (ages 5–8) found the *initial unknown*-type story problems (for both addition and subtraction) particularly difficult, in comparison with the other problem types (result unknown and change unknown), because these problems were not easy to model and "act out."

For multiplication and division, several problem types were also identified, including *equal groups* multiplication problems (e.g., groups of objects, total price of items, rates) and two types of division problems based on the *measurement* (number of groups unknown) and *partitive* (size of the group) models of division. After the problem types were identified, the project conducted numerous clinical interviews with children (ages 5–10), to identify their developmental stages, problem-solving strategies, and levels of thinking for the CGI problems.

Three levels of mathematical thinking were found to be most prevalent among children for solving (CGI) problems. Level I, *direct modeling*, involved children representing each number in the problem with concrete objects. Children used various strategies, including (but not limited to) creating two sets of objects and *joining all* of the objects together by counting them (using manipulatives or drawings); *separating from* the total number of objects the minuend and then counting the remaining objects; and *matching* the objects in two sets, one to one until one set is used up, and then counting the number of unmatched objects remaining in the larger set. Level I, primarily, involves children counting and using one-to-one correspondence and cardinality (Carpenter and Fennema 1992).

At Level II, however, children were no longer in need of representing all the quantities in the problem concretely. They were able to keep track of one quantity in the problem, by either stating it (rather than representing it concretely) or keeping it in mind, while performing the operation. At this level, children were not only able to count but also were able to make sense and make use of different counting strategies, including (but not limited to) *doubles plus/minus one*, *counting on/back* from the first number, and *counting on/back* from the larger number.

Similarly, at Level III, children were using strategies; however, their strategies mirrored arithmetic rather than counting strategies. For example, some children decomposed the addend to use a *nine plus one* strategy (to make ten) and then *added on* the remaining amount from the decomposed addend (e.g., 9 plus 4 is 13 because 9 and 1 is 10 and 3 more is 13). Children were also able to use *mental math* strategies, deriving facts and/or combining familiar quantities when the "math fact" was not at the recall level. Level III suggests that children understand the relationships between numbers, their sets, and subsets.

One of the unique characteristics of the CGI framework is that children's cognitive levels of thinking, even though hierarchical, are not age-specific. For example, very young children (ages 6–7) can solve low-number multiplication and division problems at Level I and, at the same time, solve low-number addition and subtraction problems (e.g., result unknown) at Level II or Level III. Thus, children's levels of thinking vary depending on the problem type, the operation, and the numbers involved in the task (Carpenter et al. 1989, 2017; Carpenter and Fennema 1992; Sarama and Clements 2009; Shumway and Pace 2017).

Research shows that engaging students in mathematical learning through (story-based) problem-solving not only develops their mathematical concepts and skills but also improves their reading and comprehension (e.g., Charles 2011; Fang and Schleppegrell 2010; Shanahan and Shanahan 2014; Sherman and Gabriel 2017). For example, Sherman and Gabriel (2017) argued that when students engaged in story-based mathematical problem-solving, they were required to articulate their thinking and work within a common language, including talking and writing about their processes (p. 474).

Furthermore, extensive research is currently emerging on early mathematical skill development, including enumeration and problem-solving, being strongly associated with students' mathematics achievement in later grades (up to age 15). Recent studies also found that whole-number knowledge in the first grade is a strong predictor of students' both fraction conceptual understanding and fraction arithmetic skill in seventh and eighth grades (Bailey et al. 2014; also see Byrnes and Wasik 2009; Claessens et al. 2009; Duncan et al. 2007; Jordan et al. 2009).

For example, in their recent analyses of the US data (between 1991 and 2002) from the National Institute of Child Health and Human Development Study of Early Child Care and Youth Development, Watts et al. (2014) concluded:

> We found that preschool mathematics ability predicts mathematics achievement through age 15, even after accounting for early reading, cognitive skills, and family [SES] and child characteristics. Moreover, we found that growth in mathematical ability between age 54 months and first grade is an even stronger predictor of adolescent mathematics achievement. These results demonstrate the importance of prekindergarten mathematics knowledge and early math learning for later achievement. (p. 352).

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Mathematical Games in Learning and Teaching

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Keywords
Games · Computer games · Visualization · Motivation · Programming · Learner-centered teaching

Definition

Literature examining the contribution of mathematical games in the learning and teaching of mathematics.

Characteristics

Piaget, Bruner, and Dienes suggest that games have a very important part to play in the learning of mathematics (Ernest 1986). In the last four decades, games have been proposed by a number of researchers as a potential learning tool in the mathematics classroom, and there are quite a few researchers who make claims about their efficacy in the learning and teaching of mathematics (e.g., Ernest 1986; Gee 2007; Kafai 1995). Some authors take a step further; Papert (1980) was among the first who suggested that students could learn mathematics effectively not only by playing (video) games but also by designing their own computer games, using, for instance, authoring programming tools like Scratch and ToonTalk (Kafai 1995; Mousoulides and Philippou 2005).

By synthesizing definitions by Harvey and Bright (1985, p. ii) and Oldfield (1991, p. 41), a task or activity can be defined as a pedagogical appropriate mathematical game when it meets the following criteria:

Has specific mathematical cognitive objectives.
Students use mathematical knowledge to achieve content-specific goals and outcomes in order to win the game.
Is enjoyable and with potential to engage students.
Is governed by a definite set of rules and has a clear underlying structure.
Involves a challenge against either a task or an opponent(s) and interactivity between opponents.
Includes elements of knowledge, skills, strategy, and luck.
Has a specific objective and a distinct finishing point.

While mathematical games have been the core of discussion of researchers since the late 1960s (e.g., Gardner 1970), the inclusion of games for the teaching and learning of school mathematics, among other subject areas, has been in the core of discussion in the 1990s (Provenzo 1991). An example of this perspective appears in Lim-Teo's (1991) work, who claimed that "there is certainly a place for games in the teaching of Mathematics . . . teacher to creatively modify and use games to enhance the effective teaching of Mathematics" (p. 53). At the same time, Ernest (1986) raised a question that is still cutting: "Can mathematics be taught effectively by using games?" (p. 3).

The answer to Ernest's question is not easy yet straightforward. The main pedagogical aim of using games in mathematics classrooms is to enhance the learning and teaching of mathematics through developing students' mathematical knowledge, including spatial reasoning, mathematical abstraction, higher-level thinking, decision-making, and problem-solving (Ernest 1986; Bragg 2012). Further, mathematical games help the teaching and learning of mathematics through the advantage of providing meaningful situations to students and by increasing learning (independent and at different levels) through rich interaction between players. There are positive results, suggesting that the appropriate mathematics games might improve mathematics achievement. A meta-analysis conducted by Vogel et al. (2006) concluded that mathematical games appear to be more effective than other instructional approaches on students' cognitive developments. The positive impact of mathematical games is further enhanced by technology. Digital mathematical games provide, for instance, a powerful environment for visualization of difficult mathematical concepts, linkage between different representations, and direct manipulation of mathematical objects (Presmeg 2006). However, Vogel et al. (2006), among others, exemplify that the positive relation between mathematics games and higher achievement is not the case in all studies that have been conducted in the field.

Games for learning mathematics are also beneficial for a number of other, frequently cited, arguments, including benefits like students' motivation, active engagement and discussion (Skemp 1993), improved attitudes toward mathematics and social skills, learning and understanding of complex problem-solving, and collaboration and teamwork among learners (Kaptelin and Cole 2002). Among these benefits of using mathematical games, the most cited one is active engagement. Papert (1980) expressed the opinion that learning happens best when students are engaged in demanding and challenging activities. In line with Papert, Ernest (1986) claimed that the nature of games demands children's active involvement, "making them more receptive to learning, and of course increasing their motivation" (p. 3). Various studies in both digital and non-digital mathematical games have shown that students are highly engaged with working in a game environment and that this milieu creates an appropriate venue for teaching and learning mathematics (e.g., Devlin 2011).

Research has highlighted various factors that should be taken into consideration as to acknowledge mathematical games as an appropriate and successful vehicle for the learning and teaching of mathematics. Games should not be faced in isolation of broader mathematical programs and approaches. Clear instructional objectives and pedagogies have to accompany the use of games, while at the same time these pedagogies should consider peer interaction, teacher-facilitator role, the access to and the use of technological tools, and the use of rich problem-solving contexts.

Cross-References

- ▶ Learner-Centered Teaching in Mathematics Education
- ▶ Motivation in Mathematics Learning
- ▶ Technology and Curricula in Mathematics Education
- ▶ Technology Design in Mathematics Education
- ▶ Visualization and Learning in Mathematics Education

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Mathematical Language

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Keywords

Algebraic notation · Communication · Genre · Language · Mathematical vocabulary · Multimodality · Objectification · Register · Representations · Semiotic systems

Introduction: What Is Mathematical Language?

Specialized domains of activity generally have their own specialized vocabularies and ways of speaking and writing; consider, for example, the language used in the practices of law or computer science, fishing, or football. The specialized language enables participants to communicate efficiently about the objects peculiar to their practice and to get things done, though it may simultaneously serve to exclude other people who are not specialists in the domain. This is certainly the case for the specialized activity of mathematics: While some aspects of mathematical language, such as its high degree of abstraction, may be an obstacle to participation for some people, doing mathematics is highly dependent on using its specialized forms of language, not only to communicate with others but even to generate new mathematics. In making this claim, we need to be clearer about what mathematical language is.

For some, the language of mathematics is identified with its systems of formal notation. Certainly, like other languages, these systems include a "vocabulary" of symbols and grammatical rules governing the construction and manipulation of well-formed statements. A significant part of mathematical activity and communication can be achieved by forming and transforming sequences of such formal statements. In recent years, however, it has been widely recognized that not only other semiotic systems, including what is sometimes called "natural" language, but also specialized visual forms such as Cartesian graphs or geometric diagrams play an equally essential role in the doing and communicating of mathematics. This recognition has been strongly influenced by the work of the linguist Halliday and his notion of specialized languages or *registers* (Halliday 1974), by research applying and developing theories of semiotics in mathematics and mathematics education, and by more recent developments in multimodal semiotics that address the roles of multiple modes of communication (including gestures and the dynamic visual interactions afforded by new technologies). In this entry, it is not possible to provide a full characterization of all these aspects of mathematical language; in what follows, some of the most significant characteristics will be discussed.

Characteristics of Mathematical Language

The most easily recognized aspect of the "natural" or verbal language component of mathematical language is the special vocabulary used to name mathematical objects and processes. This vocabulary was the focus of much of the early research conducted into language in mathematics education (see Austin and Howson 1979 for an overview of this research). This vocabulary includes not only some uniquely mathematical words (such as *hypotenuse*, *trigonometry*, and *parallelogram*) but, in addition, many words that are also used in everyday language, often with subtly different meanings. In English, words such as *prime*, *similar*, *multiply*, and *differentiate* originated in non-mathematical contexts and, in being adopted for mathematical use, have acquired new, more restrictive or precise definitions. The difficulties that learners may have in using such words in appropriately mathematical ways have been a focus of research; David Pimm's seminal book "Speaking Mathematically: Communication in Mathematics Classrooms" (Pimm 1987) provides a useful discussion of issues arising from this aspect of mathematical vocabulary. In national languages other than English, the specific relationships between mathematical and everyday vocabularies may vary, but similar issues for learners remain.

Another characteristic of mathematical vocabulary is the development of dense groups of words such as *lowest common denominator* or *topological vector space* or *integrate with respect to x*. Such expressions need to be understood as single units; understanding each word individually may not be sufficient. The formation of such lengthy locutions serves to pack large quantities of information into manageable units that may then be combined into statements with relatively simple grammatical structure. To consider a relatively simple example: if we wished to avoid using the complex locution *lowest common denominator*, the simple statement.

The lowest common denominator of these three fractions is 12. would need to be unpacked into a grammatically more complex statement such as.

If we find fractions with different denominators equivalent to each of these three fractions, the lowest number that can be a denominator for all three of them is 12.

The condensation of information achieved by complex locutions makes it possible to handle complex concepts in relatively simple ways. This is not unique to mathematics but is also a feature of the language of other scientific domains (Halliday and Martin 1993).

A further characteristic with a significant function in mathematics is the transformation of processes into objects; linguistically this is achieved by forming a noun (such as *rotation* or *equation*) out of a verb (*rotate* or *equate*). Like many of the special characteristics of mathematical language, this serves at least two functions that we may think of as relating to the nature of mathematical activity and to the ways in which human beings may relate to mathematics. In this case, by forming objects out of processes, the actors in the processes are obscured, contributing to an apparent absence of human agency in mathematical discourse. At the same time, however, changing processes (verbs) into objects (nouns) contributes to the construction of new mathematical objects that encapsulate the processes; the ability to think about ideas such as *function* both as a process and as an object that can itself be subject to other processes (e.g., addition or differentiation) is an important aspect of thinking mathematically. Sfard (2008) refers to these characteristics of mathematical language as *objectification* and *reification*, arguing that they both contribute to alienation – the distancing of human beings from mathematics. It is possible that alienation contributes to learners' difficulties in seeing themselves as potential active participants in mathematics. However, it is important to remember that many of the characteristics of mathematical language that seem to cause difficulties for learners are not arbitrary complexities but have important roles in enabling mathematical activity. Indeed, in Sfard's communicative theory of mathematical thinking, she makes no distinction between communicating and thinking: Thinking and doing mathematics are identified with participating in mathematical discourse, that is, communicating mathematically with others or with oneself.

Variations in Language and Thinking Mathematically

Considering the relationship between language and thinking mathematically or doing mathematics also raises questions about the possible effects of using different national languages, especially those that do not share the structures and assumptions of the European languages that have dominated the development of modern academic mathematics. Even relatively simple linguistic differences, such as the ways in which number words are structured, have been argued to make a difference to children's learning of mathematics. Barton (2008) suggests that more substantial linguistic differences such as those found in some indigenous American or Australasian languages are related to different ways of thinking about the world that have the potential to lead to new forms of mathematics.

In focusing on features of verbal language, it is important not to forget the roles played by other semiotic systems in the doing and development of mathematics. A prime example to consider is the way in which Descartes' algebraization of geometry has transformed the development of the field. A powerful characteristic of algebraic notation is that it can be manipulated according to formal rules in order to form new statements that provide new insights and knowledge. In contrast, graphical forms tend not to allow this kind of manipulation, though they may instead enable a more holistic or dynamic comprehension of the objects represented. The different affordances for communication of verbal, algebraic, and graphical modes, analyzed in detail by O'Halloran (2005), mean that, even when dealing with the "same" mathematical object, different modes of communication will enable different kinds of messages. Consider, for example, which aspects you focus on and what actions you may perform when presented with a function expressed in verbal, algebraic, tabular, or graphical form.

Duval (2006) has argued that the differences between the affordances of different modes (which he calls registers) have an important consequence for learning: Converting from one mode to another (e.g., drawing the graph of a function given in algebraic form or determining the algebraic equation for a given graph) entails understanding and coordinating the mathematical structures of both modes and is hence an important activity for cognitive development. The design of environments involving making connections between different forms of representation has been a focus of researchers working with new technologies in mathematics education.

By speaking of mathematical language, as we have so far in this entry, it might seem that there is only one variety of mathematical language that has identical characteristics in all circumstances. This is clearly not the case; young children studying mathematics in the early years of schooling encounter and use specialized language in forms that are obviously different from the language of academic mathematicians. Even among academic mathematicians writing research papers, Burton and Morgan (2000) identified variation in the linguistic characteristics of publications, possibly relating to such variables as the status of the writers as well as to the specific field of mathematics. Researchers using discourse analytic approaches have studied the language used in a number of specific mathematical and mathematics education contexts. One way of thinking about the variation found across contexts is suggested by Mousley and Marks (1991): Different kinds of purpose in communicating mathematically demand the use of different forms of language or *genres*. Thus, for example, recounting what has been done in order to solve a problem will use language with different characteristics from that required in order to present a rigorous proof of a theorem. It may be that mathematical language should be thought of in terms of a cluster of forms of language with a family resemblance, differing in the extent to which they use the characteristics identified in this entry but sharing enough specialized features to enable us to recognize them all as mathematical. An important implication of recognizing the contextual variation in mathematical language is that research into the role of language in teaching and learning mathematics needs to be sensitive to the specificity of the practice being studied and cautious in its generalizations.

Cross-References

▶ Bilingual/Multilingual Issues in Learning Mathematics
▶ Discourse Analytic Approaches in Mathematics Education
▶ Language Background in Mathematics Education
▶ Mathematical Language
▶ Semiotics in Mathematics Education

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Mathematical Learning Difficulties and Dyscalculia

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Keywords
(Developmental) Dyscalculia (DD) · Mathematical Learning Difficulty/Disability/Disorder (MLD) · Specific Learning Disorder (SLD) · Special educational needs

Introduction

Terms such as "Mathematical Learning Disability," "Developmental Dyscalculia (DD)," but also "Mathematical Learning Disorder" and "Mathematical Learning Difficulty"[1] are originated in the field of cognitive psychology in order to investigate the development of basic number processing (e.g., Passolunghi and Siegel 2004; Rousselle and Noël 2007; Piazza et al. 2010). These terms are introduced referring to atypical situations, defined as a presence of various *cognitive deficits* in a student's processing of numerical information that lead to persistent and pervasive difficulties with mathematics (e.g., Butterworth and Reigosa-Crespo 2007; Shalev 2007; Geary 2010). In early studies, these cognitive deficits were inferred by low performance, for example, in his entry in this Encyclopedia, Jeremy Kilpatrick – recalling that the term dyscalculia (Rechenschwache) was introduced in Budapest in 1916 by Paul Ranschburg – underlines how the new term was coined during Ranschburg's study of differences in calculation performance between *normal* children and low achievers in arithmetic.

In the clinical context, where these situations are diagnosed, "Mathematics Disorder" (MD) was introduced as one of the "Learning Disorders." In particular, in the fourth version of the *Diagnostic and Statistical Manual of Mental Disorders* (DSM-IV) (American Psychiatry Association 1994), MD is identified and diagnosed using discrepancy criteria: "[in the case of Mathematical Disorder] ability, as measured by individually administered standardized tests, is substantially below that expected given the person's chronological age, measured intelligence, and age-appropriate education" (ibid., Sect. 315.1). The more recent fifth version of the *Diagnostic and Statistical Manual of Mental Disorders* (2013) takes a more holistic approach. In particular, a "Specific Learning Disorder" (SLD) is described as a developmental disorder that begins by school age, but that may not be recognized until later; it involves ongoing problems in learning key academic skills, including reading, writing, and math, that provide the foundations for other academic subjects. The manual also highlights the consequences of the nontreatment of a SLD: it can potentially cause problems throughout a person's life, including lower academic achievement, lower self-esteem, higher rates of dropping out of school, higher psychological distress, and poor overall mental health, as well as higher rates of unemployment/underemployment.

The issues of diagnosis of a Mathematical Learning Disorder and instruction for the students with a positive diagnosis are getting increasing research attention; however research in this area is still lagging behind compared with other academic subjects such as reading (Verschaffel et al. 2018). In the fields of psychology and neuroscience, there is still lack of consensus on how to identify the central characteristics of MLD or even on what these are (Szücs 2016). Indeed, some definitions refer to a biologically based disorder, others to the discrepancy between mathematical achievement and general intelligence, and others yet focus on the response to intervention. Consensus is also lacking about the comorbidity and heterogeneity of the populations supposedly affected with MLD (Bartelet et al. 2014; Szücs and Goswami 2013; Watson and Gable 2013).

Generally, the clinical context lacks attention toward the important theoretical perspectives that should guide any form of educational support aimed at prevention or remediation of MLD.

In the following sections, we will introduce the main perspectives, other than the purely cognitive ones, taken in mathematics education to study MLD, focusing in particular on findings on prevention and remediation. We will conclude with considerations on the possibility of fostering more constructive collaboration across the research communities studying MLD.

The Mathematics Education Perspectives on MLD: The Issues of Prevention and Remediation

Recently, Lewis and Fischer (2016) carried out and published a review of 164 studies on MLD of a 40-year period. The review, appeared in the

[1] In the literature, the acronym MLD has several meanings: the "D" may refer to any of at least three nouns (Disability, Disorder, Difficulty); here it refers to "Difficulty."

Journal for Research in Mathematics Education, systematically analyzes the methodological criteria used to identify MLD, highlighting three main findings: there was great variability in the classification methods used; studies rarely reported demographic differences between the MLD and typically achieving groups; studies overwhelmingly focused on elementary-aged students engaged in basic arithmetic calculation. From an educational perspective, the authors argue for the necessity of standards for methodology and reporting. Lewis and Fischer, in agreement with the view of other researchers in the field, have argued that not only arithmetic but also other more complex and equally important mathematical domains (algebra, geometry, calculus, etc.) and forms of reasoning should be taken into account when studying MLD, such as spatial and geometrical reasoning, mathematical relations and patterns, and other forms of mathematical thinking with more potential toward abstraction and generalization (e.g., Hord and Xin 2015; Mulligan 2011). Moreover, consistently with the picture in the introduction, Lewis and Fisher underline how too little is yet known about the contributing factors of MLD, which, for example, are likely to include not only cognitive but also emotional and social factors.

The complexity of this scenario also emerged clearly during a panel on "Special Needs in Research and Instruction in Whole Number Arithmetic" at ICMI Study 23 on whole numbers in the primary grades (Verschaffel et al. 2018). The panel explored and discussed many open issues and challenges, with a strong emphasis on the instructional goals and interventions for children (in primary school) with MLD.

While acknowledging the importance of the purely cognitive perspectives advanced in psychology and neuroscience, in this section we focus on more sociocultural perspectives of Vygotskian inspiration that have been taken on MLD.

A solid lens through which "low achievement" and "failure" in mathematics have been observed and analyzed is the "Commognition" (▶ "Commognition"), according to which mathematics is a form of communication and learning mathematics is developing this special discourse (Sfard 2008). Within this frame, "disability" (or "learning difficulty" due to a "cognitive deficit" as seen from other perspectives) is reconceptualized in terms of persistent failure to participate in canonic mathematical discourse and failure to cope with meta-level learning (Sfard 2008, 2017; Heyd-Metzuyanim 2013). Within this perspective, all students experience difficulty whenever a transition is to be made to mathematical discourse governed by rules different from those with which the student is familiar. The need for such meta-level learning appears whenever a new type of mathematical object, e.g. a new type of number, is introduced; in most cases, however, neither teachers nor students are aware of the required meta-level change. The tacitness of this change is one reason why the necessary transition is difficult to make. Another challenge comes from the paradoxical nature of the situation, in which in order to construct a new mathematical object the student must already participate in the discourse on this object. While the resulting difficulty is inevitable and universal, students differ in their their readiness and ability to cope. If the difficulty remains unresolved, it is often because of emotional, social and educational factors rather than of cognitive ones. Indeed, messages about the students' identities, coming from teachers, peers and the learners themselves may be a critical factor in these students' approach to the difficulty and in their readiness to grapple with it. By translating one's actions into properties of the actor, identities extend a local, potentially only temporary lack of success into a universal, permanent "disability". Those labeled as 'having MLD' are only too likely give up any genuine attempt to participate in the canonic mathematical discourse; if they ever 'talk mathematics', they will feel that they are merely 'parroting' the teacher (Heyd-Metzuyanim and Sfard 2012; Heyd-Metzuyanim 2013; Heyd-Metzuyanim, et al. 2016; Lewis 2017). This perspective is coherent with the solid finding in mathematics education related to the need to go beyond a purely cognitive interpretation of students' difficulties (Schoenfeld 1983).

Moreover, Vygotsky's work with disabled learners has inspired a significant branch of research on "remediation." The underlying idea he advanced is that instead of associating disability with deficit, it is preferable to adopt a qualitative perspective to research how access to different mediating resources impacts upon development. This perspective has been successfully used both in the context of visual impairments (Healy and Fernandes 2014) and in that of remediation of a MLD. Specifically, a study by Lewis (2017) illustrates the potential utility of a bridging discourse to help students who have a history of failure gain access to the canonical mathematics discourse and content. Lewis' work draws on Vygotsky's framing of disability and uses Sfard's conceptualization of mathematics as a discourse to design a fraction remediation. The methodology used involved a fine-grained analysis of the remediation sessions, which lead to tracing out the ways in which the student's discourse shifted over time, enabling her to solve problems she had previously been unable to solve.

Another study framed within a Vygotskian lens was the Italian PerContare project (Baccaglini-Frank 2017). Here, teaching strategies and activities were developed for 1st and 2nd grades with the aim of preventing and addressing early low achievement in arithmetic using appropriately designed artifacts, grounded upon a kinesthetic and visual-spatial approach to part-whole relationships. Findings of a longitudinal study, involving ten experimental classes and ten control classes, were that the percentage of students in experimental classes who scored below the cutoff on a standard diagnostic battery used in Italy was about half of that of the children in the control classes (7% vs. 13%). Moreover, on a separate test on topics in arithmetic, the children of the experimental classes showed a greater variety of strategies when carrying out calculations, and many fewer omissions in their answers, compared to the students in the control classes (Baccaglini-Frank 2015; Verschaffel et al. 2018). Such results suggest that a careful design and implementation of teaching materials can have a significant effect on the population of students testing positive to MLD, which further shakes the fragile grounds of defining and diagnosing MLD.

Communicating Across Fields

We believe that research on MLD and DD – in particular regarding the possibilities in terms of prevention and remediation – would highly benefit from a constructive dialogue between neighboring fields. Indeed, attempts should be made in neighboring disciplines to reinterpret methodologies and findings from studies stemming from mathematics education perspectives like the ones described; this could lead to insights both at an applied level (design of material for prevention and remediation of MLD) and at a theoretical level (conceptualization of MLD or DD). Vice versa, we highly recommend staying open to perspectives from cognitive psychology and neuroscience, attempting to reinterpret and reinvest key findings.

Within this direction, the recent study by Karagiannakis et al. (2014) reorganized the main hypothesis advanced in the cognitive psychology and neuroscience fields into four domains (core number, memory, reasoning, visual-spatial) with the aim of developing a theoretical model for defining and studying "mathematical learning profiles." The developed model suggests a transition from the one-dimensional approach to dyscalculia to the four-dimensional construct of Mathematical Learning Difficulties. On one hand this transition is in line with the shift of focus we highlighted in the DSM-5; on the other hand, it brings into the picture mathematical domains other than the ones typically considered by the MLD literature until today. Based on such model, an experimental computer-based battery of mathematical tasks was designed to elicit abilities from each domain, and it was administered to a sample of 165 typical population 5th and 6th grade students. Results from explanatory and confirmatory factor analysis indicated strong evidence for supporting the solidity of the model. Moreover, K-means cluster analysis leads to identification of six performance groups with distinct characteristics, supporting the recurrent finding that the population of students labeled as MLD is quite heterogeneous (Karagiannakis et al. 2017). The model is expected to have direct implications for the field of mathematics education, because it shall allow to identify cognitive characteristics (either intrinsic or culturally developed) of mathematical profiles of students; these can be used to design more effective and comprehensive intervention programs, focusing on the students' strengths to compensate weaknesses and provide motivation.

Cross-References

- ▶ Autism, Special Needs, and Mathematics Learning
- ▶ Commognition
- ▶ 22q11.2 Deletion Syndrome, Special Needs, and Mathematics Learning
- ▶ History of Research in Mathematics Education
- ▶ Learning Difficulties, Special Needs, and Mathematics Learning
- ▶ Mathematical Ability
- ▶ Students' Attitude in Mathematics Education

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Mathematical Literacy

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Keywords
Numeracy · Quantitative literacy · Critical mathematical literacy · Mathemacy · Matheracy · Statistical literacy

Definition

The neologism "mathematical literacy" belongs to an array of related terms that have been used in English language mathematics education research and policy discourses in the context of suggestions for the improvement of mathematics teaching and learning. While diagnosis of some apparent shortcomings seems to coexist with formal mathematics education since its inception in the USA, "mathematical literacy" is linked to the reform narratives of 1980s (Craig 2018). One of the first written occurrences of the term in the USA was in 1944, when a Commission of the National Council of Teachers of Mathematics (NCTM) on Post-War Plans (NCTM 1970/2002, p. 244) required that the school should ensure mathematical literacy for all who can possibly achieve it. Shortly after (in 1950), the term was used again in the Canadian Hope Report (NCTM 1970/2002, p. 401). In more recent times, the NCTM 1989 Standards (NCTM 1989, p. 5) in the USA spoke about mathematical literacy and mathematically literate students. Apparently, no definition of the term was offered in any of these texts. The 1989 Standards did, however, put forward five general goals serving the pursuit of mathematical literacy for all students: "(1) That they learn to value mathematics, (2) that they become confident with their ability to do mathematics, (3) that they become mathematical problem solvers, (4) that they learn to communicate mathematically, and (5) that they learn to reason mathematically" (op. cit., p. 5).

In the context of international comparisons, the IEA's Third International Mathematics and Science Study (TIMSS), first conducted in 1995, administered a mathematics and science literacy test to students in their final year of secondary school in 21 countries that aimed "to provide information about how prepared the overall population of school leavers in each country is to apply knowledge in mathematics and science to meet the challenges of life beyond school." The first attempt at an explicit definition appears to be found in the initial OECD framework for PISA (Programme for International Student Assessment) in 1999 (OECD 1999). The definition has been slightly altered a number of times for subsequent PISA cycles (for the evolution of the mathematics framework over the years, see Stacey and Turner 2015). The version for PISA 2015 reads (OECD 2016, p. 65) as follows:

> Mathematical literacy is an individual's capacity to formulate, employ and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts,

> procedures, facts, and tools to describe, explain and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make well-founded judgments and decisions needed by constructive, engaged and reflective citizens.

In mathematics education research and policy texts, one finds an array of related terms, such as "numeracy," "quantitative literacy," "critical mathematical literacy," "mathemacy," "matheracy," and "statistical literacy." While some of these notions more clearly differ in extension and intension, some authors use "numeracy," "quantitative literacy," and "mathematical literacy" synonymously, whereas others distinguish also between these. While the term "mathematical literacy" appears to be of US descent, the term "numeracy" was coined in the UK, although the neologism "innumeracy" spread through a popular science publication in the USA (Paulos 1989). According to Brown et al. (1998, p. 363), "numeracy" appeared for the first time in the so-called Crowther Report in 1959, meaning scientific literacy in a broad sense, and later obtained wide dissemination through the Cockcroft Report (DES/WO 1982), which stated that its meaning had considerably narrowed by then. There have been further shifts in interpretation since then. A recent, rather wide, definition of "numeracy" can be found in OECD's PIAAC (Programme for the International Assessment of Adult Competencies) "numeracy" framework: "Numeracy is the knowledge and skills required to effectively manage and respond to the mathematical demands of diverse situations" (PIAAC Numeracy Expert Group 2009, p. 20).

