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# Calculus Teaching and Learning

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**Keywords**

Calculus key concepts · Intuitive representations · Formal definitions · Intuition of infinity · Notion of limit · Cognitive difficulties · Theoretical dimensions · Epistemological dimension · Research in teaching and learning calculus · Role of technology · Visualization · Coordination between semiotic registers · Role of historical perspective · Sociocultural approach · Institutional approach · Teaching practices · Role of the teacher · Transition between secondary school and university

## Definition: What Teaching and Learning Calculus Is About

The differential and integral calculus is considered as one of the greatest inventions in mathematics. Calculus is taught in secondary school and in university. Learning calculus includes the analysis of problems of changes and motion. Previous related concepts, like the concept of a variable and the concept of function, are necessary for the understanding of calculus concepts. However, the learning of calculus includes new notions like the notion of limit and limiting processes, which intrinsically contain changing quantities. The differential and integral calculus is based upon the fundamental concept of limit. The mathematical concept of limit is a particularly difficult notion, typical of the kind of thought required in advanced mathematics.


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## Calculus Curriculum

There have been efforts in many parts of the world to reform the teaching of calculus. In France, for example, the syllabus changed in the 1960s and 1970s, due to the influence of the Bourbaki group. The limit concept, on a rigorous basis, has penetrated even into the secondary school curriculum: in 1972, the classical definition of the derivative as the limit of a quotient of differences was introduced. Another change occurred in the French calculus curriculum in 1982, this time influenced by the findings of mathematics education research, and the curriculum focused on more intuitive approaches. As a result, the formalization of the limit has been omitted at the secondary school level. This is the situation in most countries today: at the high school level, there is an effort to develop an initial approach to calculus' concepts without relying on formal definitions and proofs. An intuitive and pragmatic approach to calculus at the senior level in high school (age 16–18) precedes the formal approach introduced at university.

On the university level, calculus is among the more challenging topics faced by new undergraduates. In the United States, the calculus reform movement took place during the late 1980s. The recommendation was that calculus courses should address fewer topics but in more depth, and students should learn through active engagement with the material. The standard course syllabus was revised, and new projects arose which incorporated technology into instruction. More recently, Bressoud et al. (2016) analyzed calculus curricula in France, Germany, the United States, Uruguay, Singapore, South Korea, and Hong Kong. They note the constant revision of the calculus curriculum and the way calculus is taught in secondary school and university in the different countries. They relate to the following questions: When does the teaching of calculus start in secondary school? Is it separated into different parts: a compulsory mathematics part for all students and an extended part for students who intend to pursue further studies, which require more mathematics? They also relate to the assessment process and to the following question: Is there an evaluation of theoretical aspects of the course on the exam and not only an evaluation of routine practical procedures? The integration of graphic technology was investigated as well. The authors differentiate between the form of work at the secondary school level, in which the activities are often devoted almost exclusively to calculation based on algebraic expressions, and the required form of work at university level, which includes more formal thinking.

The book by Bressoud et al. (2015) presents a report of selected findings from the Mathematical Association of America's (MAA's) study of *Characteristics of Successful Programs in College Calculus*. The report combines both large-scale survey data and in-depth case study analysis. The report concerns college and university students and highlights the very challenging environment students encounter, as they make the transition to postsecondary education, in their learning of calculus.

In most countries, the transition toward more formal approach that takes place at university is accompanied by conceptual difficulties.

## Early Research in Learning Calculus: The Cognitive Difficulties

The cognitive difficulties that accompany the learning of central notions like functions, limit, tangent, derivative, and integral at the different stages of mathematics education are well reported in the research literature on learning calculus. These concepts are key concepts that appear and reappear in different contexts in calculus. The students meet some of these central topics in high school, and then the same topics appear again, with a different degree of depth, at university. We might attribute the high school students' cognitive difficulties to the fact that the notions are presented to them in an informal way. In other words, we might expect that the difficulties will disappear when the students learn the formal definition of the concepts. However, undergraduate mathematics education research suggests otherwise. The cognitive difficulties that accompany the key concepts in calculus are well described in Sierpinska (1985), Davis and Vinner (1986), Cornu (1991), Williams (1991), and Tall (1992), as well as in the book *Advanced Mathematical Thinking* edited by Tall (1991). The main source of difficulty resides in the fact that many students' intuitive ideas are in conflict with the formal definition of the calculus concepts, such as the notion of limit.

In these early studies of learning calculus, the theoretical dimensions are essentially cognitive and epistemological. The cognitive difficulties that accompany the learning of the key concepts in calculus, such as the limit concept, are inherent to the epistemological nature of the mathematics domain. In the following, we consider some facets of the dynamic interaction between the formal and intuitive representations, as they were discussed in these early studies. We encounter the first expression of the dynamic interaction between intuition and formal reasoning in the terms *concept definition* and *concept image*. For example, the intuitive thinking, the visual intuitions, and the verbal descriptions of the limit concept that precede its formal definition are necessary for understanding the concept. However, research on learning calculus demonstrates that there exists a gap between the mathematical definition of the limit concept and the way one perceives it. In this case, we may say that there is a gap between the concept definition and the concept image (Tall and Vinner 1981; Vinner 1983). Vinner also found that students' intuitive ideas of the tangent to a curve are in conflict with the formal definition. This observation might explain students' conceptual difficulties in visualizing a tangent as the limiting case of a secant.

Conceptual problems in learning calculus are also related to infinite processes. Research demonstrates that some of the cognitive difficulties that accompany the understanding of the concept of limit might be a consequence of the learners' intuition of infinity. Fischbein et al. (1979) observed that the natural concept of infinity is the concept of *potential infinity*, for example, the non-limited possibility to increase an interval or to divide it. The *actual infinity*, for example, the infinity of the number of points in a segment and the infinity of real numbers as *existing*, as *given*, is, according to Fischbein, more difficult to grasp and leads to contradictions. For example, "If one looks at 1/3, it is easy to accept the equality 1/3 = 0.33… The number 0.333…represents a potential (or dynamic) infinity. On the other hand, students questioned whether 0.333… is equal to 1/3 or tends to 1/3 usually answer that 0.333…tends to 1/3."

Among the theoretical constructs that accompany the early strands in research on learning calculus, we mention the *process-object duality*. The lenses offered by this framework highlight the students' dynamic process view in relation to concepts such as limit and infinite sums and help researchers to understand the cognitive difficulties that accompany the learning of the limit concept. Gray and Tall (1994) introduced the notion of *procept*, referring to the manner in which learners cope with symbols representing both mathematical processes and mathematical concepts. Function, derivative, integral, and the fundamental limit notion are all examples of procepts. The limit concept is a procept because the same notation represents both the process of tending to the limit and also the value of the limit.

## Research and Alternative Approaches to Teaching and Learning Calculus

Different directions of research were investigated in the last decades. The use of technology offered a new resource in the effort to overcome some of the conceptual difficulties: the power of technology is particularly important in facilitating students' work with epistemological double strands like discrete/continuous and finite/infinite. Visualization and especially dynamic graphics were also used. Some researchers based their research on the historical development of the calculus. Other researchers used additional theoretical lenses that include the sociocultural approach, the institutional approach, or the semiotic approach. In the following sections, we relate to these different directions of research.

## The Role of Technology

A key aspect of nearly all the reform projects has been the use of graphics calculators, or computers with graphical software, to help students develop a better intuitive understanding of calculus. Since learning calculus includes the analysis of changing quantities, technology has a crucial role in enabling dynamic graphical representations and animations. Technology was first incorporated as a support for visualization and coordination between semiotic registers. The possibility of computer magnification of graphs allows the limiting process to be implicit in the computer magnification, rather than explicit in the limit concept. In his plenary paper, Dreyfus (1991) analyzed the powerful role of visual reasoning in learning several mathematical concepts and processes. With introduction of the new technologies, there was a rapid succession of new ideas for use in teaching calculus. Calculus uses numerical calculations, symbolic manipulations, and graphical representations, and the introduction of technology in calculus allows these different registers. Research on the role of technology in teaching and learning calculus is described, for example, in Artigue (2006), Robert and Speer (2001), and Ferrera et al. in the 2006 handbook of research on the psychology of mathematics education (pp. 256–266). In the study by Ferrera et al., research that relates to using CAS toward the conceptualization of *limit* is described. For example, Kidron and Zehavi use symbolic computation and dynamic graphics to enhance students' ability to pass from visual interpretation of the limit concept to formal reasoning. In this research, a balance between the conception of an infinite sum as a process and as an object was supported by the software. The research by Kidron, as reported in the study by Ferrera et al. (2006), describes situations in which the combination of dynamic graphics, algorithms, and historical perspective enabled students to improve their understanding of concepts such as limit, convergence, and the quality of approximation. Most studies offer an analysis of teaching experiments that promote the conceptual understanding of key notions such as limits, derivatives, and integral. For example, in a research project by Artigue (2006), the calculator was used toward conceptualization of the notion of derivative. One of the aims of the project was to enable 11th grade students to enter the interplay between local and global points of view on functional objects.

Thompson (1994) investigated the concept of rate of change and infinitesimal change, which are central to understanding the fundamental theorem of calculus. Thompson's study suggests that students' difficulties with the theorem stem from impoverished concepts of rate of change. In the last two decades, Thompson published several studies which demonstrate that a reconstruction of the ideas of calculus is made possible by using computing technology. The concept of accumulation is central to the idea of integration and therefore is at the core of understanding many ideas and applications in calculus. Thompson et al. (2013) describe a course that approaches introductory calculus with the aim that students build a reflexive relationship between concepts of accumulation and rate of change, symbolize that relationship, and then extend it. In a first phase, students develop accumulation functions from rate of change functions. In the first phase, students "restore" the integral to the fundamental theorem of calculus. In the second phase, students develop rate of change functions from accumulation functions. The main idea is that accumulation and rate of change are never treated separately: the fundamental theorem of calculus is present all the time. Rate is an important, but difficult, mathematical concept. Despite more than 20 years of research, especially with calculus students, difficulties are still reported with this concept.

Tall (2010) reflects on the ongoing development of the teaching and learning calculus since his first thinking about the calculus 35 years ago. Tall's research described how the computer can be used to show dynamic visual graphics and to provide remarkably powerful numeric and symbolic computation. As a consequence of the cognitive difficulties that accompany the conceptual understanding of the key notions in calculus, Tall's quest is for a "sensible approach" to the calculus which builds on the evidence of our human senses and uses these insights as a meaningful basis for later development from calculus to analysis and even to a logical approach in using infinitesimals. Reflecting on the many years in which reform of calculus teaching has been considered around the world and the different approaches and reform projects using technology, Tall points out that *what has occurred is largely a retention of traditional calculus ideas, now supported by dynamic graphics for illustration and symbolic manipulation for computation.*

The research on the role of technology in teaching and learning calculus is still developing, and, as pointed by Bressoud et al. (2016), the role of technology is generally the main theme discussed in the topic study group of learning and teaching calculus in the last three International Congresses on Mathematical Education (ICME).

## The Role of Historical Perspective and Other Approaches

The idea of using a historical perspective in approaching calculus was also demonstrated in other studies, not necessarily in a technological environment. Taking into account the long way in which the calculus concepts were developed and then defined, appropriate historically inspired teaching sequences were elaborated.

Recent approaches in learning and teaching calculus refer to the social dimension, such as the approach to teaching calculus called "scientific debate," which is based on a specific form of discussion among students regarding the validity of theorems. The increasing influence of sociocultural and anthropological approaches toward learning processes is well expressed in research on learning and teaching calculus. Even the construct *concept image* and *concept definition*, which was born in an era where the theories of learning were essentially cognitive, was revisited (Bingolbali and Monaghan 2008) and used in interpreting data in a sociocultural study. This was done in a study which investigated students' conceptual development of the derivative, with particular reference to rate of change and tangent aspects.

In more recent studies, the role of different theoretical approaches in research on learning calculus was analyzed. Kidron (2008) describes a research process on the conceptualization of the notion of limit by means of the discrete continuous interplay. This paper reflects many years of research on the conceptualization of the notion of limit, and the focus on the complementary role of different theories reflects the evolution of this research.

## The Role of the Teacher

In the previous section, different educational environments were described. Educational environments depend on several factors, including teaching practices. As mentioned by Artigue (2001), reconstructions have been proved to play a crucial role in calculus, especially at the secondary/tertiary transition. Some of these reconstructions deal with mathematical objects already familiar to students before the teaching of calculus at university. In some cases, reconstructions result from the fact that only some facets of a mathematical concept can be introduced at the first contact with it. The reconstruction cannot result from a mere presentation of the theory and formal definitions. Research shows that teaching practices underestimate the conceptual difficulties associated with this reconstruction and that teaching cannot leave the responsibility for most of the corresponding reorganization to students.

Research shows that alternative strategies can be developed fruitfully, especially with the help of technology; however, successful integration of technology at a large-scale level is still a major problem (Artigue 2010). Technology cannot be considered only as a kind of educational assistant; it was demonstrated how it deeply shapes what we learn and the way we learn it.

Artigue points out the importance of the teacher's dimension. Kendal and Stacey (2001) describe teachers' practices in technology-based mathematics lessons. The integration of technology into mathematics teachers' classroom practices is a complex undertaking (Monaghan 2004; Lagrange 2013). Monaghan wrote and co-wrote a number of papers in which teachers' activities in using technology in their calculus classrooms were analyzed, but there were still difficulties that the teachers had experienced in their practices that were difficult to explain in a satisfactory manner. Investigating the reasons for the discrepancy between the potentialities of technology in learning calculus and the actual uses in the classroom, Lagrange (2013) searched for theoretical frameworks that could help to focus on the teacher using technology; the research on the role of the teacher strengthened the idea of a difficult integration, in contrast with research which centered on epistemological and cognitive aspects. An activity theory framework seems helpful to provide insight on how teachers' activity and professional knowledge evolve during the use of technology in teaching calculus.

## The Transition Between Secondary and Tertiary Education

A detailed analysis of the transition from secondary calculus to university analysis is offered by Thomas et al. (2014). A number of researchers have studied the problems of the learning of calculus in the transition between secondary school and university. Some of these studies focus on the specific topics of real numbers, functions, limits, continuity, and sequences and series. They were carried out in several different countries (Brazil, Canada, Denmark, France, Israel, Tunisia) and use different frameworks. Some have shown that calculus conflicts that emerged from experiments with 1st-year students could have their roots in a limited understanding of the concept of function, as well as suggesting the need for a more intensive exploration of the dynamic nature of the differential calculus. Results of the survey suggest that there is some room for improvement in secondary school preparation for university study of calculus.

The transition to advanced calculus as taught at the university level has been extensively investigated within the Francophone community, with the research developed displaying a diversity of approaches and themes, but a shared vision of the importance to be attached to epistemological and mathematical analyses.

Analyzing the transition between the secondary school and the university, French researchers reflect on approaches to teaching and learning calculus in which the consideration of sociocultural and institutional practices plays an essential role. These approaches offer complementary insights into the understanding of teaching and learning calculus. The theoretical influence of the theory of didactic situations, which led to a long-term Francophone tradition of didactical engineering research, has been designed in the last decade to support the transition from secondary school calculus to university analysis.

## New Directions of Research

New directions of research in teaching and learning calculus were investigated in the last decades. We observe the need for additional theoretical lenses, as well as a need to link different theoretical frameworks in the research on learning and teaching calculus. In particular, we observe the need to add additional theoretical dimensions, such as the social and cultural dimensions, to the epistemological analyses that were done in the early research. It is important to note that the "new" theoretical dimensions do not replace the cognitive and epistemological theoretical approaches that dominated the early research. These early theoretical constructs are necessary and coexist with additional theoretical lenses offered by different theories. In some cases, we notice the evolution of research in the course of many years, with the same researchers facing the challenging questions concerning the cognitive difficulties in learning calculus. The questions are still challenging, and the researchers use different theoretical frameworks in their research. For example, González-Martín et al. (2014) use the theory of didactic situations to analyze research cases from the study of calculus. The authors discuss the roles of the students and the teacher and the use of epistemological analyses. In one of these research cases, González-Martín et al. (2014, p. 125) analyze an activity that "fosters an epistemological change in students' conception, allowing them to consider real numbers as conceptual objects in relation to other objects- i.e., limits- within a mathematical theory." In the last decade, we also note discursive approaches into research, including studies using the commognitive framework for the analysis of teachers' and students' discursive practices in calculus courses. The commognitive framework, with its hybrid term "commognition," emphasizes the interrelatedness of "cognition" and "communication." Nardi et al. (2014) used the commognitive approach in three studies which explore fundamental discursive shifts often occurring in the early stages of studying calculus. They illustrate, for example, the variation of discursive patterns in practices that can be perceived initially as quite similar – as in the case of introductory calculus lectures during which they observed the construction of the object of function. More examples of theoretical frameworks used in research in teaching and learning calculus are described in Bressoud et al. (2016).

The theoretical dimension is essential for research on calculus teaching and learning, but we should not neglect practice. As pointed out by Robert and Speer (2001), there are some efforts being made toward a convergence of theory-driven and practice-driven researches. More recent studies describe research on how to consider meaningfully theoretical and pragmatic issues. Biza et al. (2016) describe the increasing interest in teaching practices at university level. The authors explore the influence of teachers' perspectives, background, and research practices on their teaching, as well as the role of resources and mathematics professional development in teaching. In the study by Bressoud et al. (2015) of characteristics of successful programs in calculus, we read how some universities coordinate calculus instruction and foster a community of practice around the teaching of calculus.

As mentioned earlier, reconstructions have been proved to play a crucial role in calculus, essentially these reconstructions that deal with mathematical objects already familiar to students before the teaching of calculus. Further research should underline the important role of teaching practices in successful reorganization of previous related concepts toward the learning of calculus.

## Cross-References


▶ Actions, Processes, Objects, Schemas (APOS) in Mathematics Education
▶ Algebra Teaching and Learning
▶ Concept Development in Mathematics Education
▶ Epistemological Obstacles in Mathematics Education
▶ Intuition in Mathematics Education
▶ Technology and Curricula in Mathematics Education
▶ Technology Design in Mathematics Education
▶ Theories of Learning Mathematics
▶ Visualization and Learning in Mathematics Education


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## Collaborative Learning in Mathematics Education

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Collaborative learning (CL) involves a team of students who learn through working together to share ideas, solve a problem, or accomplish a common goal. In mathematics education, CL's popularity surged in the 1980s, but it has since continued to evolve (Artzt and Newman 1997; Davidson 1990). The terms collaborative/cooperative learning are often used interchangeably, although some claim the former requires giving students considerable autonomy (more appropriate for older students), while the latter is more clearly orchestrated by the teacher (appropriate for all ages) (Panitz 1999).

Three dimensions seem to define collaborative learning (CL) and help distinguish among its many different models: the *structure* of the CL environment (including assessments and rewards), the *teacher and student roles*, and the types of *tasks*.

The CL structure defines how student groups are formed (usually by teacher assignment) and how group members are expected to interact. Research generally recommends mixed ability grouping. Carefully designed assessment and reward structures document student learning and provide incentives for students to work productively together. All models of CL involve group accountability, but some models also include some individual rewards, while others may pit groups against each other in a competitive reward structure.

The teacher's role is to determine the CL structure and task, then serve as facilitator. In some CL models, students are assigned specific group roles (e.g., recorder, calculator); other models require students to tackle portions of the task independently, then pool their efforts toward a common solution. Individual accountability requires that each student be responsible not only for his/her own learning but also for sharing the burden for all group members' learning.

CL tasks must be carefully chosen: amenable to group work and designed so that success depends on contributions from all group members. Particular attention to task difficulty ensures all students can engage at an appropriate level.

CL is grounded in a social constructivist model of learning (Yackel et al. 2011). Some CL models involve peer tutoring (e.g., *Student Team Learning*: Slavin 1994). In the more common investigative CL models (e.g., *Learning Together*: Johnson and Johnson 1998), the emphasis is on learning through problem solving, but higher-order skills such as interpretation, synthesis, or investigation are also required.

Project-based learning (PBL) – a twenty-first-century group-investigation CL model – involves cross-disciplinary, multifaceted, open-ended tasks, usually set in a real-world context, with results presented via oral or written presentation. PBL tasks often take several weeks because students must grapple with defining, delimiting, and planning the project; conducting research; and determining both the solution and how best to present it (Buck Institute 2012). A stated PBL goal is to help students develop “twenty-first-century skills” relating to collaboration, time management, self-assessment, leadership, and presentation concurrently with engaging in critical thinking and mastering traditional academic concepts and skills (e.g., mathematics).

Research has found student learning is accelerated when students work collaboratively on tasks that are well structured, carefully implemented, and have individual accountability. There is also evidence that affective outcomes, such as interest in school, respect for others, and self-esteem, are also positively impacted (Slavin 1992).

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# Commognition

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## Definition

*Commognition*, the portmanteau of *communication* and *cognition*, is the focal notion of the approach to learning grounded in the assumption that thinking can be usefully conceptualized as one’s communication with oneself. This foundational tenet goes against the famous Cartesian split between the bodily and the mental. According to the resulting non-dualist vision of human cognition, *mathematics* is a historically established discourse, and *learning mathematics* means becoming a participant in this special form of communication. The basic assumption about thinking as communicating has multiple entailments that combine into a comprehensive non-dualist theory of learning.

## Origins

The idea of commognition was born within the context of mathematics education in response to certain weaknesses of traditional visions of human development. Whereas learning has always been seen as a process of change, proponents of the various conceptualizations that emerged in the twentieth century differed in their answers to the question of what it was that changed when learning took place. According to behaviorists, learning was a change in the learner’s *behavior*, whereas cognitivist thinkers proposed to conceptualize learning as a process of acquiring – receiving or constructing – mental entities called *concepts*, *knowledge*, or *mental* *schemes*. One common weakness of such "acquisitionist" approaches was that being focused exclusively on the individual, they fell short of fathoming the mechanisms of the historical change in human ways of acting.

In the second half of the twentieth century, the acquisitionist stance was countered by the claim that in those processes of learning that are unique to humans, the learner becomes a participant of well-defined historically established forms of activity (Vygotsky 1987; Cole 1996). This "participationist" thinking on learning was taken one step further when different domains of human knowing, with mathematics among them, have been recognized as *discursive activities*. This latter idea, which constitutes the foundation of commognitive vision of learning, arrived almost simultaneously from two directions. On the one hand, it was an inevitable conclusion from the work of psychologists and philosophers who claimed the untenability of any attempt to separate thought from its expression (Vygotsky 1987; Wittgenstein 1953). On the other hand, the statement about the discursive nature of human knowing has been made explicitly by postmodern philosophers interested in societal-historical rather than individual-ontogenetic change of the activity known as *science*, *research*, or *knowledge building* (Lyotard 1979; Foucault 1972; Rorty 1979). With its double focus on individual and collective discursive processes, which are now seen as different aspects of the same phenomenon, the commognitive approach made it possible to account for historical transformation of human activities (Sfard 2008).

Although discursive activities constitute the main source of data in almost all types of learning sciences, the commognitive approach may be the only one that rests on the explicit claim on the unity of thinking and communication. Tacitly, this tenet seems also to be present in the branch of psychology known as *discursive* (Lerman 2001).

## Foundations

According to the basic commognitive assumption, *thinking mathematically* means participating in a historically developed discourse known as *mathematical*. Here, the term *discourse* applies to a form of communication made distinct by a number of interrelated characteristics: its special *keywords* (for instance, "three," "triangle," "set," or "function" in mathematics); its unique *visual mediators* (e.g., numerals, algebraic symbols, and graphs); its distinctive *routines*, that is, patterned ways in which its characteristic tasks (e.g., defining or proving) are being performed; and its generally *endorsed narratives* (in mathematics, theorems, definitions, and computational rules, among others). The descriptor "generally endorsed," used in this last sentence, is to be understood as referring to endorsement by the community of the discourse, with this latter term signifying all those who are recognized as able to participate in that discourse.

In tune with this conceptualization, *learning of mathematics* becomes the process of individualizing mathematical discourse. Here, the term *individualizing* refers to the process as a result of which learners gradually become capable of employing the discourse agentively, in response to their own needs.

People develop specialized discourses, such as mathematical or scientific, so as to be able to generate potentially useful stories on chosen aspects of the world around them and of their own experiences. Just as biologists narrate the worlds of living things and physicists tell stories about unanimated objects, so do participants of mathematical discourse tell stories about the universe of *mathematical objects*. Unlike the majority of other discourses, however, mathematics is a genuinely *autopoietic* system: it creates all those entities its participants talk about. In this special discourse, introduction of new nouns or symbols, rather than being an act of signifying existing mathematical entities, is the initiation of the process of *objectification*, in which new objects are constructed. At least one of the following discursive devices is used in this latter process:

- *Saming*, that is, giving a common name to things that, although seemingly unrelated, can be seen in certain contexts as equivalent (this is what happens, for instance, when the term *the*

*basic quadratic function* is introduced to refer simultaneously to things as different as the expression $x^2$, a certain curve called *parabola*, the set of numbers paired with their squares, etc.)
- *Encapsulating*, that is, replacing the talk about separate objects with the talk about a single entity (this takes place, when several objects are referred to collectively as a single *set*; for instance, when numerous ordered pairs of elements are claimed to constitute *a function*)
- *Reifying*, that is, turning talk about a mathematical process with talk about an object (this is the case, e.g., when we replace "When I add 5 to 7, I get 12" with "the sum of 5 and 7 is 12")

Once a new noun is introduced in one or more of these ways, the *alienation* of the new object gradually occurs: the noun will eventually be used in impersonal narratives, implying that its referent exists independently of the discourse. The discursive construct thus created becomes an object of mathematical explorations, as a result of which new mathematical narratives will eventually emerge.

Our actions with mathematical objects at large, and our mathematical storytelling in particular, are governed by discourse-specific routines. These relatively stable patterns of action reflect our human tendency for repetition: while in a situation in which we feel a need to act (task-situation, for short), we usually recapitulate what was usefully done in those past situations that we deem similar enough to the present one to justify such repetition. Thus, the routine performed by a person P in task-situation TS may thus be seen as a pair of elements: (1) the *task*, which is P's vision of all those elements of the precedent events that must be repeated in TS, and (2) *procedure*, which is the prescription for action that aptly describes both the present and precedent performances. The same procedure may become a basis for different types of routines, depending on the performer's vision of the task.

Expert participants of mathematical discourse interpret most task-situations as requiring a (re) formulation and endorsement of a particular type of mathematical narrative. Such outcome-oriented routines can be called *explorations*. In contrast, if these are the actions of the previous performers, not just their outcome, that the person considers as requiring exact recapitulation, it is justified to describe her process-oriented routine as *ritual*. Since the ritual performance does not count in the eyes of the performer as an act of production, it can only be motivated by this person's expectation of social rewards. Of course, most routines people actually perform are neither pure rituals nor perfect explorations, and between these two extremes, there is a wide spectrum of possibilities.

## Method

Mathematical discourses are the principal object of commognitive research, and the development of these discourses is its main theme. In contrast to psychological studies that tend to analyze learning as the process of change *in the learner*, commognitive investigations seek transformations in mathematical *discourse*. As a form of communicational activity, learning is now conceived as inherently collective, or social, rather than individual phenomenon (and it is so even if it is practiced in solitude).

Detailed records of multimodal interactions and their meticulously prepared transcriptions constitute the main type of data in commognitive research on learning. Among the rules that govern data analysis, there is the *principle of wholeness*, according to which the discourse as a whole, rather than its particular objects (or concepts), constitutes the unit of analysis; the *principle of operationality*, which requires defining the keywords with the help of perceptually accessible properties of the discourse; and the *principle of alternating perspectives*, which states that analysts have to constantly alternate between the perspectives of insiders and of outsiders to their own discourse. Although each study requires its own analytic scheme, effective heuristics are available for constructing such scheme. Finally, when reporting their findings, commognitive writers favor direct quotations from data over reported speech, and they are always wary of "ontological collapse," which is the case whenever the participant's vision of reality is offered as the researcher's own narrative on that reality.

## Commognitive Theory of the Development of Mathematical Discourses

One of the main strands in commognitive research is the study of the development of mathematical discourses, with the word *development* pertaining to both ontogenetic and historical growth of this special form of communication. Although these two types of development are quite distinct – the former is mainly productive (creative) and the other mainly reproductive – there are reasons to believe that they share some basic mechanisms and are subject to a number of comparable constraints.

Objectification, the first common feature to mention, is widely practiced across mathematics as a means of compressing the discourse and thus of making it possible to say more with less. The periodic compression allows for practically unbounded growth of mathematical discourse. This growth happens in cycles of objectifying and formalizing of the current meta-discourse and then annexing it as a new layer of the full-fledged mathematical discourse. Elementary algebra, which constitutes a formalized meta-discourse of arithmetic (Caspi and Sfard 2012), may be seen as a prototypical product of this process.

Another common feature of historical and ontogenetic developments of mathematical discourse is that they involve changes on both object level and meta-level. *Object-level developments* result in extending the existing sets of endorsed narratives about already constructed mathematical objects. This type of growth is mainly accumulative. *Meta-level developments* are those that involve changes in meta-rules of the discourse. This type of transformation is not a matter of a simple accretion: it usually results in a discourse incommensurable with its predecessor. This means that within the new discourse, some of the endorsed narratives of the old one will be considered as "misconceptions." Incommensurable discourses, therefore, rather than being mutually exclusive, complement each other in their applicability. In encounters between incommensurable discourses, such as those occasioned, for instance, by successive extensions of the number system, the old discourse (e.g., that of integers) may become subsumed within the new one (that of rational numbers). This, of course, will happen at the price of losing some of the old endorsed narratives (for instance, it will no longer count as true that "multiplications makes bigger") and of modified word uses.

*Historical Development* To get a sense of their historical development, it is necessary to consider discursive activities within the context of other ones, especially of those that result in changes, reorganization, or repositioning of objects, and can thus be called *practical*. One of the main commognitive assumptions is that practical and discursive activities have always been spurring each other's development. Thus, for instance, it is reasonable to hypothesize that the emergence of numerical discourse was prompted by our ancestors' wish to extend the practical activity of making quantitative choices. This task was initially performed by putting small finite sets in one-to-one correspondence. Once numbers were introduced, it became possible to compare also sets that were too large or too distant in space or time to be physically mapped one into another. The invention of counting opened opportunities for new types of practical activities, which, in turn, gave rise to further discursive extensions. More generally, practical and discursive activities coevoved in cycles, functioning like two legs, each of which was making a constant attempt to get ahead of the other one, thus moving the whole system forward, toward ever greater complexity.

This vison of the coevolution of practical and discursive activities has been recently corroborated by findings of a cross-cultural research on the learning of mathematics in the Polynesian state of Tonga (Morris 2017). The study has shown that discourses developed in one culture to support practical activities specific to this culture may not be easily transferrable to a culture, in which these special activities are absent. Commognitive approach has also been found useful in mapping shorter-term historical changes, such as those that happened over the period of a few decades in the discourse of school mathematics in England (Morgan and Sfard 2016).

*Ontogenetic Development* Although it is reasonable to expect some parallels between the historical and ontogenetic developments, it is just as justified to expect differences. Rather than being brought into being by some practical, genuinely felt need, new discourses may appear in the life of a learner as ready-made patterns of communicating, widely practiced in the community. For instance, in today's societies, children are taught to count prior to being properly exposed to the quantitative discourse, recognizable by descriptive keywords such as *more*, *less*, *greater*, *large*, etc., and long before they are aware of how the resulting numerical discourse may be applied in any activity (Lavie and Sfard 2016). Similarly, the development of the discourse on rational numbers begins with an introduction of the calculus of fractions. In both these cases, the new discourse, if successfully developed, will be incommensurable with its predecessor, and this means that there will be a need for a meta-level learning.

In contrast to object-level learning that, theoretically, can happen without the teacher's deliberate intervention, meta-level learning requires interacting with a person who is already adept in the new discourse. This type of learning cannot be motivated or guided by the learner's own genuine interest in the outcome. For the student, the only way to enter the discourse is to imitate teacher's expert performances. At this point, the routines she performs cannot yet constitute true mathematical explorations, because the learner, not being acquainted with the focal objects, cannot judge the success of her performance by the endorsability of the mathematical narrative produced in the process. Meta-level learning is thus bound to begin with rituals.

The rituals, which are arguably inevitable at the earliest stages of meta-level learning, may later morph into explorations. For this to happen, the learner must keep participating in the new discourse while also making persistent efforts to figure out its usefulness. In the progress of de-ritualization, the performer's attention gradually shifts from the performance as such to its outcome. This shift may manifest itself, among others, in the strengthening of such characteristics of routines as *flexibility* or *applicability*. With time, the routine will become *vertically bonded*: every step in its procedure will build on the outcome of the previous ones. It will also be *horizontally bonded* with other routines: its procedure will branch into a number of alternative paths as a result of realization that other routines perform the same task. As found in research, the process of de-ritualization may be gradual and slow (Sfard and Lavie 2005; Lavie and Sfard 2016) and only too often is not being completed in school. The question of what it is that fuels or obstructs processes of de-ritualization is being addressed in numerous commognitive studies.

## Commognitive Theory of Factors that Shape the Learning of Mathematics

*Conditions for Learning* Commognitive approach offers its own vision of circumstances under which learning of mathematics becomes possible. Object-level learning requires no more than the ability to deduce new narratives from those already endorsed and thus can, in principle, be attained by learners on their own, without the help from a more experienced participant. For meta-level mathematics learning to occur, however, some special conditions are necessary. The opportunity for meta-level learning offers itself when the learners encounter a discourse incommensurable with their own. Three conditions must be fulfilled to turn such *commognitive conflict* into a genuine opportunity for learning: (1) all the participants have to agree on the question of which discourse should be the leading one, that is, common to all the participants; (2) the experienced participants of the leading discourse must accept their role as leaders (teachers), whereas other ones must be willing to act as followers (learners); and (3) the participants need to have shared expectations with regard to the possible form and pace of the learning process. Together, these three conditions constitute a *learning-teaching agreement*. Commognitive theory offers a vision of factors likely to support or counter this kind of agreement, thereby shaping the learning of mathematics.

*Culture* Any mathematical discourse, when taught in different institutional or cultural settings, may give rise to different learning processes. That this is the case has been corroborated in a study that compared mathematics learning of native Israelis to that of immigrants from the former Soviet Union (Sfard and Prusak 2005), in the commognitive research on the learning about infinity and limits by Korean-speaking students and by English speakers from the United States (Kim et al. 2012), and in a study on the learning of fractions and probability in Tonga (Morris 2017).

*Identity* While *mathematizing*, that is, participating in a discourse on mathematical objects, we tend to be simultaneously involved in the discourse of *subjectifying*, that is, in an overt or covert talk about participants. Clearly, the activity of subjectifying, unless tightly related to the performance of mathematical tasks, may reduce the participants' engagement in mathematical discourse, thereby undermining the effectiveness of their mathematics learning. Particularly strong may be effects of subjectifying that takes the form of *identification*, that is, of telling stories on the properties of the learner rather than of her actions. Identity-constituting narratives, offered directly or indirectly by their protagonists, the learners, and by the people around them, tend to function as self-fulfilling prophecies and may thus have a long-term effect on learning: the student identified as "weak" will now be more likely to fail, and the one labeled as "strong" will be more determined to achieve success. The result will reinforce the previously constructed identities, reducing the chances for a change in a reverse direction (Ben-Yehuda et al. 2005; Sfard and Prusak 2005; Heyd-Metzuyanim 2015).

*Teaching* In our society, young people enter the world of formalized mathematics mainly through opportunities for learning created for them by mathematics teachers. The teacher models the discourse for the learners and issues invitations for their active co-participation. One of the main questions to ask while trying to figure out possible outcomes of the teacher's efforts is whether the students are offered an access to explorative mathematics or are rather encouraged to satisfy themselves with ritualized discourse (Adler and Sfard 2017).

## Contributions of Commognitive Research: Past and Future

The commognitive approach may be claimed to have a number of strengths. First, research methods grounded in its underlying non-dualist onto-epistemology make it possible to investigate learning on both individual and collective levels and lead to a high-resolution picture of the relevant processes. Commognitive analyses reveal the highly consequential nature of even the tiniest of the teachers' moves. Second, the constantly expanding commognitive theory brings its own insights about mathematics learning and informs the teaching of mathematics in ways that often go against widely endorsed pedagogical principles. Last but not least, the disappearance of the thought-communication dichotomy dissolves some of the time-honored dilemmas that proved untreatable within the confines of the traditional dualist approaches. The non-duality implies that both types of phenomena can be researched, at least in principle, with the same set of conceptual tools, even if not in the same ways and not with an equal ease. One time-honored quandary that becomes treatable with these unified tools is the question of our uniquely human capacity for changing our ways of doing things from one generation to another (for *societal learning*). Unaccounted for by the traditional theorizations of learning, this special capacity for accumulating the complexity of our actions can now be explained by taking a close look at processes of development, in which discourses remain in a co-constitutive interaction with physical tools. With the tools together, they function as practically unbounded compressors, repositories, and disseminators of complexity. Since societal learning is the signature feature of the human species, commognition may be said to have made a tentative contribution to solving the puzzle of human uniqueness.

Whereas some of the old quandaries may now be regarded as dissolved, some other ones invite further commognitive study. In spite of the progress already made, figuring out the mechanisms of discourse development, whether ontogenetic or historical, is nowhere close to disappearing from the researcher's to-do list. The same may be said about the task of mapping the co-constitutive relations between our discursive and practical activities or about the project of fathoming mutual influences of mathematics and other discourses practiced in different societies. If successful in tackling these and similar issues, commognitive researchers may produce insights, the relevance and impact of which are likely to go beyond the practice of learning and teaching mathematics.