The term "quantitative literacy" is yet another term of US descent, going back to the work of Steen (e.g., Madison and Steen 2003). As to countries where English is an official language, Geiger et al. (2015) observe that "numeracy" is still more commonly used in the UK, Canada, South Africa, Australia, and New Zealand, while in the USA, "mathematical literacy" appears to be the privileged term. In South Africa, the pursuit of mathematical literacy has motivated the introduction of a new stand-alone school mathematics subject area available for learners in grades 10–12, which aims at allowing "individuals to make sense of, participate in and contribute to the twenty-first century world – a world characterized by numbers, numerically based arguments and data represented and misrepresented in a number of different ways. Such competencies include the ability to reason, make decisions, solve problems, manage resources, interpret information, schedule events and use and apply technology" (DoBE 2011, p. 8). One motivation for introducing this mathematical subject was to increase student engagement with mathematics.

Characteristics and Delimitation

Even though the notions above are interpreted differently by different authors (which suggests a need to pay serious attention to clear terminology), they do have in common that they stress awareness of the usefulness of and the ability to use mathematics in a range of different areas as an important goal of mathematics education. Furthermore, these notions are associated with education for the general public rather than with specialized academic training while at the same time stressing the connection between "mathematical literacy" and democratic participation. As in other combined phrases, such as "statistical literacy" or "computer literacy," the addition of "literacy" may suggest some level of critical understanding.

While "mathematical literacy," "quantitative literacy," and "numeracy" focus on mathematics as a tool for solving nonmathematical problems, the "mathematical competence" (and "competencies") and "mathematical proficiency" focus on what it means to master mathematics at large, including the capacity to solve mathematical as well as nonmathematical problems. The notion of "mathematical proficiency" (Kilpatrick et al. 2001) is meant to capture what successful mathematics learning means for everyone and is defined indirectly through five strands (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition). Furthermore, by referring to individuals' mental capacities, dispositions, and attitudes, the last two of these strands go beyond

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mastery of mathematics and include personal characteristics.

The notion of "mathematical competence" has been developed, explored, and utilized in the Danish KOM Project (KOM is an abbreviation for "competencies and mathematics learning" in Danish) and elsewhere since the late 1990s (Niss and Højgaard 2011). Mathematical competence is an individual's capability and readiness to act appropriately, and in a knowledge-based manner, in situations and contexts in which mathematics actually plays or potentially could play a role. While mathematical competence is the overarching concept, its constituent components are, perhaps, the most important features. There are eight such constituents ("mathematical competencies"): mathematical thinking, problem posing and solving, mathematical modeling, mathematical reasoning, handling mathematical representations, dealing with symbolism and formalism, communicating mathematically, and handling mathematical aids and tools. The description of mathematical competencies does not specifically focus on learners of mathematics nor on mathematics teaching. Also, no personal characteristics such as capacities, dispositions, and attitudes are implicated in these notions.

Motivations for Introducing Mathematical Literacy

There have always been endeavors among mathematics educators to go against the idea that the learning of basic or fundamental mathematics could be characterized solely in terms of facts and rules that have to be known (by rote) and procedures that have to be mastered (by rote). Mathematics educators have found this view reductionist, since it overlooks the importance of understanding when, and under what conditions, it is feasible to activate the knowledge and skills acquired, as well as the importance of flexibility in putting mathematics to use in novel intra- or extra-mathematical contexts and situations. For example, in the first IEA study on mathematics, which later became known as the First International Mathematics Study (FIMS), published in 1967, we read that in addition to testing factual and procedural knowledge and skills related to a set of mathematical topics, it was important to also look into five "cognitive behaviors": (1) knowledge and information (recall of definitions, notations, concepts), (2) techniques and skills (solutions), (3) translation of data into symbols or schema and vice versa, (4) comprehension (capacity to analyze problems and to follow reasoning), and (5) inventiveness (reasoning creatively in mathematics (our italics)). Another example is found in the NCTM document An Agenda for Action: Recommendations for School Mathematics of the 1980s (NCTM 1980). The document is partly written in reaction to the so-called back-to-basics movement in the USA in the 1970s, which in turn was a reaction to the "new mathematics" movement in the 1950s and 1960s. The document states:

> We recognize as valid and genuine the concern expressed by many segments of society that basic skills be part of the education of every child. However, the full scope of what is basic must include those things that are essential to meaningful and productive citizenship, both immediate and future. (p. 5)

The document lists six recommendations, including:

> 2.1. The full scope of what is basic should contain at least the ten basic skill areas [. . .]. These areas are problem solving; applying mathematics in everyday situations; alertness to the reasonableness of results; estimation and approximation; appropriate computational skills; geometry; measurement; reading, interpreting, and constructing tables, charts, and graphs; using mathematics to predict; and computer literacy. (pp. 6–7)

> 2.6. The higher-order mental processes of logical reasoning, information processing, and decision making should be considered basic to the application of mathematics. Mathematics curricula and teachers should set as objectives the development of logical processes, concepts, and language [. . .]. (p. 8)

These examples show that mathematics educators have been concerned with capturing "something more" (in addition to knowledge and skills regarding mathematical concepts, terms, conventions, rules, procedures, methods, theories, and results), which resembles what is indicated by the notion of mathematical literacy as it is, for example, used in the PISA. On the one hand, the arguments for broadening the scope of school mathematics have been utility oriented, based on the observation of students' lack of ability to use their mathematical knowledge for solving problems that are contextualized in extra-mathematical contexts, in school as well as out of school, an observation corroborated by a huge body of research. On the other hand, the constitution of mathematics as a school discipline in terms of "products" – concepts (definitions and terminology), results (theorems, methods, and algorithms), and techniques (for solving sets of similar tasks) – became challenged. Product-oriented curricula were complemented by, or contrasted with, a conception of mathematics that includes mathematical processes, such as heuristics for mathematical problem solving, mathematical argumentation, constructive and critical mathematical reasoning, and communicating mathematical matters.

There are different views about the amount of mathematical knowledge and basic skills needed for engagement in everyday practices and non-mathematically specialized professions, although it has been stressed that a certain level of proficiency in mathematics is necessary for developing mathematical literacy. The role of general mathematical competencies that transcend school mathematical subareas also has been stressed in the newer versions of conceptualizing mathematical literacy, most prominently in the versions promoted by the OECD-PISA (see above).

Critique and Further Research

Even though the notion of "mathematical literacy" has gained momentum and is now widely invoked and used in various contexts, it has also attracted different sorts of conceptual and politico-educational criticism.

Some reservations against using the very term "mathematical literacy" concern the fact that it lacks counterparts in several languages. No suitable translation exists, for example, into German and Scandinavian languages, where there are only words for "illiteracy," which stands for the fundamental inability to read or write any text. Indeed, the term "literacy" (both mathematical and quantitative literacy) has been interpreted by some to connote the most basic and elementary aspects of arithmetic and mathematics, in the same way as linguistic literacy is often taken to mean the very ability to read and write, an ability that is seen to transcend the social contexts and associated values, in which reading and writing occurs. However, the demands for reading and writing substantially vary across a spectrum of texts and contexts, as do the social positions of the speakers or readers. The same is true for a range of contexts and situations in which mathematics is used. People's private, professional, social, occupational, political, and economic lives represent a multitude of different mathematical demands. So, today, for most mathematics educators, the term mathematical literacy signifies a competency far beyond a set of basic skills.

Another critique, going against attempts at capturing mathematical literacy in terms of transferable general competencies or process skills, consists in the observation that such a conception tends to ignore the interests and values involved in posing and solving particular problems by means of mathematics. Jablonka (2003) sees mathematical literacy as a socially and culturally embedded practice and argues that conceptions of mathematical literacy vary with respect to the culture and values of the stakeholders who promote it. Also, de Lange (2003) acknowledges the need to take into account cultural differences in conceptualizing mathematical literacy. There is no general agreement among mathematics educators as to the type of contexts with which a mathematically literate citizen will or should engage and to what ends. However, there is agreement that mathematical literate citizens include nonexperts and that mathematical literacy is based on knowledge that is/should be accessible to all.

In the same vein, mathematics educators have empirically and theoretically identified a variety of intentions for pursuing mathematical literacy. For example, Venkat and Graven (2007) investigated pedagogic practice and learners' experiences in the contexts of South African classrooms, in which the subject mathematical literacy is taught. They identified four different pedagogic agendas (related to different pedagogic challenges) that teachers pursued in teaching the subject. Jablonka (2003), through a review of literature, identified five agendas on which conceptions of mathematical literacy are based. These are as follows: developing human capital (exemplified by the conception used in the OECD-PISA), maintaining cultural identity, pursuing social change, creating environmental awareness, and evaluating mathematical applications. Some terms have been introduced as alternatives to "mathematical literacy" in order to make the agenda visible. Frankenstein (e.g., 2010) uses critical "mathematical numeracy," D'Ambrosio (2003) writes about "matheracy," and Skovsmose (2002) refers to "mathemacy."

Relations of mathematical literacy to scientific and technological literacy have also been discussed (e.g., Keitel et al. 1993). Challenging questions include the role of mathematics in digital technology and the implications for the development of critical competence to counterbalance the demathematizing effect of mathematics-based technologies that operate as black boxes (e.g., Gellert and Jablonka 2009). This question becomes particularly relevant if the question of interpretability is not based on the lack of expertise of the user of such a black box, but rather is a consequence of the complexity or flexibility of the underlying mathematical model (such as in the context of machine learning).

As to the role of mathematical literacy in assessment, discrepancies between actual assessment modes and the intentions of mathematical literacy have been pointed out by researchers in different contexts (Jahnke and Meyerhöfer 2007; North 2010; Jablonka 2015). In the assessment literature, the contexts in which mathematically literate individuals are meant to engage are often referred to in vague or general terms, such as the "real-world," "everyday life," "personal life," "society," and attempts to categorize contexts often lack a theoretical foundation. Identifying the demands and knowledge bases for mathematically literate behavior in different contexts remains a major research agenda.

As far as the teaching of mathematical literacy is concerned, the transition between unspecialized context-based considerations and problem solutions that employ specialized mathematical knowledge is a continuing concern. Ethnographic studies of how use of (school-)mathematical notions and techniques is made within other practices (e.g., workplaces) show that (school-) mathematics becomes subordinated to the motives or objects characteristic of these practices. Conversely, out-of-school experiences and knowledge often become a mere springboard for developing school mathematical notions and techniques. Studies of curricula associated with teaching mathematics through and for exploring everyday practices have, for example, usefully drawn on theories of knowledge recontextualization.

These observations suggest that the meanings and usages associated with the notion of mathematical literacy and its relatives have not yet reached a stage of universally accepted conceptual clarification nor of general agreement about their place and role. Future theoretical and empirical research and development are needed for that to happen.

Cross-References

- ▶ Adults Learning Mathematics
- ▶ Competency Frameworks in Mathematics Education
- ▶ Interdisciplinary Approaches in Mathematics Education
- ▶ International Comparative Studies in Mathematics: An Overview
- ▶ Mathematical Modelling and Applications in Education
- ▶ Realistic Mathematics Education
- ▶ Recontextualization in Mathematics Education
- ▶ Word Problems in Mathematics Education

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Mathematical Modelling and Applications in Education

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Keywords
Modelling examples · Modelling cycle · Modelling competencies · Metacognition · Scaffolding

Characteristics

The relevance of promoting applications and mathematical modelling in schools is currently consensus all over the world. The promotion of modelling competencies, i.e., the competencies to solve real-world problems using mathematics, is accepted as central goal for mathematics education worldwide, especially if mathematics education aims to promote responsible citizenship. In many national curricula, modelling competencies play a decisive role pointing out that the importance of mathematical modelling is accepted at a broad international level. However, beyond this consensus on the relevance of modelling, it is still disputed how to integrate mathematical modelling into the teaching and learning processes; various approaches are discussed and there is still a lack of strong empirical evidence on the effects of the integration of modelling examples into school practice.

Theoretical Debate on Mathematical Modelling: Historical Development and Current State

Applications and modelling play an important role in the teaching and learning of mathematics; already in the nineteenth century, famous mathematics educator made a strong plea for the inclusion of contextual problems in mathematics education, mainly in elementary schools for the broad majority. At the turn to the twentieth century, Felix Klein – the first president of ICMI – laid out in the so-called syllabus from Meran the necessity to include applications in modelling in mathematics education for higher achieving children in grammar schools; however, he requested a strong balance between applications and pure mathematics. During and after the Second World War, applications lost significantly importance in many parts of the world. The claim to teach mathematics in application-oriented way has been put forth another time with the famous symposium "Why to teach mathematics so as to be useful" (Freudenthal 1968; Pollak 1968) which has been carried out in 1968. Why and how to include applications and modelling in mathematics education has been the focus of many research studies since then. This high amount of studies has not led to a unique picture on the relevance of applications and modelling in mathematics education; in contrast the arguments developed since then remained quite diverse. In addition the discussion, how to teach mathematics so as to be useful did not lead to a consistent argumentation. There have been several attempts to analyze the various theoretical approaches to teach mathematical modelling and applications and to clarify possible commonalities and differences; a few are described below.

Nearly twenty years ago, Kaiser-Meßmer (1986, p. 83) discriminated in her analysis of the applications and modelling discussion of that time various perspectives, namely, the following two main streams:

- A **pragmatic perspective**, focusing on utilitarian or pragmatic goals, i.e., the ability of learners to apply mathematics for the solution of practical problems. Henry Pollak (see, e.g., 1968) can be regarded as a prototypical researcher of this perspective.
- A **scientific-humanistic perspective**, which is oriented more towards mathematics as a science and humanistic ideals of education focusing on the ability of learners to create relations between mathematics and reality. The "early" Hans Freudenthal (see, e.g., 1973) might be viewed as a prototypical researcher of this approach.

The various perspectives of the discussion vary strongly due to their aims concerning application and modelling; for example, the following goals can be discriminated (Blum 1996; Kaiser-Meßmer 1986):

- **Pedagogical goals**: imparting abilities that enable students to understand central aspects of our world in a better way
- **Psychological goals**: fostering and enhancement of the motivation and attitude of learners towards mathematics and mathematics teaching
- **Subject-related goals**: structuring of learning processes, introduction of new mathematical concepts and methods including their illustration

- **Science-related goals**: imparting a realistic image of mathematics as science, giving insight into the overlapping of mathematical and extra-mathematical considerations of the historical development of mathematics

In their extensive survey on the state of the art, Blum and Niss (1991) focus a few years later on the arguments and goals for the inclusion of applications and modelling and discriminate five layers of arguments such as the formative argument related to the promotion of general competencies, critical competence argument, utility argument, picture of mathematics argument, and the promotion of mathematics learning argument. They make a strong plea for the promotion of three goals, namely, that students should be able to perform modelling processes, to acquire knowledge of existing models, and to critically analyze given examples of modelling processes.

Based on this position, they analyze the various approaches on how to consider applications and modelling in mathematics instruction and distinguish six different types of including applications and modelling in mathematics instruction, e.g., the *separation approach*, separating mathematics, and modelling in different courses or the *two-compartment approach* with a pure part and an applied part. A continuation of integrating applications and modelling into mathematics instruction is the *islands approach*, where small applied islands can be found within the pure course; the *mixing approach* is even stronger in fostering the integration of applications and modelling, i.e., newly developed mathematical concepts and methods are activated towards applications and modelling; whenever possible, however, in contrast to the next approach, the mathematics used is more or less given from the outset. In the *mathematics curriculum-integrated approach*, the problems come first and mathematics to deal with them is sought and developed subsequently. The most advanced approach, the *interdisciplinary-integrated approach*, operates with a full integration between mathematics and extra-mathematical activities where mathematics is not organized as separate subject.

At the beginning of the twenty-first century, Kaiser and Sriraman (2006) pointed out in their classification of the historical and more recent debate on mathematical modelling in school that several perspectives on mathematical modelling have been developed within the international discussion on mathematics education, partly new and different from the historical ones. Despite several commonalities, these strands of the discussion framed modelling and its pedagogical potential in different ways. In order to enhance the understanding of these different perspectives on modelling, Kaiser and Sriraman (2006) proposed a framework for the description of the various approaches, which classifies these conceptions according to the aims pursued with mathematical modelling, their epistemological background, and their relation to the initial perspectives.

The following perspectives were described, which continue positions already emphasized at the beginning of the modelling debate:

- *Realistic or applied modelling* fostering pragmatic-utilitarian goals and continuing traditions of the early pragmatically oriented approaches
- *Epistemological or theoretical modelling* placing theory-oriented goals into the foreground and being in the tradition of the scientific-humanistic approach
- *Educational modelling* emphasizing pedagogical and subject-related goals, which are integrating aspects of the realistic/applied and the epistemological/theoretical approaches taking up aspects of a so-called integrated approach being developed at the beginning of the nineties of the last century mainly within the German discussion

In addition the following new approaches have been developed:

- *Model eliciting and contextual approaches*, which emphasize problem-solving and psychological goals
- *Socio-critical and sociocultural modelling* fostering the goal of critical understanding of the surrounding world connected with the

recognition of the sociol-cultural dependency of the modelling activities

As kind of a meta-perspective, the following perspective is distinguished, which has been developed in the last decade reflecting demands on more detailed analysis of the students' modelling process and their cognitive and affective barriers.

- *Cognitive modelling* putting the analysis of students' modelling process and the promotion of mathematical thinking processes in the foreground

This classification points on the one hand to a continuity of the tradition on the teaching and learning of mathematical modelling; there still exist many commonalities between the historical approach already developed amongst others by Felix Klein and the new approaches. On the other hand, it becomes clear that new perspectives on modelling have been developed over the last decades emphasizing new aspects such as metacognition, the inclusion of socio-critical or socio-cultural issues, a more process-oriented view on modelling, and the modelling cycle.

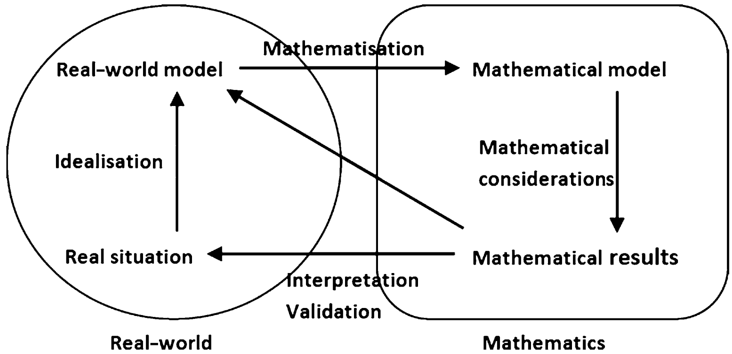
The Modelling Process as Key Feature of Modelling Activities

A key characteristic of these various perspectives is the way how the mathematical modelling process is understood, how the relation between mathematics and the "rest of the world" (Pollak 1968) is described. Analyses show that the modelling processes are differently used by the various perspectives and streams within the modelling debate, already since the beginning of the discussion. The perspectives described above developed different notions of the modelling process either emphasizing the solution of the original problem, as it is done by the realistic or applied modelling perspective, or the development of mathematical theory as it is done by the epistemological or theoretical approach. So, corresponding to the different perspectives on mathematical modelling, there exist various modelling cycles with specific emphasis, for example, designed primarily for mathematical purposes, research activities, or usage in classrooms (for an overview, see Borromeo Ferri 2006).

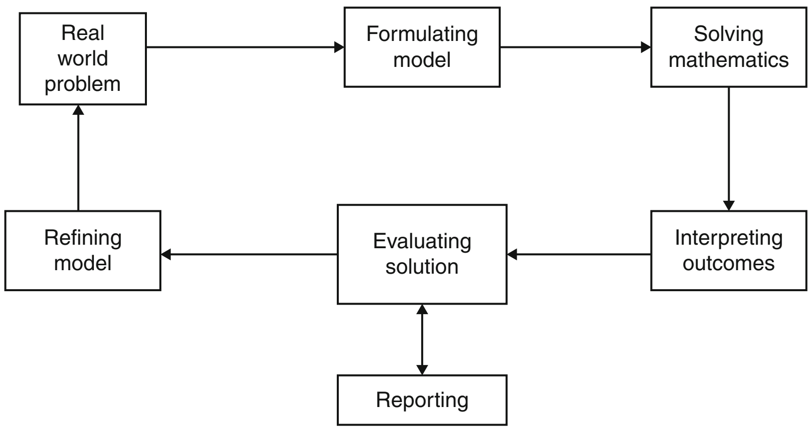
Although at the beginning of the modelling debate, a description of the modelling process as linear succession of the modelling activities was common or the differentiation between mathematics and the real world was seen more statically (e.g., by Burkhardt 1981), nowadays, despite some discrepancies, one common and widespread understanding of modelling processes has been developed. In nearly all approaches, the idealized process of mathematical modelling is described as a cyclic process to solve real problems by using mathematics, illustrated as a cycle comprising different steps or phases.

The modelling cycle developed by Blum (1996) and Kaiser-Meßmer (1986) is based amongst others on work by Pollak (1968, 1969) and serves as exemplary visualization of many similar approaches. This description contains the characteristics, which nowadays can be found in various modelling cycles: The given real-world problem is simplified in order to build a real model of the situation, amongst other many assumptions have to be made, and central influencing factors have to be detected. To create a mathematical model, the real-world model has to be translated into mathematics. However, the distinction between a real-world and a mathematical model is not always well defined, because the process of developing a real-world model and a mathematical model is interwoven, amongst others because the developed real-world model is related to the mathematical knowledge of the modeller. Inside the mathematical model, mathematical results are worked out by using mathematics. After interpreting the mathematical results, the real results have to be validated as well as the whole modelling process itself. There may be single parts or the whole process to go through again (Fig. 1).

The shown cycle idealizes the modelling process. In reality, several mini-modelling cycles occur that are worked out either in linear sequential steps like the cycle or in a less ordered way. Most modelling processes include frequent switching between the different steps of the modelling cycles.



Mathematical Modelling and Applications in Education, Fig. 1 Modelling process from Kaiser-Meßmer (1986) and Blum (1996)



Mathematical Modelling and Applications in Education, Fig. 2 Modelling process from Haines et al. (2000)

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Other descriptions of the modelling cycle coming from applied mathematics, such as the one by Haines et al. (2000), emphasize the necessity to report the results of the process and include more explicitly the refinement of the model (Fig. 2).

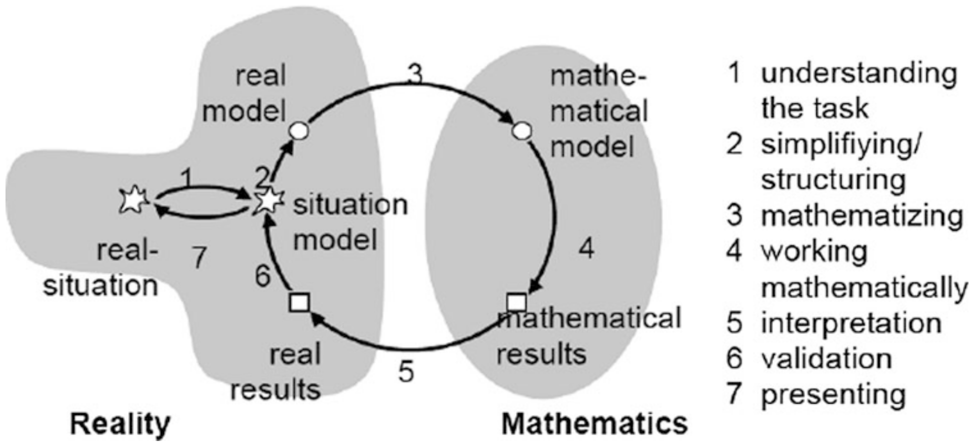
Perspectives putting cognitive analyses in the foreground include an additional stage within the modelling process, the understanding of the situation by the students. The students develop a situation model, which is then translated into the real model; Blum in more recent work (e.g., 2011) and others (e.g., Leiß, Borromeo Ferri) have described modelling activities in such a way (Fig. 3).

Detailed Description of One Modelling Cycle Based on the Lighthouse Example

The problem how far a ship is away from a lighthouse, when the crew sees the fire of the lighthouse the first time, is a well-known sea navigation problem with high relevance in former times, before most ships were equipped with GPS. This problem is proposed by protagonists of the educational modelling perspective for the teaching of mathematical modelling in school – especially Blum and Leiß – due to its mathematical richness and its easy accessibility and is adapted in the following to a local situation, namely, a lighthouse at the Northsea in Germany.

Westerhever Lighthouse

The Westerhever lighthouse was built in 1906 at the German coast of the Northsea and is 41 m high. The lighthouse should in former times inform ships, which were approaching the coast, about their position against the coastline. How far off the coast is a ship when the crew is able to see the light fire for the very first time over the horizon? (Round off whole kilometers) (Fig. 4).



Mathematical Modelling and Applications in Education, Fig. 3 Modelling process by Blum (2011)

Mathematical Modelling and Applications in Education, Fig. 4 Task on Westerhever lighthouse. (Photo by Thomas Raupach)

Development of a Real-World Model

The students have to develop a real-world model based on different assumptions, i.e., they have to simplify the situation and idealize and structure it, taking into account the curvature of the earth as key influential factor.

Development of a Mathematical Model

The first step can comprise the translation of the real-world model into a two-dimensional mathematical model describing the earth as a circle and then using the Pythagorean Theorem to calculate the required distance from the ship to the lighthouse. Another attempt refers to the definition of the cosine, which can be used instead of the Theorem of Pythagoras.

An extension of this simple model takes into account that the observer who sees the lighthouse at first is not at the height of the waterline, but a few meters higher, e.g., in a look-out. A possible approach uses the Pythagorean Theorem twice, firstly with the right-angled triangle from the geocenter to the top of the lighthouse to the boundary point, where the line of sight meets the sea surface.

Interpretation and Validation

Afterwards the results need to be interpreted and validated using knowledge from other sources. The results need to be transferred back to reality and need to be questioned.

Further Explorations and Extensions

The example of the lighthouse allows many interesting explorations, for example, the reflection on the reverse question, how far away is the horizon?

That well-known problem is similar to the problem of the lighthouse, and its solution is mathematically equivalent to the first elementary model. However, from a cognitive point of view, the real-world model is much more difficult to develop, because the curvature and its central role are psychologically difficult to grasp.

The example above is a typical modelling example showing that there exists a rich variety of modelling examples ranging from small textbook examples to complex, authentic modelling activities. Many extracurricular materials have been developed in the last decades amongst others by COMAP or the Istron Group; many examples are nowadays included in textbooks for school teaching.

Modelling Competencies and Their Promotion

A central goal of mathematical modelling is the promotion of modelling competencies, i.e., the ability and the volition to work out real-world problems with mathematical means (cf. Maaß 2006). The definition of modelling competencies corresponds with the different perspectives of mathematical modelling and is influenced by the taken perspective. A distinction is made between global modelling competencies and sub-competencies of mathematical modelling. Global modelling competencies refer to necessary abilities to perform the whole modelling process and to reflect on it. The sub-competencies of mathematical modelling refer to the modelling cycle; they include the different competencies that are essential for performing the single steps of the modelling cycle (Kaiser 2007). Based on the comprehensive studies by Maaß (2006) and Kaiser (2007), extensive work by Haines et al. (2000), and further studies and by referring to the various types of the modelling cycle as described above, the following sub-competencies of modelling competency can be distinguished (Kaiser 2007, p. 111):

- Competency to solve at least partly a real world problem through a mathematical description (that is, a model) developed by oneself;
- Competency to reflect about the modelling process by activating meta-knowledge about modelling processes;
- Insight into the connections between mathematics and reality;
- Insight into the perception of mathematics as process and not merely as product;
- Insight into the subjectivity of mathematical modelling, that is, the dependence of modelling processes on the aims and the available mathematical tools and students competencies;
- Social competencies such as the ability to work in groups and to communicate about and via mathematics.

This list is far from being complete since more extensive empirical studies are needed to receive well-founded knowledge about modelling competencies.

Obviously the sub-competencies are an essential part of the modelling competencies. In addition metacognitive competencies play a significant role within the modelling process (Maaß 2006; Stillman 2011). Missing metacognitive competencies may lead to problems during the modelling process, for example, at the transitions between the single steps of the modelling cycle or in situations where cognitive barriers appear (cf. Stillman 2011).

In the discussion on the teaching and learning of mathematical modelling, two different approaches of fostering mathematical modelling competencies can be distinguished: the holistic and the atomistic approach (Blomhøj and Jensen 2003). The holistic approach assumes that the development of modelling competencies should be fostered by performing complete processes of mathematical modelling, whereby the complexity and difficulty of the problems should be matched to the competencies of the learners. The atomistic approach, however, assumes that the implementation of complete modelling problems, especially at the beginning, would be too time-consuming and not sufficiently effective at fostering the individual modelling competencies. It is nowadays consensus that both approaches need to be integrated, although no secure empirical evaluation on the efficiency of both approaches or an integrated one has been carried out so far.

Obviously these two different approaches necessitate different ways of organizing the inclusion of modelling examples in schools: The atomistic approach seems to be more suitable for a "mixing approach," i.e., "in the teaching of mathematics, elements of applications and modelling are invoked to assist the introduction of mathematical concepts etc. Conversely, newly developed mathematical concepts, methods and results are activated towards applicational and modelling situations whenever possible" (Blum and Niss 1991, p. 61). The holistic approach can either be realized in a "separation approach," i.e., instead "of including modelling and applications work in the ordinary mathematics courses, such activities are cultivated in separate courses specially devoted to them" (Blum and Niss 1991, p. 60). Of course variations of these approaches. Like the "two-compartment approach" or the "islands approach" described by Blum and Niss (1991) seem to be possible as well.