## Cross-References


- ▶ Discourse Analytic Approaches in Mathematics Education
- ▶ Discursive Approaches to Learning Mathematics
- ▶ Mathematization as Social Process
- ▶ Theories of Learning Mathematics


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# Communities of Inquiry in Mathematics Teacher Education

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**Keywords**
Mathematics teacher education · Community · Inquiry

## Definition

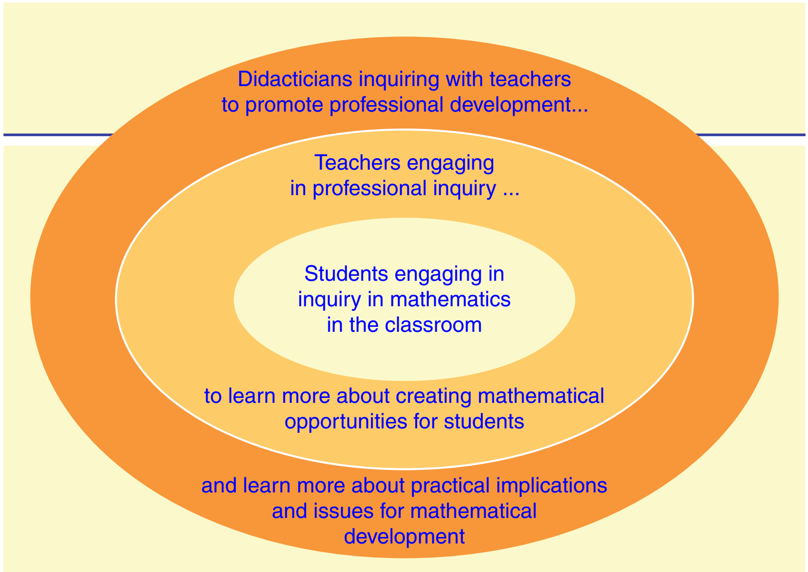
Mathematics teacher education (MTE) consists of processes and practices through which teachers or student teachers learn to teach mathematics. It involves as participants, primarily, student teachers, teachers, and teacher educators; other stakeholders such as school principals or policy officials with regulatory responsibilities can be involved to differing degrees. Thus a *community* in MTE consists of people who engage in these processes and practices and who have perspectives and knowledge in what it means to learn and to educate in mathematics and an interest in the outcomes of engagement. An *inquiry community*, or *community of inquiry*, in MTE is a community which brings *inquiry* into *practices* of teacher education in mathematics – where inquiry implies questioning and seeking answers to questions, problem solving, exploring, and investigating – and in which inquiry is the basis of an epistemological stance on practice, leading to "metaknowing" (Wells 1999; Jaworski 2006). The very nature of a "community" of inquiry rooted in communities of practice (Wenger 1998) implies a sociohistorical frame in which knowledge grows and learning takes place through participation and dialogue in social settings (Wells 1999).

## Characteristics

Rather than seeing knowledge as objective, pre-given and immutable (an *absolutist* stance: Ernest 1991) with learning as a gaining of such knowledge and teaching as a conveyance of knowledge from one who knows to one who learns, an *inquiry stance* sees knowledge as fluid, flexible and *fallible* (Ernest 1991). These positions apply to mathematical knowledge and to knowledge in teaching: teachers of mathematics need both kinds of knowledge. Knowledge is seen variously as formal and external, consisting of general theories and research-based findings to be gained and put into practice; or as craft knowledge, intrinsic to the knower, often tacit, and growing through action, engagement, and experience in practice; or yet again as growing through inquiry in practice so that the knower and the knowledge are inseparable. Cochran Smith and Lytle (1999) call these three ways of conceptualizing knowledge as knowledge *for* teaching, *in* teaching, and *of* teaching. With regard to knowledge-*of*-teaching, they use the term "inquiry as stance" to describe the positions teachers take towards knowledge and its relationships towards practice. This parallels the notion of "inquiry as a way of being" in which teachers take on the mantle of inquiry as central to how they think, act, and develop in practice and encourage their students to do so as well (Jaworski 2006).

An inquiry community in mathematics teacher education therefore involves teachers (including student teachers who are considered as less experienced teachers) engaging together in inquiry into teaching processes to promote students' learning of mathematics and, moreover, involving students in inquiry in mathematics. The main purpose of inquiry is to call into question aspects of a source (such as mathematics) which encourages a deeper engagement as critical questioning takes place and knowledge grows within the community. When the source is mathematics, inquiry in mathematics allows students to address mathematical questions in ways that seek out answers and lead to new knowledge. Thus mathematics itself becomes accessible, no longer perceived as only right or wrong, and its revealed fallibility is an encouragement to the learner to explore further and understand more deeply. Similarly as teachers explore into aspects of mathematics teaching – for example, the design of inquiry-based mathematical tasks for students – their critical attitude to their practice generates new knowledge in practice and new practice-based understandings (Jaworski 2006).

**Communities of Inquiry in Mathematics Teacher Education, Fig. 1** Three layers of inquiry in mathematics teaching development






In a *community of practice*, Wenger (1998) suggests that "belonging" to the community involves "engagement," "imagination," and "alignment." Participants *engage* with the practice, use imagination in weaving a personal trajectory in the practice and *align* with norms and expectations within the practice. The transformation of a community of practice to a community of inquiry requires participant to *look critically* at their practices as they engage with them, to question what they do as they do it, and to explore new elements of practice. Such inquiry-based forms of engagement have been called "critical alignment" (Jaworski 2006). Critical alignment is a necessity for developing an inquiry way of being within a community of inquiry.

Like teachers, teacher educators in mathematics (sometimes called didacticians, due to their practices in relation to the didactics of mathematics) are participants in communities of inquiry in which they too need to develop knowledge in practice through inquiry. Their practices are different from those of teachers, but there are common layers of engagement in which teachers and teacher educators side by side explore practices in learning and teaching of mathematics in order to develop practice and generate new knowledge. Teacher educators also have responsibilities in linking theoretical perspectives to development of practice and to engaging in research formally for generation of academic knowledge. Thus it is possible to see three (nested) layers of inquiry community in generating new understandings of teaching to develop the learning of mathematics: inquiry by students into mathematics in the classroom, inquiry by teachers into the processes and practices of creating mathematical learning in classrooms, and inquiry by teacher educators into the processes by which teachers learn through inquiry and promote the mathematical learning of their students (Jaworski and Wood 2008) (Fig. 1).

## Cross-References

▶ Communities of Practice in Mathematics Education
▶ Inquiry-Based Mathematics Education
▶ Mathematics Teacher Education Organization, Curriculum, and Outcomes

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# Communities of Practice in Mathematics Education

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**Keywords**

Ethnomathematics · Professional learning communities · Social theory of learning · Equity

## Definition and Originators

Communities of practice (CoP) are an important component of an emerging social theory of learning. Lave and Wenger (1991) originally envisioned this social learning theory as a way to deepen and extend the notion of situated learning that occurs in traditional craft apprenticeships, contexts in which education occurs outside of formal schools ("Anthropological Approaches"). Drawing upon evidence from ethnographic investigations of apprenticeships in a range of settings (e.g., tailoring), they have frequently argued that it is important to separate learning from formal school contexts to understand that most human activities involve some form of teaching and learning. Wenger (1998) argued that three dimensions inherently connect CoP's two components (community and practice): "1) mutual engagement; 2) a joint enterprise; 3) a shared repertoire" (p. 73). One important aim of a CoP is the negotiation of meaning among participants. Groups of people who live or work in the same location do not create a CoP unless they are actively involved in communicating with each other about important issues and working together toward common goals. Another important aspect of CoP is that learning may be demonstrated by changes in the personal identities of the community members. Changes in identity are accompanied by increasing participation in the valued practices of this particular CoP as newcomers become old-timers in the community.

## How CoP Connects to Developments in Theories of Learning Mathematics

Social theories of learning have a long history in psychology (Cole 1996). Nevertheless, more experimental and reductionist theories were the predominant form of psychology until the late twentieth century. The reemergence of social theories of learning has occurred in numerous places, such as discursive psychology (Harré and Gillett 1994), as well as in mathematics education (Lerman 2001; van Oers 2001). Sfard (1998) has outlined the reasons why we need a social learning theory in mathematics education. She contrasted two key metaphors: learning as acquisition versus learning as participation. Most research conducted during the last century in mathematics education used the acquisition metaphor. In contrast, the participation metaphor shifts the focus from individual ownership of skills or ideas to the notion that learners are fundamentally social beings who live and work as members of communities. Teaching and learning within CoP depend upon social processes (collaboration or expert guidance) as well as social products (e.g., tools) in order to help newcomers master the important practices of their community (► "Theories of Learning Mathematics"). In addition, we need social theories of learning to address some of the fundamental quandaries of educational research and practice (Sfard 2008). These enduring dilemmas include the puzzling discrepancy in performance on in-school and out-of-school mathematical problems.

## History of Use

Lave's (1988, 2011) own empirical research began with a focus on mathematical proficiency in out-of-school settings (e.g., tailoring garments). She initially chose situated cognition tasks that required mathematical computations so that she could more easily compare them with school-like tasks (▶ "Informal Learning in Mathematics Education"). Other investigators in ethnomathematics conducted similar studies for a range of cultural activities (e.g., selling candy on the street) (Nunes et al. 1993) (▶ "Ethnomathematics"). One recurrent finding of this research has been that children, adolescents, and adults can demonstrate higher levels of mathematical proficiency in their out-of-school activities than in school, even when the actual mathematical computations are the same (Forman 2003). Another finding was that social processes (e.g., guided participation) and cultural tools (e.g., currency) were important resources for people as they solved mathematical problems outside of school (Saxe 1991, 2012). This research forces one to question the validity of formal assessments of mathematical proficiency and to wonder how mathematical concepts and procedures are developed in everyday contexts of work and play (▶ "Situated Cognition in Mathematics Education"). Many of these investigators began to question the basic assumptions of our individual learning theories and turn their attention to developing new social theories of learning.

Social theories of learning have had a greater impact on school-based research in the last 10 years. Research in teacher education, for example, has embraced the idea of CoP because it allows us to go beyond a bifurcated focus: either on individual teachers or on the organizational structure of schools (Cobb and McClain 2006; Cobb et al. 2003) (▶ "Communities of Practice in Mathematics Teacher Education"). Cobb and his colleagues used an expanded version of Wenger's (1998) CoP framework to view teachers' practices as part of the "lived organizations" (2003, p. 13) of schools and districts (▶ "Mathematics Teacher as Learner"). This expanded framework has allowed them to understand the multiple communities in which teachers and administrators participate (e.g., as mathematics leaders, as school leaders, or as members of a professional teaching group) (▶ "Education of Mathematics Teacher Educators"). Each community may have its tacit norms and practices, requiring individuals to serve as brokers during boundary encounters and to create boundary objects that allow them to mediate between groups. Wenger (1998) argued that people create boundary objects through a process of reification. For example, boundary objects in teacher education can be common planning tools or agreed-upon student characteristics. Although boundary objects may not embody identical meanings for all groups of participants, they can allow for coordination of activity between communities.

As applications by Cobb and his colleagues of CoP to teacher education were widely disseminated in the mathematics education community, other investigators worked on expanding the theoretical and empirical knowledge base (▶ "Professional Learning Communities in Mathematics Education"). For example, Bannister (2015) combined Goffman's notion of frame analysis with Wenger's CoP to conduct a microanalysis of changes in the pedagogical reasoning of one team of high school mathematics teachers over several months. Her analysis of ethnographic data focused on both participation patterns (e.g., turn taking) and reification (e.g., boundary objects such as "struggling students") (▶ "Discourse Analytic Approaches in Mathematics Education"). She was able to document distinct changes in the ways that this group of teachers characterized struggling students: from attending to static attributes to focusing on classroom interventions to support those students (▶ "Frameworks for Conceptualizing Mathematics Teacher Knowledge").

## Perspectives on Issues in Different Cultures/Places

The earliest research about CoP was conducted in diverse cultural settings: Brazil, Liberia, and

Papua New Guinea (▶ "Cultural Diversity in Mathematics Education"). In addition to a broad range of national settings, this ethnographic work focused on the mathematical reasoning that occurred in the daily lives of people outside of schools. More recently, research sites were located in schools in Europe or North America (e.g., Cobb and McClain 2006; Corbin et al. 2003). Thus, unlike many educational innovations, the study of CoP began in impoverished locations and later spread to wealthy settings.

## Gaps That Need to Be Filled

Forms of mutual engagement change over time within any community (Wenger 1998). Collective goals evolve as different interpretations clash and new understandings are negotiated. New boundary objects are created and modified, new vocabulary developed, and new routines and narratives invented when this happens. As investigators such as Cobb and others follow teacher communities of practice over periods of months or years, we are able to understand the tensions and struggles that occur in different school districts as they attempt to change teachers' practices to be more standards-based. Their application of Wenger's CoP has allowed them to keep a dual focus on the learning of individual teachers and the institutional constraints and affordances presented by their schools, districts, and government entities. This dual focus can be seen in the Railside School project, originally documented by Boaler and Staples (2008) (▶ "Equity and Access in Mathematics Education"). After several years of successful implementation, Railside was derailed due to national policy changes that increased standardization and accountability requirements (Nasir et al. 2014). Thus, CoP provides a framework for confronting the realities of maintaining a successful teacher learning community over long periods.

Finally, several investigators in mathematics education are now asking us to transcend the limitations of the CoP perspective in order to understand race, power, and identity in mathematical practices. They refer to this expansion of CoP as the sociopolitical turn (Gutierrez 2013; Nasir and McKinney de Royston 2013). These authors draw on critical race theory to characterize the dynamics that occur during interactions among members of dominant (white, middle-class adults) and nondominant communities (working class parents and students of color) (▶ "Urban Mathematics Education"). These investigators and others question the narratives in which the underachievement of students of color is an individual failure and not a systemic devaluing of their cultural capital. This new direction allows us to situate communities in economic and political hierarchies that serve to maintain the status quo of systemic inequality at individual and collective levels. And it may permit us to construct counter-narratives of positive identity development by recognizing the cultural funds of knowledge of these students and offering different ways to access the power of mathematics (Quintos et al. 2011) (▶ "Language Background in Mathematics Education"). In their own way, these investigators are returning to the roots of CoP in situated practice in order to re-examine the enduring dilemmas of mathematics education.

## Cross-References

▶ Communities of Practice in Mathematics Teacher Education
▶ Cultural Diversity in Mathematics Education
▶ Discourse Analytic Approaches in Mathematics Education
▶ Education of Mathematics Teacher Educators
▶ Equity and Access in Mathematics Education
▶ Ethnomathematics
▶ Frameworks for Conceptualizing Mathematics Teacher Knowledge
▶ Informal Learning in Mathematics Education
▶ Language Background in Mathematics Education
▶ Mathematics Teacher as Learner
▶ Professional Learning Communities in Mathematics Education
▶ Situated Cognition in Mathematics Education
▶ Theories of Learning Mathematics
▶ Urban Mathematics Education

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C

# Communities of Practice in Mathematics Teacher Education


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**Keywords**

Mathematics teacher education · Communities of practice · Identity


## Definition

**Communities of practice in mathematics teacher education** are informed by a theory of learning as social participation, in which teacher learning and development are conceptualized as increasing participation in social practices that develop an identity as a teacher.

## Background

The idea of learning in a community of practice grew from Jean Lave's and Etienne Wenger's research on learning in apprenticeship contexts (Lave 1988; Lave and Wenger 1991). Drawing on their ethnographic observations of apprentices learning different trades, Lave and Wenger developed a theory of learning as social practice to describe how novices come to participate in the practices of a community. These researchers introduced the term "legitimate peripheral participation" to explain how apprentices, as newcomers, are gradually included in the community through modified forms of participation that are accessible to potential members working alongside master practitioners. Although social practice theory aimed to offer a perspective on learning in out-of-school settings, Lave (1996) afterwards argued that apprenticeship research also has implications for both learning and teaching in schools and for students and teachers as participants in social practices that shape identities.

To further analyze the concepts of identity and community of practice, Wenger (1998) proposed a more elaborated social theory of learning that integrates four components – meaning (learning as experience), practice (learning as doing), community (learning as belonging), and identity (learning as becoming). Wenger explained that communities of practice are everywhere – in people's workplaces, families, and leisure pursuits, as well as in educational institutions. Most people belong to multiple communities of practice at any one time and will be members of different communities throughout their lives. His theory has been applied to organizational learning as well as learning in schools and other formal educational settings.

## Communities of Practice as a Framework for Understanding Mathematics Teacher Learning and Development

Social theories of learning are now well established in research on mathematics education. Lerman (2000) discussed the development of "the social turn" in mathematics education research and proposed that social theories drawing on community of practice models provide insights into the complexities of teacher learning and development. From this perspective, learning to teach involves developing an identity as a teacher through increasing participation in the practices of a professional community (Lerman 2001). At the time of publication of Lerman's (2001) review chapter on research perspectives on mathematics teacher education, there were few studies drawing on Lave's and Wenger's ideas. Reviewing the same field 5 years later, Llinares and Krainer (2006) noted increasing interest in using the idea of a community of practice to conceptualize learning to teach mathematics. Such studies can be classified along several dimensions, according to their focus on:

1. Preservice teacher education or the professional learning and development of practicing teachers
2. Face-to-face or online interaction (or a combination of both)
3. Questions about how a community of practice is formed and sustained compared with questions about the effectiveness of communities of practice in promoting teacher learning

Research has been informed by the two key conceptual strands of Wenger's (1998) social practice theory. One of these strands is related to the idea of learning as increasing *participation in socially situated practices* and the other to learning as developing an *identity* in the context of a community of practice.

### Learning as Participation in Practices

With regard to *participation in practices*, Wenger describes three dimensions that give coherence to communities of practice: mutual engagement of participants, negotiation of a joint enterprise, and development of a shared repertoire of resources for creating meaning. Mutuality of engagement need not require homogeneity, since productive relationships arise from diversity and these may involve tensions, disagreements, and conflicts. Participants negotiate a joint enterprise, finding ways to do things together that coordinate their complementary expertise. This negotiation gives rise to regimes of mutual accountability that regulate participation, whereby members work out who is responsible for what and to whom, what is important and what can safely be ignored, and how to act and speak appropriately. The joint enterprise is linked to the larger social system in which the community is nested. Such communities have a common cultural and historical heritage, and it is through the sharing and reconstruction of this repertoire of resources that individuals come to define their relationships with each other in the context of the community.

This aspect of Wenger's theory has been used to investigate discontinuities that may be experienced in learning to teach mathematics in the different contexts in which prospective and beginning teachers' learning occurs – the university teacher education program, the practicum, and the early years of professional experience (Llinares and Krainer 2006). One of the more common discontinuities is evident in the difficulty many beginning teachers experience in sustaining the innovative practices they learn about in their university courses. This observation can be explained by acknowledging that prospective and beginning teachers participate in separate communities – one based in the university and the other in school – which often have different regimes of accountability that regulate what counts as "good teaching."

Researchers have also investigated how participation in online communities of practice supports the learning of prospective and practicing teachers of mathematics, and insights into principles informing the design of such communities are beginning to emerge (Goos and Geiger 2012). Some caution is needed in interpreting the findings of these studies, since few present evidence that a community of practice has actually been formed: for example, by analyzing the extent of mutual engagement, how a joint enterprise is negotiated, and whether a shared repertoire of meaning-making resources is developed by participants (Goos and Bennison 2008). Nevertheless, studies of online communities of practice demonstrate that technology-mediated collaboration does more than simply increase the amount of knowledge produced by teachers; it also leads to qualitatively different forms of knowledge and different relationships between participants.

### Learning as Developing an Identity

With regard to *identity development*, Wenger wrote of different modes of belonging to a community of practice through engagement, imagination, and alignment. Beyond actually engaging in practice, people can extrapolate from their experience to imagine new possibilities for the self and the social world. Alignment, the third mode of belonging, refers to coordinating one's practices to contribute to the larger enterprise or social system. Alignment can amplify the effects of a practice and increase the scale of belonging experienced by community members, but it can also reinforce normative expectations of practice that leave people powerless to negotiate identities.

Research into teacher *identity* development in communities of practice is perhaps less advanced than studies that analyze evidence of changing participation in the *practices* of a community. This may be due to a lack of well-developed theories of identity that can inform research designs and provide convincing evidence that identities have changed. Jaworski's (2006) work on identity formation in mathematics teacher education proposes a conceptual shift from learning within a community of practice to forming a community of inquiry. The distinguishing characteristic of a community of inquiry is reflexivity, in that participants critically reflect on the activities of the community in developing and reconstructing their practice. This requires a mode of belonging that Jaworski calls "critical alignment" – adopting a critically questioning stance in order to avoid perpetuating undesirable normative states of activity.

## Issues for Future Research

Elements of Wenger's social practice theory resonate with current ways of understanding teachers' learning, and this may explain why his ideas have been taken up so readily by researchers in mathematics teacher education. Nevertheless, the notion of situated learning in a community of practice composed of experts and novices was not originally focused on school classrooms, nor on pedagogy, and so caution is needed in applying this perspective on learning as an informal and tacit process to learning in formal education settings, including preservice and in-service teacher education (Graven and Lerman 2003). Wenger's model was developed from studying learning in apprenticeship contexts, where teaching is incidental rather than deliberate and planned, as in university-based teacher education. It remains to be seen whether community of practice approaches can be applied to understand the role of teacher educators in shaping teachers' learning.

## Cross-References

- ▶ Communities of Inquiry in Mathematics Teacher Education
- ▶ Communities of Practice in Mathematics Education
- ▶ Mathematics Teacher as Learner
- ▶ Mathematics Teacher Identity
- ▶ Professional Learning Communities in Mathematics Education

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# Competency Frameworks in Mathematics Education

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**Keywords**
Competence · Conceptual framework · Taxonomy · Subject matter · Mental process

## Definition

A structural plan for organizing the cognitive skills and abilities used in learning and doing mathematics.

## Characteristics

The concept of competence is one of the most elusive in the educational literature. Writers often use the term *competence* or *competency* and assume they and their readers know what it means. But arriving at a simple definition is a challenging matter. Dictionaries give such definitions as "the state or quality of being adequately or well qualified"; "the ability to do something successfully or efficiently"; "possession of required skill, knowledge, qualification, or capacity"; "a specific range of skill, knowledge, or ability"; and "the scope of a person's or group's knowledge or ability." *Competence* seems to possess a host of near synonyms: *ability, capability, cognizance, effectuality, efficacy, efficiency, knowledge, mastery, proficiency, skill, and talent* – the list goes on.

Arriving at a common denotation across different usages in social science is even more difficult. "There are many different theoretical approaches, but no single conceptual framework" (Weinert 2001, p. 46). Weinert identifies seven different ways that "competence has been defined, described, or interpreted theoretically" (p. 46). They are as follows: general cognitive competencies, specialized cognitive competencies, the competence-performance model, modifications of the competence-performance model, cognitive competencies and motivational action tendencies, objective and subjective competence concepts, and action competence. Competency frameworks in mathematics education fall primarily into Weinert's specialized-cognitive-competencies category, but they also overlap some of the other categories.

The progenitor of competency frameworks in mathematics education is Bloom's (1956) *Taxonomy of Educational Objectives*, which attempted to lay out, in a neutral way, the cognitive goals of any school subject. The main categories were *knowledge, comprehension, application, analysis, synthesis*, and *evaluation*. These categories were criticized by mathematics educators such as Hans Freudenthal and Chris Ormell as being especially ill suited to the subject of mathematics (see Kilpatrick 1993 on the critiques as well as some antecedents of Bloom's work). Various alternative taxonomies have subsequently been proposed for school mathematics (see Tristán and Molgado 2006, pp. 163–169, for examples). Further, Bloom's taxonomy has been revised (Anderson and Krathwohl 2001) to separate the knowledge dimension (*factual, conceptual, procedural*, and *metacognitive*) from the cognitive process dimension (*remember, understand, apply, analyze, evaluate*, and *create*), which does address one of the complaints of mathematics educators that the original taxonomy neglected content in favor of process. But the revision nonetheless fails to address such criticisms as the isolation of objectives from any context, the low placement of understanding in the hierarchy of processes, and the failure to address important mathematical processes such as representing, conjecturing, and proving.

Whether organized as a taxonomy, with an explicit ordering of categories, or simply as an arbitrary listing of topics, a competency framework for mathematics may include a breakdown of the subject along with the mental processes used to address the subject, or it may simply treat those processes alone, leaving the mathematical content unanalyzed. An example of the former is the model of outcomes for secondary school mathematics proposed by James Wilson (cited by Tristán and Molgado 2006, p. 165). In that model, mathematical content is divided into *number systems, algebra*, and *geometry*; cognitive behaviors are divided into *computation, comprehension, application, and analysis*; and affective behaviors are either *interests and attitudes* or appreciation. Another example is provided by the framework proposed for the Third International Mathematics and Science Study (TIMSS; Robitaille et al. 1993, Appendix A). The main content categories are *numbers; measurement; geometry* (position, visualization, and shape; symmetry, congruence, and similarity); *proportionality; functions, relations, and equations; data representation, probability, and statistics; elementary analysis; validation and structure;* and *other content* (informatics). The performance expectations are *knowing, using routine procedures, investigating and problem solving, mathematical reasoning*, and *communicating*.

Other competency frameworks, like that of Bloom's (1956) taxonomy, do not treat different aspects of mathematical content separately but instead attend primarily to the mental processes used to do mathematics, whether the results of those processes are termed *abilities, achievements, activities, behaviors, performances, practices, proficiencies*, or *skills*. Examples include the five strands of mathematical proficiency identified by the Mathematics Learning Study of the US National Research Council – *conceptual understanding, procedural fluency, strategic competence, adaptive reasoning*, and *productive disposition* – and the five components of mathematical problem-solving ability identified in the Singapore mathematics framework: *concepts, skills, processes, attitudes*, and metacognition (see Kilpatrick 2009, for details of these frameworks).

A final example of a competency framework in mathematics is provided by the KOM project (Niss 2003), which was charged with spearheading the reform of mathematics in the Danish education system. The KOM project committee addressed the following question: What does it mean to master mathematics? They identified eight competencies, which fell into two groups. The first four address the ability to ask and answer questions in and with mathematics:

1. Thinking mathematically
2. Posing and solving mathematical problems
3. Modeling mathematically
4. Reasoning mathematically

The second four address the ability to deal with and manage mathematical language and tools:

5. Representing mathematical entities
6. Handling mathematical symbols and formalisms
7. Communicating in, with, and about mathematics
8. Making use of aids and tools

Niss (2003) observes that each of these competencies has both an analytic and a productive side. The analytic side involves understanding and examining the mathematics, whereas the productive side involves carrying it out. Each competency can be developed and used only by dealing with specific subject matter, but the choice of curriculum topics is not thereby determined. The competencies, though specific to mathematics, cut across the subject and can be addressed in multiple ways.

The KOM project also found it necessary to focus on mathematics as a discipline. The project committee identified three kinds of "overview and judgment" that students should develop through their study of mathematics: its actual application, its historical development, and its special nature. Like the competencies, these qualities are both specific to mathematics and general in scope.

Niss (2003) observes that the competencies and the three kinds of overview and judgment can be used: (a) normatively, to set outcomes for school mathematics; (b) descriptively, to characterize mathematics teaching and learning; and (c) metacognitively, to help teachers and students monitor and control what they are teaching or learning. These three usages apply as well to the other competency frameworks developed for mathematics.

Regardless of whether a competency framework is hierarchical and regardless of whether it addresses topic areas in mathematics, its primary use will be normative. Competency frameworks are designed to demonstrate to the user that learning mathematics is more than acquiring an array of facts and that doing mathematics is more than carrying out well-rehearsed procedures. School mathematics is sometimes portrayed as a simple contest between knowledge and skill. Competency frameworks attempt to shift that portrayal to a more nuanced portrait of a field in which a variety of competences need to be developed.

## Cross-References


- ▶ Bloom's Taxonomy in Mathematics Education
- ▶ Competency Frameworks in Mathematics Education
- ▶ Frameworks for Conceptualizing Mathematics Teacher Knowledge
- ▶ International Comparative Studies in Mathematics: An Overview


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## Complexity in Mathematics Education

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**Keywords**
Complexity theory · Complexity modeling · Design-based research · Mathematical modelling · Systems thinking

### Definition/Introduction

Over the past half-century, "complex systems" perspectives have risen to prominence across many academic domains in the sciences, engineering, and the humanities. Mathematics was among the originating domains of complexity research. Education has been a relative latecomer, and so perhaps not surprisingly, mathematics education researchers have been leading the way in the field.

There is no unified definition of complexity, principally because formulations emerge from the study of specific phenomena. One thus finds quite focused definitions in such fields as mathematics and software engineering, more indistinct meanings in chemistry and biology, and quite flexible interpretations in the social sciences (cf. Mitchell 2009). Because mathematics education reaches across several domains, conceptions of complexity within the field vary from the precise to the vague, depending on how and where the notion is taken up. Diverse interpretations do collect around a few key qualities, however. In particular, complex systems adapt and are thus distinguishable from complicated systems. A complicated system is one that comprises many interacting components and whose global character can be adequately described and predicted by specifying the rules of operation of the individual parts. A complex system comprises many interacting agents, and *emergence* of global behaviors that cannot be adequately predicted by simply specifying the rules of the individual agents is a central characteristic of such systems. Some popularly cited examples of complex, emergent phenomena include anthills, economies, and brains, which are more than the linear sum of behaviors of individual ants, consumers, and neurons. In brief, whereas the opposite of complicated is simple, opposites of complex include reducible and decomposable. Hence, prominent efforts toward a coherent, unified description of complexity revolve around such terms as emergent, noncompressible, multilevel, self-organizing, context-sensitive, and adaptive.

This entry is organized around four categories of usage within mathematics education – namely, complexity as: an epistemological discourse, a historical discourse, a disciplinary discourse, and a pragmatic discourse.

### Complexity as an Epistemological Discourse

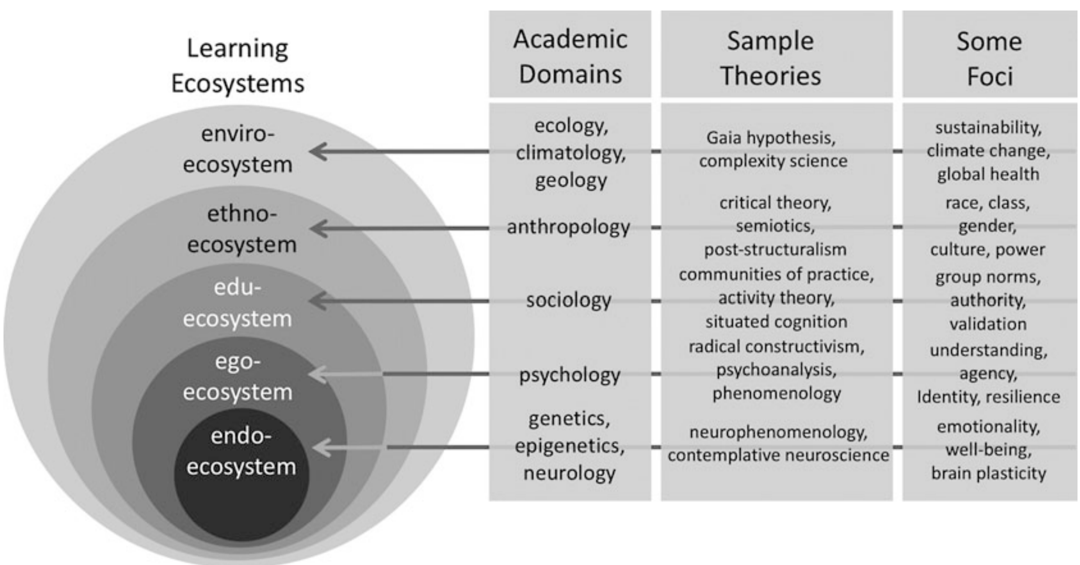
Among educationists interested in complexity, there is frequent resonance with the notions that a complex system is one that knows (i.e., perceives, acts, engages, develops, etc.) and/or learns (adapts, evolves, maintains self-coherence, etc.). This interpretation reaches across many systems that are of interest among educators, including physiological, personal, social, institutional, epistemological, cultural, and ecological systems. Unfolding from and enfolding in one another, it is impossible to study one of these phenomena without studying all the others.

This is a sensibility that has been well represented in the mathematics education research literature for decades in the form of varied theories of learning. Among others, radical constructivism, socio-cultural theories of learning, embodied, and critical theories share essential characteristics of complexity. That is, they all invoke bodily metaphors, systemic concerns, evolutionary dynamics, emergent possibilities, and self-maintaining properties. Of particular relevance is the recent emphasis on intersectionality as a key element of critical race and gender theories, which explicitly situates our experiences of knowing and learning in mathematics classrooms as emergent from our simultaneous positions of marginalization and privilege, as well as the interplay between historical, institutional, and social forces and individual desires (Leyva 2017).

As illustrated in Fig. 1, when learning phenomena of interest to mathematics educators are understood as nested systems, a range of theories become necessary to grapple with the many issues the field must address. A pedagogy for knowing and doing mathematics that is epistemologically committed to complexity necessitates insights in the form of multilevel and diverse models of the complex dynamics of knowing and learning (Mowat and Davis 2010). More significantly, perhaps, by introducing the systemic transformation into discussions of individual knowing and collective knowledge, complexity not only enables but compels a consideration of the manners in which knowers and systems of knowledge are co-implicated (Davis and Simmt 2006).

## Complexity as a Historical Discourse

School mathematics curricula are commonly presented as a-historical and a-cultural. Contra this perception, complexity research offers an instance of emergent mathematics that has arisen and that is evolving in a readily perceptible time frame. As an example of what it describes – a self-organizing, emergent coherence – complexity offers a site to study and interrogate the nature of mathematics, interrupting assumptions of fixed and received knowledge.




**Complexity in Mathematics Education, Fig. 1** Some of the nested complex systems of interest to mathematics educators

To elaborate, the study of complexity in mathematics reaches back the late nineteenth century when Poincaré conjectured about the three-body problem in mechanics. Working qualitatively, from intuition Poincaré recognized the problem of thinking about complex systems with the assumptions and mathematics of linearity (Bell 1937). The computational power of mathematics was limited the calculus of the time; however, enabled by digital technologies of the second half of the twentieth century, such problems became tractable and the investigation of dynamical systems began to flourish. With computers, experimental mathematics was born and the study of dynamical systems led to new areas in mathematics. Computational modeling made it possible to model and simulate the behavior of a function over time by computing thousands and hundreds of thousands of iterations of the function. Numerical results were readily converted into graphical representations (the Lorenz attractor, Julia sets, bifurcation diagrams) which in turn inspired a new generation of mathematicians, scientists, and human scientists to think differently about complex dynamical systems. Further advances in computing in the form of parallel and distributed computing and multiagent modeling enabled scientists and mathematicians to simulate emergent phenomena by modeling simultaneous interactions between thousands of interacting agents (Mitchell 2009). Through such efforts, since the mid-twentieth century, as mathematicians, physical and computer scientists were exploring dynamical systems (e.g., Smale, Prigogine, Lorenz, Holland), their work and the work of biologists, engineers, and social scientists became progressively more intertwined and interdisciplinary (Gilbert and Troitzsch 2005; MacLeod and Nersessian 2016).

In brief, the emergence of complexity as a field of study foregrounds that mathematics might be productively viewed as a humanity. More provocatively, the emergence of a mathematics of implicatedness and entanglement alongside the rise of a more sophisticated understanding of humanity's relationship to the more-than-human world might be taken as an indication of the ecological character of mathematics knowledge.

## Complexity as a Disciplinary Discourse

A common criticism of contemporary grade school mathematics curriculum is that little of its content is reflective of mathematics developed after the sixteenth or seventeenth centuries, when publicly funded and mandatory education spread across Europe. A deeper criticism is that the mathematics included in most pre-university curricula is fitted to a particular worldview of cause–effect and linear relationships. Both these concerns might be addressed by incorporating complexity-based content into programs of study.

Linear mathematics held sway at the time of the emergence of the modern school – that is, during the Scientific and Industrial Revolutions – because it lent itself to calculations that could be done by hand. Put differently, linear mathematics was first championed and taught for pragmatic reasons, not because it was seen to offer accurate depictions of reality. Descartes, Newton and their contemporaries were well aware of nonlinear phenomena. However, because of the intractability of many nonlinear calculations, when they arose they were routinely replaced by linear approximations. As textbooks omitted nonlinear accounts, generations of students were exposed to oversimplified, linearized versions of natural phenomena. Ultimately that exposure contributed to a resilient worldview of a clockwork reality.

However, recent advances in computational modeling have made it possible for complex phenomena that are traditionally taught in post-secondary levels, to be easily accessible to much younger learners. With the ready access to similar technologies in most school classrooms within a culture of ubiquitous computation, there is now a growing call for deep, curricular integration of computer-based modeling and simulation in K–12 mathematics and science classrooms (Wilkerson-Jerde and Wilensky 2015; Sengupta et al. 2015). Efforts for such integration fundamentally rely on learners iteratively designing, evaluating and re-designing mathematical models as the pedagogical approach, using agent-based modeling languages and platforms (e.g., Scratch, Agentsheets, NetLogo, ViMAP). In agent-based modeling, learners can simulate the relevant mathematical behaviors by programming the on-screen behavior of computational agents (e.g., the Logo turtle) using body-syntonic commands (e.g., move forward, turn). Emergence, in such computational models, is simulated as the aggregate-level outcome that arises from the interactions between many individual-level computational agents. The creator of the first such modeling language (Logo), Papert (1980) argued that agent-based modeling can create space in secondary and tertiary education for new themes such as recursive functions, fractal geometry, and modeling of complex phenomena with mathematical tools such as difference equations, iterations. Others (e.g., English 2006; Lesh and Doerr 2003) have advocated for similarly themed content, but in a less calculation-dependent format, arguing that the shift in sensibility from linearity to complexity is more important than the development of the computational competencies necessary for sophisticated modeling (Davis and Renert 2013). In either case, the imperative is to provide learners with access to the tools of complexity, along with its affiliated domains of fractal geometry, chaos theory, and dynamic modeling.

New curriculum in mathematics is emerging. More profoundly, when, how, who, and where we teach are also being impacted by the presence of complexity sensibilities in education because they are a means to nurture emergent possibility.

## Complexity as a Pragmatic Discourse

To recap, complexity has emerged in education as a set of mathematical tools for analyzing phenomena; as a theoretical frame for interpreting activity of adaptive and emergent systems; as a new sensibility for orienting oneself to the world; and for considering the conditions for emergent possibilities leading to more productive, "intelligent" classrooms. In the last of these roles, complexity might be regarded as the pragmatic discourse – and of the applications of complexity discussed here, this one may have the most potential for affecting school mathematics by offering guidance for structuring learning contexts and re-shaping disciplinary pedagogies. Three key insights have emerged in the literature that can guide pragmatic action in the K–12 classroom. First, complexity offers direct advice for organizing classrooms to support the individual-and-collective generation of insight – by, for example, nurturing the common experiences and other redundancies of learners while making space for specialist roles, varied interpretations, and other diversities. For example, participatory simulations, in which each learner can themselves play the role of an agent in complex system using embodied, physical, and computational forms of modeling, have been shown to be effective pedagogical approaches for modeling emergent mathematical behaviors by highlighting and integrating both individual and collective insight (e.g., Colella 2000). Second, the emphasis on such participatory forms of mathematical modeling, in the context of modeling complex phenomena, can act as a bridge across disciplines (e.g., biology and mathematics education, see Dickes et al. 2016). A third key insight is the notion of reflexivity across disciplines – that is, conceptual development within each scientific, engineering, and mathematical discipline can be deepened further when relevant phenomena are represented as complex systems using mathematical modeling in ways that also highlight key practices of engineering design such as design thinking (Sengupta et al. 2013).