Results of Empirical Studies on the Implementation of Mathematical Modelling in School

Several empirical studies have shown that each step in the modelling process is a potential cognitive barrier for students (see, e.g., Blum 2011, as overview). Stillman et al. (2010) describe in their studies these potential "blockages" or "red flag situations," in which there is either no progress made by the students, errors occur and are handled, or anomalous results occur. Stillman (2011) in her overview on the cognitively oriented debate on modelling emphasizes the importance of reflective metacognitive activity during mathematical modelling activities especially within transitions between phases in the modelling process. She identifies productive metacognitive acts promoting students' metacognitive competences at various levels and distinguishes routine metacognition responding to blockages or red flag situations from meta-metacognition being brought in by teachers trying to promote students' development of independent modelling competencies leading to reflective metacognition.

So far the role of the teacher within modelling activities has not been researched sufficiently: Until now not enough secure empirical evidence exists, how teachers can support students in independent modelling activities, how can they support them in overcoming cognitive blockages, and how can they foster metacognitive competencies. It is consensus that modelling activities need to be carried out in a permanent balance between minimal teacher guidance and maximal students' independence, following well-known pedagogical principles such as the principal of minimal help. Research calls for individual, adaptive, independence-preserving teacher interventions within modelling activities (Blum 2011), which relates modelling activities to the approach of scaffolding. Scaffolding can be according to well-known definitions described as a metaphor for tailored and temporary support that teachers offer students to help them solve a task that they would otherwise not be able to perform. Although scaffolding has been studied extensively in the last decades, it was found to be rare in classroom practice. Especially for modelling processes, which comprise complex cognitive activities, scaffolding seems to be especially necessary and appropriate. But scaffolding has to be based on a diagnosis of students' understanding of the learning content, which most teachers did not ascertain; in contrast most teachers provided immediate support or even favoured their own solution.

In the future, learning environments for modelling need to be established, which support independent modelling activities, for example, by sense-making using meaningful tasks, model-eliciting activities based on challenging tasks, or the usage of authentic tasks.

Cross-References

▶ Interdisciplinary Approaches in Mathematics Education
▶ Mathematization as Social Process
▶ Realistic Mathematics Education
▶ Word Problems in Mathematics Education

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Mathematical Proof, Argumentation, and Reasoning

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Keywords

Argumentation · Logic · Proof · Reasoning · Visualization

Definition

Argumentation, reasoning, and proof are concepts with ill-defined boundaries. More precisely, they are words that different people use in different ways. What one can perhaps say is that reasoning is the concept with the widest compass. Logic is usually taken to mean a more structured form of reasoning, with its own subset, formal logic, which is logic in its most rigidly structured form. Though people most closely associate logic with mathematics, all forms of reasoning have had, and continue to have, valuable roles in mathematical practice. For that reason and, perhaps even more important, because of their usefulness in teaching, the many forms of reasoning have also found their place in the mathematics curriculum.

Characteristics

This entry will explore in more detail the concepts of argumentation, reasoning, and proof as understood by mathematicians and educators and present some of their implications for mathematics education. It will go on to describe some more recent thinking in mathematics education and in the field of mathematics itself.

Mathematical Proof

Mathematics curricula worldwide aim at teaching students to understand and produce proofs, both to reflect proof's central position in mathematics and to reap its many educational benefits. Most documents addressed to teachers, such as that by the National Council of Teachers of Mathematics (NCTM 2000), give the following reasons for teaching proof: (1) to establish certainty; (2) to gain understanding; (3) to communicate ideas; (4) to meet an intellectual challenge; (5) to create something elegant, surprising, or insightful; and (6) to construct a larger mathematical theory.

This list encompasses not only justification but also considerations of understanding, insight, and aesthetics and in so doing further reflects mathematics itself. These additional considerations are important not only in the classroom but in mathematical practice as well: for mathematicians, too, a proof is much more than a sequence of logical steps that justifies an assertion.

Proof also plays other significant roles in mathematical practice. Proof can serve to present new methods and demonstrate their value, to inspire new hypotheses, and to show connections between different parts of mathematics. For practicing mathematicians, these too are valuable aspects of proof; yet the mathematics curricula, by and large, have failed to explore their educational potential.

Proof pervades all mathematical work. Unless it is considered an axiom, a mathematical assertion without a proof must remain a conjecture. To justify an assertion is the role of a proof. In the purest sense, a mathematical proof is a logical derivation of a given statement from axioms through an explicit chain of inferences obeying accepted rules of deduction. A "formal proof" will employ formal notation, syntax, and rules of inference ("axiomatic method"). Thus, strictly formal derivations will consist of unambiguous strings of symbols and conform to a mechanical procedure that will permit the correctness of the proof to be checked. Such proofs are considered highly reliable.

However, proofs in mathematical journals rarely conform to this pattern. As Rav (1999) pointed out, mathematicians express "ordinary" proofs in a mixture of natural and formal language, employing passages of explicit formal deductions only where appropriate. They bridge between these passages of formal deduction using passages of informal language in which they provide only the direction of the proof, by making reference to accepted chains of deduction. Consequently, most mathematicians would characterize ordinary proofs as informal arguments or "proof sketches."

Nevertheless, these ordinary informal proofs do provide a very high level of reliability, because the bridges are "derivation indicators" that are easily recognized by other mathematicians and provide enough detail to allow easy detection and repair of errors (Azzouni 2004). In this way, the social process by which such proofs are scrutinized and ultimately accepted improves their validity. In fact, most accepted mathematical proofs consist of valid arguments that may not lend themselves to easy formalization (Hanna 2000; Manin 1998; Thurston 1994).

To reflect mathematical practice, then, a mathematics curriculum has to present both formal and informal modes of proof. If they wish to teach students how to follow and evaluate a mathematical argument, make and test a conjecture, and develop and justify their own mathematical arguments and proofs, educators have to provide the students with the entire gamut of mathematical tools, including both the formal and informal ones. Without this important double approach, students will lack the body of mathematical knowledge that enables practicing mathematicians to communicate effectively by using "derivation indicators" and other mathematical shorthand (cf. Hanna and de Villiers 2012).

Reasoning and Proof

Most mathematics curricula recognize that reasoning and proof are fundamental aspects of mathematics. In fact, much of the literature on mathematics teaching refers to them as one entity called "reasoning and proof."

We may take reasoning, in the broadest sense, to mean the common human ability to make inferences, deductive or otherwise. As Fischbein (1999) noted, everyday reasoning may differ from explicit mathematical reasoning in both process and result. In everyday reasoning, for example, we may even accept a statement without any type of proof at all, because we judge it to be self-evident or intuitively plausible, or at least more plausible than its contradiction. However, in many realms, including mathematics, such everyday reasoning provides little help (e.g., it is not intuitively clear that the sum of the angles in any triangle is always 180°). In all such cases we would need defined rules of reasoning in order to reach a valid conclusion. We would need to construct a correct chain of inference – that is, to construct a proof.

Thus, all mathematics educators aim to teach students the rules of reasoning. In the Western tradition, the rules of reasoning are derived from classical mathematics and philosophy and include, for example, the syllogism and such elementary rules as *modus ponens, modus tollens*, and *reductio ad absurdum*. Students typically first encounter these basic concepts of logic in the axiomatic proofs of Euclidean geometry.

Here the teacher's role is crucial. In addition to concepts specific to the mathematical topic, the teacher must make the students familiar with rules of reasoning, patterns of argumentation, and appropriate terms (e.g., assumption, conjecture, example, refutation, theorem, and axiom). How students actually learn these concepts is unfortunately a question of cognition that educators have yet to resolve, though researchers investigating this issue have proposed a number of promising models of cognition. One such model, the "cognitive development of proof," combines three worlds of mathematics: the conceptual/embodied, the proceptual/symbolic, and the axiomatic/formal (cf. Tall et al., chapter 2 in Hanna and de Villiers 2012). Another, based on extensive observations of college-level students learning mathematics, uses a psychological framework of "proof schemes" (Harel and Sowder 1998). Yet another (Balacheff 2010) aims at analyzing the learning of proof by considering how three "dimensions" – the subject, the milieu, and the problem – can be used to build a bridge between knowing and proving. Duval (2007) model stresses that the cognitive processes needed to understand and devise a proof depend on students' learning "how proof really works" (learning its syntactic and deductive elements) and "how to be convinced by proof." Stylianides (2008) proposes that the processes of reasoning and proving encompass three "components" – mathematical, psychological, and pedagogical – while Reid and Knipping (2010) discuss still other variations.

Argumentation and Proof

Many researchers in mathematics education have chosen to use the term "argumentation," which encompasses the various approaches to logical disputation, such as heuristics, plausible, and diagrammatic reasoning, and other arguments of widely differing degrees of formality (e.g., inductive, probabilistic, visual, intuitive, and empirical). Essentially, argumentation includes any technique that aims at persuading others that one's reasoning is right. As used by its proponents, the concept also implies exchange and cooperation in forming and criticizing arguments so as to arrive at the best conclusion despite imperfect knowledge. Evidently, the broad concept of argumentation encompasses mathematical proof as a special case.

In recent years, however, mathematics educators have been accustomed to use "argumentation" to mean "not yet proof" and "proof" to mean "mathematical proof." Consequently, opinion remains divided on the usefulness of encouraging students to engage in "argumentation" as a step in learning proof. Boero (in La lettre de la preuve 1999) and others see a great benefit in having students engage in conjecturing and argumentation as they develop an understanding of mathematical proof. Others take a quite different view, claiming that argumentation, because it aims only to establish plausibility, can never be more than a distraction from the task of teaching proof (e.g., Balacheff 1999; Duval – in La lettre de la preuve 1999). Despite these differences of opinion, however, the practice of teaching students the techniques of argumentation has recently been gaining ground in the classroom.

Durand-Guerrier et al. (Chapter 15 in Hanna and de Villiers 2012) reported on over 100 recent studies on argumentation in mathematics education that discuss the complex relationships between argumentation and proof from various mathematical and educational perspectives. Most of these studies reported that students can benefit from argumentation's openness of exploration and flexible validation rules as a prelude to the stricter uses of rules and symbols essential in constructing a mathematical proof. They also showed that appropriate learning environments can facilitate both argumentation and proof in mathematics classes.

Furthermore, some studies provided evidence that students who initially embarked upon heuristic argumentation in the classroom were nevertheless capable of going on to construct a valid mathematical proof. By way of explanation, Garuti et al. (1996) introduced the notion of "cognitive unity," referring to the potential continuity between producing a conjecture through argumentation and constructing its proof. Several other researchers have provided support for this idea and for other benefits or limitations of argumentation, particularly argumentation based on Toulmin's (1958) model of argument.

Toulmin's model, the one now most commonly used in mathematics education, proposes that an argument is best seen as comprising six elements: the Claim (C), which is the statement to be proved as a theorem or the conclusion of the argument; the Data (D), the premises; the Warrant (W) or justification, which is the connection between the Claim and the Data; the Backing (B), which gives authority to the Warrant; the Qualifier (Q), which indicates the strength of the Warrant by terms such as "necessarily," "presumably," "most," "usually," "always," and so on; and the Rebuttal (R), which specifies conditions that preclude the Claim (e.g., if the Warrant is not convincing).

Clearly, Toulmin's model reflects practical and plausible reasoning. It includes several types of inferences, admits of both inductive and deductive reasoning, and makes explicit both the premises and the conclusion, as well as the support that led from premises to conclusion. It is particularly relevant to mathematical proof in that it can include formal derivations of theorems by logical inference.

Practical Classroom Approaches

In addition to argumentation, a number of other approaches have been investigated for their value in teaching mathematical reasoning. Educators have debated, for example, whether the study of symbolic logic, more particularly the propositional calculus, would help students understand and produce proofs. Durand-Guerrier et al. (Chapter 16 in Hanna and de Villiers 2012) have examined this question and provide some evidence for the value of integrating techniques of symbolic logic into the teaching of proof.

Visualization, and diagrammatic reasoning in particular, is another technique whose value in teaching mathematics, and especially proof, has been discussed extensively in the literature and in conferences, albeit inconclusively. After examining numerous research findings, Dreyfus et al. (Chapter 8 in Hanna and de Villiers 2012) concluded that the issue required further research; in fact, both philosophers of mathematics and mathematics educators are still debating the contribution of visualization to the production of proofs. Current computing technologies have offered mathematicians an array of powerful tools for experiments, explorations, and visual displays that can enhance mathematical reasoning and limit mathematical error. These techniques have classroom potential as well. Borwein (Chapter 4 in Hanna and de Villiers 2012) sees several roles for computer-assisted exploration, many of them related to proof: graphing to expose mathematical facts, rigorously testing (and especially falsifying) conjectures, exploring a possible result to see whether it merits formal proof, and suggesting approaches to formal proof. Considerable research has demonstrated that the judicious use of dynamic geometry software can foster an understanding of proof at the school level (de Villiers 2003; Jones et al. 2000).

Physical artifacts (such as abaci, rulers, and other ancient and modern tools) provide another technique for facilitating the teaching of proof. Arzarello et al. (Chapter 5 in Hanna and de Villiers 2012) demonstrate how using such material aids can help students make the transition from exploring to proving. In particular, they show that students who use the artifacts improve their ability to understand mathematical concepts, engage in productive explorations, make conjectures, and come up with successful proofs.

Trends in Proof

In mathematical practice, as we have seen, ordinary informal proofs are considered appropriate and suitable for publication. Still, mathematicians would like to have access to a higher level of certainty than those informal proofs afford. For this reason, contemporary mathematical practice is trending toward the production of proofs much more rigorous and formal than those of a century ago (Wiedijk 2008). In practice, however, one cannot write out in full any formal proof that is not trivial, because it encompasses far too many logical inferences and calculations.

The last 20 years have seen the advent of several computer programs known as "automatic proof checkers" or "proof assistants." Because computers are better than humans at checking conformance to formal rules and making massive calculations, these new programs can check the correctness of a proof to a level no human can match. According to Wiedijk (2008), such programs have been successful in confirming the validity of several well-known theorems, such as the Fundamental Theorem of Algebra (2000) and the Prime Number Theorem (2008).

Mathematics educators and students have already benefitted greatly from educational software packages in areas other than proof, such as Dynamic Geometric Software (DGS) and Computer Algebra Systems (CAS), and researchers are working on advanced proof software specifically for mathematics education. For example, there is now a fully functional version of Theorem-Prover System (TPS) appropriate for the school and undergraduate levels, named eduTPS (Maric and Neuper 2011). The role of Artificial Intelligence in mathematics education, and in particular that of automated proof assistants, has already been the subject of several doctoral dissertations. Unfortunately, mathematics educators have not yet tested the proof software or tried it in the classroom, so its usefulness for teaching mathematics has not yet been firmly established.

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Cross-References

▶ Argumentation in Mathematics
▶ Argumentation in Mathematics Education
▶ Deductive Reasoning in Mathematics Education
▶ Logic in Mathematics Education

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Mathematical Representations

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Keywords
Cognitive configurations · Concrete embodiments · Diagrams · External representations · Gestures · Graphs · Imagery · Inscriptions · Interpretation · Internal representations · Language · Manipulatives · Meanings · Models · Neuroscience · Productions · Representational systems · Semiotics · Signification · Symbols · Symbolization · Visualization

Definitions

As most commonly interpreted in education, *mathematical representations* are visible or tangible productions – such as diagrams, number lines, graphs, arrangements of concrete objects or manipulatives, physical models, written words, mathematical expressions, formulas and equations, or depictions on the screen of a computer or calculator – that encode, stand for, or embody mathematical ideas or relationships. Such a production is sometimes called an *inscription* when the intent is to focus on a specific instance without referring, even tacitly, to any interpretation of it. To call something a *representation* thus includes reference to some meaning or signification it is taken to have. Such representations are called *external* – i.e., they are external to the individual who produced them and accessible to others for observation, discussion, interpretation, and/or manipulation. Spoken language, interjections, gestures, facial expressions, movements, and postures may sometimes function as external representations carrying mathematical meaning.

The term *representation* is also used very importantly to refer to a person's *mental*, *cognitive*, or brain constructs, concepts, or configurations. Then the mathematical representation is called *internal* to the individual. Examples include individuals' visual and/or spatial cognitive representation of geometrical objects or mathematical patterns, operations, or situations; their kinesthetic encoding of operations, shapes, and motions; their internal conceptual models of mathematical ideas; the language that they use internally to describe mathematical situations; their heuristic plans and strategies for problem solving; and their affective and motivational states in relation to mathematical problems and situations. The idea of external representation is expressible in German as *Darstellung* and that of internal representation as *Vorstellung*.

Representation also refers to the *act* or *process* of inventing or producing representations – so that "mathematical representation" is something that students and others *do*. Reference may be to the physical production of external representations as well as to the *cognitive*, *mental*, or *neurological processes* involved in constructing internal or external representations. The term also describes the semiotic relation between external productions and the internal mathematical ideas they are said to represent. Finally, it may refer specifically to the mathematical encoding of *non-mathematical* patterns – i.e., using the ideas and notations of mathematics as a *language* to represent concepts in physics, chemistry, biology, and economics, to describe quantitatively the laws that govern phenomena, to make predictions, and to solve problems.

Characteristics

Representations are considered to be mathematically *conventional*, or standard, when they are based on assumptions and conventions shared by the wider mathematical community. Examples of such conventional mathematical representations include configurations of base ten numerals, abaci, number lines, Cartesian graphs, and algebraic equations written using standard notation. In contrast, mathematical representations created on specific occasions by students are frequently *idiosyncratic*. Examples may include verbal utterances, pictures, diagrams, illustrative gestures, physical movements, and original or nonstandard notations invented by the individual.

Even when they are unconventional, mathematical representations can be *shared* and not simply personal. That is, the forms and meanings of representations may be *negotiated* during class discussions or group problem solving. Concrete structured manipulative materials such as geoboards, Cuisenaire rods, base ten blocks, pegboards, and attribute blocks, as well as calculators, graphing calculators, and a wide variety of computer environments, facilitate students' construction, discussion, interpretation, and sharing of many different kinds of external representations – both standard and idiosyncratic. Likewise, internal mathematical representations, depending on their degree of consistency with the internal representations of others, can be characterized as conventional or idiosyncratic, shared or personal.

In discussion, one often refers to a mathematical representation "in the abstract." For instance, to talk about "examining the graph of the equation $y = 3x - 2$" is to suggest (among other things) a kind of *idealized* or *generic* external representation in which a straight line has been drawn intersecting the horizontal *x*-axis at the point $x = 2/3$ and the vertical *y*-axis at the point $y = -2$. This stands in contrast to discussing a specific *instance* of the graph as it might occur in a textbook illustration (with particular scales, ranges of values, and so forth), in a blackboard drawing (perhaps quite imperfect), or on a graphing calculator. Internal representations are also frequently considered "in the abstract," as one refers, for example, to idealized mathematical ideas, concept images, or visualized symbol configurations.

An essential feature of most mathematical representations is that they not only have signification, but they belong to or are situated within *structured systems of representation* within which other configurations have similar signifying relationships. This is analogous to the way words and sentences occur, not as discrete entities in isolation from each other but within natural languages endowed with grammar, syntax, and networks of semantic relationships. Furthermore, representational systems (like languages) *evolve*. And previously developed systems of mathematical representation serve (up to a point) as "scaffolds" or "templates" for the development of new systems: relationships between configurations in the new system refer to their meanings in the prior system, but may be more general and more abstract.

For example, algebra as a representational system entails the interpretation of letters as variables that can assume numerical values. But it also involves algebraic expressions, operational symbols, and equality and inequality symbols, configured according to fairly precise syntactic rules, as well as processes for manipulating and transforming them. Up to a point, the prior arithmetic system of representation serves as a kind of template for the development of algebraic notation. The system evolved historically, and it evolves within learners in interaction with their external environments. As mathematics is learned, the structured nature of the mathematical representations creates a certain tension between a student's interpretation of meanings, acquisition of procedures, and eventual apprehension of underlying structures (e.g., Gravemeijer et al. 2010).

Characteristics of conventional structured mathematical representational systems can often be described in considerable detail. A written or printed numeral may represent a natural number, but it does so within our base ten Hindu-Arabic system of notation, a representational system of numeration involving the conventional signs {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}, rules for writing multidigit numerals, and conventions for interpreting "place value." A specific Cartesian graph of an equation in two variables occurs within the wider conventional system of graphical representation based on two orthogonal coordinate axes in the plane, a method of locating points in the plane corresponding to ordered pairs of coordinates, the use of certain letters to signify variables that can take on numerical values, and conventions involving positive and negative directions.

The precision of such characterizations is, of course, a prized feature of mathematics. Furthermore, an important aspect of the power of abstract mathematics is that mathematical constructs map to other constructs (i.e., can be represented) in ways that *respect or preserve the mathematical structure*. When the structure thus respected is algebraic, such maps are called homomorphisms or isomorphisms. When the structure is topological, it is called a homeomorphism. For example, in mathematics a *group representation* is a precisely defined notion: a homomorphism from a given, abstract mathematical group to a group of linear operators acting on a vector space.

However, the mathematical representations that occur in educational contexts, even when conventional, are extremely varied. They are most often incomplete and almost always highly ambiguous. Indeed, *ambiguity* and *context-dependence* are characteristic features of the interpretation of mathematical representations and systems of representation. Resolution of ambiguity in the process of interpreting a representation often entails making use of contextual and/or tacit information that is outside the representational system within which the ambiguity has occurred.

Mathematical representations and systems of representation are frequently characterized according to the nature of the representing configurations – e.g., internal or external; enactive, iconic, or symbolic; verbal, visual, spatial, auditory, or kinesthetic; concrete or abstract/symbolic; and static or dynamic. They may also be characterized according to the medium in which they are encoded – e.g., pencil and paper, chalkboard or smartboard, and tablet- or computer-based. Mathematical metaphors are representations that typically involve words or phrases, visual imagery, and some enactive or kinesthetic encoding of mathematical ideas. Different representational systems may be *linked*; and (with today's interactive communications technology) external, dynamic systems of representation may be multiply linked for purposes of mathematics teaching.

Research

Contrasting philosophical views that have greatly influenced mathematics education sometimes exclude or limit the study of representations as such within their respective paradigms. Behaviorism was based on the idea that mental states of any kind are inadmissible as explanations of observable learning or problem solving. External productions or configurations and their manipulation could be discussed, but could not be regarded as representing internal mathematical conceptualizations or as being represented by them. External configurations might only have observable relationships with other external ones. In contrast, radical constructivism was based on the tenet that individuals have access only to their own worlds of experience and none to the "real world." With exclusive emphasis placed on "experiential reality," internal configurations could only be understood to "re-present" other internal mathematical experiences in different ways. Still other viewpoints are based on the idea that the external-internal distinction itself entails a Cartesian mind-body dualism that is not tenable.

Nevertheless, research on representations and systems of representation in mathematics education has been ongoing for well over half a century and continues apace. Jerome Bruner, whose thinking contributed to some of the visionary ideas proposed by advocates of the "new mathematics" during the 1960s, characterized and discussed three kinds of representation by learners – enactive, iconic, and symbolic – seen as predominant during successive stages in a child's learning a concept (Bruner 1966). Semiotic and cognitive science approaches to mathematics education incorporated mathematical representation in its various interpretations. Artificial intelligence models for problem solving sought to simulate human internal representations and heuristics (e.g., Newell and Simon 1972; Palmer 1978; Skemp 1982; Davis 1984).

During the 1980s and 1990s, continuing research on representation by many (e.g., Janvier 1987; Goldin and Kaput 1996; Goldin and Janvier 1998) helped lay the groundwork for the inclusion by the National Council of Teachers of Mathematics (NCTM) in the United States of "Representations" as one of the major strands in its *Principles and Standards for School Mathematics* (NCTM 2000). The NCTM also devoted its 2001 Yearbook to the subject (Cuoco and Curcio 2001).

The NCTM's standards included many of the different meanings of mathematical representation described here:

> "The term representation refers both to process and to product – in other words, to the act of capturing a mathematical concept or relationships in some form and to the form itself. . . . Moreover, the term applies to processes and products that are observable externally as well as those that occur 'internally,' in the minds of people doing mathematics." (NCTM 2000, p. 67)

The US Common Core State Standards in Mathematics (CCSS-M), adopted with federal incentives by a large majority of states between 2011 and the present, include "Standards for Mathematical Practice." The discussion refers to the NCTM process standards, but accords representation and other processes less explicit focus. However, "Model with Mathematics" appears as a CCSS-M mathematical practice standard, and specific mathematical representations such as graphs occur in the CCSS-M content standards at various grade levels (CCSS Initiative 2018).

Continuing research on mathematical representation in education has included work on cognition and affect, on the affordances for mathematics learning offered by technology-based dynamic representation and linked representations, on mathematical representation in special education, on sociocultural contexts and their influences, on models for mathematical learning and problem solving, on the role of representations in particular conceptual domains of mathematics, and on the role of touch (haptic representation) and gesture in children's learning of mathematics (e.g., Goldin 1998, 2008; Hitt 2002; Kaput et al. 2002; Lesh and Doerr 2003; Duval 2006; Moreno-Armella et al. 2008;

Anderson et al. 2009; Roth 2009; Gravemeijer et al. 2010; Novack et al. 2014; van Garderen et al. 2018).

The emerging field of cognitive neuroscience research is potentially transformative for our understanding of mathematics learning and for mathematics education, as the brain encoding of number, mathematical expressions, and their spatial representation is explored. Current perspectives bring to bear models based on parallel distributed processing of information, as distinct from the sequential processing (e.g., search and subgoal decomposition algorithms) central to Newell and Simon's approach. Neural networks and evolutionary models provide new ways to simulate mathematical learning and problem solving and to describe internal representations (e.g., McClelland et al. 2016).

Teachers and researchers try to infer features of students' internal representations from the external representations they produce or with which they are presented. The representing relationship is usually understood in research to be in principle two-way, "bridging" the external and the internal. In addition, distinct external representations can represent each other (e.g., equations, graphs, and tables of values) in a student's thinking, and distinct internal representations can do likewise (e.g., as the student visualizes or imagines a function of a real variable as a formula, a graph, a machine generating outputs from inputs, or a set of ordered pairs satisfying some conditions). However, in any specific situation, one cannot simply assume a close or one-to-one correspondence between external and internal representations or between distinct external or internal ones. Different researchers have offered different perspectives on what it is that representations actually represent and the nature of the representing relationship.

Much research on mathematical representation in education is devoted to the study of specific conceptual domains such as number, fractions or rational numbers, integers (positive and negative), algebra, geometry and spatial concepts, functions and graphs, and statistics. The goal is frequently to study, in such a domain, how students generate representations, interact and move within various representations, translate between representations, or interpret one representation using another. Researchers seek to characterize students' understandings in terms of multiple representations, to infer students' thinking from the representations they produce and manipulate, to identify the affordances and obstacles associated with particular kinds of representation, and to develop new representational teaching methods using new media.

When representations are embodied in different media, different features of a conceptual domain of mathematics may become the most salient. Thus, the mathematical meanings may be regarded as *distributed* across various representational media in which they are encoded. With the advent of increasingly diverse and sophisticated technological environments, *dynamic* and *linked* mathematical representations are becoming increasingly important. These are built to respond to learners' actions, touches, or gestures according to preestablished structures and may eventually lead not only to novel teaching methods but to quite new interpretations of what it means to understand mathematics.

When a mathematical representation is first introduced, it is typically assigned a definite meaning or signification. For instance, a specific number-word may correspond to the result of counting fingers or objects; a positive whole number exponent may be defined as a conventional abbreviation of repeated multiplication; or the letter x may stand for an unknown number in a problem. Sometimes the initial signification is taken to be so fundamental that it poses a cognitive or epistemological obstacle to reinterpretation or later generalization. Certain misconceptions or alternative conceptions can be understood in this way. But as relationships develop, their meanings evolve, transfer to new contexts, and eventually may change profoundly. Such processes occur across the history of mathematics, within particular cultures, and within individual learners. Characterization of mathematical thinking and learning as fundamentally representational continues to be an important theoretical and empirical research perspective in mathematics education (e.g., Moreno-Armella et al. 2008; Anderson et al. 2009; Heinze et al. 2009; Moreno-Armella and Sriraman 2010).

Cross-References

- ▶ Affect in Mathematics Education
- ▶ Algebra Teaching and Learning
- ▶ Constructivism in Mathematics Education
- ▶ Functions Learning and Teaching
- ▶ History of Mathematics and Education
- ▶ Manipulatives in Mathematics Education
- ▶ Mathematical Language
- ▶ Metaphors in Mathematics Education
- ▶ Misconceptions and Alternative Conceptions in Mathematics Education
- ▶ Number Lines in Mathematics Education
- ▶ Number Teaching and Learning
- ▶ Problem-Solving in Mathematics Education
- ▶ Semiotics in Mathematics Education
- ▶ Teaching Practices in Digital Environments
- ▶ Visualization and Learning in Mathematics Education

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Mathematics Classroom Assessment

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Keywords
Formative assessment · Assessment tasks · Questioning · Assessment rubrics · Feedback · Self-assessment

Definition

Classroom assessment refers to the activities undertaken by teachers in eliciting and interpreting evidence of student learning and using this evidence to inform subsequent action.

Classroom assessment can be distinguished from *external assessment*, which often involves standardized tests carried out on a large scale. The most important difference between classroom assessment and external assessment arises from their different purposes. Wiliam (2007) summarizes the main purposes of assessment as:

1. Certifying the achievement or level of performance of individual students (summative)
2. Supporting students' learning and informing teachers' instructional decisions (formative)
3. Evaluating the quality of educational programs or institutions (evaluative)

Although teachers may design *classroom assessments* for both summative and formative purposes, it is more common to use this term to refer to assessment that is intended to support learning and teaching, in other words, formative assessment (Van den Heuvel-Panhuizen and Becker 2003; Wilson and Kenney 2003). On the other hand, *external assessment* is most often used for summative or evaluative purposes.

Background

Throughout the twentieth century, educational assessment was increasingly associated with externally administered tests that measure the performance of students, as well as teachers, schools, and whole school systems. This measurement paradigm continues to influence classroom assessment practices, despite the emergence of new theories of learning and curriculum that require new approaches to assessment. Shepard (2000) argues that classroom assessment should be epistemologically consistent with instruction, and indeed this was the case for much of the twentieth century when social efficiency models of curriculum and associationist and behaviorist theories of learning informed educational thinking and practice. These psychological theories assumed that learning is most efficient when knowledge and skills are broken into small steps and accumulated sequentially. Closely aligned with such theories is the idea of scientific measurement of skill mastery, which led to development of the "objective" test as the dominant method of assessing student achievement.

Time-restricted objective tests that require only recall of previously learned facts and rehearsed procedures are still a common form of mathematics classroom assessment in many countries. However, this traditional approach to assessment is out of alignment with the broadly social-constructivist conceptual frameworks that shape current understandings of learning and curriculum. Learning mathematics is now viewed as a process of constructing knowledge within a social and cultural context, and deep understanding, problem solving, and mathematical reasoning have become valued curricular goals. As the goals of mathematics education change, along with understanding of how students learn mathematics, new approaches to classroom assessment are called for that make students' thinking visible while enhancing teachers' assessment abilities (Van den Heuvel-Panhuizen and Becker 2003).

A Social-Constructivist Approach to Classroom Assessment

Work on developing a social-constructivist approach to mathematics classroom assessment is less advanced than research on mathematics learning, but key principles informing a new approach to assessment are well established and have been promulgated via research reports (Shepard 2000; Wiliam 2007; Wilson and Kenney 2003), curriculum documents (National Council of Teachers of Mathematics 1995, 2000), and professional development resources (Clarke 1997). Three overarching principles that correspond to each of the elements of the definition of assessment provided above are shown in Table 1, with particular reference to classroom assessment in mathematics.

Mathematics Classroom Assessment, Table 1 Classroom assessment principles

Definition of classroom assessment	Assessment principle	Assessment examples
Classroom assessment involves teachers in . . . eliciting evidence of student learning	Assessment should model good mathematical practice	Tasks
		Classroom discussion and questioning
Classroom assessment involves teachers in . . . interpreting evidence of student learning	Assessment should promote valid judgments of the quality of student learning	Alignment
		Multiple forms of evidence
		Explicit criteria and standards
Classroom assessment involves teachers in . . . acting on evidence of student learning	Assessment should enhance mathematics learning	Feedback
		Self-assessment

Eliciting Evidence of Student Learning

The principle of modeling good mathematical practice in classroom assessment is consistent with curriculum goals that value sophisticated mathematical thinking (abstraction, contextualization, making connections between concepts and representations) and appropriate use of mathematical language and tools.

Classroom assessment can provide insights into students' mathematical thinking through tasks that have more than one correct answer or more than one solution pathway, require application of knowledge in familiar and unfamiliar contexts, and invite multiple modes of communication and representation for demonstrating understanding. Time-restricted tests are usually unsuitable for revealing students' thinking in these ways. While investigative projects and mathematical modeling tasks provide rich opportunities for students to demonstrate understanding of significant mathematics, so too do more modest tasks such as "good" questions (Sullivan and Clarke 1991). Good questions are open-ended, elicit a range of responses, and can reveal what a student knows before and after studying a topic. These questions can easily be adapted from more conventional tasks that have only one correct answer, as demonstrated in Table 2.

Assessment is something that teachers are doing all the time, not only through tasks designed for assessment purposes but also in classroom discussion. In mathematics education, social-constructivist research carried out by Cobb,

Mathematics Classroom Assessment, Table 2 Converting conventional questions to "good" questions

Conventional question	Open-ended "good" question
Find the mean of these three numbers: 12, 16, 26	The mean age of three people is 18. What might their ages be?
Find the area of a rectangle with length 3 units and width 4 units	Draw a triangle with an area of 6 square units
Find the equation of the line passing through the points (2, 1) and (−1, 3)	Write the equations of at least five lines passing through the point (2, 1)

Forman, Lampert, O'Connor, and Wood has investigated the teacher's role in initiating students into mathematical discourse and practices (Lampert and Cobb 2003; Forman 2003). From an assessment perspective, a teacher purposefully orchestrating classroom discussion is collecting evidence of students' understanding that can inform subsequent instruction.

Interpreting Evidence of Student Learning

Teachers do not have direct access to students' thinking, and so assessment relies on interpretation of observable performance to enable judgments to be made about the quality of students' learning. Shepard (2000) notes that teachers are often reluctant to trust qualitative judgments because they believe that assessment needs to be "objective", requiring formula-based methods that rely on numerical marks or scores. This is a reductionist approach more consistent with the scientific measurement paradigm of assessment than the social-constructivist paradigm, where the goal of assessment is to provide a valid portrayal of students' learning (Clarke 1997).

The validity of teachers' assessment judgments can be strengthened by ensuring that assessment practices are aligned with curriculum goals and instruction. This means that the form and content of mathematics classroom assessments should reflect the ideas about good mathematical practice envisioned in curriculum documents and (ideally) enacted in classrooms. Assessment promotes valid judgments when it draws on multiple forms of evidence, as no single assessment tool can reveal the full range of student learning.

Validity is also enhanced when teachers explicitly communicate to students the criteria and standards that will be used to judge the quality of their performance (Wiliam 2007). Sadler's (1989) work on ways of specifying achievement standards has been influential in stimulating the development of assessment rubrics that use verbal descriptors to communicate the characteristics of task performance that will be assessed (criteria) and the benchmarks for describing the quality of performance (standards). A well-constructed rubric can make explicit the mathematical practices that teachers value, but students will not necessarily understand the verbal descriptors in the same way as the teacher. There is an opportunity here for teachers to engage students in discussion about the meaning of the criteria and what counts as good quality performance. Some researchers suggest that teachers can involve students in the development of rubrics in the process of looking at samples of their own or other students' work (Clarke 1997; Wiliam 2007). In this way, students can become familiar with notions of quality and develop the metacognitive ability to judge the quality of their own mathematical performances.

Acting on Evidence of Student Learning

One of the most important ways in which assessment can enhance mathematics learning is through the provision of feedback that can be used by students to close the gap between actual and desired performance. The notion of feedback had its origins in engineering and cybernetics, but finds extensive application in education. Ramaprasad's (1983) definition of feedback makes it clear that feedback is only formative if the information provided to the student is used in some way to improve performance. Reviews of research on feedback have identified characteristics of effective formative feedback in relation to quantity, timeliness, and strategies for engaging students in task-related activities that focus on improvement (Bangert-Drowns et al. 1991). However, Shepard (2000), arguing from a social-constructivist perspective, points out that these studies are mostly of little value because they are informed by behaviorist assumptions about learning and assessment. Drawing on Vygotsky's idea of the zone of proximal development, she calls for more research on dynamic assessment where the teacher uses scaffolded feedback to guide students through the solution process for a problem.

Involving students in self-assessment can enhance metacognitive self-regulation and help students become familiar with the criteria and standards that will be used to judge their performance. Controlled experiments have shown that structured self-assessment improves students' mathematics performance, but classroom self-assessment can also be used informally to gain insights into how students experience mathematics lessons. The IMPACT (Interactive Monitoring Program for Accessing Children's Thinking) procedure described by Clarke (1997) is one such approach. It invites students to write about important things they have learned in mathematics in the past month, problems they have found difficult, what they would like more help with, and how they feel in mathematics classes at the moment. This is a self-assessment tool that makes assessment a more open process and recognizes the important role of student affect in mathematics learning.

Issues in Classroom Assessment

A social-constructivist approach to classroom assessment places significant demands on mathematics teachers' knowledge and expertise. This includes knowing:

- How to design tasks and orchestrate classroom discussions that elicit students' mathematical thinking
- How to formulate assessment criteria and standards that reflect valued mathematical activity
- How to make balanced judgments about the quality of student performance across a range of different tasks
- How to provide contingent, "real-time" feedback that moves students forward in their learning
- How to encourage students to share ownership of the assessment process

Teachers' beliefs about what counts as "fair" or "objective" assessment also need to be taken into consideration, since the scientific measurement paradigm still exerts a strong influence on teachers' assessment practices. Although there are many research studies investigating social-constructivist mathematics teaching, the possibilities for introducing new approaches to mathematics classroom assessment require further research focusing in particular on supporting teacher development and change.

Cross-References

▶ Competency Frameworks in Mathematics Education
▶ External Assessment in Mathematics Education
▶ Mathematical Modelling and Applications in Education
▶ Metacognition
▶ Questioning in Mathematics Education
▶ Scaffolding in Mathematics Education
▶ Zone of Proximal Development in Mathematics Education

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Mathematics Curriculum Evaluation

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Keywords
Mathematics curricula · Curriculum evaluation · Assessment · Curriculum coherence · Alignment · Math standards · Mathematical competencies

Definition

Mathematics curriculum evaluation is the process of collecting and analyzing data with the purpose of making decisions about whether to keep, modify, or completely change a mathematics curriculum or some of its components.

Notions and Meanings

Though the definition above provides a sense of what *mathematics curriculum evaluation* means, the fact is that because of evasive meanings of the terms involved, it is difficult to adopt one agreed-upon definition. Defining *mathematics curriculum evaluation* draws on the more general concepts of *curriculum* and *curriculum evaluation*, taking into consideration the specific characteristics of mathematics as a discipline.

Curriculum

Historically, the term *curriculum* has been used in different meanings, including one or more of the following: goals and objectives determining the expectations of learning that are set by policy makers, textbooks used to guide teaching, instructional methods, plan of experiences, and/or actual experiences that learners go through in order to reach the specified learning goals. Larger meanings of *curriculum* include, in addition, the pedagogical framework or philosophy underlying the teaching practices and materials, training programs for supporting teachers, and/or guidelines for assessing students' learning. There is, however, a wide agreement that a curriculum may not be limited to a syllabus or list of topics set for teaching and learning.

The different processes involved at any point in the design, development, and implementation of a curriculum affect the ways the intentions of the curriculum are conceptualized, actualized, and implemented (Stein et al. 2007). As a result, educators distinguish different manifestations of a curriculum. Bauersfeld (1979) introduced the distinction between three entities, *the matter meant, the matter taught*, and *the matter learnt*, the first referring to the expectations set for learning mathematics, usually reflected in official documents such as a curriculum plan, standards, and/or textbooks; the second referring to the curriculum as taught and actualized by teachers through their classroom practices; and the third referring to what is actually learned by students. This distinction has later been used under different names and sometimes with added curriculum manifestations. The International Association for the Evaluation of Educational Achievement (IEA) used the names *intended, implemented*, and *attained curricula*, which have subsequently been widely used in mathematics education (e.g., Akker 2003; Cai 2010). The *assessed* curriculum came to be added to the threesome, to refer to the contents and mathematical processes that are addressed in assessments such as achievement tests.

Akker (2003) identifies two more specific aspects for the *intended* curriculum, which are the *ideal* curriculum (philosophical foundations) and the *formal/written* curriculum (intentions as specified in curriculum documents); two for the *implemented* curriculum, the *perceived* curriculum (interpretations by users, e.g., teachers) and the *operational* curriculum (as enacted in the classroom); and two for the *attained* curriculum, the *experiential* curriculum (learning experiences as pupils perceive them) and the *learned* curriculum (achieved learner outcomes).

Curriculum Evaluation

This complexity and the manifold nature of the notion of *curriculum* make it even more difficult to capture the notion of *curriculum evaluation.* It is frequently found in implicit or informal forms, inherent to making decisions about daily teaching practices, interpretations of students' results on tests, and actions of developing or supplementing teaching materials. Such actions may be taken by individuals (e.g., teacher, school principal) or groups (e.g., teachers in a math department, parents, employers). More explicit and formal aspects of evaluation are adopted when decisions need to be made about more general curriculum components at the institutional or national level (e.g., school board, educational committees, Ministry of Education). With such actions, "there is a need to convince the community, educators, teachers, parents, etc." (Howson et al. 1981), hence the need for explicit and evidence-based curriculum evaluation.

Curriculum evaluation always has, to various extents, dimensions of institutional, social, cultural, and political nature. Designing, developing, implementing, and evaluating a curriculum involve different actors and are affected by social, economical, and political forces as well as by different cultural groups in the community. This is, for instance, made clear in Artigue and Bednarz (2012) where the authors compare the results of several case studies of math curriculum design, development, and follow-up in some French-speaking countries or regions, namely, Belgium, Burkina Faso, France, Québec, Romand Switzerland, and Tunisia, using as a filter the notion of *social contract* due to Rousseau. The social contract considered here is determining, explicitly but also partly implicitly, the relationships between school and nation (or region), by fixing the authorities and obligations of the different institutions involved in the educational endeavor, the rights and duties of the different actors, as well as the respective expectations.

Though the terms *evaluation* and *assessment* are sometimes used interchangeably, their meanings came gradually to be more precisely defined and distinguished. Niss (1993) refers to the Discussion Document of the 1990 ICMI study on *Assessment in mathematics education and its effects* to highlight this distinction: "Assessment in mathematics education is taken to concern the judging of the mathematical capacity, performance and achievement – all three notions to be taken in their broadest sense – of students whether as individuals or in groups (. . .). Evaluation in mathematics education is taken to be the judging of educational or instructional systems, in its entirety or in parts, as far as mathematics teaching is concerned." (p. 3).

Evaluation is often perceived as an integral phase of the curriculum development process seen as a cycle. Sowell (2005) identifies four phases: (1) planning, that is, determining curriculum aims and objectives, naming the key issues and trends as global content areas, and considering the needs; (2) developing curriculum content or subject matter according to specific criteria or standards; (3) implementing, through teaching strategies that convert the written curriculum into instruction; and (4) evaluating, based on criteria that help in identifying the curriculum's strengths and weaknesses.

When a curriculum evaluation action is to be taken, the complexity of the curriculum, its numerous components and actors involved, leads to raising many questions as to the aspects to be evaluated, for example, the quality of textbooks, students' learning, teaching practices, and consistency between specific components. For evaluating these different aspects, different techniques, tools, and instruments are needed. Other questions would be about the criteria on which to base the evaluation. Talmage (1985) identified five types of "value questions" to be considered for the evaluation of a curriculum: (a) the question of intrinsic value, related to the appropriateness and worth of the curriculum; (b) the question of *instrumental* value, related to whether the curriculum is achieving what it is supposed to achieve, and concerned with the consistency of the program components with its goals and objectives and with its philosophical or psychological orientation; (c) the question of *comparative* value, asked when comparing a new program to the old one or comparing different curricula; (d) the question of *idealization* value posed throughout the delivery of the new program and concerned with finding ways to make the program the best possible; and (e) the question of *decision* value asking about whether to retain, modify, or eliminate the curriculum.

Particularly, the concept of *curriculum alignment* is used in many sources and evaluation studies (e.g., Romberg et al. 1991; Schmidt et al. 2005; Osta 2007). According to Schmidt et al. (2005), alignment is the degree to which various "policy instruments," such as standards, textbooks, and assessments, accord with each other and with school practice. Curriculum alignment may also be defined as the consistency between the various manifestations of a curriculum: the intended, the implemented (also called enacted), the assessed, and the attained curriculum. Porter (2004) defines curriculum assessment as "measuring the academic content of the intended, enacted, and assessed curricula as well as the content similarities and differences among them. (. . .) To the extent content is the same, they are said to be aligned" (p. 12).

Alignment is also referred to as *curriculum coherence*. The term coherence received more attention with the studies motivated by TIMSS results, especially in the USA. Schmidt and Prawat (2006) claim that the term *curriculum coherence* was defined as *alignment* in most of the studies that were conducted before the release of TIMSS results in 1997. In their study on "curriculum coherence and national control of education," several types of alignment were measured: "Alignment between content standards and textbooks, alignment between textbooks and teacher coverage, and alignment between content standards and teacher coverage" (p. 4). Globally, a curriculum is said to be coherent if its components are aligned with one another.

Evaluation may be *formative* or *summative*. Formative evaluation takes place during the process of development of the curriculum. It includes pilot studies of teaching units, interviews with teachers, and/or tests to assess students' learning from those units. Its aim is to adjust the process of development based on the results. Procedures used for formative evaluation are usually informal, unsystematic, and sometimes implicit. Summative evaluation is conducted to determine the worth or quality of a curriculum that is completely developed and implemented. Its main purpose is to make decisions about the continuation, alteration, or replacement of the curriculum or some of its components.

Models of Mathematics Curriculum Evaluation

Many types of activities conducted throughout the years, in formal and/or informal ways, in different regions of the world, have aimed at the evaluation of mathematics curricula. Such activities contributed to shaping the meaning of math curriculum evaluation as used today and to the development and refinement of techniques and instruments used. As this process evolved in different places of the world and in different societies and communities, different models emerged that may be distinguished by their level of formality, the level of rigidity of the tools or instruments they use, and the scope of factors and actors they involve in the analysis. The following examples may provide a sense of these differences:

Since the first large-scale projects of curricular reform and evaluation in the USA and other Anglo-Saxon societies, the experiences in mathematics curriculum evaluation tended toward more and more systematization and control by sets of criteria and detailed guidelines. Guides for curriculum evaluation are abundant. In their guide for reviewing school mathematics programs, for example, Blume and Nicely (1991) provided a list of criteria that characterize "an exemplary mathematics program" (p. 7), which should systematically develop mathematical concepts and skills; be sequential, articulated, and integrated; help students develop problem-solving skills and higher-order thinking; encourage students to develop their full potential in mathematics; promote a belief in the utility and value of mathematics; relate mathematics to students' world; use technology to enhance instruction; and be taught by knowledgeable, proficient, and active professionals. The guide then provides rubrics that help in determining the extent to which each one of those criteria is met by the mathematics curriculum under evaluation. Similarly, Bright et al. (1993) insist, in their "guide to evaluation," on the importance of examining the quality of curricula in a systematic and an ongoing way, based on selected criteria. For specific aspects of mathematics – problem solving, transition from arithmetic to algebra, materials for teaching statistics, and manipulative resources for mathematics instruction – the guide provides ways to focus the evaluation, pose evaluation questions, collect and analyze data, and report results.

Other models of math curriculum evaluation use more flexible approaches that take into consideration the rapport that the different actors (teachers, principals, educational authorities, etc.) have with the curriculum. For instance, the curricular reform in Québec, started in 1995 and presented by Bednarz et al. (2012), is qualified by these authors as a *hybrid* model, characterized by its long-term span, the involvement of actors with different perspectives, creating multiple interactions among them, and the involvement of teachers and school personnel. The evaluation model presented is formative and rather informal, regulated by the roles assigned to the actors, and perceiving the curriculum as being in continuous development, according to the experiences lived by different groups of practice. Concurrently, programs for raising teachers' awareness of the major directions and principles are created, aiming at teachers' appropriation of, and adherence to, a curriculum that is "alive" and open to debate.

The examples above show the richness and complexity of tasks of curriculum evaluation. They also show that these tasks cannot be separated from the culture and the characteristics of societies in which they emerge and develop.

Mathematics Curriculum Evaluation and Large–Scale Reforms

The notion of *mathematics curriculum evaluation* has been, since its first-known instances in the history of mathematics education, associated with major reforms in mathematics contents, teaching materials, and methods. When stakeholders, decision makers, governmental or non-governmental agencies, educators, or mathematicians start questioning mathematics teaching practices and materials currently in effect, actions are usually undertaken for evaluating their worth and developing alternative programs, which in turn call for evaluation.

Following are briefly some of the major landmark reforms and evaluation initiatives that had a considerable international impact.

The 1960s witnessed the wave of *New Mathematics* curricula, based on the Bourbakist view of mathematics. *New Math* programs were worldwide taught in schools in most countries. They resulted in a proliferation of textbooks to support instruction. They were also paralleled with large projects for piloting those textbooks as they were developed, especially in the USA (e.g., SMSG, School Mathematics Study Group) and in the UK (e.g., SMP, School Mathematics Project). Those projects resulted in a considerable body of research, widely disseminating a culture of evidence-based evaluation of mathematics curriculum materials. But serious problems of credibility and validity were raised, since many of the evaluative studies were conducted by the same groups which participated in the development of the curriculum materials. SMSG, for example, undertook a large enterprise of curriculum development and conducted a large-scale evaluation in the context of the National Longitudinal Study of

Mathematical Abilities (NLSMA). The NLSMA study adopted a model that was based on two dimensions of analysis. The first is by *categories of mathematical content* (number systems, geometry, and algebra), and the second is by *levels of behavior* (namely, computation, comprehension, application, and analysis). Such two-entry model will later, with different extents of modification, guide many of the mathematics curriculum evaluation studies around the world.

According to Begle and Wilson (1970), the major research design adopted for the pilot studies was the experimental design, by which student achievement in experimental classes, where the tested materials were used, was compared to achievement in control classes that used "traditional" materials. Two types of tests were used and administered to both groups, standardized tests and tests to evaluate mathematical knowledge according to the new math content. Major concerns about the validity of those comparisons were raised, especially because they use, with both groups, tests developed to assess the learning of the new content, which privileged the experimental group. The use of standardized tests was also contested, as these only provide scores which don't uncover the real learning problems, and which focus on recalling information and computation skills rather than mathematical thinking.

During and after their implementation, New Math curricula motivated debates and evaluation actions, formal as well as informal, in various parts of the world, because of their elitism and extreme mathematical formalism and because of the difficulties faced by teachers who were not prepared to cope with them. Most of those evaluation actions were motivated by the two opposing positions that arose in the mathematics education community. While one position advocated the *New Math* curricula as improving student learning, the other maintained that they were causing a drastic loss of students' basic mathematical skills.

Other landmarks that motivated many studies for evaluating mathematics curricula worldwide were the NCTM's Standards (NCTM 1989, 1995). These documents were influential, not only in the USA but in the conception of mathematics curricula in many other countries. Many research studies were conducted that tried to evaluate the alignment of mathematics curriculum materials and textbooks with the Standards.

The beginnings of the twenty-first century witnessed a new wave of calls for reform, characterized by increase of state control, core requirements, and systematic evidence-based evaluation of mathematics curricula, because of the international assessments and studies. An extensive body of worldwide research for evaluating mathematics curricula was motivated by the Third International Mathematics and Science Study (TIMSS), later known as Trends in International Mathematics and Science Study, conducted since 1995 on a regular 4-year cycle, and the Program for International Student Assessment (PISA), conducted since 2000 on a regular 3-year cycle. Many of those studies used the rich cross-national data to compare and evaluate participating countries' curricula. Schmidt et al. (2005) advocated that "the presence of content standards is not sufficient to guarantee curricula that lead to high-quality instruction and achievement" (p. 525). The lack of coherence between the *intended* and the *enacted* curricula was found to be one of the main reasons for relatively low scores in international comparative tests. Houang and Schmidt (2008) present the 1995 TIMSS ICA (International Curriculum Analysis) cross-national study which "captures" the curriculum from the participating countries, using the tripartite model of curriculum: the intended, implemented, and attained curricula. The study established methodological procedures and instruments to encode curriculum documents and textbooks (Houang and Schmidt 2008).

As a reaction to the results of international assessments in mathematics and science (TIMSS and PISA), we see many countries tending to more standardization and centralization in their math curricular procedures and practices. Central governments are taking more and more control in countries where more freedom and authority used to be left to states, districts, cantons, or even smaller communities. The concern of accountability of educational systems and the pressure of international assessments are prevalent. A compulsory common core is imposing itself as a solution in countries where no central curriculum was adopted before. For example, many USA states have already started implementing the Common Core State Standards (CCSS 2010).

The international assessments, especially PISA, motivated, on the other hand, an increasing trend in many countries toward designing mathematics curricula, according to a set of mathematical competencies, to be used for student learning assessment. This influence is made clear in the study by Artigue and Bednarz (2012) already mentioned. In Denmark as well, the eight mathematical competencies set by the KOM project (Niss 2003) aiming at an "in-depth reform of mathematics education" are very close, almost identical for some, to the PISA framework's cognitive competencies.

The increase of governmental control and the rise of calls for evidence-based judgments of educational systems' performance, added to the increasing pressure of the international assessments, are expected to motivate new waves of curriculum monitoring and evaluative procedures. Crucial questions and new problems will be awaiting investigation. Particularly the rise of the "evaluation by competencies" trend for assessing students' learning will lead to changes in the ways the evaluation of mathematics curricula is approached. These changes will raise new types of research questions and create a need for rethinking the different techniques, categories, and criteria used for mathematics curriculum evaluation.

Cross-References

- ▶ Competency Frameworks in Mathematics Education
- ▶ Curriculum Resources and Textbooks in Mathematics Education
- ▶ Mathematics Teacher Education Organization, Curriculum, and Outcomes
- ▶ Mathematics Teachers and Curricula
- ▶ Technology and Curricula in Mathematics Education

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Mathematics Learner Identity

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Definition

A socially produced way of being, as enacted and recognized in relation to learning mathematics. It involves stories, discourses and actions, decisions, and affiliations that people use to construct who they are in relation to mathematics, but also in interaction with multiple other simultaneously lived identities. This incorporates how they are treated and seen by others, how the local practice is defined and what social discourses are drawn upon regarding mathematics and the self.

The concept of identity in relation to learning mathematics has become increasingly evident in the mathematics education literature since before the turn of the century. Lave and Wenger (1991) introduced identity to mathematics education and conceptualized it in relation to learning in communities of practice. In the following decade other theories were introduced, notably those of Holland and Colleagues (1998) and Gee (2000) and early influential work within mathematics education included Martin (2000), Boaler and Greeno (2000), and Sfard and Prusak (2005) (see Darragh 2016). Initially identity was often associated together with *attitudes* and *beliefs*, however, over these past two decades our use of the concept has increasingly split from this domain to become seen as very much embedded and produced in the social and political context. It has been found to have high explanatory value in understanding students' participation and experiences in mathematics and how power is enacted through the production of the subject, be it the individual learner of mathematics or particular social groups within mathematics education.

Many authors have highlighted how learner identity has been poorly defined, conceptualized, and operationalized in the mathematics education literature (e.g., Bishop 2012; Cobb and Hodge 2009; Darragh 2016; Radovic et al. 2018; Sfard and Prusak 2005). Compounding this problem is the fact that identity has been used by authors coming from contrasting paradigms and trying to explain diverse aspects of students' relationship with the subject, from individual decision making to social influences and relations. Two recent literature reviews have mapped its use in applied research, showing definitions which are participative, narrative, discursive, psychoanalytic, positioned, or performative, come from differing theoretical underpinnings (Darragh 2016), and how subjective/social and representational/enacted aspects are emphasized in contrasting definitions (Radovic et al. 2018). In an effort to bring together the commonalities in this diversity of approaches, we draw attention to certain features of the concept of mathematics learner identity which appear to be agreed upon by a majority of researchers in our domain. The definition of mathematical learner identity as a *socially produced way of being, enacted and recognized in relation to mathematics learning* is an attempt that we hope will be recognizable broadly by researchers who use the concept, albeit from varying theoretical perspectives; we further explain this definition in the following paragraphs drawing from literature in the field to illustrate.

Identity is ***socially produced***. This means it is not something that belongs to an individual in isolation, rather it is inherently bound to social contexts; identity depends on the physical, temporal, and interpersonal context, and correspondingly it is a fluid and a constantly changing process. Different approaches have conceptualized and explored specific aspects of this process and do so by defining the social in different ways. The social may be seen as relationships and interactive moments between students, their peers and their teachers (e.g., Bishop 2012; Heyd-Metzuyanim and Sfard 2012; Wood 2013); as social definitions of competence that structure what is valued in a specific local community (Boaler and Greeno 2000; Cobb and Hodge 2009; Nasir 2002); or as social discourses that produce what is mathematics and defines the learner (Mendick 2005). All of these different levels of the social appear to be connected in complex ways, with social structures living in interactions, and local definitions of competence and shared practices mirroring both micro relationships and larger social discourses. Following this, mathematics identities are produced and reproduced in ways that can be explored by zooming-in and zooming-out from individual to social realities (Lerman 2001).

Different perspectives have engendered or use different metaphors to explain the "substance" of learners' identities. The more general and comprehensive metaphor is understanding identities as learners' ***ways of being*** in the social activity of doing mathematics. On the one hand, this involves the learner's private "sense" of being, including conscious and cognitive appraisals about oneself (i.e., self-concepts and self-efficacy beliefs) (Eccles 2009), self-understandings in relation to what is valued in mathematics (Cobb and Hodge 2009), and also embodied or unconscious feelings of who one is (Bibby 2009; Walshaw 2011). This private aspect gives the individual a sense of continuity and of connection or belonging to a community. On the other hand, identities are expressed and recognized in the public sphere as different kinds of people (Gee 2000) or as spaces where social discourses work and are worked (Mendick 2005). These social products are attached to contexts, moments in time and purposes. This implies that although there is a sense of who one is and a sense of continuity, this is not something static or essential, but something that is fluid and in constant negotiation with the social.

Accordingly, the literature within mathematics education sees identity as a process rather than an object: Identity is something you *do* (Gutiérrez 2013) or something that involves *identity work* (Chronaki 2011; Mendick 2005) in negotiating tensions in the production of the self. Identity may be ***enacted*** in different ways and therefore research can also operationalize it focusing on these different enactments. Sociocultural theories of identity tend to operationalize identity according to narratives/stories (Sfard and Prusak 2005), acts of positioning (Esmonde 2009; Turner et al. 2013), or as participation in classroom activity (Oppland-Cordell and Martin 2015) and in post-compulsory mathematics courses and careers (Black et al. 2009). Post-structural theories of identity look at the discursive constructions of identity, considering how the subject may perform themselves in relation to wider discourses in society (Chronaki 2011; Mendick 2005; Stentoft and Valero 2009). Psychoanalytic approaches consider relationships, desire, fantasy, and unconscious decisions and emotions (Bibby 2009; Walshaw 2011). Finally, psychological and sociocognitive theories emphasize conscious decisions and perceptions that guide learners' actions (Eccles 2009). All of these perspectives engender contrasting operationalizations including decisions and affiliations made by individuals; acts of positioning during micro-interactions, actions of individuals and groups during classroom observations, stories such as in interview transcripts in which self-authoring or broader social discourses are attended to (see also for comparison Langer-Osuna and Esmonde 2017). Following these diverse operationalizations, identity research requires multiple, sometimes complementary, methodologies, data, and approaches to analysis (Radovic et al. 2018).