As complexity becomes more prominent in educational discourses and entrenches in the infrastructure of "classrooms" mathematics education can move from an individualistic culture to one of cooperation and collaboration and from mono-disciplinarity towards inter- and trans-disciplinarity. These, in turn, have entailments for the outcomes of schooling as evident in movements from disciplinary ideas to crosscutting practices, from independent workers to team-based workplaces, and from individual knowing to social action.

## Cross-References

▶ Design Research in Mathematics Education
▶ Mathematical Modelling and Applications in Education

▶ Technology and Curricula in Mathematics Education
▶ Technology Design in Mathematics Education
▶ Theories of Learning Mathematics

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# Computational/Algorithmic Thinking

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Algorithms · Algorithmic thinking · Computational thinking · Mathematics curriculum · Technology

## Introduction

In many countries, the curricular relationship with digital technologies is moving very rapidly (Stephens 2018). These technologies are not only seen as learning and teaching tools for existing disciplines such as mathematics but are also associated with new forms of literacy to be developed for scientific, societal, and economic reasons (Bocconi et al. 2016). Computational thinking, a term coined by Papert (1980), a key element of the new digital literacy, has been described by Wing (2011) as a fundamental personal ability like reading, writing, and arithmetic which enables a person to recognize aspects of computations in various problem situations and to deal appropriately with those aspects by applying tools and techniques from computer science (The Royal Society 2011).

To support an appropriate integration of digital technology in mathematics education, research must focus on the way in which the use of this technology can mediate the learning of mathematics (Drijvers 2018), including relating procedural and conceptual mathematical knowledge (e.g., Artigue 2010). In this entry, we present some prevailing definitions of computational thinking and connect them to the closely related construct of algorithmic thinking. We comment on the current limited but growing research relating computational thinking to mathematics education and argue for research in mathematics education and computer sciences education to explore common ground and disclose opportunities for a more explicit and dynamic relationship.

In the current context, there ought to be a two-way relationship whereby mathematics contributes to digital literacy and computational/algorithmic thinking (CT/AT) can contribute to the development of deeper mathematical understanding at all stages of school education. In considering some emerging implications for mathematics education, we outline different models and practices currently used in different countries to integrate CT/AT into the curriculum more generally. Most importantly, we identify a rich interface between algorithms and mathematics, for example, in the areas of proof and conjecture, where this mutual dynamism might be cultivated more effectively in the mathematics curriculum.

## Defining CT and AT and Relating Them

Despite its widespread use, a widely accepted definition of CT is lacking (Mouza et al. 2017). CT has been defined in terms of its main facets, practices, concepts, components, and dimensions, with a focus that ranged from specific subject area (s), such as programming or STEM education, to a general educational setting such as K-12 subjects.

In a context of programming, Brennan and Resnick (2012) used a three-component framework, comprising *CT concepts* (e.g., loops that specify a repetition of the same instruction(s)), *CT practices* (e.g., testing and debugging that are practiced to identify and remove program errors and malfunctions), and *CT perspectives* (e.g., connecting that promotes a view of computation as a means to interact and work with others). In a high school STEM (Science, Technology, Engineering, and Mathematics) context, Weintrop et al. (2016) proposed a four-category taxonomy, comprising the following categories of practices: *data practices* (e.g., preparing, visualizing), *modeling and simulation practices* (e.g., building and using computational models), *computational problem-solving practices* (e.g., programming, troubleshooting), and *system-thinking practices* (e.g., defining systems, managing complexity). For K-12 subjects, Shute et al. (2017) assume that main CT facets are decomposition, abstraction (data collection and analysis, pattern recognition, modeling), algorithms (algorithm design, parallelism, efficiency, automation), iteration, debugging, and generalization.

Despite an evident diversity in defining CT in the literature, there are common CT components, such as decomposition, abstraction, and algorithms (Shute et al. 2017). These common components are present in a model proposed by Hoyles and Noss (2015) who, in an attempt to enhance mathematics learning through revisiting programming, assumed that CT is based upon decomposition, abstraction, pattern recognition, and algorithmic thinking.

Algorithmic thinking (AT), on the other hand, is one form of mathematical reasoning, which may take many forms, such as algebraic, spatial and geometric, and statistical. AT is required whenever one has to comprehend, test, improve, or design an algorithm, which may, in brief, be defined as "a precisely described routine procedure that can be applied and systematically followed through to a conclusion" (The Concise Oxford Dictionary of Mathematics, fourth ed., p. 11). In somewhat more detail, algorithms may be defined as solutions to a mathematical problem expressed in a sequence of clearly defined instructions that process some numeric, symbolic, or geometric data. To deal with algorithms successfully, AT calls for distinct cognitive abilities, including decomposition (breaking a problem down into subproblems) and abstraction (making general statements summarizing particular examples regarding underlying concepts, procedures, relationships, and models).

AT also calls for pattern recognition, but because this recognition may be viewed as an instance of abstraction and generalization (Scantamburlo 2013), we assume in this entry that there are three AT cornerstones, namely, decomposition, abstraction, and algorithmization. Bearing in mind that CT deals with solutions in representations that could be efficiently processed by information-processing agents (Wing 2011) and that these agents are nowadays computers mostly, we assume that four CT cornerstones are decomposition, abstraction, algorithmization, and automation (Kadijevich 2018). In other words, it is precisely the application of automation that separates AT from CT in our view, having the former not being equal to the latter but rather included in it.

## State of Research on CT/AT

CT originated from learning mathematics with technology. The term was introduced by Seymour Papert in his well-known book *Mindstorms: Children, Computers, and Powerful Ideas* published 40 years ago. It was used to denote specific thinking children applied in learning mathematics (i.e., Turtle Geometry) through LOGO programming.

The term has then been mostly taken over by computer science specialists, who carried out many studies that link CT and computer science topics, mostly programming (e.g., Hickmott et al. 2018). Consequently, CT has become a critical curricular component in computer science (informatics) education (e.g., Webb et al. 2017).

CT has not had a similar status in mathematics education. The reason may be that studies explicitly linking CT and learning mathematics are rather rare, mostly dealing with areas that are traditionally connected to programming, for example, numbers and operations, algebra, and geometry (Hickmott et al. 2018). There are other areas of mathematics suitable for technology-supported problem-solving that should be explored, the above researchers underlined, such as functions, probability, and statistics explored through modeling, simulations, and data analysis, respectively.

Although suitable learning paths for these explorations have not been proposed by researchers in mathematics education (e.g., in a programming context, apply an understand-debug-extend learning trajectory (Brennan and Resnick 2012); in a STEM context, follow a use-modify-create learning path (Lee et al. 2011)), a CT pedagogy for the work with various conceptual or digital objects in mathematical classes has been proposed by Kotsopoulos et al. (2017). This pedagogy makes use of four overlapping activities: *unplugging*, not using computers; *tinkering*, taking objects apart and changing/modifying their components; *making*, constructing new objects; and *remixing*, appropriating of objects or their components to produce new objects. As examples of these activities, consider, respectively, sorting mathematical expressions, modifying the content of a spreadsheet, developing an interactive geometry presentation, and combining and modifying existing interactive reports to visualize data with a dashboard – a set of interactive reports.

Research is also limited with respect to AT in mathematics education. However, valuable findings are reported by Abramovich (2015) and Lockwood et al. (2016), for example. Lockwood et al. (2016) found that procedural knowledge may be developed through implementing procedures, especially through designing procedures and algorithms, which would result in knowledge that is rich in connections. Their study also suggested that mathematicians may prefer to use term AT even when computer tools are used to support their thinking. According to Abramovich (2015), AT may be used to develop conceptual knowledge (i.e., a deeper conceptual understanding) if a special case of a formula, or an algorithm in general, is used as a means for asking advanced questions about the result obtained by applying it.

Despite a limited research on CT/AT in mathematics education at present, the application of the CT/AT lens in mathematics education may be beneficial to mathematics learning because it may result in a more focused instruction on AT and its core components (decomposition, abstraction, and algorithmization), possibly supported by a concrete automation (the use of particular computer tools). The relevance of these three core components to mathematics learning can, for example, be found in a model of mathematical thinking comprising the triad abstraction-modeling-problem-solving (Drijvers et al. 2019). We give closer attention to exploring these potentialities for the mathematics curriculum in the following section.

## Curricular Aspects and Emerging Implications for Mathematics Education

Given the historical and epistemological proximity between mathematics and computer sciences, mathematics education at all stages is expected to contribute to this new area of literacy (Stephens 2018). Moreover, the increased use of digital technologies throughout the school years will likely influence the teaching and learning of mathematics in new ways. Several examples relevant at different stages of school mathematics illustrate the richness of this two-way relationship:

- Using the language of algorithms to exemplify and unpack mathematical concepts and procedures (e.g., in the primary school where the language of algorithms can be used to highlight the relationship between very closely related procedures such as multiplication and division or addition and subtraction).
- Identifying and refining the mathematical variables and parameters to use a given algorithm (e.g., in data analysis)
- Using an algorithmic thinking to solve a mathematical problem in order to identify its mathematical structure and to generalize the solutions (e.g., in computational problem-solving)
- Using algorithms to provide accessible introductions to modeling, optimization, operations research, and experimental mathematics
- Generating examples of problems for which the algorithm works, and similarly generating counterexamples (i.e., problems for which the algorithm does not work)
- Using the iterative process of algorithmic design to highlight the iterative process of conjecturing and proving
- Using algorithmic thinking to highlight the distinction between branches of mathematics which seek to explore whether a solution exists and other branches of mathematics which seek to determine how a solution (if it exists) can be found

These examples illustrate and endorse the two-way relationship between algorithmic thinking and mathematics, moving to a richer conception of algorithms as entities or objects that can be investigated from a mathematical point of view, rather than merely tools or sets of procedures that need to be expressed in syntactically correct form. For example, algorithmic thinking is of critical importance to the processes of conjecturing and proving. Modeste (2016) argues that areas such as discrete mathematics (graph theory, combinatorics) can provide rich opportunities for students to explore relations between proof, language, algorithm, programming, and logic in mathematics and informatics, *requiring the exploration of specific concepts at the informatics-mathematics interface* (our emphasis). There is a need for further research to explore these possibilities in specific areas of the mathematics curriculum.

However, the affordances that digital technologies offer for the teaching and learning of mathematics depend in large measure on how these technologies are integrated into the school curriculum and by the degree of involvement of teachers of mathematics in that enterprise. Embedding CT/AT in the mathematics curriculum is complex due to the long "lead time" needed to change national curricula and the difficulties teachers face in dislodging or reorienting current content. Different models and instances of curriculum implementation are being tried, each with advantages and disadvantages to fostering creative interfaces between mathematics and informatics. Examples of these models are provided below:

1. A cross-curriculum model produces least disruption to the existing school subjects. It is especially attractive to the elementary school, because it allows possibilities for integration across subject areas. On the negative side as the Finnish experience shows (Prime Minister's Office 2016), take-up of implementation can be slow and uneven across schools, and real integration may be shallow.
2. Taught within the Information/Digital Technologies Curriculum where teachers are likely to be well disposed toward taking on algorithmic thinking – a prevalent model in England (Department of Education 2013) and Australia (ACARA 2016). On the downside,

mathematical connections may be passed over in favor of a focus on mastering and using technology.

3. Gradualist model. This is evident in Japan where "programming thinking" will be introduced in several subject areas, including mathematics, starting in 2020 in Grade 5 and moving into other grades in subsequent years (see Kanemune et al. 2017). This model takes account of teacher anxiety and allows time to prepare guidance for teachers and teaching resources. The challenge will be to create dynamic interfaces between typically entrenched subject boundaries.
4. A separate subject in the middle years taught by mathematics and information technology teachers. This is the French model of *Algorithmique et Programmation* (Ministere de l'Education 2016). Having a formal school subject allows for the development of teaching resources and curriculum materials as well as assessments. This model can provide opportunities for exploring interfaces with mathematics of the kind discussed above.
5. A senior secondary subject specifically devoted to the study of *Algorithmics*. This model is being followed in the Victorian Certificate of Education (Victorian Curriculum and Assessment Authority 2017) in Australia and in France. A separate senior school subject can articulate easily with university courses and generally assumes that students have had rich prior experiences with coding and algorithmic thinking.

## Conclusion

The focus on *Algorithmics* as a formal area of study directs attention away from equating algorithmic thinking with using tools and procedures necessary to correctly construct algorithms. For example, *Algorithmics* (Victorian Curriculum and Assessment Authority 2017) not only requires students to construct and use algorithms in solving problems, for example, in graph theory, but it also examines theoretical issues such as computational complexity and models for computation. These issues require the exploration of specific concepts at the informatics-mathematics interface and support a stronger and richer sense of algorithmic thinking.

One of these concepts is computational thinking, which is, in brief, algorithmic thinking supported by some automation, whereby, among other things, theoretical issues mentioned above can be made alive as well. Research studies examined in sections "Defining CT and AT and Relating Them" and "State of Research on CT/AT" support the position that to cultivate computational thinking, educators need to support students in practicing its main steps (e.g., abstraction, automation) and their sub-steps (e.g., identification of entities in abstraction; debugging and iteration in automation) and relating them. To this end, educators need to create interfaces between informatics and mathematics, for example, by modeling using approximate solutions (Kenderov 2018), assist students in using rich computational environments as a means of automation, and encourage them to progress in their learning by following suitable learning paths.

For this practice to emerge in the school mathematics curriculum, appropriately rich problems and resources are needed, along with specifically focused research. These potentialities are more likely to be realized if algorithmic thinking is situated within both the mathematics and the information technology curricula and thus taught by the teachers of these two disciplines.

## Cross-References

- ▶ Algorithmics
- ▶ Algorithms
- ▶ Data Handling and Statistics Teaching and Learning
- ▶ Information and Communication Technology (ICT) Affordances in Mathematics Education
- ▶ Mathematical Representations
- ▶ Technology and Curricula in Mathematics Education

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## Concept Development in Mathematics Education

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**Keywords**

Notion · Concept · Concept formation in babies · Concrete object · Similarities · Generalization · Ostensive definitions · Mathematical definitions · Intuitive · Concept image · Concept definition · Stereotypical examples · System 1 and system 2 · Pseudo-analytical · Pseudo-conceptual · Mathematical objects · Mathematical mind

### Characteristics

Concept formation and development in general is an extremely complicated topic in cognitive psychology. There exists a huge literature about it, classical and current. Among the classical works on it, one can mention for instance, Piaget and Inhelder (1958) and Vygotsky (1986). However, this issue is restricted to concept formation and development in mathematics. Nevertheless, it is suggested not to isolate mathematical concept formation and development from concept formation and development in general.

One terminological clarification should be made before the main discussion. When dealing with concepts, very often, also the term “notion” is involved. A *notion* is a lingual entity – a word, a word combination (written or pronounced); it can also be a symbol. A *concept* is the meaning associated in our mind with a notion. It is an idea in our mind. Thus, *a notion is a concept name.* There might be concepts without names and for sure there are meaningless notions, but discussing them requires subtleties which are absolutely irrelevant to this context. In many discussions people do not bother to distinguish between notions and concepts, and thus the word “notion” becomes ambiguous. The ambiguity is easily resolved by the context.

As recommended above, it will be more useful not to disconnect mathematical concept formation from concept formation in general, and therefore, let us start our discussion with an example of concept formation in babies. How do we teach them, for instance, the concept of chair? The common practice is to point at various chairs in various contexts and to say “chair.” Amazingly enough, after some repetitions, the babies understand that the word “chair” is supposed to be related to chairs, which occur to them in their daily experience, and when being asked “what is this?” they understand that they are supposed to say “chair.” Later on, they will imitate the entire ritual on their own initiative. They will point at chairs and say “chair.” I would like to make a theoretical claim here by saying that, *seemingly*, they have constructed in their mind the class of all possible chairs. Namely, a concept is formed in their mind, and whenever a concrete object is presented to them, they will be able to decide whether it is a chair or not. Of course, some mistakes can occur in that concept formation process. It is because in this process, two *cognitive mechanisms* are involved. The first mechanism is the one that identifies similarities. The mind distinguishes that one particular chair presented to the

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baby is similar to some particular chairs presented to her or him in the past. The second mechanism is the one which *distinguishes differences*. The mind distinguishes that a certain object is not similar to the chairs which were presented to the baby in the past, and therefore, the baby is not supposed to say "chair" when an object that is not a chair is presented to him or her by the adult. Mistakes about the acquired concept might occur because of two reasons. An object, which is not a chair (say a small table), appears to the baby (or even to an adult) like a chair. In this case, the object will be considered as an element of the class of all chairs while, in fact, it is not an element of this class. The second reason for mistakes is that an object that is really a chair will not be identified as a chair because of its weird shape. Thus, an object which was supposed to be an element of the class is excluded from it. More examples of this type are the following: sometimes, babies consider dogs as cats and vice versa. These are intelligent mistakes because there are some similarities between dogs and cats. They are both animals; sometimes they even have similar size (in the case of small dogs) and so on.

The above process which leads, in our mind, to the construction of the set of all possible objects to which the concept name can be applied is a kind of *generalization.* Thus, generalizations are involved in the formation of any given concept. Therefore, concepts can be considered as generalizations.

The actions by means of which we try to teach children concepts of chair are called *ostensive definitions*. Of course, only narrow class of concepts can be acquired by means of ostensive definitions. Other concepts are acquired by means of *explanations* which can be considered at this stage as definitions. Among these concepts I can point, for instance, at a forest, a school, work, hunger and so on. When I say definitions at this stage, I do not mean definitions which are similar, or even seemingly similar to rigorous mathematical definitions. The only restriction on these definitions is that familiar concepts will be used in order to explain a non-familiar concept. Otherwise, the explanation is useless. (This restriction, by the way, holds also for mathematical definitions, where new concepts are defined by means of previously defined concepts or by primary concepts.) In definitions which we use in non-technical context in order to teach concepts, we can use examples. For instance, in order to define furniture, we can say: A chair is furniture, a bed is furniture, and tables, desks, and couches are furniture.

The description which was just given deals with the primary stage of concept formation. However, concept formation in ordinary language is by far more complicated and very often, contrary to the mathematical language, ends up in a vague notion. Take, for instance, again, the notion of furniture. The child, when facing an object which was not previously introduced to him or to her as furniture, should decide whether this object is furniture or not. He or she may face difficulties doing it. Also adults might have similar difficulties. This is only one example out of many which demonstrates the complexity of concept formation in the child's mind as well as in the adult's mind. There are even greater complexities when concept formation of abstract nouns, adjectives, verbs, and adverbs is involved. Nevertheless, despite that complexity, the majority of children acquire language at an impressive level by the age of six (an elementary level is acquired already at the age of three). The cognitive processes associated with the child's acquisition of language are discussed in details in *cognitive psychology, linguistics, and philosophy of language*. One illuminating source which is relevant to this issue is Quine's (1964) "Word and object." However, a detailed discussion of these processes is not within the scope of this issue.