Identity must also be ***recognized*** by others in order to be legitimate. Sfard and Prusak (2005) capture this in their diagrammatical depiction of identity as including who identifies whom. Educational systems identify the mathematics learner through labels such as "high ability," "good student," "failure," "learning disabled," among others. An individual may appropriate such a label into their own identity performance or they may engage in considerable identity work to enact their identity differently. In addition, it should be acknowledged that much research about mathematical learner identity are in part the researchers' own identifications of the subject. We apply labels such as "positive mathematics identity" (Stentoft and Valero 2009), "good at maths" (Mendick 2005), "fragile identities" (Solomon 2009), "oppositional" (Cobb and Hodge 2009), or identity as a "doer of mathematics" (Boaler and Greeno 2000) or define micro identities such as using the authoritarian voice or making statements of superiority/inferiority (Bishop 2012). In many cases these identifications enable us greater understanding of students' experiences of learning mathematics, but we should keep in mind the author of the label/identity.

A final aspect of identity generally agreed upon is that identity is not singular but ***multiple.*** We talk of *identities* in the plural and any mathematics learner identity interacts with the many other identities an individual may simultaneously perform. Researchers have captured this idea using the term "hybrid" identities (Chronaki 2011), multiple identities (Nasir 2002), and increasingly using the notion of intersectionality (Leyva 2017). This latter term has been utilized particularly in recent years to call attention to the variation of experiences within social groups. Studies examining multiple identities have produced findings about students' experiences of learning mathematics which have important implications for issues of equity.

Research exploring the relationship between students' mathematical identities and social membership identities such as gender, ethnicity, race has allowed us more complete understanding of the way that mathematics produces students differently. For example, certain classroom and wider practices make it more difficult for some students to identify with mathematics, as demonstrated in relation to gender (Radovic et al. 2017), ethnicity (Chronaki 2011), and race (Martin 2000; Nasir 2002). Studies on MLI have also shown that acts of power are not only wielded from "the powerful" (e.g., institutions) but are reproduced in practice, including reproduction in social interactions between students, their peers, and teachers and in how mathematics is presented in different contexts. In this sense, identity has provided another way of looking at the practices within the mathematics classroom, beyond a focus on teaching and learning, to consider how the classroom culture may enable (or discourage) students from identifying with the subject of mathematics, how it distributes and reproduces power and may even propose an "alternative politics of possibility" (Chronaki 2011, p. 210).

Cross-References

- ▶ Affect in Mathematics Education
- ▶ Discourse Analytic Approaches in Mathematics Education
- ▶ Equity and Access in Mathematics Education
- ▶ Gender in Mathematics Education
- ▶ Mathematics Teacher Identity
- ▶ Poststructuralist and Psychoanalytic Approaches in Mathematics Education
- ▶ Students' Attitude in Mathematics Education

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Mathematics Teacher as Learner

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Keywords
Teacher as learner · Design-based learning · Sources and strategies · Reflection and enaction · Individual learning versus institutional learning

Rationale

Analogous to mathematics power as goal for student learning, mathematics teachers learn to increase their pedagogical power of identifying challenges in a specific classroom environment and properly applying strategies to solve those challenges. Nurturing the power requires a complex and lifelong learning process through which teachers gradually go beyond themselves as they dig into the essences of mathematics learning and have ability to structure lessons for students to experience the learning accordingly.

Research on teacher learning can be generally categorized into two trends. One is tied to person, inheriting the research in psychology. Fuller (1969) conceptualized teacher concern into three major phases: nonconcern, concern with self, and concern with pupils. Clarke and Hollingsworth (2002) elaborated teacher growth as a nonlinear and interconnected learning process involving personal attributes, teaching experimentation, perception of professional communities, and the observation of salient outcomes.

Another trend originates from Vygotsky's work, focusing on interpersonal relationships and identities in teaching and learning interactions as well as the modes of thinking linked to forms of social practices. Learning inherently is viewed as increasing participation in socially organized practices (Lave and Wenger 1991). The conception, Zone of Proximal Development (ZPD), is also adopted to describe teacher learning in relation to the social setting and the goals and actions of tiers of participants (e.g., Goos and Geiger 2010). Additionally, Putnam and Borko (2000) combined both psychology and sociocultural perspectives, stating that teacher learning involves a process of enculturation and *construction*, which can be investigated by lines of research with roots in various disciplines (e.g., anthropology).

Reflection and enaction have been treated as crucial and inseparable mechanisms for teacher growth. Reflective thinking instead of routine thinking can effectively help teachers to overcome challenges (Dewey 1933). The distinction between reflection-in-action and reflection-on-action further presents how both mechanisms interact and lead to the learning (Schön 1983). Specifically, the power of institutional learning where school teachers work together as a term for their growth should be highlighted because school-based environments entail the norms and rationality for teachers to frequently implement new ideas into teaching practices and have ample opportunities to learn from each other in their daily-life teaching.

Sources and Strategies

A variety of sources and strategies have been proposed to facilitate teacher learning. Narrative cases offer teachers opportunities to situate their teaching for detecting and challenging the pedagogical problems. Analyzing mathematics tasks allows teachers to evaluate cognitive complexity of the tasks, converse the tasks into lesson structures, and properly enact them with students in class. Research findings can be materials as well to facilitate teachers' understanding of students' cognitive behaviors and improve the teaching quality. Strategies such as peer coaching or lesson study also make possible the learning of teachers by observing and analyzing peers' teaching experiences.

Of importance are the design-based professional development programs in which teachers can learn from educators, peer teachers, and students. Design-based approach has the capacity of encompassing all strengths for the facilitation of teacher learning listed above. By participating in designing tasks, teachers actively challenge the pedagogical problems that they concern. Designing tasks and enacting them with students also develops teachers' competence in coordinating experiences from different learning environments into the refinement of the tasks and the teaching. Particularly, as any of the existing instructional materials (e.g., test items) can be the sources to initiate new designs for promoting students' active thinking, this strategy is powerful to engender the ongoing learning journeys of teachers.

Teacher Learning Theory

Theories of student learning have been used to construct models and frameworks to facilitate teacher learning. Nevertheless, fundamental theories for teacher learning have not been well established yet. In light with the perspective viewing mathematics as the core for the learning of educators, teachers, and students (Mason 2008), it is particularly important to develop teachers' mathematical pedagogical thinking, the notion created by making analogy to mathematical thinking, and use the pedagogical thinking as principles to solve teachers' teaching problems (e.g., the use of specializing and generalizing thinking for probing students' error patterns across different mathematics topics).

Cross-References

▶ Mathematics Teacher Education Organization, Curriculum, and Outcomes
▶ Models of In-Service Mathematics Teacher Education Professional Development
▶ Professional Learning Communities in Mathematics Education
▶ Teacher as Researcher in Mathematics Education

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Mathematics Teacher Education Organization, Curriculum, and Outcomes

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Introduction

Tatto et al. (2010) stated: "We know little about the organization of the opportunities to learn mathematics and mathematics pedagogy offered to prospective and practicing teachers across the world and their relative effectiveness" (p. 313). The quote comes from a paper based on reports from 20 participating countries collected as part of the 2005 Conference of the International Commission on Mathematics Instruction (ICMI-15) (see Tatto et al. 2009). Since then the Teacher Education and Development Study in Mathematics or TEDS-M (see Tatto et al. 2012) was implemented in 2008 to begin to answer such questions.

Structure and Characteristics

In the 7 years between the ICMI-15 and the TEDS-M studies, the education of teachers has become an important policy issue. While we know more about the structure and characteristics of teacher education, the image that emerges is one of increased complexity. On the one hand, there are efforts by supranational institutions (e.g., European Union) to unify the system of teacher education, while on the other hand, countries and regions under the influence of globalization forces struggle to implement fast-paced reforms that may threaten or end up reaffirming more traditional teacher preparation systems. The fast development of information technologies, growth of multiculturalism, economic development, and globalization – all these place a great deal of pressure on education systems and also on teacher education. Educators, politicians, sociologists, as well as the general public all over the world ask the same questions: what skills, knowledge, attitudes, and values should be passed on to the new generation? How can children, young people, and their teachers be prepared for what they can expect in their future everyday life and career? (see e.g., Sarrazy and Novotná 2014) More specifically regarding teachers, what are the characteristics of teacher education programs that can prepare their graduates effectively for what is now needed? How can the outcomes of teacher education programs for teachers of mathematics be measured in ways that are reliable and valid? What kinds of policies are effective in recruiting qualified teachers of mathematics from diverse backgrounds?

In contrast to the above-quoted studies, this text is an encyclopedia entry which only outlines the main ideas but can never be exhaustive. The reader is advised to consult the sources we cite here and other relevant sources to obtain more exhaustive information on a whole range of questions concerning mathematics teacher education.

Institutions

The range of institutions preparing future teachers is large and includes secondary as well as tertiary schools (universities, national teacher colleges, both public and private). In some countries, it is also possible to read a course in mathematics and, only after having graduated and having made the decision to teach, to take a course in pedagogy and pedagogical content designed for in-service teachers who lack pedagogical education.

In many countries, teachers can also achieve credentials in practice (such as the notable *Teach for America* program in the USA and its variants now making inroads in many other countries). In some countries, it is possible to begin to teach without a proper teacher credential, but the situation is changing rapidly. For example, in England there are the “school direct” routes which allow teachers without a teaching credential to begin to teach. They do have to have some knowledge of the subject, but they lack knowledge of pedagogy including PCK. Some states in the USA, notably Arizona, are also passing laws that allow individuals to teach without a proper credential arguing teacher shortages. Indeed and different from England, there are teacher shortages in a number of the most populated states in the USA. So there is a new trend that seems to give more importance to content knowledge over and above pedagogy knowledge (for the English and US situation across the different subjects, including mathematics for future secondary school teachers, see, e.g., Tatto et al. 2018).

In some countries, preservice and in-service teachers can also attend distance courses (increasingly offered on-line), usually organized by universities. They may be attended either by in-service unqualified teachers or by in-service teachers who make the decision to extend their qualification by another subject. These courses may also be selected by people who do not work as teachers but are planning to change their profession and become teachers later on.

In-service training is necessary also for practicing teachers who have already achieved credentials but want professional development and support. In many countries, these development programs are supported by the government and authorities as it is understood that in the teachers’ professional lifetime, they cannot be expected to teach the same contents using the same methods (see Schwille and Dembélé 2007; Tatto 2008). Just as doctors are expected to follow the newest trends and technologies, teachers must be expected to keep up with the latest developments, both in content and pedagogical content. That is why some countries financially motivate their teachers to develop, offering better salaries to those who are willing to learn by engaging in further study. It is also true that many in-service teachers welcome the possibility for further training as it has the potential to give them.

Study Programs

In general, there are two possibilities of organization of teacher education. Programs may be concurrent which means that the preservice teacher takes at the same time mathematics, didactics, and general pedagogy seminars and lectures of. This system is sometimes criticized because it may fail to provide future teachers with in-depth content knowledge, considered as a prerequisite to mastering teaching methods, and by an overly formal pedagogical training. The other possible model is consecutive, which means that the preservice teachers first study the content and only subsequently methodology, psychology, and pedagogy. This may work well if it does not result in neglect of pedagogy and pedagogical content knowledge, which is sometimes the case especially among preservice teachers for secondary schools. This also depends on who teaches the future teachers, which will be discussed later.

The advantage of some consecutive programs is that it enables the structuring of university studies to include a bachelor's and master's degree, where the preservice teachers spend their time in the bachelor's studies focusing only on mathematics and the master's course focusing on the study of pedagogical content knowledge. This organization may be a way of preventing recent reform efforts emerging in some countries to shorten the study time of preservice preparation (e.g., to 3 years) or eliminate it all together, claiming that a bachelor's degree is sufficient to become a teacher.

The preparation of primary school teachers, on the other hand, tends to be concurrent as the general belief is that teachers for this stage should be real experts in pedagogical disciplines. The scope of subjects future primary school teachers study often results on superficial knowledge across all the disciplines. The TEDS-M study however uncovered that in some countries primary teachers are prepared as specialists and that in these cases their knowledge of the subject is significantly increased as reflected in their overall performance in the TEDS-M assessments (Tatto et al. 2012).

Teacher education programs typically include teaching practice or practicum, which may take various forms. It may be one semester spent on an affiliated school supervised by an accredited practicing teacher. It may include a couple of hours a week for a longer period of time. Or it may be few years following graduation, the so-called induction, when the fresh teacher is supervised and supported until he/she gets more experience of classroom and school practice (see Britton et al. 2003). This part of teacher education is considered very important under the assumption that only hands-on experience and advice of an experienced practitioner would enable mastering the necessary skills and that theoretical knowledge, albeit of pedagogical content and pedagogy, will never make a complete teacher (Grossman et al. 2011).

Who Teaches Future Teachers?

For the most part, future teachers of mathematics are taught by mathematicians, mathematics educators (usually with a degree in mathematics and pedagogy), and teacher educators. In practical experience, future teachers are often supervised by experienced practitioners. Comprehensive teacher education requires the combination of all these aspects.

Countries that offer in-service teacher professional development sometimes organize them outside university walls in various kinds of pedagogical centers. They hire trainers (from pedagogical centers, experience practitioners, etc.) to deliver different seminars and courses. One must stress that even these trainers must be trained too. The value of trainer training through formal programs of professional development and support has emerged as an area of concern. It may seem strange, even unnecessary, to suggest that the training of trainers ("trainer education" or "formal professional development" for trainers) needs to be justified. But while the value of the professional development opportunities for teacher educators is significant, it is rarely done or documented. If in-service teachers report the need of growing self-esteem, the team spirit, it would follow that the same must apply to teacher trainers and educators. While the academic world of universities and many international conferences and projects offer university teacher educators the chance to grow, develop, exchange information, and cooperate at the international level, teacher trainers still need other more formal avenues of professional development, specially at times of constant and demanding curricular change.

Who Enters the Profession?

The study programs offered by universities and national teacher preparation institutions may be selective or nonselective. This means that some institutions require from their participants to pass entrance exams or to have passed certain school leaving exams at the secondary school level. We have no knowledge that the candidates would be asked to pass any aptitude tests to show their *predisposition* for the profession in any countries although they are asked to demonstrate academic proficiency in the disciplines. It is a question whether or not it would help the education systems if only candidates of certain skills and talents were accepted to study education programs. It would definitely not be easy to specify which predispositions are essential for success in future work with pupils.

In case of nonselective admittance to universities, personal choice is what matters, but even if admittance is restrictive, only people with talent for the subject are likely to enroll. The problem in many countries is that teaching is not the most glamorous career, the job is poorly paid, and the reputation of teachers is low. The unfortunate consequence then is that education programs are entered only by those candidates who failed in other entrance exams to more demanding and desirable fields of study.

The TEDS-M study found that different countries' policies designed to shape teachers' career trajectories have a very important influence on who enters teacher education and eventually who becomes a teacher. These policies can be characterized as of two major types (with a number of variations in between): career-based systems where teachers are recruited at a relatively young age and remain in the public or civil service system throughout their working lives and position-based systems where teachers are not hired into the civil or teacher service but rather are hired into specific teaching positions within an unpredictable career-long progression of assignments. In a career-based system, there is more investment in initial teacher preparation, knowing that the education system will likely realize the return on this investment throughout the teacher's working life. While career-based systems have been the norm in many countries, increasingly the tendency is toward position-based systems. In general, position-based systems, with teachers hired on fixed, limited-term contracts, are less expensive for governments to maintain. At the same time, one long-term policy evident in all TEDS-M countries is that of requiring teachers to have university degrees, thus securing a teaching force where all its members have higher education degrees. These policy changes have increased the individual costs of becoming a teacher while also increasing the level of uncertainty of teaching as a career.

Professional Teacher's Competences

What skills, abilities, knowledge, and attitudes should graduates of teacher preparation programs master? For a long time, designers of teacher preparation programs have struggled to balance the theoretical with the practical knowledge and skills (Ball and Bass 2000). However, there is no consensus on the proportion of the different teacher preparation "ingredients."

It is clear that a good teacher of mathematics must understand more than the mathematical discipline. They must master other skills in order to be able to plan and manage their lessons, to transmit knowledge, and especially to facilitate their pupils' learning. They must get introduced to various types of classroom management (whole class, group work, pair work, individual work) and understand the advantages and disadvantages in different activities; they must learn how to work with pupils with specific learning needs and problems and how to work with mixed-ability classes to answer the needs of the talented as well as below average students. They must learn to pose motivating and challenging questions, learn how to facilitate pupils' work, must be aware of the difference in pupils' learning styles, and must be experts in efficient communication and appropriate language use. They must be able to work with mistakes. They must also know the demands in the output, what the pupils will be expected to master, and in what form they will be expected to show their knowledge and skills. They must be able to mediate the increasing demands for excelling in examinations and developing deep and relevant learning. They should be able to manage the development and the administration of summative or formative assessments to inform and plan their teaching; they should be able to understand the advantages of each of these types of assessments (Even and Ball 2009). These of course cannot be acquired in purely mathematical courses, and preservice teachers must undergo more extensive preparation.

According to Shulman (1987), the knowledge that teachers must master consists of content knowledge (in this case mathematics), pedagogical content knowledge (didactics and methodology of the studied subject, the ability to act adequately directly in the course of lessons) and pedagogy (philosophy of education, history of education, educational psychology, sociology of education), knowledge of pupils, and knowledge of context. In several studies, knowledge, beliefs, and attitudes toward mathematics and practical skills are highlighted (see, e.g., Nieto 1996).

Whatever classification or division we choose, the fact remains that it is at this point impossible to give one answer to the question of how much time and attention should be paid to each of the components. The problem is that it is impossible to state objectively which part of this knowledge makes a really good teacher. In general terms, it can be said that usually future primary school teachers get much more training in pedagogy and psychology, while future secondary school teachers get more training in the mathematics itself. The problem of the first situation is the lack of the teacher's knowledge of mathematics which often results in lack of self-confidence. Unaware of the underlying mathematical structures, the teacher may be hardly expected to identify the sources of pupils' mistakes and misconceptions, let alone correct them. Primary school teachers report that this lack of self-confidence in the discipline prevents them from adequate reactions to their pupils' questions and problems. If it is true that mathematics that has already been discovered is "dead" mathematics and is brought to life by teachers (Sarrazy and Novotná to be published), the teacher must know it and be able to assist in this rediscovery.

In contrast, if teachers are not trained adequately in pedagogy and pedagogical content knowledge, they may fail to pinpoint the sources of their pupils' problems as they may be related to their cognitive abilities, age, and methods used in lessons, among others.

The problem with mathematical content knowledge is that there is wide disagreement regarding the extent and depth of the mathematical content pupils should be taught to make use of in their future life. If there is disagreement regarding what pupils need to know, there is also disagreement on the mathematics their teachers need. The current trend emphasizing transversal- horizontal skills (learning to learn, social competences, cross-curricular topics) seems to put more emphasis on everything but the mathematical content. However, there is no doubt that pupils must learn also mathematics as they will be using it in many everyday situations in their future. Calculators and computers will never really substitute human mathematical thinking.

This problem of lack of agreement of what mathematics to teach and how much of it to teach is well known to those involved in mathematics education at all levels (Tatto and Hordern 2017). One of the strategies recently introduced to solve this problem is the development of content standards currently implemented in a considerable number of countries. They might differ in form, in the degree of obligation, and in the level of details included, but they certainly share one characteristic: they define the framework for the volume of mathematics that teachers will have to teach and consequently the bases for the mathematical content to be included in the teacher education curriculum.

The TEDS-M study shows that there are topics and areas that can be found in the curricula of teacher education programs in a considerable number of countries and may therefore be regarded as the cornerstones of mathematics education. These topics are numbers; measurement; geometry; functions, relations, and equations; data representation, probability, and statistics; calculus; and validation, structuring, and abstracting. The opportunity to learn these topics varies according to the grade levels future teachers are prepared to teach with primary teachers predominantly studying topics such as numbers, measurement, and geometry. As programs prepare teachers for higher grades, the proportion of areas reported as having been studied increases. Importantly TEDS-M found that the Asian countries and other countries whose future teachers did well on the TEDS-M assessments did offer axiomatic geometry, analytic geometry, and to a lesser degree – and only among those preparing to teach upper-level secondary grades – linear algebra, calculus, and probability (Tatto and Hordern 2017).

Cross-References

- ► Communities of Inquiry in Mathematics Teacher Education
- ► Communities of Practice in Mathematics Teacher Education
- ► Education of Mathematics Teacher Educators
- ► Mathematics Teacher as Learner
- ► Mathematics Teachers and Curricula
- ► Models of In-service Mathematics Teacher Education Professional Development
- ► Models of Preservice Mathematics Teacher Education
- ► Teacher Education Development Study-Mathematics (TEDS-M)

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Mathematics Teacher Educator as Learner

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Action research · Intervention research · Lifelong learning · Professional development · Reflection · Role model · Teacher education

Mathematics Teacher Educators: Definition

Mathematics teacher educators in postsecondary institutions are academics educating prospective or practicing teachers; in many cases they do both. Thus, teacher educators initiate, guide, and support teacher learning across the teacher's professional lifespan (see also the entry ▶ "Education of Mathematics Teacher Educators" and, Even and Ball 2009). Most teacher educators have the task not only to teach (and to evaluate their teaching) but also to do research (including systematic and self-critical evaluation) and to do organizational administrative work. The quality of teaching, research, and organization is based on teacher educators' attitude towards and competence in continuous learning. The more complex teacher education activities are (e.g., running a challenging master's program or leading a professional development program for a couple of schools), the more the components of teaching, research, and organization are interwoven and influence each other.

Since teachers also have the task to teach, to critically reflect on their work (and maybe to do or be involved in research), and to do administrative work, observing teacher educators' actions may serve as a learning opportunity for teachers. Thus, teacher educators can be seen as role models for teachers. This makes teacher education a complex endeavour (see Krainer and Llinares 2010) since a serious teacher educator needs to live the goals he or she is claiming to his or her participants: it would be inconsistent and an obstacle for the learning process if, for example, a teacher educator stresses students' active learning but mainly designs his or her courses in a way where passive learning (listening to lectures) is dominating. This affords teacher educators to reflect the (explicit or implicit) "learning theory" underlying their teaching and – in best case – to make it transparent and discussable in the teacher education process. One of the challenges of teacher educators is to create genuine learning situations for teachers, often through carefully designed tasks, in which teachers experience as learners the kind of learning that the mathematics teacher educator wishes to convey (Zaslavsky 2007).

Mathematics Teacher Educators' Learning Through Research

Research on teacher educators' learning as practitioners is sparse, however increasing (see, e.g., in general: Russell and Korthagen 1995; Cochran-Smith 2003; Swennen and van der Klink 2009; directly related to mathematics teacher education: Zaslavsky and Leikin 2004; Even 2005; Jaworski and Wood 2008) with growing interest in the mathematics education community evidenced by discussion groups in recent international mathematics education conferences (e.g., PME 35 proceedings and ICME 12 preconference proceedings). Most opportunities for teacher educators to learn are not offered as formal courses. Such formats are discussed in the entry ► "Education of Mathematics Teacher Educators." The emphasis here is on teacher educators' autonomous efforts to learn, in particular, through reflection and research on their practice.

Teachers' ability to critically reflect on their work is a crucial competence (see, e.g., Llinares and Krainer 2006). Teacher educators need to evoke this inquiry stance (link to entry ► "Inquiry-Based Mathematics Education") of teachers as a basis of their learning. From this perspective, teacher educators learn from their practice through ongoing reflection on their thinking and actions as an inherent aspect of their work with teacher (i.e., as reflective practitioners – Schön 1983) and/or through systematic, intentional inquiry of their teaching in order to create something new or different in terms of their knowledge, "practical theories" (see Altrichter et al. 2008, pp. 64–72), and teaching. However, this dual role of researcher and instructor when educators inquire into their own practice puts a special focus on the question of how teacher education and research are interwoven.

A survey of recent research in mathematics teacher education published in international journals, handbooks, and mathematics education conference proceedings (see Adler et al. 2005) claims that most teacher education research is conducted by teacher educators studying the teachers with whom they are working. Such studies could involve studying characteristics of their students or the instructional approaches in which they engage their students. This presents a challenging situation for educator-researchers who need to reflect on their dual role to guard against unintentional biases that could influence the outcome of the research and their learning. For example, "success stories" that dominate the research literature may suggest that teacher educators' learning generally involves situations that improve teachers' learning and knowledge. However, this could be explained at least by two reasons: such published research of teacher education projects might be planned more carefully than others, and the readability to publish successful projects is higher than to publish less successful ones.

In spite of this challenge, there are good reasons for teacher educators to study teachers' learning through their own courses and programs. In system theory it is taken for granted that we only have a chance to understand a system (e.g., teachers in a mathematics teacher education course) if we try to bring about change in this system. This means that trying to understand is important to achieve improvement, and trying to improve is important to increase understanding. However, the researcher needs to reflect carefully on the strengths and weaknesses of distance and nearness to the practical field being investigated. For example, telling a "rich story", taking into account systematic self-reflection on one's own role as a teacher educator and researcher in the process, being based on a viable research question and building on evidence and critical data-analysis, is an important means to gather relevant results in teacher education research.

Mathematics Teacher Educators' Learning Through Action Research and Intervention Research

Action research and intervention research are two of the common methods mathematics teacher educators might engage in when conducting research as a basis of their learning. These methods allow them to investigate their own practice in order to improve it. This investigation process might be supported by other persons, but it is the teacher educators who decide which problem is chosen, which data are gathered, which interpretations and decisions are taken, etc. Action research challenges the assumption that knowledge is separate from and superior to practice. Thus, through it, teacher educators' production of "local knowledge" is seen as equally important as general knowledge, and "particularization" (e.g., understanding a specific student's mathematical thinking) is seen as equally important as "generalization" (e.g., working out a classification of typical errors).

"Intervention research" (see, e.g., Krainer 2003) done by teacher educators to investigate teachers' learning can take place in their classrooms influenced by interventions of their colleagues or often – as research shows – by their own interventions (e.g., see Chapman 2008) or in the field where it does not only apply knowledge that has been generated within the university, but much more, it generates "local knowledge" that could not be generated outside the practice. Thus, this kind of research is mostly process oriented and context bounded, generated through continuous interaction and communication with practice. Intervention research tries to overcome the institutionalized division of labor between science and practice. It aims both at balancing the interests in developing and understanding and at balancing the wish to particularize and generalize. Action research as intervention research done by practitioners themselves (first-order action research) can also provide a basis for teacher educators to investigate their own intervention practice (second-order action research, see, e.g., Elliott 1991).

Worldwide, there is an increasing number of initiatives in mathematics education based on action research or intervention research. However, most of them are related to teachers' action research (see, e.g., Chapman 2011; Crawford and Adler 1996; papers in JMTE 6(2) and 9(3); Benke et al. 2008; Kieran et al. 2013). In some cases, even the traditional role names (teachers vs. researchers) are changed in order to express that both, individual learning and knowledge production for the field, are a two-way street. For example, in the Norwegian Learning Communities in Mathematics (LCM) project (Jaworski et al. 2007), the team decided to replace "researchers and practitioners" with "teachers and educators" ("both of whom are also researchers"). There are a lot of projects in which teachers document their (evidence-based) experiences in reflecting papers. In Austria, for example, nearly 1000 papers – written by teachers for teachers – have been gathered since the 1980s within the context of programs like PFL (see, e.g., Krainer 1998) and IMST (Pegg and Krainer 2008; Krainer and Zehetmeier 2013) and can be searched by key word in an Internet database (http://imst.ac.at). The most extensive and nationally widespread version of action research by teachers is practiced in Japan within the framework of "lesson study" (see, e.g., Hart et al. 2011). In general, teacher educators who participate directly or indirectly in such cases of teachers' action research are afforded opportunities to learn in and from these experiences.

Cross-References

▶ Education of Mathematics Teacher Educators
▶ Inquiry-Based Mathematics Education

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Mathematics Teacher Identity

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Identity · Mathematics identity · Teacher identity

Definition

Mathematics teacher identity (MTI) is commonly "defined" or conceptualized in recent publications of the mathematics education research community as ways of being, becoming, and belonging; as developing trajectories, and in narrative and discursive terms.

A Brief History

The concept of identity can be traced to Mead (1934) and Erikson (1968), the former seeing identity as developed in interaction with the environment, and thus multiple, though it appears more unified to the individual (Lerman 2012). The latter saw identity as something that develops throughout one's life and is seen as more unified. The study of teacher identity is more recent. Perspectives focus on images of self (Nias 1989) as determining how teachers develop, or on roles (Goodson and Cole 1994). One can argue that societal expectations and perceptions and at the same time the teacher's own sense of what matters to them play key roles in teachers' professional identity. Beijaard et al. (2004) argue, in their review, that 1988 saw the emergence of teacher identity as a research field. Special issues of teacher education journals focusing on teacher identity attest to this (e.g., Teaching and Teacher Education 21, 2005; Teacher Education Quarterly, June 2008).

Identity Research in Mathematics Education and MTI

Darragh's (2016) examination of literature on identity within *mathematics* education journals over the past two decades indicates "an explosion" (p. 19) of papers relating to identity. She notes that "the largest outside influences on identity appear to be Wenger (1998) and/or Lave and Wenger (1991)" (p. 23). Indeed Lave and Wenger (1991, p. 115) argued that "learning and a sense of identity are inseparable: They are aspects of the same phenomenon." Most identity research in mathematics education however is clustered in a few regions or countries. Darragh (2016, p. 23) notes that "The largest number of studies were located in the US (36%); 15% came from the UK, 11% from Europe and 5% from each of Australia, South Africa and New Zealand." See *Mathematics Learner Identity* entry for other theorists influencing identity research focused on mathematics learners.

Within this explosion of identity research in mathematics education, *teacher* identity research has gained prominence. Darragh's (2016) review indicates that just under half of all identity articles she reviewed focused on teachers or pre-service teachers' professional identities as mathematics teachers or their mathematical identities as teachers in general.

Despite increasing engagement with mathematical learning and identity, many have argued that the notion of identity is not operationalized. See for example, Sfard and Prusak (2005) for a critique on identity literature and their subsequent provision of a narrative and operationalized definition. MTI is increasingly accepted as a dynamic rather than a fixed construct even while debates continue as to whether an individual has one identity with multiple aspects or multiple identities (see Grootenboer and Ballantyne 2010). Such interpretations of identity point to teacher agency to reconstruct or re-author her story through participation in various mathematics education practices, particularly in the context of mathematics teacher support (e.g., Hodgen and Askew 2007; Lerman 2012). In contexts where mathematics teacher morale is low, and teachers are identified as mathematically deficient, identity research encourages teacher education programs to focus on the re-authoring of negative and damaging narratives (e.g., Graven 2012).