In addition to the language acquisition, the child acquires also broad knowledge about the world. He or she knows that when it rains, it is cloudy, they know that dogs bark and so on and so forth. In short, they know infinitely many other facts about their environment. And again, it is obtained in a miraculous way, smoothly without any apparent difficulties. Things, however, become awkward when it gets to mathematics. One possible reason for things becoming awkward in mathematics is that, in many cases, *mathematical thinking is essentially different from the natural intuitive mode of thinking* according to which the child's intellectual development takes place. The major problem is that mathematical thinking is shaped by rigorous rules, and in order to think mathematically, children, as well as adults, should be aware of these rules while thinking in mathematical contexts. One crucial difficulty in mathematical thinking is that *mathematical concepts are strictly determined by their definitions*. In the course of their mathematical studies, children, quite often, are presented to mathematical notions with which they were familiar from their past experience. For instance, in Kindergarten they are shown some geometrical figures such as squares and rectangles. The adjacent sides of the rectangle which are shown to the children in Kindergarten have always different length. Thus, the set of all possible rectangles which is constructed in the child's mind includes only rectangles, the adjacent sides of which have different length. In the third grade, in many countries, a definition of a rectangle is presented to the child. It is a quadrangle which has four right angles. According to this definition, *a square is also a rectangle*. Thus, a *conflict* may be formed in the child's mind between the suggested definition and the concept he or she already has about rectangles. The concept the child has in mind was formed by the set of examples and the properties of these examples which were presented to the child. It was suggested (Vinner 1983) to call it the *concept image* of that notion. Thus, in the above case of the rectangle, there is a conflict between the *concept image* and the *concept definition*. On the other hand, quite often some concepts are introduced to the learner by means of formal definitions. For instance, an altitude in a triangle. However, a formal definition, generally, remains meaningless unless it is associated with some examples. The examples can be given by a teacher or by a textbook, or they can be formed by the learners themselves. The first examples which are associated with the concept have a crucial impact on the concept image. Unfortunately, quite often, in mathematical thinking, when a task is given to students, in order to carry it out, they consult their concept image and forget to consult the concept definition. It turns out that, in many cases, there are critical examples which shape the concept image. In some cases, these are the first examples which are introduced to the learner. For instance, in the case of the altitude (a segment which is drawn from one vertex or the triangle and it is perpendicular to the opposite side of this vertex *or to its continuation*), it is pedagogically reasonable to give examples of altitudes in acute angle triangles. Later on, in order to form the appropriate concept image of an altitude, the teacher, as well as the textbook, should give examples of altitudes from vertices of acute angles in an obtuse angle triangle. However, before this stage of the teaching takes place, the concept image of the altitude was shaped by the stereotypical examples of altitudes in an acute angle triangle (sometimes, even by the stereotypical examples of altitudes which are perpendicular to a horizontal side of a triangle). Thus, when the learners face a geometrical problem about altitudes which do not meet the stereotypes in their concept image, they are stuck. It does not occur to them to consult the concept definition of the altitude, and if it does occur, they usually recall the first part of the definition ("a segment which is drawn from one vertex or the triangle and it is perpendicular to the opposite side of this vertex") and forget the additional phrase in the definition ("or to its continuation"). Two additional examples of this kind are the following: (1) At the junior high level, in geometry, when a quadrangle is defined as a particular case of a polygon (a quadrangle is a polygon which has four sides), the learners have difficulties to accept a concave quadrangle or a quadrangle that intersects itself as quadrangles. (2) At the high school level, when a formal definition of a function is given to the students, eventually, the stereotypical concept image of a function is that of an algebraic formula. A common formal definition of a function can be the following one: a correspondence between two non-empty sets which assigns to every element in the first set (the domain) exactly one element in the second set (the range). Even if some nonmathematical examples are given to the students (for instance, the correspondence which assigns to every living creature its mother), even then, the stereotypical concept image of a function is that of an algebraic formula, as claimed above.

A plausible explanation to these phenomena can be given in terms of the psychological theory about *system 1* and *system 2*. Psychologists, nowadays, speak about two cognitive systems which they call system 1 and system 2. It sounds as if there are different parts in our brain which produce different kinds of thinking. However, this interpretation is wrong. The correct way to look at system 1 and system 2 is to consider them as *thinking modes*. This is summarized very clearly in Stanovich (1999, p. 145). System 1 is characterized there by the following adjectives: *associative, tacit, implicit, inflexible, relatively fast, holistic, and automatic*. System 2 is characterized by: *analytical, explicit, rational, controlled, and relatively slow*. Thus, notions that were used by mathematics educators in the past can be related now to system 1 or system 2, and therefore this terminology is richer than the previously suggested notions. Fischbein (1987) spoke about *intuition* and this can be considered as system 1. Skemp (1979) spoke about two systems which he called *delta one* and *delta two*. They can be considered as *intuitive* and *reflective* or using the new terminology, system 1 and system 2, respectively. Vinner (1997) used the notions *pseudo-analytical* and *pseudo-conceptual* which can be considered as system 1.

*In mathematical contexts the required thinking mode is that of system 2*. This requirement presents some serious difficulties to many people (children and adults) since, most of the time, thought processes are carried out within system 1. Also, in many people, because of various reasons, system 2 has not been developed to the extent which is required for mathematical thinking in particular and for rational thinking in general. Nevertheless, in many contexts, learners succeed in carrying out mathematical tasks which are presented to them by using system 1. This fact does not encourage them to become aware of the need to use system 2 while carrying out mathematical tasks.

When discussing concept development in mathematical thinking, it is worthwhile to mention also some concepts which can be classified as *metacognitive* concepts. Such concepts are *algorithm, heuristics*, and *proof*. While studying mathematics, the learners face many situations in which they or their teachers use algorithms, heuristics, and proof. However, usually, the notions "algorithm" and "heuristics" are not introduced to the learners in their school mathematics. Some of them will be exposed to them in college, in case they choose to take certain advanced mathematics courses. As to the notion of proof, in spite of the fact that this notion is mentioned a lot in school mathematics (especially in geometry), the majority of students do not fully understand it. Many of them try to identify mathematical proof by its *superficial characteristics*. They do it without understanding the logical reasoning associated with these characteristics. A meaningless use of symbols and verbal expressions as "therefore," "it follows," and "if... then" is considered by many students as a mathematical proof (See for instance Healy and Hoyles 1998). It turns out that it takes a lot of mathematical experience until meaningless verbal rituals (as in the case of the baby acquiring the concept of chair) become *meaningful thought processes*. And how do we know that the learners use the above verbal expressions meaningfully? We assume so because their use of these expressions is in absolute agreement with the way we, mathematicians and mathematics educators, use them.

Another important aspect of mathematical concept development is the understanding that certain mathematical concepts are related to each other. Here comes the idea of structure. For instance, from triangles, quadrangles, pentagons, and hexagons, we reach the concept of a polygon. From the general concept of quadrangles, we approach to trapezoids, parallelograms, rhombus, rectangles, and squares, and we realize there all kinds of class inclusions. Thus, we distinguish partial order in the set of mathematical concepts. Finally, and this is perhaps the ultimate stage of mathematical concept development, we conceive mathematics as a collection of various *deductive structures* (Peano's Arithmetic, Euclidean Geometry, Set Theory, Group Theory, etc.). Also, in more advanced mathematical thinking, we conceive mathematical objects (numbers, functions, geometrical figures in Euclidean geometry, etc.) as abstract objects. All these require thought processes within system 2. However, it should be emphasized that all the above concept developments do not occur simultaneously. They also do not occur in all students who study mathematics. One should take many mathematics courses and solve a lot of mathematical problems in order to achieve that level. Those who do it should have special interest in mathematics or what can be called mathematical curiosity. It requires, what some people call, a mathematical mind. Is it genetic (Devlin 2000) or acquired? At this point we have reached a huge domain of psychological research which is far beyond the scope of this particular encyclopedic issue.

## Cross-References


- ▶ Abstraction in Mathematics Education
- ▶ Concept Development in Mathematics Education
- ▶ Critical Thinking in Mathematics Education
- ▶ Intuition in Mathematics Education
- ▶ Mathematical Proof, Argumentation, and Reasoning
- ▶ Metacognition
- ▶ Problem-Solving in Mathematics Education
- ▶ Theories of Learning Mathematics
- ▶ Values in Mathematics Education
- ▶ Zone of Proximal Development in Mathematics Education


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# Constructivism in Mathematics Education


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## Background

Constructivism is an epistemological stance regarding the nature of human knowledge, having roots in the writings of Epicurus, Lucretius, Vico, Berkeley, Hume, and Kant. Modern constructivism also contains traces of pragmatism (Peirce, Baldwin, and Dewey). In mathematics education the greatest influences are due to Piaget, Vygotsky, and von Glasersfeld. See Confrey and Kazak (2006) and Steffe and Kieren (1994) for related historical accounts of constructivism in mathematics education.

There are two principle schools of thought within constructivism: radical constructivism (some people say individual or psychological) and social constructivism. Within each there is also a range of positions. While radical and social constructivism will be discussed in a later section, it should be noted that both schools are grounded in a strong skeptical stance regarding reality and truth: *Knowledge cannot be thought of as a copy of an external reality, and claims of truth cannot be grounded in claims about reality.*

The justification of this stance toward knowledge, truth, and reality, first voiced by the skeptics of ancient Greece, is that to verify that one's knowledge is correct, or that what one knows is true, one would need access to reality by means other than one's knowledge of it. The importance of this skeptical stance for mathematics educators is to remind them that students have their own mathematical realities that teachers and researchers can understand only via models of them (Steffe et al. 1983, 1988).

Constructivism did not begin within mathematics education. Its allure to mathematics educators is rooted in their long evolving rejection of Thorndike's associationism (Thorndike 1922; Thorndike et al. 1923) and Skinner's behaviorism (Skinner 1972). Thorndike's stance was that learning happens by forming associations between stimuli and appropriate responses. To design instruction from Thorndike's perspective meant to arrange proper stimuli in a proper order and have students respond appropriately to those stimuli repeatedly. The behaviorist stance that mathematics educators found most objectionable evolved from Skinner's claim that all human behavior is due to environmental forces. From a behaviorist perspective, to say that children participate in their own learning, aside from being the recipient of instructional actions, is nonsense. Skinner stated his position clearly:

> Science . . . has simply discovered and used subtle forces which, acting upon a mechanism, give it the direction and apparent spontaneity which make it seem alive. (Skinner 1972, p. 3)

Behaviorism's influence on psychology, and thereby its indirect influence on mathematics education, was also reflected in two stances that were counter to mathematics educators' growing awareness of learning in classrooms. The first stance was that children's learning could be studied in laboratory settings that have no resemblance to environments in which learning actually happens. The second stance was that researchers could adopt the perspective of a universal knower. This second stance was evident in Simon and Newell's highly influential information processing psychology, in which they separated a problem's "task environment" from the problem solver's "problem space."

> We must distinguish, therefore, between the task environment – the omniscient observer's way of describing the actual problem "out there" – and the problem space – the way a particular subject represents the task in order to work on it. (Simon and Newell 1971, p. 151)

Objections to this distinction were twofold: Psychologists considered themselves to be Simon and Newell's omniscient observers (having access to problems "out there"), and students' understandings of the problem were reduced to a subset of an observer's understanding. This stance among psychologists had the effect, in the eyes of mathematics educators, of blinding them to students' ways of thinking that did not conform to psychologists' preconceptions (Thompson 1982; Cobb 1987). Erlwanger (1973) revealed vividly the negative consequences of behaviorist approaches to mathematics education in his case study of a successful student in a behaviorist individualized program who succeeded by inventing mathematically invalid rules to overcome inconsistencies between his answers and an answer key.

The gradual release of mathematics education from the clutches of behaviorism, and infusions of insights from Polya's writings on problem solving (Polya 1945, 1954, 1962), opened mathematics education to new ways of thinking about student learning and the importance of student thinking. Confrey and Kazak (2006) described the influence of research on problem solving, misconceptions, and conceptual development of mathematical ideas as precursors to the emergence of constructivism in mathematics education.

Piaget's writings had a growing influence in mathematics education once English translations became available. In England, Skemp (1961, 1962) championed Piaget's notions of schema, assimilation, accommodation, equilibration, and reflection as ways to conceptualize students' mathematical thinking as having an internal coherence. Piaget's method of clinical interviews also was attractive to researchers of students' learning. However, until 1974 mathematics educators were interested in Piaget's writings largely because they thought of his work as "developmental psychology" or "child psychology," with implications for children's learning. It was in 1974, at a conference at the University of Georgia, that Piaget's work was recognized in mathematics education as a new field, one that leveraged children's cognitive development to study the growth of knowledge. Smock (1974) wrote of *constructivism's* implications for instruction, not *psychology's* implications for instruction. Glasersfeld (1974) wrote of Piaget's genetic epistemology as a theory of knowledge, not as a theory of cognitive development. The 1974 Georgia conference is the first occasion this writer could find where "constructivism" was used to describe the epistemological stance toward mathematical knowing that characterizes constructivism in mathematics education today.

Acceptance of constructivism in mathematics education was not without controversy. Disputes sometimes emerged from competing visions of desired student learning, such as students' performance on accepted measures of competency (Gagné 1977, 1983) versus attendance to the quality of students' mathematics (Steffe and Blake 1983), and others emerged from different conceptions of teaching effectiveness (Brophy 1986; Confrey 1986). Additional objections to constructivism were in reaction to its fundamental aversion to the idea of truth as a correspondence between knowledge and reality (Kilpatrick 1987).

## Radical and Social Constructivism in Mathematics Education

Radical constructivism is based on two tenets: "(1) Knowledge is not passively received but actively built up by the cognizing subject; (2) the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality" (Glasersfeld 1989, p. 114). Glasersfeld's use of "radical" is in the sense of fundamental – that cognition is "a constitutive activity which, alone, is responsible for every type or kind of structure an organism comes to know" (Glasersfeld 1974, p. 10).

Social constructivism is the stance that history and culture precede and preform individual knowledge. As Vygotsky famously stated, "Every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first *between* people . . ., then *inside* the child" (Vygotsky 1978, p. 57).

The difference between radical and social constructivism can be seen through contrasting interpretations of the following event. Vygotsky (1978) illustrated his meaning of *internalization* – "the internal reconstruction of an external operation" – by describing the development of pointing:

> The child attempts to grasp an object placed beyond his reach; his hands, stretched toward that object, remain poised in the air. His fingers make grasping movements. At this initial state pointing is represented by the child's movement, which seems to be pointing to an object – that and nothing more. When the mother comes to the child's aid and realizes his movement indicates something, the situation changes fundamentally. Pointing becomes a gesture for others. The child's unsuccessful attempt engenders a reaction not from the object he seeks but *from another person* [sic]. Consequently, *the primary meaning of that unsuccessful grasping movement is established by others* [italics added]. (Vygotsky 1978, p. 56)

Vygotsky clearly meant that meanings originate in society and are transmitted via social interaction to children. Glasersfeld and Piaget would have listened agreeably to Vygotsky's tale – until the last sentence. They instead would have described the child as making a connection between his attempted grasping action and someone fetching what he wanted. Had it been the pet dog bringing the desired item, it would have made little difference to the child in regard to the practical consequences of his action. Rather, the child realized, in a sense, "Look at what I can make others do with this action." This interpretation would fit nicely with the finding that adults mimic infants' speech abundantly (Fernald 1992; Schachner and Hannon 2011). Glasersfeld and Piaget might have thought that adults' imitative speech acts, once children recognize them as imitations, provide occasions for children to have a sense that they can influence actions of others through verbal behavior. This interpretation also would fit well with Bauersfeld's (1980, 1988, 1995) understanding of communication as a reflexive interchange among mutually oriented individuals: "*The [conversation] is constituted at every moment through the interaction of reflective subjects*" (Bauersfeld 1980, p. 30 italics in original).

Paul Ernest (1991, 1994, 1998) introduced the term social constructivism to mathematics education, distinguishing between two forms of it. One form begins with a radical constructivist perspective and then accounts for human interaction in terms of mutual interpretation and adaptation (Bauersfeld 1980, 1988, 1992). Glasersfeld (1995) considered this as just radical constructivism. The other, building from Vygotsky's notion of cultural regeneration, introduced the idea of mathematical objectivity as a social construct.

Social constructivism links subjective and objective knowledge in a cycle in which each contributes to the renewal of the other. In this cycle, the path followed by new mathematical knowledge is from subjective knowledge (the personal creation of an individual), via publication to objective knowledge (by intersubjective scrutiny, reformulation, and acceptance). Objective knowledge is internalized and reconstructed by individuals, during the learning of mathematics, to become the individuals' subjective knowledge. Using this knowledge, individuals create and publish new mathematical knowledge, thereby completing the cycle. (Ernest 1991, p. 43).

Ernest focused on objectivity of adult mathematics. He did not address the matter of how children's mathematics comes into being or how it might grow into something like an adult's mathematics.

Radical and social constructivists differ somewhat in the theoretical work they ask of constructivism. Radical constructivists concentrate on understanding learners' mathematical realities and the internal mechanisms by which they change. They conceive, to varying degrees, of learners in social settings, concentrating on the sense that learners make of them. They try to put themselves in the learner's place when analyzing an interaction. Social constructivists focus on social and cultural mathematical and pedagogical practices and attend to individuals' internalization of them. They conceive of learners in social settings, concentrating, to various degrees, on learners' participation in them. They take the stances, however, of an observer of social interactions and that social practices predate individuals' participation.

Conflicts between radical and social constructivism tend to come from two sources: (1) differences in meanings of truth and objectivity and their sources and (2) misunderstandings and miscommunications between people holding contrasting positions. The matter of (1) will be addressed below. Regarding (2), Lerman (1996) claimed that radical constructivism was internally incoherent: How could radical constructivism explain agreement when persons evidently agreeing create their own realities? Steffe and Thompson (2000a) replied that interaction was at the core of Piaget's genetic epistemology and thus the idea of intersubjectivity was entirely coherent with radical constructivism. The core of the misunderstanding was that Lerman on the one hand and Steffe and Thompson on the other had different meanings for "intersubjectivity." Lerman meant "agreement of meanings" – same or similar meanings. Steffe and Thompson meant "nonconflicting mutual interpretations," which might actually entail nonagreement of meanings of which the interacting individuals are unaware. Thus, Lerman's objection was valid relative to the meaning of intersubjectivity he presumed. Lerman on one side and Steffe and Thompson on the other were in a state of intersubjectivity (in the radical constructivist sense) even though they publicly disagreed. They each presumed they understood what the other meant when in fact each understanding of the other's position was faulty.

Other tensions arose because of interlocutors' different objectives. Some mathematics educators focused on understanding individual's mathematical realities. Others focused on the social context of learning. Cobb et al. (1992) diffused these tensions by refocusing discussions on the work that theories in mathematics education must do – they must contribute to our ability to improve the learning and teaching of mathematics. Cobb et al. first reminded the field that, from any perspective, what happens in mathematics classrooms is important for students' mathematical learning. Thus, a theoretical perspective that can capture more, and more salient, aspects for mathematics learning (including participating in practices) is the more powerful theory. With a focus on the need to understand, explain, and design events within classrooms, they recognized that there are indeed social dimensions to mathematics learning and there are psychological aspects to participating in practices and that researchers must be able to view classrooms from either perspective while holding the other as an active background: "[W]e have proposed the metaphor of mathematics as an evolving social practice that is constituted by, *and does not exist apart from*, the constructive activities of individuals" (Cobb et al. 1992, p. 28, italics added).

Cobb et al.'s perspective is entirely consistent with theories of emergence in complex systems (Schelling 1978; Eppstein and Axtell 1996; Resnick 1997; Davis and Simmt 2003) when taken with Maturana's statement that "anything said is said by an observer" (Maturana 1987). Practices, as stable patterns of social interaction, exist in the eyes of an observer who sees them. The theoretician who understands the behavior of a complex system as entailing simultaneously both microprocesses and macrobehavior is better positioned to affect macrobehavior (by influencing microprocesses) than one who sees just one or the other. It is important to note that this notion of emergence is not the same as Ernest's notion of objectivity as described above.

## Truth and Objectivity

Radical constructivists take the strong position that children have mathematical realities that do not overlap an adult's mathematics (Steffe et al. 1983; Steffe and Thompson 2000b). Social constructivists (of Ernest's second type) take this as pedagogical solipsism.