Clusters of Research in MTI

Several clusters of issues can be identified in MTI research. These include the following:

1. Discipline specialization is considered to be highly significant in teacher identity both generally and in mathematics teacher research specifically (Hodgen and Askew 2007). Connecting teacher identity and teacher emotion is argued by some to be particularly important in relation to mathematics teacher identity where many teachers teach the subject without disciplinary specialization in their teacher training and with histories of negative and/or very procedural/traditional experiences of learning mathematics within their own schooling (Hodgen and Askew 2007; Grootenboer and Zevenbergen 2008; Lerman 2012).
2. Research into mathematics teacher identities often deals separately with primary nonspecialist teachers, who teach across subjects, and with secondary teachers, who teach only or predominantly mathematics and who may or may not have specialized in mathematics in their pre-service education. The nature of the way in which the discipline specificity of mathematics influences teacher identities differs in relation to whether one is identified as a generalist or a mathematics teacher. While it may be internationally accepted that many more secondary mathematics teachers have discipline-specific training in their pre-service studies, the extent to which this is the case differs across countries. As Grootenboer and Zevenbergen (2008) point out, depending on the extent of the shortage of qualified mathematics teachers, secondary school mathematics classes are often taught by nonspecialist teachers. Shortage of qualified mathematics teachers can be particularly high for developing countries. In this respect, research into supporting such teachers to strengthen their mathematics teacher identities becomes important. Graven (2004) describes how out-of-field teachers participating in a long-term mathematics teacher in-service program transformed their identities from accidental "teachers of mathematics" to "professional mathematics teachers" with trajectories of further studies in the subject. Research also tends to deal separately with either pre-service, pre-service and beginner teachers, or in-service teachers, as the way in which identities evolve for these groups of teachers differs in relation to the different practices in which they participate. For example, in the Australian context, Goos and Bennison (2008) research the development of a communal identity as beginning teachers of mathematics through the emergence of an online community of practice.
3. MTI has also been foregrounded in relation to studies researching mathematics teacher retention. The ICME-12 (2012) Discussion Group (DG11) on teacher retention included as a key theme the notion of identity and mathematics teacher retention. Several of the papers presented in this DG highlighted the role of strengthened professional identities, increased sense of belonging, and development of leadership identities as enabling factors contributing to teacher retention. Presenters in this discussion group were from the USA, South Africa, Israel, New Zealand, Norway, and India. Research on mathematics teacher identity seems to be of particular interest in these countries as well as in the UK and Australia (see reference list). Similarly research into the relationship between teacher identity and sustaining commitment to teaching (more generally than only for mathematics teaching) has been argued across USA and Australian contexts (e.g., Day et al. 2005).
4. Another cluster of research focuses on teacher identities in relation to curriculum specificities. In the Australian context, where numeracy is to be taught by all teachers across the curriculum, Bennison (2015) developed an analytic lens for researching identity of mostly non-

mathematics teachers as embedders-of-numeracy. A growing area of research in MTI explores the relationship between mathematics "teacher change/learning" and radical curriculum change. This research often points to disjuncture (contradiction) between mathematics teacher identities and expectations of reform mandates (Schifter 1996; Van Zoest and Bohl 2005; Lasky 2005; Westaway and Graven 2018). Research also investigates the relationship between teacher identity and assessment policy and in particular the increasing use of national standardized assessments across various contexts (Morgan et al. 2002; Pausigere and Graven 2013a).

5. Connected with the cluster above that researches mathematics teacher identities in relation to curriculum and assessment policies is a cluster of work that draws on Bernstein's macro perspective on the way policy, curriculum, and assessment practices shape teacher identity. His work has been used to complement localized analyses of identity within teacher communities with a broader concept of identity connected to macro structures of power and control. Bernstein first introduced the concept of identity in 1971 (Bernstein and Solomon 1999). This analysis did not focus on identity in terms of regulation and realization in practice but rather on identity in terms of the "construction of identity modalities and their change within an institutional level" (p.271). Thus Bernstein approaches identity from a broader systemic level, which of course impacts on enabling and constraining the emergence of localized individual teacher identities. Bernstein's notion of "Projected Pedagogic Identities" (Bernstein and Solomon 1999) provides a way of analyzing macro-promoted identities within contemporary curriculum change, which is the context within which teacher roles are elaborated in curriculum documents. South African and British Mathematics Educators have particularly drawn on the work of Bernstein to analyze positions available to teachers within often contradictory and shifting "official" discourses. (See for example Morgan et al. 2002; Naidoo and Parker 2005; Pausigere and Graven 2013b).

Concluding Comments

Identity research in mathematics education seems to focus on either learners or teachers. Darragh's (2016) review indicated only 2% of articles focused on the mathematical identities of both teachers and their learners, which would indicate that research into the relationship between teacher and learner identities is under-researched. The works of Heyd-Metzuyanim (2013) and Heyd-Metzuyanim and Graven (2016) are two examples of recent attempts to examine *the relationship* between mathematics learner identities and mathematics teacher identities.

The references suggest that as in the case of identity research in education more broadly, mathematics teacher identity research is not necessarily a global perspective. Research on international interpretations of the relevance of the notion is needed. At the same time the notion is ubiquitous in the social sciences and mathematics education researchers working with "identity" need to specify how they are using the term, what the sources are for their perspectives, and the relevance for the teaching and learning of mathematics.

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Cross-References

► Mathematics Learner Identity

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Mathematics Teachers and Curricula

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Definition and Historical Background

The word curriculum has had several meanings over time and has been interpreted broadly not only as a project about *what* should be learned by students but, in the context of teachers and curriculum, as all the experiences which occur within a classroom. These different meanings are grounded in different assumptions about teaching and the nature of interactions of the teacher with ideas that support curriculum guidelines (Clandinin and Connelly 1992). These different meanings have defined several roles of teachers in mathematics curriculum development that can be described as the history of a shift from teachers as curriculum users to teachers as curriculum interpreters and/or curriculum makers. Whereas the former view assumes curricula to be "teacher-proof," the latter includes teachers' activities like reflecting, negotiating issues of curricula and disseminating to their peers. This shift mirrors acknowledgment of the centrality of the teacher in curricula issues (Clarke et al. 1996; Hershkowitz et al. 2002; Lappan et al. 2012) and viewing teachers as key stakeholders of educational change (Kieran et al. 2013). These meanings are located along a continuum from a view of curricula as fixed, embodying discernible and complete images of practice to a view of curricula guidelines as influencing forces in the construction of practice.

In the 1970s, Stenhouse (1975) defined curriculum as "an attempt to communicate the essential principles and features of an educational proposal in such a form that it is open to critical scrutiny and capable of effective translation into practice" (p. 4). The teacher is central to this translation into practice. A model that is commonly used for analysis in mathematics education sees curricula as located at three levels: the *intended curriculum* (at the system level, the proposal), the *implemented curriculum* (at the class level, the teacher's role), and the *attained curriculum* (at the student level, the learning that takes place) (Clarke et al. 1996).

Focusing on the implemented curriculum, Stenhouse began the "teachers as researchers" movement. He believed that the "development of teaching strategies can never be *a priori*. New strategies [principled actions] must be worked out by groups of teachers collaborating within a research and development framework [. . .] grounded in the study of classroom practice" (p. 25). The development of this idea in the mathematics education field illustrated the complexity of teaching and the key roles played by teachers, underlining the importance of teachers' processes of interpretation of curricula materials (Zack et al. 1997). This role of mathematics teachers in the development of curricula has been highlighted by the recent technological advances favoring cooperative work among teachers in design tasks (e.g., e-textbooks) that have been seen as interfaces between policy and practice. This new position underscores the role of teachers' authority in the curriculum design process (Pepin et al. 2016). As a consequence, new perspectives are being generated to understand how the relationships between teachers and curriculum change when teachers gain experience through professional learning opportunities (Remillard et al. 2009).

Different Cultures Shaping Different Forms of Interaction Between Teachers and Curricula

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The relation between teacher and curricula depends on internal and external influences. Teachers frame their approach to curricula differently, dependent on their conceptions of different components of curricula and/or through the different structures of professional development initiatives (Remillard et al. 2009). Locally, teachers' knowledge and pedagogical beliefs are influences as they engage with curricula materials. Furthermore, the content and form of curricula materials influence the ways in which teachers interpret, evaluate, and adapt these materials considering their students' responses and needs in a specific institutional context.

Globally, countries have different curricular traditions shaping different conditions for teachers' roles in curriculum development. Thus, the diversity of cultures and features of each country's system generate different modes of interaction between teachers and curricula, as well as different needs and trends in teacher professional development (Clarke et al. 1996). However, results from international comparison assessments such as TIMSS and PISA are producing moves of mathematics curricula between countries (e.g., the translation of the Singapore curriculum to different countries due to good scores). This does not, then, reflect the cultural idiosyncrasy in different global regions in the world.

The main elements which have been proved to affect the relation between teachers and curricula, are, for instance, the distance that usually exists between the intended curriculum and the implemented curriculum; whatever the level of detail and prescription of the curriculum description, the implemented curriculum remains a subtle composition of the old and the new. In this sense, curricula are related with teacher practice, and curricula change is linked to how teachers continuously further develop or change their current practice, in particular with regard to teaching and assessment and professional development initiatives (Krainer and Llinares 2010).

Teachers and Curricula Within a Collaborative Perspective

From this view of interaction between teacher and curriculum, curriculum development initiatives are a context for teacher professional development reconstructing wisdom through inquiry. There is a long tradition of teachers developing curriculum materials in collaborative groups.

In the United Kingdom in the late 1970s and early 1980s, Philip Waterhouse's research (2001, updated by Chris Dickinson), supported by the Nuffield Foundation, led to the founding of a number of curriculum development organizations called Resources for Learning Development Units. In these units, the mathematics editor (one of a cross-curricular team of editors) worked with groups of not more than ten teachers, facilitating their work on either developing materials related to government initiatives or from perceived needs of teachers themselves. The explicit focus for the teachers was on the development and then production of materials that had been tried out in their classrooms. However, the implicit focus of the editor was on the professional development of those teachers in the groups. Also, in France, since the 1970s, the IREM network has functioned on the basis of mixed groups of academics, mathematicians, and teachers inquiring, experimenting in classrooms, producing innovative curriculum material, and organizing teacher professional development sessions relying on their experience (e.g., www.univ-irem.fr/). In recent views of how teachers interact with, draw on, refer to, and are influenced by curriculum resources, teachers are challenged to express their professional knowledge keeping a balance between the needs of their specific classrooms and their conceptions. In many countries, as mathematics education research has matured, there is increasing development of curriculum materials by teachers themselves working collaboratively and the organization of teacher professional development, for example, Sésamath, a French online association of mathematics teachers to design curriculum materials collaboratively.

Barbara Jaworski, working in Norway from 2003 to 2010, has led research projects in partnership with teachers to investigate "Learning Communities in Mathematics" and "Teaching Better Mathematics" (see, e.g., Kieran et al. 2013). In Canada, led by Michael Fullan, there is a large-scale project supporting professional development of teachers through curriculum reform in literacy and numeracy based on in-school collaborative groupings of teachers attending a central "fair" to present their inquiry work once a year. This project, *Reach Every Student, energizing Ontario Education*, works on the attained curriculum through the implemented one and has led to Fullan's (2008) book *Six Secrets of Change.* In the Latin-American context, the "praxis perspective" adopted in development of curricula in Costa Rica from 2012 to 2015 underlines the role played by different factors such as defining opportunities of teachers' professional development, the strategic role played by the online interaction, and the influence of different forms of assessment on teacher practice.

With the spread of ideas through international conferences, meetings and research collaborations, ideas such as the Japanese "lesson study" have spread widely (Alston 2011). Lesson study is a professional development process where teachers engage in systematically examining their practice. It is considered to be a means of supporting the dissemination of documents like standards, benchmarks, and nationally validated curricula. These multiple views define distinctive professional development pathways through curricula reforms. These pathways influence teachers' professional identities and work practices. Another example is "learning study" where teachers collaborate (with or without a researcher) with the aim of enhancing student learning of a particular topic (Runesson 2008). By carefully and systematically studying their classroom teaching and students' learning, teachers explore what students must learn in order to develop a certain capability. Learning study is based on an explicit learning theory (variation theory, Lo 2012).

Social perspectives on the role of teachers in curricula reforms are being reported by Kieran and others (2013), where the major focus is on the role and nature of teachers' interactions within a group of teachers. From this perspective, teachers are motivated by collaborative inquiry activities (teams, communities, and networks) aiming at interpreting and implementing curricula materials, as a way of "participation with" (Remillard et al. 2009, Pegg and Krainer 2008). These engagements must be understood in light of their particular local and global contexts. Teachers' learning through collaborative inquiry activities, contextualized in curriculum development initiatives, has allowed the contextual conditions in which curriculum is implemented in different traditions to be made explicit. Pegg and Krainer (2008) reported examples of large-scale projects involving national reform initiatives in mathematics where the focus was initiating purposeful pedagogical change through involving teachers in rich professional learning experiences. The motivation for these initiatives was a perceived deficiency in students' knowledge of mathematics (and science) understood as the attained curriculum. In all of these programs, collaboration, communication, and partnerships played a major role among teachers and university staff members of the program. In these programs, the teachers were not only seen as participants but crucial change agents who were regarded as collaborators and experts (Pegg and Krainer 2008). This view of teachers as change agents emerged from the close collaboration among groups of stakeholders and the different forms of communications that developed. From all those variables defining the relationships between teachers and curricula, how curricula principles move between cultures have begun to appear as key issues (e.g., comparison and analysis of textbooks from different cultures, Leung et al. 2006).

Open Questions

The relationship between teacher and curricula defines a set of open questions in different realms. These questions are linked to the fact that the relationship between teachers and curricula is moving, due to a diversity of factors: the increasing autonomy and power given to teachers regarding curriculum design and implementation in some countries at least, the development of collaborative practices and networks in teachers' communities, the evolution of relationships between researchers and teachers, the explosion of curriculum resources and their easier accessibility thanks to the internet, the impact of international comparisons favoring the moving of curricular principles between cultures, etc. Thus, some open questions are:

1. What are the implications of the school-based partial transfer of power in curriculum decision-making to teachers based on teachers' practical, personal reflective experience and networks?
2. What role do collegial networks play in how ideas about curricula change are shared

including when the design uses the affordance of digital curriculum resources (e.g., using electronic communications and online platforms to share the curriculum resources)?

3. How do new kinds of practices and teaching objectives emerge as a consequence of new resources influencing the relation between teacher and curricula?
4. How can reform initiatives cope with the balance between national frameworks for curricula (e.g., educational standards as expressions of societal demands) and local views on curricula as negotiated between the teachers of one school?
5. How does the exchange between cultures influence the curriculum-teacher relationships and how could sociocultural theories explain these influences?
6. What role do students play in ideas related to curricula (e.g., starting topics based on students' interests, questions, and so on)?

Cross-References

- ▶ Communities of Practice in Mathematics Education
- ▶ Curriculum Resources and Textbooks in Mathematics Education
- ▶ Education and Professional Development of Teacher Educators
- ▶ Learning study in Mathematics Education
- ▶ Lesson Study in Mathematics Education
- ▶ Mathematics Teacher as Learner
- ▶ Models of In-service Mathematics Teacher Education Professional Development

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Mathematization as Social Process

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Keywords
Mathematization · Demathematization · Mathematics in action · Technological imagination · Hypothetical reasoning · Justification · Legitimation · Realization · Elimination of responsibility · Critical mathematics education

Definition

Mathematization refers to the formatting of production, decision-making, economic management, means of communication, schemes for surveilling and control, war power, medical techniques, etc., by means of mathematical insight and techniques.

Mathematization provides a particular challenge for mathematics education as it becomes important to develop a critical position to mathematical rationality as well as new approaches to the construction of meaning.

Characteristics

The notions of mathematization and demathematization, the claim that there is mathematics everywhere, and mathematics in action are addressed, before we get to the challenges that mathematics education is going to face.

Mathematization and Demathematization

It is easy to do shopping in a supermarket. One puts a lot of things into the trolley and pushes it to the checkout desk. Here an electronic device used by the cashier makes a pling-pling-pling melody, and the total to be paid is shown. One gets out the credit card, and after a few movements by the fingers, one has bought whatever. No mathematics in this operation.

However, if we look at the technologies that are configuring the practice of shopping, one finds an extremely large amount of advanced mathematics being brought in operation: The items are coded and the codes are read mechanically; the codes are connected to a database containing the prices of all items; the prices are added up; the credit card is read; the amount is subtracted from the bank account associated to the credit card; security matters are observed; schemes for coding and decoding are taking place.

We have to do with a mathematized daily practice, and we are immersed in such practices. We live in a mathematized society (see Keitel et al. 1993, for an initial discussion of such processes). Gellert and Jablonka (2009) characterize the mathematization of society in the following way: "Mathematics has penetrated many parts of our lives. It has capitalised on its abstract consideration of number, space, time, pattern, structure, and its deductive course of argument, thus gaining an enormous descriptive, predictive and prescriptive power" (p. 19).

However, most often the mathematics that is brought into action is operating beneath the surface of the practice. At the supermarket, there is no mathematics in sight. In this sense, as also emphasized by Jablonka and Gellert (2007), a demathematization is accompanying a mathematization.

There Is Mathematics Everywhere

Mathematization and the accompanying demathematization have a tremendous impact on all forms of practices. Mathematics-based technology is found everywhere.

One can see the modern computer as a materialized mathematical construct. Certainly the computer plays a defining part of a huge range of technologies. It is defining for the formation of databases and for the processing of information and knowledge.

Processes of production are continuously taking new forms due to new possibilities for automatization, which in turn can be considered a materialized mathematical algorithm. Any form of production – being of TV sets, mobile phones, kitchen utensils, cars, shoes, whatever – represents a certain composition of automatic processes and manual labor. However, this composition is always changing due to new technologies, new needs for controlling the production process, new conditions for outsourcing, and new salary demands. Crucial for such changes is not only the development of mathematics-based technologies of automatization but also of mathematics-based procedures for decision-making.

In general mathematical techniques have a huge impact on management and decision-making (see, for instance, O'Niel 2016). As an indication, one can think of the magnitude of cost-benefit analyses. Such analyses are crucial, in order not only to identify new strategies for production and marketing but to decision-making in general. Complex cost-benefit analyses depend on the calculation power that can be executed by the computer. The accompanying assumption is that a *pro et contra* argumentation can be turned into a straightforward calculation. This approach to decision-making often embraces an ideology of certainty claiming that mathematics represents objectivity and neutrality. Thus, in decision-making we find an example not only of a broad application of mathematical techniques but also an impact of ideological assumptions associated with mathematics.

Mathematics-based technologies play crucial roles in different domains, and we can think of medicine as an example. Here we find mathematics-based technologies for making diagnoses, for defining normality, for conducting a treatment, and for completing a surgical operation. Furthermore, the validation of medical research is closely related to mathematics. Thus, any new type of medical treatment needs to be carefully documented, and statistics is crucial for doing this.

Not only medicine but also modern warfare is mathematized. As an example one can consider the drone, the unmanned aircraft, which has been used by the USA, for instance, in the war in Afghanistan. The operation of the drone includes a range of mathematics brought in action. The identification of a target includes complex algorithms for pattern recognition. The operation of a drone can only take place through the most sophisticated channels of communication, which in turn must be protected by advanced cryptography. Channels of communication as well as cryptography are completely mathematized. The decision of whether to fire or not is based on cost-benefit analyses: Which target has been identified? How significant is the target? What is the probability that the target has been identified correctly? What is the probability that other people might be killed? What is the price of the missile? Mathematics is operating in the middle of this military logic.

Mathematics in Action

The notion of mathematics in action – that can be seen as a further development of "formatting power of mathematics" (Skovsmose 1994) – can be used for interpreting processes of mathematization (see, for instance, Christensen et al. 2009; Skovsmose 2009, 2014, 2016; Yasukawa et al. 2016). Mathematics in action can be characterized in terms of the following issues:

Technological imagination refers to the conceptualization of technological possibilities. We can think of technology of all kinds: design and construction of machines, artifacts, tools, robots, automatic processes, networks, etc.; decision-making concerning management, advertising, investments, etc.; and organization with respect to production, surveillance, communication, money processing, etc. In all such domains, mathematics-based technological imagination has been put into operation. A paradigmatic example is the conceptualization of the computer in terms of the Turing machine. Even certain limits of computational calculations were identified before any experimentation was completed. One can also think of the conceptualization of the Internet, of new schemes for surveilling and robotting (see, for instance, Skovsmose 2012), and of new approaches in cryptography (see, for instance, Skovsmose and Yasukawa 2009). In all such cases, mathematics is essential for identifying new possibilities.

Hypothetical reasoning addresses consequences of not-yet-realized technological constructions and initiatives. Reasoning of the form "if p then q, although p is not the case" is essential to any kind of technological enterprise. Such hypothetical reasoning is most often model based: one tries to grasp implications of a new technological construct by investigating a mathematical representation (model) of the construct. Hypothetical reasoning makes part of decision-making about where to build an atomic power plant, what investment to make, what outsourcing to make, etc. In all such cases one tries to provide a forecasting and to investigate possible scenarios using mathematical models. Naturally a mathematical representation is principally different from the construct itself, and the real-life implication might turn out to be very different from calculated implications. Accompanied by (mischievous) mathematics-based hypothetical reasoning, we are entering the risk society.

Legitimation or *justification* refers to possible validations of technological actions. While the notion of justification includes an assumption that some degree of logical honesty has been exercised, the notion of legitimation does not include such an assumption. In fact, mathematics in action might blur any distinction between justification and legitimation. When brought into effect, a mathematical model can serve any kind of interests.

Realization refers to the phenomenon that mathematics itself comes to be part of reality, as was the case at the supermarket. A mathematical model becomes part of our environment. Our life-world is formed through techniques as well as through discourses emerging from mathematics. Real-life practices become formed through mathematics in action. It is this phenomenon that has been referred to as the formatting power of mathematics.

Elimination of responsibility might occur when ethical issues related to implemented action are removed from the general discourse about technological initiatives. Mathematics in action seems to be missing an acting subject. As a consequence, mathematics-based actions easily appear to be conducted in an ethical vacuum. They might appear to be determined by some "objective" authority as they represent a logical necessity provided by mathematics. However, the "objectivity" of mathematics is a myth that needs to be challenged.

Mathematics in action includes features of imagination, hypothetical reasoning, legitimation, justification, realization including a dematematization of many practices, as well as an elimination of responsibility. Mathematics in action represents a tremendous knowledge-power dynamics.

New Challenges

Mathematics in action brings about several challenges to mathematics education of which I want to mention some.

Over centuries mathematics has been celebrated as crucial for obtaining insight into nature, as being decisive for technological development, and as being a pure science. Consistent or not, these assumptions form a general celebration of mathematics. This celebration can be seen as almost a defining part of modernity. However, by acknowledging the complexity of mathematics in action, such celebration cannot be sustained. Mathematics in action has to be addressed critically in all its different instantiations. Like any form of action, mathematics in action may have any kind of qualities, such as being productive, risky, dangerous, benevolent, expensive, dubious, promising, and brutal. It is crucial for any mathematics education to provide conditions for reflecting critically on any form of mathematics in action.

This is a challenge to mathematics education both as an educational practice and research. It becomes important to investigate mathematics in action as part of complex sociopolitical processes. Such investigations have been developed with reference to ethnomathematical studies, but many more issues are waiting for being addressed (see, for instance, D'Ambrosio's 2012 presentation of a broad concept of social justice).

Due to processes of mathematization and not least to the accompanying processes of dematematization, one has to face new challenges in creating meaningful activities in the classroom. Experiences of meaning have to do with experiences of relationships. How can we

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construct classroom activities that, on the one hand, acknowledge the complex mathematization of social practices and, on the other hand, acknowledge the profound demathematization of such practices? This general issue has to be interpreted with reference to particular groups of students in particular sociopolitical contexts (see, for instance, Gutstein 2012).

To break from any general celebration of mathematics, to search for new dimensions of meaningful mathematics education, and to open for critical reflections on any form of mathematics in action are general concerns of critical mathematics education (see also ► "Critical Mathematics Education" in this Encyclopedia).

Cross-References

- ► Critical Mathematics Education
- ► Critical Thinking in Mathematics Education
- ► Dialogic Teaching and Learning in Mathematics Education
- ► Mathematical Literacy

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Metacognition

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Keywords
Metamemory · Metacognitive knowledge · Metacognitive experiences · Metacognitive strategies

Definition

Any knowledge or cognitive activity that takes as its object, or monitors, or regulates any aspect of cognitive activity; that is, knowledge about, and thinking about, one's own thinking.

Characteristics

Although the construct, metacognition, is used quite widely and researched in various fields of psychology and education, its history is relatively short beginning with the early work of John Flavell on *metamemory* in the 1970s. Metamemory was a global concept encompassing a person's knowledge of "all possible aspects of information storage and retrieval" (Schneider and Artelt 2010). Flavell's (1979) model of metacognition and cognitive monitoring has underpinned much of the research on metacognition since he first articulated it. It was a revised version of his taxonomy of metamemory that he had developed with Wellman (Flavell and Wellman 1977). According to his model, a person's ability to control "a wide variety of cognitive enterprises occurs through the actions and interactions among four classes of phenomena: (a) metacognitive knowledge, (b) metacognitive experiences, (c) goals (or tasks), and (d) actions (or strategies)" (p. 906). *Metacognitive knowledge* incorporates three interacting categories of knowledge, namely, personal, task, and strategy knowledge. It involves one's (a) *sensitivity* to knowing how and when to apply selected forms and depths of cognitive processing appropriately to a given situation (similar to subsequent definitions of partly what is called *procedural metacognitive knowledge*), (b) intuitions about intra-individual and inter-individual differences in terms of beliefs, feelings, and ideas, (c) knowledge about task demands which govern the choice of processed information, and (d) a stored repertoire of the nature and utility of cognitive strategies for attaining cognitive goals. The first of these is mostly implicit knowledge, whereas the remaining three are explicit, conscious knowledge. *Metacognitive experiences* are any conscious cognitive or affective experiences which control or regulate cognitive activity. Achieving *metacognitive goals* are the objectives of any metacognitive activity. *Metacognitive strategies* are used to regulate and monitor cognitive processes and thus achieve metacognitive goals.

In the two decades that followed when Flavell and his colleagues had initiated research into metacognition (Flavell 1976, 1979, 1981), the use of the term became a buzzword resulting in an extensive array of constructs with subtle differences in meaning all referred to as metacognition (Weinert and Kluwe 1987). This work was primarily in the area of metacognitive research on reading; however, from the early 1980s, work in mathematics education had begun mainly related to problem solving (Lester and Garofalo 1982) particularly inspired by Schoenfeld (1983, 1985, 1987) and Garofalo and Lester (1985). Cognition and metacognition were often difficult to distinguish in practice, so Garofalo and Lester (1985) proposed an operational definition distinguishing cognition and metacognition which clearly demarcates the two, namely, cognition is "involved in doing," whereas metacognition is "involved in choosing and planning what to do and monitoring what is being done" (p. 164). This has been used subsequently by many researchers to be able to delineate the two.

Today, the majority of researchers in metacognitive research in mathematics education have returned to the roots of the term and share Flavell's early definition and elaborations (Desoete and Veenman 2006). The field has firmly established the foundations of the construct and by building on these foundations, several researchers have extended Flavell's work usefully and there is an expanding body of knowledge in the area. The elements of his model have been extended by others (e.g., elaborations of metacognitive experiences, see Efklides 2001, 2002) or are the subject of debate (e.g., motivational and emotional knowledge as a component of metacognitive knowledge, see Op 't Eynde et al. 2006). Subsequently, it has led to many theoretical elaborations, interventions, and ascertaining studies in mathematics education research (Schneider and Artelt 2010).

Flavell did not expect metacognition to be evident in students before Piaget's stage of formal operational thought, but more recent work by others has shown that preschool children already start to develop metacognitive awareness. Work in developmental and educational psychology as well as mathematics education has shown that metacognitive ability, that is, the ability to gainfully apply metacognitive knowledge and strategies, develops slowly over the years of schooling and there is room for improvement in both adolescence and adulthood. Furthermore, studying the developmental trajectory of metacognitive expertise in mathematics entails examining both frequency of use and the level of adequacy of utilization of metacognition. Higher frequency of use does not necessarily imply higher quality of application, with several researchers reporting such phenomena as *metacognitive vandalism*, *metacognitive mirage* and *metacognitive misdirection*. Metacognitive vandalism occurs when the response to a perceived metacognitive trigger ("red flag") involves taking drastic and destructive actions that not only fail to address the difficulty but also could change the nature of the task being undertaken. Metacognitive mirage results when unnecessary actions are engaged in, because a difficulty has been perceived, but in reality, it does not exist. Metacognitive misdirection is the relatively common situation where there is a potentially relevant but inappropriate response to a metacognitive trigger that is purely inadequacy on the part of the task solver not deliberate vandalism. Recent research shows that as metacognitive abilities in mathematics develop, not only is there increased usage but also the quality of that usage increases.

The popularity of the metacognition construct stems from the belief that it is a crucial part of everyday reasoning, social interaction as occurs in whole class and small group work and more complex cognitive tasks such as mathematical problem solving, problem finding and posing, mathematical modeling, investigation, and inquiry based learning.

Cross-References

► Problem-Solving in Mathematics Education

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Meta-didactical Transposition

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Keywords
Meta-didactical transposition · Teacher education · Praxeologies · Broker · Professional development

Definition

Meta-Didactical Transposition is a theoretical framework used to describe mathematics teachers' professional development as a process, comprising a number of variables and their possible changes over time. It was first introduced in Italy at the National Seminar in Didactics of Mathematics (http://www.seminariodidama.unito.it/mat12.php) and then more widely for the international community (Arzarello et al. 2014). It is based on Chevallard's Anthropological Theory of Didactics (Chevallard 1985 and Chevallard, this Encyclopedia) and the framework takes into account the relationships and reciprocal influences of two communities – the community of teachers and that of researchers, involved in professional development – with respect to their professional practices.

Meta-Didactical Transposition involves these intertwined features:

1. *Institutional aspects*
2. *Meta-didactical praxeologies*
3. The dynamics between *internal and external components*
4. The role of the *broker*
5. *Double dialectics*

Meta-Didactical Transposition has some consequences and applications in other countries, such as France and Australia (Aldon et al. 2013; Prodromou et al. 2017), and it has led to other studies, such as two research fora at conferences convened by the International Group for the Psychology of Mathematics Education (Aldon et al. 2013; Clark-Wilson et al. 2014) and other papers (e.g., Taranto et al. 2017).

Why Do We Need a Dynamic Framework Relating to Teachers' Practices?

Teacher education is a complex phenomenon encompassing different variables and contexts: recently, the theme of "teachers working and learning in collaboration" has been studied in a survey on international literature in the last years (Robutti et al. 2016). This theme can be approached from different perspectives: institutional, cognitive, didactical. The *Meta-Didactical Transposition* framework was developed by a team of researchers in 2012 to describe the processes involved in teacher education. It has been presented at the national seminar in didactics of mathematics (Arzarello et al. 2012[1]) and then disseminated on various occasions: Psychology of Mathematics Education, cross-countries seminars (PME), International Congress on Mathematics Education (ICME), The International Commission for the Study and Improvement of Mathematics Teaching (CIEAEM).

The framework was introduced as an attempt to describe teachers' practices in educational programs in a dynamic way, namely, as processes evolving over time (Arzarello et al. 2014) as a mean to capture the theoretical choices taken by academics involved in national programs of teacher education directed by the Italian Ministry of Education. While designing these teacher education programs, the necessity to introduce the framework emerged, as there was a sense of some-

[1] http://www.seminariodidama.unito.it/mat12.php

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thing missing from other frameworks found in literature (e.g., the content knowledge for teachers, by Ball and Bass 2003), which although strong and valued by the international community, did not completely fit with the Italian situation.

Our framework had to:

(a) Take into account the importance of the institutions in a way that considered not only the educational programs for teachers, but also the teachers' work in the classrooms. In Italy, as in many other European countries, the whole educational system (from kindergarten to university) is public and is governed by multiple institutions at different levels (national, regional, local). Alongside this, the institutional dimension has importance within the politics of the European Union. As lifelong education is considered a strategic element for development in Europe, educational programs are promoted for prospective or in-service teachers. These programs assume a clear cooperation between the research world and the institutional-political world (see http://ec.europa.eu/education/llp/official-documents-on-the-llp_en.htm). This led us to the Anthropological Theory of Didactics (Chevallard, this Encycopedia) as a theoretical basis for the development of Meta-Didactical Transposition, grounded by the assumption that the teaching of mathematics is considered to be contextualized within multiple institutions.

(b) Value the work of teachers in communities: Many teachers' educational programs in Italy (e.g., m@t.abel,[2] Piano Lauree Scientifiche,[3] MOOC[4]) are organized in small/large communities of teachers working together within a professional development project. These communities are more than communities of practice (Wenger 1998) as teachers who are involved in institutional programs have formal tasks to accomplish and practices to follow. In many cases, they can be considered as communities of inquiry, in the sense of Jaworski (2008, p. 313): "In terms of Wenger's (1998) theory, that *belonging to a community of practice* involves *engagement, imagination* and *alignment*, we might see the normal desirable state as *engaging* students and teachers in forms of practice and ways of being in practice with which they *align* their actions and conform to expectations . . . in an inquiry community, we are not satisfied with the normal (desirable) state, but we approach our practice with a questioning attitude, not to change everything overnight, but to start to explore what else is possible; to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to provide answers to them. In this activity, if our questioning is systematic and we set out purposefully to inquire into our practices, we become researchers." (see also ► "Communities of Inquiry in Mathematics Teacher Education", This Encyclopedia).

(c) Consider also the community of researchers who are involved in the educational program and who not only take the role of designers of the tasks for teachers, but also as trainers of the teachers and as academics who research the topic of teacher education, as happened to the Italian team in the development of Meta-Didactical Transposition.

(d) Acknowledge the fact that teachers work alongside researchers in such programs and that interactions between the two communities are at the core of the teachers' professional development, with a deep influence of the community of researchers on the community of teachers. Vice versa, the importance of the influence of the teachers' community on the researchers' community is also to be considered. In fact, what has characterized the Italian academic context in mathematics education over many years is exactly this productive interplay between teachers and researchers at the level of teacher education, research, and the implementation of teaching

[2] http://mediarepository.indire.it/iko/uploads/allegati/M7PWITOE.pdf

[3] http://www.dipmatematica.unito.it/do/home.pl/View?doc=pls.html

[4] http://www.difima.unito.it/mooc

experiments in the classes. In Italy, many universities welcome teachers' participation in research groups and researchers go to schools to work with students, alongside the teachers. The productive interaction between teachers (at different school levels) and researchers is one of the distinctive features in the development of Italian mathematics education, in terms of its theoretical and experimental approaches (Arzarello and Bartolini Bussi 1998).

(e) Last but not least, we needed to capture the professional development phenomena in a dynamic way, as they occur in process and not only giving snapshots at certain moments. During an educational program, teachers encountering a new didactical paradigm are changed by the experience, and this change is evident if we compare their attitude at the end of the program to that of the starting points. They evolve, embrace new ideas, viewpoints, practices, or simply gain awareness of the content they have met during the program. On the other hand, researchers may evolve too, resulting in changes in their practices and/or awareness. The interaction between the two communities is not neutral as it results in effects on both. A framework that emphasizes this evolution is a framework that takes into account the professional development as a process, not only a product, and describes it in a dynamic way (as a movie, not as a snapshot).

The previous points were the main reasons, in the Italian institutional context, that directed academics towards the development of the Meta-Didactical Transposition framework in 2012, along with a sense of missing something, if using other frameworks (e.g., Ball and Bass 2003) even strong and valued by the international community – but not completely fitting with the Italian situation.

The Meta-Didactical Transposition framework is constructed to highlight the need to take the complexity of teacher education into account with respect to the institutions in which the teachers operate, alongside the relationships that teachers must have with these institutions.

The team of researchers involved in developing the framework was composed by experienced and newcomer members, coming from two Universities: Torino and Modena. The experienced ones were Ferdinando Arzarello, Ornella Robutti, Nicolina Malara, and Rossella Garuti, while the newcomers were Cristina Sabena, Annalisa Cusi, and Francesca Martignone.

Meta-Didactical Transposition

The Meta-Didactical Transposition framework is introduced to describe the practices of both researchers and teachers, when they work together in the institutions (schools, Universities), within an educational program. We are referring typically to a community of teachers involved in a professional development (it could be an educational program of mathematics, or technology integrated in mathematics teaching, or other, at national or local level), planned and carried out by researchers with the role of designers of the program, and also as teacher educators.

This framework is based on the Chevallard's Anthropological Theory of Didactics (Chevallard 1985, 1992, and This Encyclopedia), which is grounded in the teaching of mathematics at school. However, Meta-Didactical Transposition extends this theory to the context of teacher education, usually fully situated within and constrained by the institutions, to take account of:

- The constraints imposed by the institutions (including schools, universities, policy makers, teachers' associations, mathematics society, and Ministry of Education) that promote teacher education in relation to some specific goals (e.g., promoting teachers' knowledge of new curricula, new teaching practices, or the integration of new technologies for the teaching of mathematics)
- The complexity of mathematics teachers' professional development situated in the institutions and involving teachers' and researchers' communities and the dialectics between the two communities

- Professional development being considered in a dynamic way, taking into account the evolutionary processes involved in the practices of members of the communities, that of researchers and that of teachers.

The *Institutional Aspects*

Chevallard's theory focuses on the institutional dimension of mathematical knowledge: it places mathematical learning and teaching firmly within the human activities related to it and in the context of social institutions. Actually, Chevallard stresses the fact that the very nature of mathematical objects in school depends on the person or the institution which it is related to: "An object exists since a person, or an institution acknowledges that it exists (for it itself)" (▶ "Didactic Transposition in Mathematics Education"). At the core of his theory are the notions of Didactic Transposition in Mathematics Education (Chevallard and Bosch, "Anthropological Theory of the Didactic (ATD) ," this Encyclopedia) and praxeology. According to Chevallard, the didactical transposition consists of the transformation of knowledge through different stages: the knowledge as it is produced and used at university level, the knowledge that is expected to be taught based on national curricula and syllabuses, and the knowledge taught by the teachers.

The Meta-Didactical Transposition (Arzarello et al. 2014; Aldon et al. 2013) framework places mathematical and professional learning – of teachers working together – in the human activities related to it and in the context of social institutions. This framework is useful to describe a process – analogous to the didactical transposition – that occurs when a *community of researchers* work with a *community of teachers* in a professional development activity. The term "meta-didactical" refers to the fact that important issues related to the didactical transposition of knowledge are faced at a meta-level. The involvement of researchers and teachers consists in:

- The researchers design and coach the educational programs, as a task commissioned by institutional authorities (e.g., school administration, Ministry of Education, teachers' associations), or as a course planned by other institutional authorities (university, research center, mathematical association, international project, or others). The program can be configured – for example – as a teachers' professional development only, or as a research project meant to collect and analyze data, or a dissemination of a research project.
- The teachers participate in the program, either on a voluntary basis or because of an official duty.

Both of these communities are in relationship with the school: the actual schools where the teachers teach, and the school as an institution with its curricula, its teaching traditions, the textbooks used, etc.

The *Meta-Didactical Praxeologies*

The main theoretical tool of the Anthropological Theory of Didactics (Chevallard 1992) is the notion of *praxeology*, a neologism made of two words derived by the Greek terms *praxis* and *logos*: *praxis* as the "know how," *logos* as the "knowledge." According to Chevallard, a praxeology consists of four interrelated *components*: task, technique, technology (used to mean justification), and theory. The given task and the technique used to solve the task are the practical counterpart of the praxeology (the *praxis*), while the technology (in the sense of justification) and the theory are the theoretical counterpart that validates the use of that technique (the *logos*). In a mathematics classroom, we can identify the mathematical type of task (e.g., T: determining the equation of the tangent to the graph of a generic function f) that students have to solve, the employed technique and the more or less explicit justification for using it, all within a specific mathematical theory. These components constitute the *mathematical praxeology*.

At the same time, there exist the teacher's questions and actions to build such a mathematical praxeology with her students, which gives birth to a *didactical praxeology*. What may occur is:

- The teacher introduces her students to a type of *task* (didactical type of task).

- The teacher has to manage *how to* organize such an approach (didactical technique).
- The teacher has to know *why she has to* organize it like that (didactical technology – in the sense of justification).
- The teacher has to explain *why she knows that she has to* organize it like that (didactical theory).

The Meta-Didactical Transposition framework includes the *meta-didactical praxeologies*, which comprise the tasks, techniques, and justifying discourses of researchers and of teachers. Referring to the four components of a praxeology for researchers:

- The researchers – as trainers – introduce the teachers – engaged in the professional development activity – to the type of *task* (meta-didactical type of task).
- The researchers have to manage *how to* organize such an approach (meta-didactical technique).
- The researchers have to know *why they have to* organize it like that (meta-didactical technology – in the sense of justification).
- The researchers have to explain *why they know that they have to* organize it like that (meta-didactical theory).

Referring to the four components of a praxeology for teachers:

- The teachers are introduced to a *task* (e.g., to design an activity of geometry for their class using a DGE software), within an institutional frame (e.g., national curriculum).
- The teachers have to solve the task using *some techniques*, according to the professional development.
- The teachers have to know *why they choose* such a solution.
- The teachers have to justify *why they know that they have* to organize it like that (to support their choices theoretically).

To exemplify, we can report a task for teachers in a teacher training course described by Sullivan (2008, p. 3) and quoted in Arzarello et al. (2014): the question "which number is bigger: 2/3 or 201/301?" is given to teachers as a basis of a lesson, and they have to design the steps to introduce the students to this task (e.g., the context of a baseball match, where a player's statistics shifts from 200/300 to 201/301). This task is a stimulus for teachers' reflection, and what they activate to solve it are meta-didactical techniques and their justifications (in mathematical and didactical terms). For example, based on one's professional experience, the teachers might discuss why the initial question presents difficulties for many students and why the baseball example makes sense in a classroom and thus might help students to overcome the associated difficulty and why it is necessary to foster the transition from every day to scientific and formal concepts, using a constructivist approach, according to a Vygotskian frame.

The praxeologies of researchers and teachers at the beginning of a professional development program can be far apart from each other, but then they can evolve towards the same *shared praxeologies* (Arzarello et al. 2014). A typical example is when a new praxeology (or some of its components) is developed by teachers in response to a stimulus in the program. As a consequence, there could be teachers' development of both a new awareness (at the cultural level) and new competences (at the methodological-didactical level, i.e., that of teaching practice), which lead them to activate, in their classrooms, a didactical transposition in line with the meta-didactical transposition. Simultaneously, a researchers' praxeology (or some components) also may evolve as a consequence of their interaction with the teachers and their reflections.

Evolution in the praxeologies does not mean that all the teachers (or researchers) involved in the educational program evolve in the same way with the same transformation of components: in fact, different teachers may evolve in different ways, with respect to their histories and experiences. Therefore, further research is necessary to investigate the factors that influence these different trajectories in the praxeologies.

The Dynamics Between *Internal and External Components*

Not only praxeologies evolve during professional development. Their components can transform as well (e.g., the technique, or the technology). The praxeological components can be considered as internal or external to a community. They are considered internal to a community when commonly shared and used by the members of the community, and external to a community when the members of the communities do not typically use them. The components can also be internal or external to one or some members of a community. The idea of external and internal (to refer to a praxeology component) is taken by Clark and Hollingsworth (2002 p. 951), who distinguish an external domain, located outside the teacher's personal world, from the internal domains, which "constitute the individual teacher's professional world of practice, encompassing the teacher's professional actions, the inferred consequences of those actions, and the knowledge and beliefs that prompted and responded to those actions."

Of course, the goal of a teacher professional development program is to promote the change of praxeological components that are initially external to the teachers' community into internal ones (e.g., activities using new technologies, such as new GeoGebra tools, or new pedagogical techniques, such as student-centered teaching approaches). Furthermore, the researchers participating within a teachers' professional development program may also benefit from transforming praxeological components that are external to their community into internal ones.

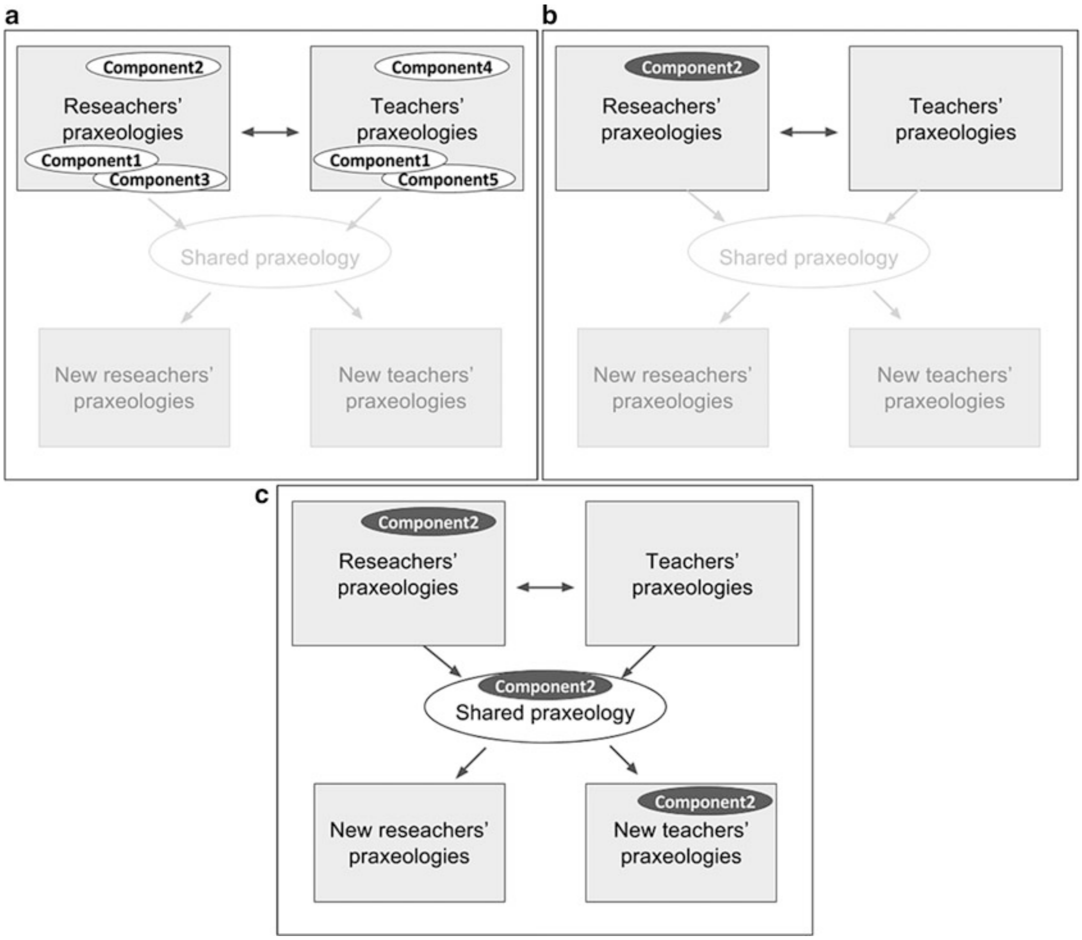
Figure 1 aims to explain this process. Within teachers' professional development, the researchers interact with the teachers, according to their praxeologies, and as a result of these interactions, the transformation of praxeological components from external to internal may occur. These components may evolve differently for different teachers, due to contextual factors, or to institutional influences, or attitudes towards teaching and mathematics, beliefs, and so on. If there is the same transformation of a component (2 in Fig. 1c) from external to internal for all the teachers, finally researchers and teachers share the corresponding praxeology that was initially internal (Fig. 1b) only to researchers (Prodromou et al. 2017).

To exemplify, a community of teachers starts a professional development program in which, due to some institutional situation (e.g., curriculum changes), a community of researchers introduces a specific ICT tool (e.g., a dynamic geometry software). At the end of the program, the initial techniques (and their theoretical counterpart) have become a new a set of shared techniques, as a result of the actions taken by researchers and teachers.

The Role of the *Broker*

The Meta-Didactical Transposition framework uses the notion of *broker* as a professional who belongs to more than one community and makes possible the exchanges between them: "Brokers [. . .] are able to make new connections across communities of practice, enable coordination, and – if they are good brokers – open new possibilities for meaning" (Rasmussen et al. 2009, p.109). In this way, brokers can facilitate the transition of mathematical concepts from one community to the other (*boundary crossing*), which is accomplished by drawing on *boundary objects*: "boundary objects are those objects that both inhabit several communities of practice and satisfy the informational requirements of each of them" (Bowker and Star 1999, p. 297). For example, a teacher belongs to the community of mathematics experts, to that of her colleagues in the school, and to her classroom community.

In the Italian community of academics in mathematics education, the role of broker is often played by a so-called teacher-researcher – who is part of the communities of researchers and of teachers – or it can be played also by a researcher, a PhD student, or a master student. The role of broker is fundamental in the exchange of information, techniques, justifications, theories, namely, all about praxeologies and their components. In fact, the role of the researchers is to organize research project in which the educational program is inserted, then to design the program with its activities and actions. The role of the teacher-researchers is to collaborate in these



Meta-didactical Transposition, Fig. 1 (**a**, **b**, **c**) The process of transformation of praxeologies

phases and to participate also in the professional development program as trainers, where the role of the teachers involved is to be learners in communities with colleagues. Participating simultaneously to the researchers' community and to the teachers' community, the teacher-researcher acts as a broker between the two communities.

The *Double Dialectic*

In the Meta-Didactical Transposition framework, the *double dialectic* represents a product of the interactions between the two communities of researchers and teachers involved in the professional development.

The first dialectic is at the *didactical level* and takes place in the classroom, involving the personal meanings that students attach to an activity they are engaged in, and its scientific, shared meaning (Vygotsky 1978). The second dialectic is at the *meta-didactical level* and lies in the personal interpretation that the teachers give to the first dialectic, as a result of both their praxeologies and the meaning of the first dialectic in the community of researchers (a result of researchers' praxeology). The second dialectic corresponds to the scientific shared meaning of the first dialectic.

Typically, the second dialectic arises from a contrast between researchers' praxeologies (or some of their components) and teachers' praxeologies. It is through this double dialectic that teachers' praxeologies can change over time, during the professional development or after it, and align with the praxeologies of the researchers. This process may trigger a significant evolution of the teacher professional competences.

Applications, Integration, and Evolution of the Meta–Didactical Transposition Framework

The Meta-Didactical Transposition framework can be applied in a variety of situations in which the interactions and mutual exchanges between the communities involved in a process of professional development give rise to an evolution in their praxeologies (or their components), which changes their status from external to internal to a community (or vice versa).

In the following some applications and evolutions of the framework:

A) The Meta-Didactical Transposition framework is helpful to analyze the mutual interactions between the communities involved in the process of design and in the process of teaching experiments in the classrooms (respectively the *community of design* and *community of experimentation*), to highlight the role of each community, the relationships with the other community, and the possible exchange of the praxeologies or components between them (Robutti 2015).
B) To gain a better and deeper understanding of the complexity of the process of teachers' professional development, the theoretical idea of *emergence* has been used in combination with the Meta-Didactical Transposition framework, firstly to take into account the various *agents* that can influence a process of professional development at a micro level, and secondly to consider the effects of these agents when changes appear in the praxeologies of teachers, at the macrolevel (Prodromou et al. 2017). As the waves on the surface of the sea are the visible phenomenon at macrolevel, resulting from many particles acting at microlevel, in the same way the change in a praxeology is the phenomenon at macrolevel, resulting of many agents acting at micro level: methodological, institutional, material and technological, and motivational (Prodromou et al. 2017). The integration of macro- and microlevel points of view gives a detailed lens to better describe the dynamics in the praxeologies.
C) The Meta-Didactical Transposition frame in itself is not sufficient to give details when a community of in-service teachers is trained within a MOOC (Massive Open On-line Course), using virtual interaction mediated by a web platform in a distance-learning approach. A theoretical integration is needed for such broader contexts (Taranto et al. 2017), because a MOOC can be considered as an artifact, namely a static set of materials that becomes dynamic when it is opened to the trainees. When open, it gives rise to a complex *ecosystem*, where the teachers involved in the community interact through the available tools. This ecosystem usually evolves as a network, thanks to the participants' contributions, and also the network-knowledge of individuals evolves, transforming the MOOC artifact into an instrument (according to Verillon and Rabardel 1995). The Meta-Didactical Transposition is used to study the community of inquiry (according to Jaworski 2008) – the trainers – and the community of practice (according to Wenger 1998) – the teachers as trainees in the MOOC. The trainers evolved in their praxeologies interacting in the MOOC, and the trainees too, making connections in the platform and reporting from experimentations in their classes. Both of them enrich the ecosystem with new reports and feedbacks. Both individuals and communities evolve in their praxeologies, via the ecosystem.

The Meta-Didactical Transposition framework is used in literature alone and integrated with other frames, having shown its robust structure and also its limits and constraints. What is invariant is the institutional context and the fact that there are praxeologies shared in communities of teachers and researchers. What changes is the kind of community according to their work (design, professional development, experimentation, or others). The frame is developing maintaining its structure and integrating other theoretical elements (emergence, instrumental genesis, connectivism, . . ., as previously shown at points A, B, C) and it is spreading around different communities of academics over the world.

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Metaphors in Mathematics Education

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Keywords

Metaphor · Conceptual metaphor · Metaphoring · Reification · Embodied cognition · Gestures · Analogy · Representations

Definition

Etymologically metaphor means "transfer," from the Greek meta (trans) + pherein (to carry). Metaphor is in fact "transfer of meaning."

Introduction

Metaphors are very likely as old as humankind. Recall Indra's net, a 2500-year-old Buddhist metaphor of dependent origination and interconnectedness (Cook 1977; Capra 1982), consisting of an infinite network of pearls, each one reflecting all others, in a never-ending process of reflections of reflections, highly appreciated by mathematicians (Mumford et al. 2002).

It was Aristotle, however, with his taxonomic genius, who first christened and characterized metaphors c. 350 BC in his Poetics: "Metaphor consists in giving the thing a name that belongs to something else; the transference being either from genus to species, or from species to genus, or from species to species, on the grounds of analogy" (Aristotle 1984, 21:1457b). Interestingly for education, Aristotle added:

> The greatest thing by far is to be a master of metaphor. It is the one thing that cannot be learned from others; it is also a sign of genius, since a good metaphor implies an eye for resemblance. (loc. Cit. 21:1459a).

But time has not passed in vain since Aristotle. Widespread agreement has been reached (Richards 1936; Black 1962, Black 1993; Ortony 1993; Ricoeur 1977; Reddy 1993; Gibbs 2008; Indurkhya 1992, 2006; Johnson and Lakoff 2003; Lakoff and Núñez 2000; Wu 2001; Sfard 1994, 1997, 2009) that metaphor serves as the often unknowing foundation for human thought (Gibbs 2008) since our ordinary conceptual system, in terms of which we both think and act, is fundamentally metaphorical in nature (Johnson and Lakoff 2003).

Characteristics

Metaphors for Metaphor

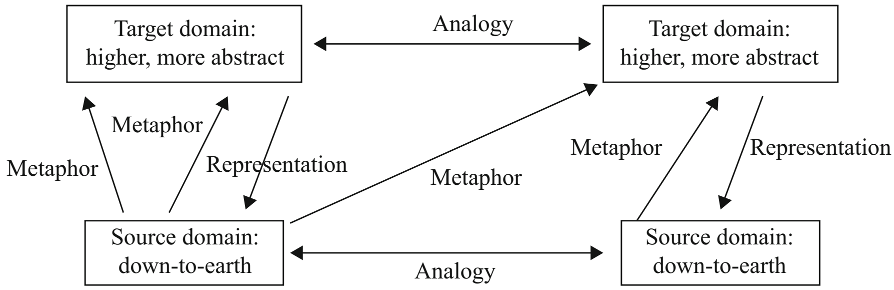
"There is no non metaphorical standpoint from which one could look upon metaphor" remarked Ricoeur (1977). To Bruner (1986) "Metaphors are crutches to help us to get up the abstract mountain," but "once up we throw them away (even hide them) . . . (p. 48). Empirical evidence suggests however that metaphor is a permanent resource rather than a temporary scaffold becoming later a "dead metaphor" (Chiu 2000). We find also *theory-constitutive metaphors* that do not "worn out" like literary metaphors and provide us with heuristics and guide our research (Boyd 1993; Lakoff and Núñez 1997). Recall the "tree of life" metaphor in Darwin's theory of evolution or the "encapsulation metaphor" in Dubinsky's APOS theory (Dubinsky and McDonald 2001).

In the field of mathematics education proper, it has been progressively recognized during the last decades (e.g., Chiu 2000, 2001; van Dormolen 1991; Edwards 2005; English 1997; Ferrara 2003; Gentner 1982, 1983; Lakoff and Núñez 2000; Parzysz et al. 2007; Pimm 1987; Presmeg 1997; Sfard 1994, 1997, 2009; Soto-Andrade 2006, 2007) that metaphors are powerful cognitive tools that help us in grasping or building new mathematical concepts, as well as in solving problems in an efficient and friendly way: "metaphors we calculate by" (Bills 2003).

According to Lakoff and Núñez (2000), (conceptual) metaphors appear as mappings from a *source* domain into a *target* domain, carrying the inferential structure of the first domain into the one of the second, enabling us to understand the latter, often more abstract and opaque, in terms of the former, more down-to-earth and transparent. In the classical example "A teacher is a gardener," the *source* is gardening, and the *target* is education.

Figure 1 maps metaphors, analogies, and representations and their relationships (Soto-Andrade 2007).

We thus see metaphor as bringing the target concept into being rather than just shedding a new light on an already existing notion, as representation usually does, whereas analogy states a similarity between two concepts already constructed (Sfard 1997). Since new concepts arise from a crossbreeding of several metaphors rather than from a single one, multiple metaphors, as well as the ability to transiting between them, may be necessary for the learner to make sense of a new concept (Sfard 2009). Teaching with multiple metaphors, as an antidote to unwanted entailments of one single metaphor, has been recommended (e.g., Low 2008; Sfard 2009; Chiu 2000, 2001).



Metaphors in Mathematics Education, Fig. 1 A topographic metaphor for metaphors, representations, and analogies

Metaphor and Reification

Sfard (1994) named *reification* the metaphorical creation of abstract entities, seen as the transition from an *operational* to a *structural* mode of thinking. Experientially, the sudden appearance of reification is an "aha!" moment, the birth of a metaphor that brings a mathematical concept into existence. Reification is however a double-edged sword: Its *poietic* (generating) edge brings abstract ideas into being, and its *constraining* edge bounds our imagination and understanding within the confines of our former experience and conceptions (Sfard 2009). This "metaphorical constraint" (Sfard 1997) explains why it is not quite true that anybody can invent anything, anywhere, anytime, and why metaphors are often "conceptual recycling." For instance, the construction of complex numbers was hindered for a long time by *overprojection* of the metaphor "number is quantity" until the new metaphor "imaginary numbers live in another dimension" installed them in the "complex plane." "To understand a new concept, I must create an appropriate metaphor. . ." says one of the mathematicians interviewed by Sfard (1994).

Metaphor, Embodied Cognition, and Gestures

Contemporary evidence from cognitive neuroscience shows that our brains process literal and metaphorical versions of a concept in the same localization (Knops et al. 2009; Sapolsky 2010). Gibbs and Mattlock (2008) show that real and imagined body movements help people create embodied simulations of metaphorical meanings involving haptic-kinesthetic experiences. The underlying mechanism of cross-domain mappings may explain how abstract concepts can emerge in brains that evolved to steer the body through the physical, social, and cultural world (Coulson 2008). It has been proposed that acquiring metaphoric items might be facilitated by acting them out, as in total physical response learning (Low 2008).