The implications of [radical constructivism] are that individual knowers can construct truth that needs no corroboration from outside of the knower, making possible any number of "truths." Consider the pedagogical puzzles this creates. What is the teacher trying to teach students if they are all busy constructing their own private worlds? What are the grounds for getting the world right? Why even care whether these worlds agree? (Howe and Berv 2000, pp. 32–33).

Howe and Berv made explicit the social constructivist stance that there is a "right" world to be got – the world of socially constructed meanings. They also revealed their unawareness that, from its very beginning, radical constructivism addressed what "negotiation" could mean in its framework and how stable patterns of meaning could emerge socially (Glasersfeld 1972, 1975, 1977). Howe and Berv were also unaware of the notion of *epistemic subject* in radical constructivism – the mental construction of a non-specific person who has particular ways of thinking (Beth and Piaget 1966; Glasersfeld 1995). A teacher need not attend to 30 mathematical realities with regard to teaching the meaning of fractions in a class of 30 children. Rather, she need only attend to perhaps 5 or 6 epistemic children and listen for which fits the ways particular children express themselves (Thompson 2000).

## A Short List: Impact of Constructivism in Mathematics Education

- Mathematics education has a new stance toward learners at all ages. One must attend to learner's mathematical realities, not just their performance.
- Current research on students' and teachers' thinking and learning is largely consistent with constructivism – to the point that articles rarely declare their basis in constructivism. Constructivism is now taken for granted.
- Teaching experiments (Cobb and Steffe 1983; Cobb 2000; Steffe and Thompson 2000b) and design experiments (Cobb et al. 2003) are vital and vibrant methodologies in mathematics education theory development.
- Conceptual analysis of mathematical thinking and mathematical ideas is a prominent and

widely used analytic tool (Smith et al. 1993; Glasersfeld 1995; Behr et al. 1997; Thompson 2000; Lobato et al. 2012).

- What used to be thought of as *practice* is now conceived as *repeated experience*. Practice focuses on repeated behavior. Repeated experience focuses on repeated reasoning, which can vary in principled ways from setting to setting (Cooper 1991; Harel 2008a, b).
- Constructivism has clear and operationalized implications for the design of instruction (Confrey 1990; Simon 1995; Steffe and D'Ambrosio 1995; Forman 1996; Thompson 2002) and assessment (Carlson et al. 2010; Kersting et al. 2012).

## Cross-References

▶ Constructivist Teaching Experiment
▶ Misconceptions and Alternative Conceptions in Mathematics Education
▶ Sociomathematical Norms in Mathematics Education

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# Constructivist Teaching Experiment

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## Introduction

The constructivist is fully aware of the fact that an organism's conceptual constructions are not fancy-free. On the contrary, the process of constructing is constantly curbed and held in check by the constraints it runs into. (Ernst von Glasersfeld 1990, p. 33).

The constructivist teaching experiment emerged in the United States circa 1975 (Steffe et al. 1976) in an attempt to understand children's numerical thinking and how that thinking might change rather than to rely on models that were developed outside of mathematics education for purposes other than educating children (e.g., Piaget and Szeminska 1952; McLellan and Dewey 1895; Brownell 1928). The use of the constructivist teaching experiment in the United State was buttressed by versions of the teaching experiment methodology that were being used already by researchers in the Academy of Pedagogical Sciences in the then Union of Soviet Socialist Republics (Wirszup and Kilpatrick 1975–1978). The work at the Academy of Pedagogical Sciences provided academic respectability for what was then a major departure in the practice of research in mathematics education in the United States, not only in terms of research methods but more crucially in terms of the research orientation of the methodology. In El'konin's (1967) assessment of Vygotsky's (1978) research, the essential function of a teaching experiment is the production of models of student thinking and changes in it:

Unfortunately, it is still rare to meet with the interpretation of Vygotsky's research as modeling, rather than empirically studying, developmental processes. (El'konin 1967, p. 36).

Similarly, the primary purpose of constructivist teaching experiments is to construct explanations of students' mathematical concepts and operations and changes in them. Without experiences of students' mathematics afforded by teaching, there would be no basis for coming to understand the mathematical concepts and operations students construct or even for suspecting that these concepts and operations may be distinctly different from those of teacher/researchers. The necessity to attribute mathematical concepts and operations to students that are independent of those of teacher/researchers has been captured by Ackermann (1995) in speaking of human relations:

In human relations, it is vital to attribute autonomy to others and to things—to celebrate their existence independently from our current interaction with them. This is true even if an attribution (of existence) is a mental construct. We can literally rob others of their identity if we deny them an existence beyond our current interests (p. 343).

Students' mathematical concepts and operations constitute first-order models, which are models that students construct to organize, comprehend, and control their own experience (Steffe et al. 1983, p. xvi). Through a process of *conceptual analysis* (von Glasersfeld 1995), teacher/researchers construct models of students' mathematical concepts and operations to explain what students say and do. These second-order models (Steffe et al. 1983, p. xvi) are called *mathematics of students* and students' first-order models are called simply *students' mathematics*. While teacher/researchers may write about the schemes and operations that constitute these second-order models as if they are identical to students' mathematics, these constructs, in fact, are a construction of the researcher that only references students' mathematics. Conceptual analysis is based on the belief that mathematics is a product of the functioning of human intelligence (Piaget 1980), so the mathematics of students is a legitimate mathematics to the extent that teacher/researchers can find rational grounds to explain what students say and do.

The overarching goal of the teacher/researchers who use the methodology is to establish the mathematics of students as a conceptual foundation of students' mathematics education (Steffe and Wiegel 1992; Steffe 2012). The mathematics of students opens the way to ground school mathematics in the history of how it is generated by students in the context of teaching. This way of regarding school mathematics casts it as a living subject rather than as a subject of being (Steffe 2007).

## Characteristics: The Elements of Constructivist Teaching Experiments

### Teaching Episodes

A constructivist teaching experiment involves a sequence of teaching episodes (Hunting 1983; Steffe 1983). A teaching episode includes a teacher/researcher, one or more students, a witness of the teaching episodes, and a method of recording what transpires during the episodes. These records can be used in preparing subsequent episodes as well as in conducting conceptual analyses of teaching episodes either during or after the experiment.

### Exploratory Teaching

Any teacher/researcher who hasn't conducted a teaching experiment but who wishes to do so should first engage in exploratory teaching (Steffe and Thompson 2000). It is important that the teacher/researcher becomes acquainted, at an experiential level, with students' ways and means of operating in whatever domain of mathematical concepts and operations are of interest. In exploratory teaching, the teacher/researcher attempts to put aside his or her own concepts and operations and not insist that the students learn what he or she knows (Norton and D'Ambrosio 2008). Otherwise, the teacher/researcher might become caught in what Stolzenberg (1984) called a "trap" – focusing on the mathematics the teacher/researcher takes as given instead of focusing on exploring students' ways and means of operating. The teacher/researcher's mathematical concepts and operations can be orienting, but they should not be regarded, initially at least, as constituting what students should learn until they are modified to include at least aspects of a mathematics of students (Steffe 1991a).

### Meanings of "Experiment"

*Testing Initial Research Hypotheses*. One goal of exploratory teaching is to identify essential differences in students' ways and means of operating within the chosen context in order to establish initial research hypotheses for the teaching experiment (Steffe et al. 1983). These differences are essential in establishing the constructivist teaching experiment as involving an "experiment" in a scientific sense. The established differences can be used to place students in experimental groups and the research hypothesis is that the differences between the students in the different experimental groups would become quite large over the period of time the students participate in the experiment and that the students within the groups would remain essentially alike (Steffe and Cobb 1988). Considerable hypothesis building and testing must happen during the course of a teaching experiment as well. However, one does not embark on the intensive work of a constructivist teaching experiment without having initial research hypotheses to test.

The research hypotheses one formulates prior to a teaching experiment guide the initial selection of the students and the teacher/researcher's overall general intentions. The teacher/researcher does his or her best to set these initial hypotheses aside during the course of the teaching episodes and focus on promoting the greatest progress possible in all participating students. The intention of teacher/researcher is for the students to test the research hypotheses by means of how they differentiate themselves in the trajectory of teaching interactions (Steffe 1992; Steffe and Tzur 1994). A teacher/researcher returns to the initial research hypotheses retrospectively after completing the teaching episodes. This method – setting research hypotheses aside and focusing on what actually happens in teaching episodes – is basic in the ontogenetic justification of school mathematics.

*Generating and Testing Working Hypotheses.* In addition to formulating and testing initial research hypotheses, another modus operandi in a teaching experiment is for a teacher/researcher to generate and test hypotheses during the teaching episodes. Often, these hypotheses are conceived "on the fly," a phrase Ackermann (1995) used to describe how hypotheses are formulated in clinical interviews. Frequently, they are formulated between teaching episodes as well. A teacher/researcher, through reviewing the records of one or more earlier teaching episodes, may formulate hypotheses to be tested in the next episode (Hackenberg 2010). In a teaching episode, the students' language and actions are a source of perturbation for the teacher/researcher. It is the job of the teacher/researcher to continually postulate possible meanings that lie behind students' language and actions. It is in this way that students guide the teacher/researcher. The teacher/researcher may have a set of hypotheses to test before a teaching episode and a sequence of situations planned to test the hypotheses. But because of students' unanticipated ways and means of operating as well as their unexpected mistakes, a teacher/researcher may be forced to abandon these hypotheses while interacting with the students and to create new hypotheses and situations on the spot (Norton 2008). The teacher/researchers also might interpret the anticipated language and actions of the students in ways that were unexpected prior to teaching. These impromptu interpretations are insights that would be unlikely to happen in the absence of direct, longitudinal interaction with the students in the context of teaching interactions. Here, again, the teacher/researcher is obliged to formulate new hypotheses and to formulate situations of learning to test them (Tzur 1999).

## Living, Experiential Models of Students' Mathematics

Through generating and testing hypotheses, boundaries of the students' ways and means of operating – where the students make what to a teacher/researcher are essential mistakes – can be formulated (Steffe and Thompson 2000). These essential mistakes are of the same nature as those Piaget found in his studies of children, and a teacher/researcher uses them for essentially the same purpose he did. They are observable when students fail to make viable adaptations when interacting in a medium. Operations and meanings a teacher/researcher imputes to students constitute what are called living, experiential models of students' mathematics. Essential mistakes can be thought of as illuminating the boundaries of what kinds of adaptations a living, experiential model can currently make in these operations and meanings. These boundaries are usually fuzzy, and what might be placed just inside or just outside them is always a source of tension and often leads to creative efforts on the part of a teacher/researcher. What students can do is understood better if what they cannot do is also understood. It also helps to understand what a student can do if it is understand what other students, whose knowledge is judged to be at a higher or lower level, can do (Steffe and Olive 2010). In this, we are in accordance with Ackermann (1995) that:

> The focus of the clinician [teacher] is to understand the originality of [the child's] reasoning, to describe its coherence, and to probe its robustness or fragility in a variety of contexts. (p. 346).

## Meanings of Teaching in a Teaching Experiment

Learning how to interact with students through effective teaching actions is a central issue in any teaching experiment (Steffe and Tzur 1994). If teacher/researchers knew ahead of time how to interact with the selected students and what the outcomes of those interactions might be, there would be little reason for conducting a teaching experiment (Steffe and Cobb 1983). There are essentially two types of interaction engaged in by teacher/researchers in a teaching experiment: responsive and intuitive interactions and analytical interactions.

### Responsive and Intuitive Interaction

In responsive and intuitive interactions, teacher/researchers are usually not explicitly aware of how or why they interact as they do. In this role, teacher/researchers are agents of interaction and they strive to harmonize themselves with the students with whom they are working to the extent that they "lose" themselves in their interactions. They make no intentional distinctions between their knowledge and the students' knowledge, and, experientially, everything is the students' knowledge as they strive to feel at one with them. In essence, they become the students and attempt to think as they do (Thompson 1982, 1991; van Manen 1991). Teacher/researchers do not adopt this stance at the beginning of a teaching experiment only. Rather, they maintain it throughout the experiment whenever appropriate. By interacting with students in a responsive and an intuitive way, the goal of teacher/researchers is to engage the students in supportive, nonevaluative mathematical interactivity.

### Analytical Interaction

When teacher/researchers turn to analytical interaction, they "step out" of their role in responsive/intuitive interaction and become observers as well. As first-order observers, teacher/researchers focus on analyzing students' thinking in ongoing interaction (Steffe and Wiegel 1996). All of the teacher/researchers' attention and energy is absorbed in trying to think like the students and produce and then experience mathematical realities that are intersubjective with theirs. The teacher/researchers probes and teaching actions are not to foment adaptation in the students but in themselves. When investigating student learning, teacher/researchers become second-order observers, which Maturana (1978) explained as "the observer's ability . . . to operate as external to the situation in which he or she is, and thus be an observer of his or her circumstance as an observer" (p. 61). As second-order observers, teacher/researchers focus on the accommodations they might engender in the students' ways and means of operating (Steffe 1991b). They become aware of how they interact and of the consequences of interacting in a particular way. Assuming the role of a second-order observer is essential in investigating student learning in a way that explicitly as well as implicitly takes into account the mathematical knowledge of the teacher/researchers as well as the knowledge of the students (Steffe and Wiegel 1996).

### The Role of a Witness of the Teaching Episodes

A teacher/researcher should expect to encounter students operating in unanticipated and apparently novel ways as well as their making unexpected mistakes and becoming unable to operate. In these cases, it is often helpful to be able to appeal to an observer of a teaching episode for an alternative interpretation of events. Being immersed in interaction, a teacher/researcher may not be able to act as a second-order observer and step out of the interaction, reflect on it, and take further action on that basis. In order to do so, a teacher/researcher would have to "be" in the interaction and outside of it, which can be difficult. It is quite impossible to achieve this if there are no conceptual elements available to the teacher/researcher from past teaching experiments that can be used in interpreting the current situation. The result is that teacher/researchers usually react to surprising behavior by switching to a more intuitive mode of interaction. When this happens, the observer may help a teacher/researcher both to understand the student and to posit further interaction. There are also occasions when the observer might make an interpretation of a student's actions that is different from that of a teacher/researcher for any one of several reasons. For example, the observer might catch important elements of a student's actions that apparently are missed by a teacher/researcher. In any case, the witness should suggest but not demand specific teaching interventions.

## Retrospective Conceptual Analysis

Conceptual analysis is intensified during the period of retrospective analysis of the public records of the teaching episodes, which is a critical part of the methodology. Through analyzing the corpus of video records, the teacher/researchers conduct a historical analysis of the living, experiential models of students' mathematics throughout the period of time the teaching episodes were conducted. The activity of model building that was present throughout the teaching episodes is foregrounded, and concepts in the core of a constructivist research program like assimilation, accommodation, scheme (von Glasersfeld 1981), cognitive and mathematical play, communication, spontaneous development (Piaget 1964), interaction (von Foerster 1984), mental operation (von Glasersfeld 1987), and self-regulation emerge in the form of specific and concrete explanations of students' mathematical activity. In this regard, the modeling process in which we engage is compatible with how Maturana (1978) regards scientific explanation:

As scientists, we want to provide explanations for the phenomena we observe. That is, we want to propose conceptual or concrete systems that can be deemed intentionally isomorphic to the systems that generate the observed phenomena. (p. 29).

However, in the case of a teaching experiment, we seek models that fit within our living, experiential models of students' mathematics without claiming isomorphism because we have no access to students' mathematical realities outside of our own ways and means of operating when bringing the students' mathematics forth. So, we cannot get outside our observations to check if our conceptual constructs are isomorphic to students' mathematics. But we can and do establish viable ways and means of thinking that fit within the experiential constraints that we established when interacting with the students in teaching episodes (Steffe 1988, 1994; Norton and Wilkins 2010).

Since the time of its emergence, the constructivist teaching experiment has been widely used in investigations of students' mathematics as well as in investigations of mathematics teaching (cf. Appendix for sample studies). It has also been adapted to fit within related research programs (e.g., Cobb 2000; Confrey and Lachance 2000; Simon et al. 2010).

## Cross-References

▶ Constructivism in Mathematics Education
▶ Discourse Analytic Approaches in Mathematics Education
▶ Early Algebra Teaching and Learning
▶ Elkonin and Davydov Curriculum in Mathematics Education
▶ Hypothetical Learning Trajectories in Mathematics Education
▶ Interactionist and Ethnomethodological Approaches in Mathematics Education
▶ Learner-Centered Teaching in Mathematics Education
▶ Mathematics Teacher as Learner
▶ Number Teaching and Learning
▶ Probability Teaching and Learning
▶ Psychological Approaches in Mathematics Education
▶ Realistic Mathematics Education
▶ Teacher as Researcher in Mathematics Education
▶ Theories of Learning Mathematics
▶ Zone of Proximal Development in Mathematics Education

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## Appendix: Example Studies Using Teaching Experiment Methodology

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## Cooperative Didactic Engineering

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Design-based research · Methodology · Epistemology · Teaching practice · Didactical engineering

### Definition

The Joint Action Theory in Didactics (Sensevy 2019) aims at theorizing a specific process of design-based research (Cobb et al. 2003) and design-based implementation research (Fishman et al. 2013), called *cooperative engineering* (Sensevy et al. 2013; Joffredo-Le Brun et al. 2018), in order to contribute to the elaboration of new forms of schooling. Cooperative engineering (CE) refers to a methodological process in which a collective of teachers and researchers engage in a joint action to codesign, implement, and re-implement a teaching sequence on a particular topic. Each stage of the process is based on an analysis and evaluation of the previous stage, and thus a crucial aspect in the building of a cooperative engineering is its iterative structure. In this respect, it is similar to the lesson studies approach (e.g., Elliott 2012). Another fundamental aspect of this methodological process, similar to a characteristic of educational action research (e.g., Kemmis 2009), is the participation of teachers in the conception of the cooperative engineering process. CE also shares some of the traits of collaborative research (e.g., Bednarz 2009), in particular its focus on the way teachers and researchers can work together. The characteristic features of CE broadly situate it within the learning science paradigm (e.g., Koschmann 2011).

### Origin

Cooperative engineering includes "the controlled design and experimentation of teaching sequences and adopting an internal mode of validation based on the comparison between the a priori and a posteriori analyses of these" (Artigue 2018). The origin of this aspect of CE can be traced back to didactical engineering (Brousseau 1997; Artigue 2015, 2018; Barquero and Bosch 2015).

In keeping with other recent developments in educational research, CE takes into account the shift of interest toward teachers' "representations and practices" and "the current evolution of vision of relationships between researchers and teachers" (Artigue 2018); this has led to a redefinition of its modes of validation as we shall see.

### CE's Background Assumptions

First and foremost, CE is based on a challenge to fundamental Western dualisms, including those between theory and practice and ends and means (Dewey 1920). As Dewey argued, such dualisms are social and inherited from political structures of domination. One of the main ends of CE, therefore, is to dilute such dualisms in a practical manner.

Another background assumption of CE is the conviction that practice is dense with problems that science has not yet even begun to tackle. Scientific knowledge of practice is lacunary, and contrary to the view that science holds answers to most problems of practice, CE adopts a stance in which practice situations have to be carefully described and studied before any attempt is made to solve them. Collectively describing and studying practice situations is the first step in the problematization process. In CE, this conception entails priority being given to a bottom-up collective inquiry, aimed at building specific theories of action (Cobb and Jackson 2011) and elements of a principled practical knowledge (Bereiter 2014).

## Principles

CE unfolds through a system of ideas that can be seen as Deweyan principles: "Principles are methods of inquiry and forecast which require verification by events" (Dewey 1922, p. 239).