The didactical chasm existing between the ubiquitous motion metaphors in the teaching of calculus and the static and timeless character of current formal definitions (Kaput 1979) is in fact bridged by the often unconscious gestures (Yoon et al. 2011) that lecturers enact in real time while speaking and thinking in an instructional context (Núñez 2008). So gestures inform mathematics education better than traditional disembodied mathematics (Núñez 2007).

Metaphors for Teaching and Learning

When confronted with the metaphor "teaching is transmitting knowledge," many teachers say: This is not a metaphor, teaching *is* transmitting knowledge! What else could it be? Unperceived here is the "Acquisition Metaphor," dominant in mathematics education, that sees learning as acquiring an accumulated commodity. The alternative, complementary, metaphor is the Participation Metaphor: learning as participation (Sfard 1998). Plutarch agreed when he said "A mind is a fire to be kindled, not a vessel to be filled" (Sfard 2009).

Educational Metaphors

Grounding and *linking* metaphors are used in forming mathematical ideas (Lakoff and Núñez 2000). The former "ground" our understanding of mathematics in familiar domains of experience, the latter link one branch of mathematics to another.

Lakoff and Núñez (1997) point out that often mathematics teachers attempt to concoct ad hoc extensions of grounding metaphors beyond their natural domain, like "helium balloons" or "anti-matter objects" for negative numbers. Although the grounding "motion metaphor" extends better to negative numbers: -3 steps means walking backwards 3 steps and multiplying by -1 is turning around, they consider this extension a forced "educational metaphor." For an explicit account of such educational metaphors, see Chiu (1996, 2000, 2001). Negative numbers arise more naturally, however, via flows in a graph: A "negative flow" of 3 units from agent A to agent B "is" a usual flow of 3 units from B to A.

Metaphoring (Metaphorical Thinking) in Mathematics Education

Presmeg (2004) studied idiosyncratic metaphors spontaneously generated by students in problem-solving as well as their influence on their sense making. Students generating their own metaphors increase their critical thinking, questioning, and problem-solving skills (Low 2008). There are however potential pitfalls occasioned by invalid inferences and overgeneralization.

Building on their embodied prior knowledge, students can understand difficult concepts metaphorically (Lakoff and Núñez 1997). Explicit examples have been given by Chiu (2000, 2001), e.g., students using their knowledge of motion to make sense of static polygons through the "polygons are paths" metaphor, and so "seeing" that the sum of the exterior angles is a whole turn and that exterior angles are more "natural" than interior angles! "Polygons are enclosures between crossing sticks" elicits different approaches. Source understanding overcomes age to determine metaphoring capacity, since 13-month infants can already metaphorize (Chiu 2000). Also, a person's prior (nonmetaphorical) target understanding can curtail or block metaphoring (loc. Cit.).

Examples of Metaphors for Multiplication

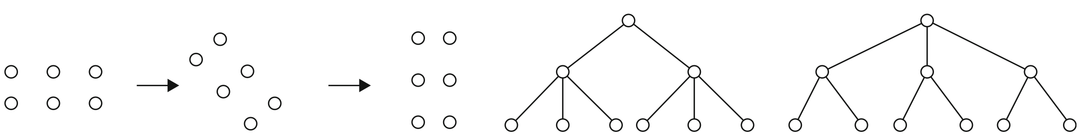
Chiu (2000) indicates the following:

"Multiplication A × B is replacing the original A pieces by B replications of them."

"Multiplication A × B is cutting each of the current A objects into B pieces."

"Area metaphor" and "Branching metaphor" for multiplication (Soto-Andrade 2007) are illustrated in Fig. 2.

In the area metaphor, commutativity is perceived as invariance of area under rotation. We "see" that $2 \times 3 = 3 \times 2$, *without counting and knowing that it is 6*. In the branching metaphor, commutativity is less obvious unless this metaphor becomes a "met-before" (McGowen and Tall 2010) because you know trees very well. Our trees also suggest a "hydraulic metaphor," useful to grasp multiplication of fractions: A litre of water drains evenly from the tree apex, through the ducts. Then 1/6 appears as 1/3 *of* 1/2 in the left tree and also as 1/2 *of* 1/3 in the right tree. Our hydraulic metaphor enables us to see the "two sides of the multiplicative coin": 2×3 is bigger but $1/2 \times 1/3$ is smaller than both factors. It also opens up the way to a deeper metaphor for multiplication: "multiplication is concatenation", a generating metaphor for category theory in mathematics.



Metaphors in Mathematics Education, Fig. 2 Two metaphors for commutativity of multiplication

On the Metaphorical Nature of Mathematics

Lakoff and Núñez's claim that mathematics consists entirely of conceptual metaphors has stirred controversy among mathematicians and mathematics educators. Dubinsky (1999) suggests that formalism can be more effective than metaphor for constructing meaning. Goldin (1998, 2001) warns that the extreme view that all thought is metaphorical will be no more helpful than earlier views that it was propositional and finds that Lakoff and Núñez's "ultrarelativistism" dismisses perennial values central to mathematics education like mathematical truth and processes of abstraction, reasoning, and proof among others (Goldin 2003).

However some distinguished mathematicians dissent. Manin (2007), referring to Metaphor and Proof, complains about the imbalance between various basic values which is produced by the emphasis on proof (just one of the mathematical genres) that works against values like "activities", "beauty" and "understanding", essential in high school teaching and later, neglecting which a teacher or professor tragically fails. He also claims that controverted Thom's Catastrophe Theory "is one of the developed mathematical metaphors and should only be judged as such". Thom himself complains that "analogy, since positivism, has been considered as a remain of magical thinking, to be condemned absolutely, being nowadays hardly considered as more than a rhetorical figure (Thom 1994). He sees catastrophe theory as a pioneering theory of analogy and points out that narrow minded scientists objecting the theory because it provides nothing more than analogies and metaphors, do not realize that they are stating its true purpose: to classify all possible types of analogical situations (Porte 2013).

The preface to Mumford et al. (2002) reads: "Our dream is that this book will reveal to our readers that mathematics is not alien and remote but just a very human exploration of the patterns of the world, one which thrives on play and surprise and beauty."

McGowen and Tall (2010) argue that even more important than metaphor for mathematical thinking are the particular mental structures built from experience that an individual has "met-before." Then one can analyze the met-befores of mathematicians, mathematics educators, and developers of theories of learning to reveal implicit assumptions that support their thinking in some ways and hinder it in others. They criticize the top-down nature of Lakoff and Núñez "mathematical idea analysis" and their unawareness of their own embodied background and implicit met-befores that shape their theory.

Open Ends and Questions

Further research is needed on methods and techniques of teaching metaphor.

Facts on how the neural substrate of perception and action is co-opted by higher-level processes suggest further research on comparing visual, auditory, and kinesthetic metaphors.

How can teachers facilitate the emergence of idiosyncratic metaphors in the students?

May idiosyncratic metaphors be voltaic arcs that spring when didactical tension is high enough in the classroom?

How and where do students learn relevant metaphors: from teachers, textbooks, or sources outside of the classroom?

How can we facilitate students' transiting between metaphors?

How can teaching trigger change in students' metaphors?

What roles should the teacher play in metaphor teaching?

What happens when there is a mismatch between teacher and student's metaphors?

Do experts continue using the same metaphors as novices? If yes, do they use them in the same way?

Cross-References

▶ Mathematical Representations

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Misconceptions and Alternative Conceptions in Mathematics Education

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Keywords
Understanding · Learning · Constructivism · Cognitive models · Child's perspective · Cognitive conflict · Child's conceptions

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Definition

The term "misconception" implies incorrectness or error due to the prefix "mis." However its connotation never implies errors from a child's perspective. From a child's perspective, it is a reasonable and viable conception based on their experiences in different contexts or in their daily life activities. When children's conceptions are deemed to be in conflict with the accepted meanings in mathematics, the term misconceptions has tended to be used. Therefore some researchers or educators prefer to use the term "alternative conception" instead of "misconception." Other terms sometimes used for misconceptions or terms related to misconceptions include students' mental models, children's arithmetic, preconceptions, naïve theories, conceptual primitives, private concepts, alternative frameworks, and critical barriers.

Some researchers avoid using the term "misconceptions," as they consider them as misapprehensions and partial comprehensions that develop and change over the years of school. For example, Watson (2011), based on an extensive program of research, identifies developmental pathways that can be observed as middle school students move towards more sophisticated understandings of statistical concepts, culminating in a hierarchical model incorporating six levels of statistical literacy (p. 202).

Characteristics

Research on misconceptions in mathematics and science commenced in the mid-1970s, with the science education community researching the area much more vigorously. This research carefully rejected the tabula rasa assumption that children enter school without preconceptions about a concept or topic that a teacher tries to teach in class. The first international seminar Misconceptions and Educational Strategies in Science and Mathematics was held at Cornell University, Ithaca, NY, in 1983, with researchers from all over the world gathering to present research papers in this area – although the majority of research papers were in the field of science education.

In mathematics education, according to Confrey (1987), research on misconceptions began with the work of researchers such as Erlwanger (1975), Davis (1976), and Ginsburg (1976), who pioneered work focusing on students' conceptions. In the proceedings of the second seminar: Misconceptions and Educational Strategies in Science and Mathematics, Confrey (1987) used constructivism as a framework for a deep analysis of research on misconceptions. Almost two decades later, Confrey and Kazak (2006) identified examples of misconceptions which have been extensively discussed by the mathematics education community – for example, "Multiplication makes bigger, division makes smaller," "The graph as a picture of the path of an object," "Adding equal amounts to numerators and denominators preserves proportionality," and "longer decimal number are bigger, so the 1.217 > 1.3" (pp. 306–307). Concerning decimals, a longitudinal study by Stacey (2005) showed that this misconception is persistent and pervasive across age and educational experience. In another extensive study, Ryan and Williams (2007) examined a variety of misconceptions among 4–15-year-old students in number, space and measurement, algebra, probability, and statistics, as well as preservice teachers' mathematics subject matter knowledge of these areas.

From the teacher's perspective, a misconception is not a trivial error that is easy to fix, but rather it is resilient or pervasive when one tries to get rid of it. The reason why misconceptions are stubborn is that they are viable, useful, workable, or functional in other domains or contexts. Therefore, it is important for teachers not only to treat misconceptions with equal importance to mathematical concepts but also to identify what exactly the misconception is in the learning context and to clarify the relationship between the misconception and the mathematical concept to be taught. In other words, the teacher needs to construct the task for the lesson taking the misconception into consideration in order to resolve the conflict between the misconception and the mathematical concept. By doing this the lesson may open up a new pathway to children's deeper and wider understanding of the mathematical concept to be taught.

So far many misconceptions have been identified at the elementary and secondary levels, however only a few of them are considered for inclusion in actual teaching situations. While very few of these are incorporated in mathematics textbooks, one exception is the misconception that figures with the same perimeter have the same area. For example, Takahashi (2006) describes an activity used in a fourth-grade Japanese textbook to introduce the formula for the area of a rectangle that asks students to compare the areas of carefully chosen figures that have the same perimeter – for example, 3×5 cm and 4×4 cm rectangles.

Further research is needed to develop how to incorporate misconceptions into textbook or teaching materials in order to not only resolve the misconception but also to deepen and expand children's understanding of mathematical concepts.

Cross-References

- ▶ Concept Development in Mathematics Education
- ▶ Constructivism in Mathematics Education
- ▶ Constructivist Teaching Experiment
- ▶ Learning Environments in Mathematics Education
- ▶ Situated Cognition in Mathematics Education

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Models of In-Service Mathematics Teacher Education Professional Development

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Keywords
Professional development · Mathematical knowledge for teaching · Teacher beliefs · Study of practice · School-based leadership

It can be assumed that ongoing improvement in learning is connected to the knowledge of the teacher. This knowledge can be about the mathematics they will teach, ways of communicating that mathematics, finding out what students know and what they find difficult to learn, and managing the classroom to maximize the learning of all students. It is evident from much of the literature that teacher professional development is complex and there are numerous models used and proposed. There are some common characteristics of various models such as examining teachers' existing theories of practice, offering multiple opportunities for them to learn, and facilitating learning from others in a community of practice (e.g., Anthony et al. 2014): pedagogical approaches, motivation, beliefs, disposition (e.g., Prodromou et al. 2018), and linking to the classroom (Goos 2014; Visnovska and Cobb 2015). Consideration can also be given to activities that bring about the interactions of these aspects which are subsequently enacted in participating teacher practice (Prodromou et al. 2018). Furthermore, Anthony et al. (2014) indicated the focus can be on empowerment to transform one's teaching beyond the focus of the professional learning. It is also suggested that researchers can identify tools needed to support teacher learning, such as time and space, professional knowledge, and resources. Within each of these approaches, there is a focus on classroom discourse, tasks, and the tools to support teacher learning.

Informed by the current literature, this entry is about approaches to in-service mathematics teacher education. The basic organizer is teacher decision-making, since effective classroom teaching is essentially about planning experiences that engage students in activities that are mathematically rich, relevant, accessible, and the management of the learning that results. As Zaslavsky and Sullivan (2011) proposed, educating practicing teachers involves facilitating growth from "uncritical perspectives on teaching and learning to more knowledgeable, adaptable, judicious, insightful, resourceful, reflective and competent professionals ready to address the challenges of teaching" (p. 1). Furthermore, Furlong (2014) indicated that the most effective forms of ongoing teacher professional learning can draw on specialist expertise, peer support, and effective leadership.

This entry is structured around an adaptation of the Clark and Peterson (1986) framework, which includes three background factors: teacher knowledge; attitudes, beliefs, and values; and the opportunities and constraints experienced. Clark and Peterson indicated that these factors influence each other and together inform teachers' intentions to act and their subsequent classroom actions. While there have been more recent models such as the interconnected model of teacher growth (Clarke and Hollingsworth 2002), Schoenfeld's (2011) goal-oriented decision-making framework, and McNeill et al. (2016) collaborative model of professional development, the Clark and Peterson framework essentially connects background considerations with practice. This makes it ideal for structuring the professional learning of practicing mathematics teachers.

The first of these background factors refers to teacher knowledge. Hill et al. (2008) proposed a model informing the design of practicing teacher education directed at improving knowledge. There were two major categories: subject matter knowledge and pedagogical content knowledge. Hill et al. described *subject matter knowledge* as consisting of common content knowledge, specialized content knowledge, and knowledge at the mathematical horizon. For each of these, the emphasis is on developing in teachers the capacity not only to learn any new mathematics they need but also to view the mathematics they know in new ways. Generally, connecting this learning to the further development of their pedagogical content knowledge facilitates both of these orientations. Hill et al. argued that *pedagogical content knowledge* includes knowledge of content and teaching, knowledge of content and students, and knowledge of curriculum. Similarly, Rowland et al. (2009) referred to knowledge of content and pedagogical content knowledge as foundation knowledge in their Knowledge Quartet. The other three dimensions include transformation (representing the mathematics), connection (e.g., coherence of planning, sequencing of instruction), and contingency (e.g., responding to student ideas, noticing teachable moments). In addressing knowledge of content and teaching, Zaslavsky and Sullivan (2011) proposed focusing teacher learning on experiences such as those involving comparing and contrasting between and across topics to identify patterns and make connections, designing and solving problems for use in their classrooms, fostering awareness of similarities and differences between tasks and concepts, and developing the capacity of teachers to adapt successful experiences to match new content. Knowledge of content and students is primarily about the effective use of data to inform planning and teaching (Roche et al. 2014). Essentially, the goal is to examine what students know as distinct from what they do not. In terms of knowledge of curriculum, Sullivan et al. (2012) described several processes as the first level of knowing the curriculum. These include where teachers evaluate resources, draw on the experience of colleagues, analyze assessment data to make judgments on what the students know, and interpret curriculum documents to identify important ideas (Charles 2005). The subsequent levels involve selecting, sequencing, and adapting experiences for the students, followed by planning the teaching (Chan et al. 2018; Smith and Stein 2011). All of these can inform the design of practicing teacher education.

The second background factor refers to the constraints that teachers anticipate they may confront. Such constraints can be exacerbated by the socioeconomic, cultural, or language background of the students, geographic factors, and gender. A further constraint is the diversity of readiness that teachers experience in all classes, even those grouped to maximize homogeneity. Sullivan et al. (2006) described a planning framework that addresses constraints such as accessible tasks, explicit pedagogies, and specific enabling prompts for students experiencing difficulty. Such prompts involve slightly lowering an aspect of the task demand. For example, simplify the form of representation, the size of the number, or the number of steps, to enable a student experiencing difficulties to proceed at that new level; and then if successful the student can proceed with the original task. Teacher educators can encourage practicing teachers to examine the existence and sources of constraints and strategies that can be effective in overcoming those constraints.

The third background factor includes teachers' beliefs about the nature of mathematics and the way it is taught and learned. It is widely accepted that teachers' beliefs about the nature of mathematics influence their pedagogical practices (e.g., Beswick 2012; Cross 2009). Particularly important is whether teachers believe that all students can learn mathematics or whether such learning is just for some (Hannula 2004). For example, Voss et al. (2013) found that teachers' beliefs impacted on their instructional practice and consequently student learning outcomes. Also important is whether teachers see their own and students' achievement as incremental and amenable to improvement through effort (Bobis et al. 2016; Dweck 2000). Teacher education can include experiences that address this by, for example, examining forms of affirmation, studying tasks that foster inclusion, and developing awareness of threats such as self-fulfilling prophecy effects (Brophy 1983). Rather than compartmentalizing the elements of the background factors described above, it is preferable that the education of practicing teachers incorporate all elements together, a suitable context for which is the study of practice.

The most famous example of teacher learning from the study of practice is Japanese Lesson Study, which is widely reported in the Japanese context (e.g., Fernández and Yoshida 2004; Inoue 2010; Watanabe et al. 2008) and has been adapted to Western contexts (e.g., Doig and Groves 2011; Lewis et al. 2004). Other examples of learning through the study of practice include realistic simulations offered by videotaped study of exemplary lessons (Clarke and Hollingsworth 2000; Clarke et al. 2009); interactive study of recorded exemplars (e.g., Merseth and Lacey 1993); case methods of teaching dilemmas that problematize aspects of teaching (e.g., Stein et al. 2000); focusing on task design in a Lesson Study approach (Fujii 2013); and Learning Study which is similar to Japanese Lesson Study but focuses on student learning (Chan et al. 2018; Runesson et al. 2011).

In contrast to the adaptations of Japanese Lesson Study, other school-based professional learning involve whole-school collaborative models in which lead teachers (mathematics specialists) work with external partners to design professional learning at a point of need (e.g., Bruce et al. 2010; Downton et al. 2018). Chan et al. (2018) suggested that understanding the teacher learning process (in situ learning) should lead to improvement in both teacher knowledge and practice. Some studies have focused on teacher noticing (e.g., Fernández et al. 2013), while others have focused on facilitating classroom discourse (e.g., Staples and King 2017) or the cultural specificity of teacher instructional choices (Leong and Chick 2011; Lepik et al. 2012).

As indicated within the research literature (e.g., Furlong 2014; Gaffney and Faragher 2010; Sexton and Lamb 2017), a related factor is the need for effective school-based leadership of the mathematics teachers. Within this role is a critical dimension of establishing "interpersonal trust" (Grootenboer et al. 2015). If the focus is on sustainable, collaborative school-based approaches to improving teaching, this needs active and sensitive leadership. Such leaders can be assisted to study processes for leadership, as well as developing their confidence to lead the aspects of planning, teaching, and assessment described above.

Cross-References

- ▶ Communities of Inquiry in Mathematics Teacher Education
- ▶ Learning Study in Mathematics Education
- ▶ Lesson Study in Mathematics Education
- ▶ Mathematics Teacher as Learner
- ▶ Noticing of Mathematics Teachers
- ▶ Pedagogical Content Knowledge Within "Mathematical Knowledge for Teaching"
- ▶ Professional Learning Communities in Mathematics Education
- ▶ Subject Matter Knowledge Within "Mathematical Knowledge for Teaching"
- ▶ Teacher Beliefs, Attitudes, and Self-Efficacy in Mathematics Education

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Models of Preservice Mathematics Teacher Education

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Keywords
Mathematics teacher education · Preservice teachers · Professional development · Prospective teachers

Definition

Models of preservice teacher education are understood as structures of professional learning set up by intention for prospective mathematics teachers.

Characteristics

Preservice teacher education is widely considered as necessary for preparing prospective mathematics teachers for mastering the challenges of the mathematics classroom. To this end, models of preservice teacher education have been developed and are subject to ongoing investigations. For the profession of teaching mathematics, specific professional knowledge is necessary. In particular, designing learning opportunities and exploring the students' understanding or adaptive strategies of fostering mathematical competency require not only mathematical knowledge and pedagogical knowledge but also pedagogical content knowledge (Shulman 1986; Ball et al. 2008; Bromme 1992). This knowledge encompasses declarative and procedural components (e.g., Baumert et al. 2010; Ball et al. 2008), as well as prescriptive views and epistemological orientations (e.g., Pajares 1992; McLeod 1989; Törner 2002); it ranges from rather global components (cf. Törner 2002) to content-specific or even classroom situation-specific components (Kuntze 2012; Lerman 1990).

The goal of developing such a multifaceted professional knowledge underpins the significance of specific and structured environments for initial professional learning. However, it is widely agreed that models of preservice teacher education have to be seen as subcomponents in the larger context of continued professional learning throughout the whole working period of teachers rather than being considered as an accomplished level of qualification. Even though these models of preservice teacher education are framed by various institutional contexts and influenced by different cultural environments (Leung et al. 2006; Bishop 1988), the following fundamental aspects which are faced by many such models of preservice teacher education may be considered:

- Theoretical pedagogical content knowledge is essential for designing opportunities of rich conceptual learning in the classroom. Hence, in models of preservice teacher education, theoretical knowledge such as knowledge about dealing with representations or knowledge about frequent misconceptions of learners (cf. Ball 1993) is being supported in particular methodological formats which may take the form, e.g., of lectures, seminars, or focused interventions accompanying a learning-on-the job phase (Lin and Cooney 2001).
- Linking theory to practice is a crucial challenge of models of preservice teacher education. The relevance of professional knowledge for acting and reacting in the classroom is asserted to be supported by an integration of theoretical knowledge with instructional practice. In models of preservice teacher education, this challenge is addressed by methodological approaches such as school internships, frequently with accompanying seminars and elements of coaching (cf. Joyce and Showers 1982; Staub 2001; Kuntze et al. 2009), and specific approaches such as lesson study (Takahashi and Yoshida 2004), video-based work (e.g., Sherin and Han 2003; Seago

2004; Dreher and Kuntze 2012; Kuntze 2006), or work with lesson transcripts. For several decades, approaches such as "microteaching" (e.g., Klinzing 2002) had emphasized forms of teacher training centered in practicing routines for specific instructional situations. Seen under today's perspective, the latter approach tends to underemphasize the goal of supporting reflective competencies of prospective teachers which tend to be transferable across contents and across specific classroom situations (Tillema 2000).

- Developing competencies of instruction- and content-related reflection is a major goal in preservice teacher education. Accordingly, learning opportunities such as the analysis and the design of mathematical tasks (e.g., Sullivan et al. 2009, cf. Biza et al. 2007), the exploration of overarching ideas linked to mathematical contents or content domains (Kuntze et al. 2011), or the analysis of video-taped classroom situations (Sherin and Han 2003; Reusser 2005; Kuntze et al. 2008) are integrated in models of preservice mathematics education, supporting preservice teachers to build up reflective competencies or to become "reflective practitioners" (e.g., Smith 2003; Atkinson 2012).

The scenarios mentioned above indicate that there are a wide variety of possible models of preservice teacher education, as it has also been observed in comparative studies of institutional frameworks (König et al. 2011; Tatto et al. 2008). In contrast, research on the effectiveness of different models of preservice teacher education is still relatively scarce. Studies like TEDS-M (Tatto et al. 2008) constitute a step into this direction and set the stage for follow-up research not only in processes of professional learning in the settings of specific models of preservice teacher education but also into effects of specific professional learning environments, as they can be explored in quasi-experimental studies. In addition to a variety of existing qualitative case studies, especially quantitative evidence about models of preservice teacher education is still needed (cf. Adler et al. 2005). Such evidence from future research should systematically identify characteristics of effective preservice teacher education. Moreover empirical research about models of preservice teacher education should give insight how characteristics of effective professional development for in-service mathematics teachers (Lipowsky 2004) may translate into the context of the work with preservice teachers, which differs from professional development of in-service teachers (da Ponte 2001).

Cross-References

- ▶ Communities of Practice in Mathematics Teacher Education
- ▶ Lesson Study in Mathematics Education
- ▶ Mathematics Teacher Education Organization, Curriculum, and Outcomes
- ▶ Models of In-Service Mathematics Teacher Education Professional Development
- ▶ Pedagogical Content Knowledge Within "Mathematical Knowledge for Teaching"
- ▶ Reflective Practitioner in Mathematics Education
- ▶ Subject Matter Knowledge Within "Mathematical Knowledge for Teaching"
- ▶ Teacher Beliefs, Attitudes, and Self-Efficacy in Mathematics Education
- ▶ Teacher Education Development Study-Mathematics (TEDS-M)

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Motivation in Mathematics Learning

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Keywords
Motivation · Affect · Self-efficacy · Math anxiety · Disposition

Definition

The impetus for and maintenance of mathematical activity. Mathematics learning, as goal-directed behavior, involves the development of expectations, values, and habits that constitute the reasons why people choose to engage and persevere on the one hand or disengage and avoid on the other, in mathematics and mathematically related pursuits.

Characteristics and Findings from Various Theoretical Perspectives

The history of motivation research applied to mathematics learning began with the study of biological drives and incentive in the first decades of the twentieth century (see Brownell 1939 for a good review of this perspective as applied to education). Following the tenets of classical and operant (instrumental) conditioning, it was found that if a reinforcer was provided for successfully completing a behavior, the probability of that behavior occurring in the future under similar circumstances would increase. Additionally, Thorndike found that the intensity of the behavior would increase as a function of the reinforcement value (1927). These general theories of the use of incentives to motivate student learning dominated educational theory roughly until the middle of the 1960s.

They are still valuable to educators today, particularly in the use of behavior modification techniques, which regulate the use of rewards and other reinforcers contingent upon the learner's successive approximation of the desired behavioral outcomes, which could be successful skill attainment or increase in positive self-statements to reduce math anxiety and so on (Bettinger 2008).

Since the mid-1960s, research on motivation in the psychology of learning has focused on six different, but not distinct, theoretical constructs: Attributions, Goal Theory, Intrinsic Motivation, Self-Regulated Learning, Social Motivation, and Affect. These factors grew out of a general cognitive tradition in psychology but recently have begun to explain the impact of social forces, particularly classroom communities and teacher-student relationships on student enjoyment and engagement in mathematical subject matter (see Middleton and Spanias 1999 for a review comparing these perspectives).

Attribution Theory

Learners' beliefs about the causes of their successes and failures in mathematics determine motivation based on the locus of the cause (internal or external to the learner) and its stability (stable or unstable). Productive motivational attributions tend to focus on internal, stable causes (like ability and effort) for success as these lead to increased persistence, self-efficacy, satisfaction, and positive learning outcomes. Lower performing demographic populations tend to show more external and unstable attributional patterns. These appear to be caused by systematic educational biases (Kloosterman 1988; Pedro et al. 1981; Weiner 1980).

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Goal Theory

Goal theories focus on the stated and unstated reasons people have for engaging in mathematical tasks. Goals can focus on *Learning* (also called *Mastery*), *Ego* (also called *Performance*), or *Work Avoidance*. People with learning goals tend to define success as improvement of their performance or knowledge. Working towards these kinds of goals shows results in the valuation of challenge, better metacognitive awareness, and improved learning than people with ego goals. Work avoidance goals are debilitating, psychologically, as they result from learned helplessness and other negative attributional patterns (Wolters 2004; Covington 2000; Gentile and Monaco 1986).

Intrinsic Motivation and Interest

The level of interest a student has in mathematics, the more effort he or she is willing to put out, the more he or she thinks the activity is enjoyable, and the more they are willing to persist in the face of difficulties (Middleton 1995; Middleton and Spanias 1999; Middleton and Toluk 1999). Intrinsic Motivation and Interest theories have shown that mathematical tasks can be designed to improve the probability that a person will exhibit task-specific interest and that this task-specific interest, over time, can be nurtured into long-term valuation of mathematics and its applications (Hidi and Renninger 2006; Köller et al. 2001; Cordova and Lepper 1996).

Self-Regulated Learning

Taken together, these primary theoretical perspectives can be organized under a larger umbrella concept: Self-Regulated Learning (SRL). Internal, stable attributions are a natural outcome of Learning Goals, and Interest is a natural outcome of internal, stable, attributions. Each of these perspectives contributes to the research on the others such that the field of motivation in general, and in mathematics education specifically, is now able to use these principles to design classroom environments, tasks, and interventions to improve mathematics motivation and performance (Zimmerman and Schunk 2011; Eccles and Wigfield 2002; Wolters and Pintrich 1998).

Social Motivation

In addition to the aforementioned psychological theories, study of students in classrooms has recently yielded principles for understanding how social groups motivate themselves. In general these theories show that needs for affiliation and relatedness with peers, fear of disapproval, and the need to demonstrate competence interact in complex ways in the classroom (Urdan and Schoenfelder 2006). Intellectual goals and social needs therefore are integrally related. Additionally, the need for social concern is a critical motivator for student prosocial learning (Jansen 2006). Students who feel a concern for the struggles of others are able to provide support for the learning of others. This is a key component of effective group work and social discourse in mathematics classrooms.

Affect

The outcomes of learning environments consist of cognitive as well as affective responses. People tend to enjoy mathematics more when they find it interesting and useful, and they tend to dislike or even fear engagement in mathematics when they believe they will not be successful (Hoffman 2010). Goldin et al. (2011) have shown that people build affective structures which allow them to predict the emotional content and probable outcomes of mathematical activity. Activity forms a physiological feedback loop between behavior and goals and therefore has both an informational role as well as a reinforcement role (Hannula 2012). These cognitive structures are integral to self-regulation and decision-making regarding when and how deeply to engage in mathematics tasks.

Cross-References

- ▶ Affect in Mathematics Education
- ▶ Communities of Practice in Mathematics Education
- ▶ Creativity in Mathematics Education
- ▶ Equity and Access in Mathematics Education
- ▶ Gender in Mathematics Education
- ▶ Mathematics Teacher Identity

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