*A principle of targeted symmetry.* Teachers and researchers are both practitioners but practitioners of a different kind. The idea is that in order to improve an educational process, teachers and researchers are viewed a priori as equally able to propose adequate manners of acting or relevant ways of conceptualizing practice in the elaborated design. Teachers and researcher participate in what is called an *epistemic cooperative relationship*, which postulates striving for an epistemic symmetry in the engineering dialogue.

*The necessity of acknowledging differences*. Cooperative Engineering requires that every agent be responsible for proposing to the collective her first-hand point of view so as to contribute what she "sees" and what she "knows" from her position. There is a fundamental link between research based on this postulate of symmetry and this acknowledging of differences. The first-hand point of view, *which every participant is able to make explicit*, concretizes differences stemming from each person's experience. Such differences are not founded on the status of someone who knows something versus someone who does not. Rather, they are the result of different experiences in/of the social world relating to the common engineering practice.

*The necessity of building a common reasoning about ends and means*, and thus the potentiality to play both as a collective and as an individual in the game of giving and asking for reasons (Brandom 2001). In such a game, each participant becomes able to give the rationale of the elaborated structures and is therefore able to understand and build a first-hand relationship with this design rationale, whether it be "practical" or "theoretical," thereby going beyond any epistemic division of labor. By building a common repertoire of described and analyzed practices, participants make themselves capable of designing ends-in-views (Dewey 1922), which emerge from practical accomplishments in the designing process.

*The Engineer Stance.* Cooperative engineering may foster a kind of *local, practical indistinguishability between teachers and researchers*. At some moments of practice, both of them share an engineer stance, which includes theoretical and concrete ways of responding to a problem of teaching practice. This principle has to be understood as being in relation to the "The necessity of acknowledging differences principle." Speaking of a "local, practical indistinguishability" between the teacher and the researcher does not mean that they fuse together within an unlikely fuzzy stance. It does not erase the differences between the two professions but rather temporally and locally reunites them together under an *engineer stance*. This stance brings all the members of the CE together in a shared epistemic responsibility.

*Cooperate to produce a work.* In many forms of "collaborative research," teachers and researchers do not work together on a common concrete object, i.e., the designing of a teaching sequence. In CE, teachers and researchers have to cooperate in order to produce a common work – an *opera* to use the Latin word for "work" or "labor." This common work lies both in the representational structure of the teaching sequence and in the concrete unfolding of the teaching–learning process itself.

This means that in a CE research project, it is the "concrete object," the teaching sequence itself, which is the *touchstone* of the research process. This "concrete object" is enacted in a practical accomplishment, which is depicted in a hypermedia system, as we will see. Such a hypermedia system is a fundamental means of regulation in that it provides evidence through a warranted assertibility process (Dewey 1938).

*Cooperate to produce knowledge*. Participating in a CE means participating in a knowledge work in a twofold way. Firstly, as in Didactic Engineering, emphasis is put on the piece of knowledge to be taught, which is jointly studied by the members of the CE. Studying a piece of knowledge means building a connoisseur's relationship with this knowledge. It is a long, collective process which precedes teaching. Secondly, the whole cooperative process of designing a teaching sequence can be seen as a production of knowledge in the form of the teaching sequence; this includes the various descriptions, depictions, comments, and analyses that enable it to be understood and mastered.

## CE as a Form of Both Anthropological and Engineering Research

The goals of designing teaching sequences and developing theories of teaching and learning are intertwined in CE. Thus, CE is first and foremost fundamental research within an anthropological approach (Chevallard and Sensevy 2014), whose object is the "Didactic Human Fact" (Cloud 2015), i.e., human being learning and human being teaching. But this human fact is always becoming, always virtually other than it is, and as it is constantly in a state of development, never final; it necessitates being transformed to be understood, as in natural science, and the whole process requires transformation for understanding and understanding for transformation.

In this respect, cooperative engineering may contribute to the building of a new research paradigm that is both anthropological and design-based: anthropological in that it aims to elaborate a theory of practice and design-based in that it aims to build better educational designs.

## CE: An Epistemology of Paradigmatic Analogy, Toward the Ascent from the Abstract to the Concrete

Sciences of culture are sciences of contexts (Passeron 2013). This means that assertions produced within the sciences of culture have to systematically be referred to the contexts they denote. A good manner in which to build such a frame of reference consists of instituting some contexts as exemplars (Kuhn 1974). We may hypothesize that a given example of practice has to be considered first as an "emblematic example" within a particular research endeavor; this then needs to further pertain to the common knowledge of a research community to become an exemplar in this research community. Such a conception radically inverses standard interpretations of the relationship between the concrete and the abstract in which the abstract is conceived of as the common area shared by some concrete elements. It is based on a Marxian dialectical vision of these relationships, in the sense that scientific activity is seen to render possible the *ascent from the abstract to the concrete* (Engeström et al. 2012; Ilyenkov 1982; Marx 2012; Davydov 1990). According to this epistemology, CE can be seen as a deliberate attempt to fundamentally give priority to the concrete of practice over the abstract ideas that may describe it.

Thus, when in the process of building designs, cooperative engineers institute certain aspects of practice as emblematic examples; this enables them to both illustrate and to understand some crucial dimensions of the teaching–learning process. CE thus puts at the forefront a documenting process, in which emblematic examples are given to be seen and understood. This is the role of PTHAS.

## A Method of Documenting Practice and Research on Practice: The PTHAS

In this way, emblematic examples can be structured and designed in hypermedia systems, picture–text–audio hybrid systems (PTAHS), cf. Sensevy et al. 2018. In such systems, films of practice, as well as various comments on and analysis of this practice, play an essential role (Sensevy 2011; Tiberghien and Sensevy 2012) in documenting its main features. Thus, the epistemology of paradigmatic analogy that we sketched above is also "an epistemology of methodology," in which the progress of knowledge relies on the building, studying, and refining of emblematic examples of practice that serve as frames of reference in the scientific inquiry.

While using PTAHS, a CE team focuses this inquiry on how practice works, in order to answer questions about how a given teaching sequence can be managed and achieved successfully. It is possible to consult an example of a PTHAS[1] elaborated within the ACE (**A**rithmetics and **C**omprehension at **E**lementary School) program (Sensevy et al. 2013; Joffredo-le Brun et al. 2018; Fischer et al. 2018). This program, which aimed at providing a curriculum for the first and second grades in mathematics, is currently based on the development of PTAHS with the twofold goal of enhancing the relevance of the research work as well as reinforcing the concreteness of the dissemination process.

## Cross-References


- ▶ Didactic Contract in Mathematics Education
- ▶ Didactic Engineering in Mathematics Education
- ▶ Didactic Situations in Mathematics Education
- ▶ Didactic Transposition in Mathematics Education
- ▶ Didactical Phenomenology (Freudenthal)
- ▶ Joint Action Theory in Didactics (JATD)


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# Creativity in Mathematics Education

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## Definition

Even though mathematics education, unlike general psychology, has not yet fully embraced creativity as a systematic research domain (Sriraman and Leikin 2017), there have been several papers, books, and special issues of journals devoted to mathematical creativity published in recent years. In this entry, we will attempt to present an up-to-date status on and understanding of creativity in mathematics. We will also try to clear up some of the confusion regarding related concepts such as creativity, giftedness and ability, and the relationship between them. First, the concepts of giftedness, ability, and creativity will be discussed and differentiated. Second, common themes from the relevant literature will be synthesized that capture the main ideas in the studies. Lastly, the synthesis will be situated into the more generally framed research in psychology.

C

## Creativity

One of the main challenges in investigating mathematical creativity is the lack of a clear and accepted definition of the term mathematical creativity and creativity itself. Previous examinations of the literature have concluded that there is no universally accepted definition of either creativity or mathematical creativity (Sriraman 2005; Mann 2005). Treffinger et al. (2002) write, for instance, that there are more than 100 contemporary definitions of mathematical creativity. So how can the scientific communities produce so much research on creativity, when there is no clear-cut definition of creativity (Sriraman 2017)? There are certain parameters agreed upon in the literature that helps narrow down the concept of creativity. Most investigations of creativity take one of two directions: extraordinary creativity, known as big C, or everyday creativity, known as little c (Kaufman and Beghetto 2009). Extraordinary creativity refers to exceptional knowledge or products that change our perception of the world. Feldman et al. (1994) writes: "the achievement of something remarkable and new, something which transforms and changes a field of endeavor in a significant way . . . the kinds of things that people do that change the world." Ordinary, or everyday, creativity is more relevant in a regular school setting. Feldhusen (1995) describes little c as: "Wherever there is a need to make, create, imagine, produce, or design anew what did not exist before – to innovate – there is adaptive or creative behavior, sometimes called 'small c'." Investigation into the concept of creativity also distinguishes between creativity as either domain specific or domain general (Kaufman and Beghetto 2009).

Whether or not creativity is domain specific or domain general, or if you look at ordinary or extraordinary creativity, most definitions of creativity include some aspect of usefulness and novelty (Sternberg 1999; Plucker and Beghetto 2004; Mayer 1999) – otherwise known as the standard definition of creativity (Runco and Jaeger 2012). What is useful and novel, however, depends on the context of the creative process of an individual. The criteria for useful and novel in professional arts would differ significantly from what is considered useful and novel in a mathematics class in lower secondary school. There is therefore a factor of relativeness to creativity. For a professional artist, some new, groundbreaking technique, product, or process that changes his or her field in some significant way would be creative, but for a mathematics student in lower secondary school, an unusual solution to a problem could be creative. Csikszentmihalyi (2014) shed further light on this by explaining how creativity is a phenomenon that results from an interaction between three parties: “a set of social institutions, or *field*, that selects from the variations produced by individuals those that are worth preserving; a stable cultural *domain* that will preserve and transmit the selected new ideas or forms to the following generations; and finally the *individual*, who brings about some change in the domain, a change that the field, will consider to be creative.” Mathematical creativity in a K-12 setting can as such be defined as the process that results in a novel solution or idea to a mathematical problem or the formulation of new questions, produced by an individual or several individuals, and considered worth preserving within the context of school mathematics (Sriraman 2005).

## Giftedness and Ability

For decades giftedness was equated with concept of intelligence or IQ (Renzulli 2005; Brown et al. 2005; Coleman and Cross 2001). Terman (1925) claimed that gifted individuals are those who score at the top 1% of the population on the Stanford-Binet test. This understanding of giftedness is still wide spread today. Researchers working in cognitive and metacognitive areas still use high IQ as a marker of giftedness. However, many researchers now include other factors as well and view giftedness as a more multifaceted concept where intelligence is one of several aspects (Renzulli 2005). One example is Renzulli’s (2005) three-ring model of giftedness. In an attempt to capture the many facets of giftedness, Renzulli presented giftedness as an interaction between above-average ability, creativity, and task commitment. He went on to separate giftedness into two categories: schoolhouse giftedness and creative productive giftedness. The former refers to the ease of acquiring knowledge and taking standardized tests. The latter involves creating new products and processes, which Renzulli thought was often overlooked in school settings. Other researchers have also proposed multidimensional models of giftedness, which integrate factors such as environmental factors, creativity, and even luck (Miller 2012).

In this entry, we will focus on giftedness in mathematics, as giftedness as a concept is dependent on the context and field (Csikszentmihalyi 2000). However, first we have to clarify a certain linguistic confusion. Mathematical ability is another term that has often been used interchangeably with mathematical giftedness. High mathematical ability has also been usually seen as equivalent to mathematical attainment (Piirto 1999), and to some degree there is some truth to that notion. There is a statistical relationship between academic attainment in mathematics and early identified high mathematical ability (Benbow and Arjmand 1990). However, Ching (1997) discovered that hidden talent go largely unnoticed in typical classrooms, and Kim et al. (2004) state that traditional tests rarely identify mathematical creativity. Hong and Aqui (2004) compared cognitive and motivational characteristics of high school students who were academically gifted in math, creatively talented in math, and nongifted. The authors found that the creatively talented students used more cognitive strategies than the academically gifted students. Similar findings have been reported elsewhere, with high ability in mathematics related to solving complex mathematical problems (Davis and Rimm 1989; Geary and Brown 1991; Lev and Leikin 2017). These findings indicate that mathematical ability and mathematical attainment in a traditional K-12 setting are not synonymous – which is also clear from the definitions themselves. Ability is defined as "the quality of being able to do something, especially the physical, mental, financial, or legal power to accomplish something." Attainment is defined as "Something, such as an accomplishment or achievement, that is attained." The key difference is that ability points to a potential to do something, while attainment refers to something that has been accomplished. In the field of mathematics, mathematical ability then refers to the ability to do mathematics. Mathematical attainment, on the other hand, is usually seen as doing well on tests and other formal assessments in school settings. The conflation between the two concepts is therefore to a large extent caused by the assumption that formal assessment in school mathematics is a valid representation of mathematical ability. As we can see from the literature, this is not necessarily the case; in particular we sometimes see a distinction between academic attainment in mathematics and creative talent in mathematics throughout the literature (see for instance Selden et al. 1994; Kim et al. 2004; Livne and Milgram 2006; Haavold 2011).

So what are mathematical ability and mathematical giftedness then? Although neither construct are precisely defined in the literature, we can say that an individual is mathematically gifted if his or her ability in mathematics is well above the norm for their age. Due to the lack of a conceptual clarity regarding giftedness and the heterogeneity of the gifted population, both in general and in mathematics, identification of gifted students has varied (Kontoyianni et al. 2011). Instead, we can see some common characteristics of giftedness in mathematics in the research literature. Krutetskii (1976) noted in his investigation of gifted students in mathematics a number of characteristic features: ability for logical thought with respect to quantitative and spatial relationships, number and letter symbols, the ability for rapid and broad generalization of mathematical relations and operations, flexibility of mental processes and mathematical memory. Similar features of mathematical giftedness have been proposed by other researchers (see for instance Sriraman 2005).

## Giftedness and Creativity

In the field of professional mathematics, the creative mathematician is a rarity. At this level, mathematical creativity implies mathematical giftedness, but the reverse is not necessarily true (Sriraman 2005). Usiskin's (2000) eight tiered hierarchy of creativity and giftedness in mathematics further shed some light of this view of the relationship between creativity and giftedness in professional mathematics. In this model, we find the productive mathematician at level five. These are your typical mathematicians who have successfully completed a Ph.D. in mathematics and are capable of publishing in the field. At level six and seven, on the other hand, we find the exceptional mathematicians who have moved their fields forward and who has made their mark on history. It is here, at level six and seven, Usiskin claims that we find the creative mathematicians. Therefore, we can say that in Usiskin's (2000) model, mathematical creativity implies mathematical giftedness, but not vice versa.

However, while Usiskin's model is interesting, it is limited to big-C type creativity and mathematical giftedness among professional mathematicians. It does not necessarily tell us much about the relationship between creativity and giftedness in school mathematics. Here, the relationship between giftedness and creativity has been the subject of much discussion (Leikin 2008; Sternberg 1999). Several studies have, for instance, found a significant correlation between mathematical creativity and mathematical attainment in various forms (see, for instance, Ganihar and Wajiha 2009; Haavold 2016; Kattou et al. 2013; Mann 2005; Prouse 1967; Sak and Maker 2006; Tabach and Friedlander 2013). In a review on the relationship between creativity and giftedness in mathematics, Sriraman and Haavold (2017) concluded that although there is a significant statistical relationship between mathematical creativity and mathematical attainment, mathematical attainment does not necessarily entail mathematical creativity. An explanation for this is found in Sriraman's (2005) argument that mathematical creativity in the K–12 setting is seen on the fringes of giftedness. This idea is intuitively appealing as traditional mathematics teaching emphasizes procedures, computation, and algorithms. There is little attention to developing conceptual ideas, mathematical reasoning, and problem-solving activities (Cox 1994).

According to Haylock (1997) and Sheffield (2009), low attaining students do not possess the sufficient mathematical knowledge for creativity to manifest. Solid content knowledge is required for individuals to make connections between different concepts and types of information. Feldhausen and Westby (2003) assert that an individual's knowledge base is the fundamental source of their creative thought. Mathematical ability therefore seems to be a necessary, but not sufficient, condition for mathematical creativity to manifest. Theoretical support for this conclusion is found in general creativity research within psychology. The foundation view of creativity suggests a positive relationship between knowledge and creativity. Since a knowledgeable individual knows what has been done within a field, he or she can move forward and come up with new and useful ideas (Weisberg 1999). Deep knowledge within a field is essential to the creative process.

In a series of studies investigating creativity and giftedness, researchers developed a model that shed further light on these relationships. Their findings suggest that mathematical creativity and mathematical abilities are the fundamental components of mathematical giftedness (Kontoyianni et al. 2011, 2013; Pitta-Pantazi et al. 2011, 2012). In these studies, mathematical ability was defined as (a) spatial abilities, (b) quantitative abilities, (c) qualitative abilities, (d) verbal abilities, and (e) causal abilities, while mathematical creativity was defined in terms of fluency, flexibility, and originality. Unlike Usiskin (2000), this model (Pitta-Pantazi 2017) indicates that mathematical creativity does not imply mathematical giftedness. In fact, the results show that mathematical ability contributes more to mathematical giftedness than mathematical creativity – supporting the foundation view of creativity. The researchers also found that more general and natural cognitive factors were also necessary, but not sufficient, conditions for predicting mathematical giftedness. All students who were identified as mathematically gifted had a high fluid intelligence, but not all students who had a high fluid intelligence were identified as mathematically gifted.

## Empirical Research

The lack of a clear definition of mathematical creativity has led to functional and pragmatic empirical approaches. Haylock (1987) summarized research on mathematical creativity into two investigative models: (1) the ability to overcome fixations in mathematical problem solving and (2) the ability for divergent production. Creativity as divergent production was first proposed by Guilford and Torrance and is based on both associative theory and Guilford's theory of the structure of intellect. Guilford (1959) considered creative thinking as involving divergent thinking, in which fluency, flexibility, originality, and elaboration were central features. Fluency is the number of solutions to a problem or situation, flexibility the number of different categories of solutions, originality the relative unusualness of the solution, and elaboration the amount of detail in the responses. Building on Guilford's work, Torrance et al. (1966) developed the Torrance Test of Creative Thinking to assess individuals' capacity for creative thinking. This, in turn, led to the use of different divergent production tests in numerous contexts, including mathematics perspectives education (e.g., Aiken 1973; Haylock 1987; Haavold 2016; Kattou et al. 2013; Krutetskii 1976; Leikin and Lev 2013). The common feature of all these tests is problems and situations with many possible responses. Unlike convergent thinking, where the subject must seek one solution, divergent thinking tasks allow for many possible solutions (Haylock 1987).

Recently, Mann et al. (2017) proposed a fifth subconstruct of the modern day construct of mathematical creativity. An iconoclast is a person who attacks settled beliefs or institutions. In mathematics, iconoclasm refers to mathematically creative individuals' tendency to oppose commonly accepted principles and solutions. Iconoclasts are often nonconformist and open to new and uncommon solution paths. However, iconoclasm is still a theoretical proposal. Mann et al. (2017) state that empirical proof of existence is still needed, and they encouraged the development of an instrument that investigates whether problem solvers will challenge commonly accepted algorithms when they are faced with a relatively inefficient solution.

Although most research into creativity is based on divergent production tests, it is also worth mentioning that the practice of accepting divergent thinking as a proxy for creativity has been subject to much criticism. The most obvious criticism is that creativity can just as well be the result of a convergent process. Tan and Sriraman (2017), for instance, propose convergent thinking as equally important in the context of mathematics. The authors argue that people can also develop their capacity for creativity in convergence (e.g., collaboration), through mathematical learning (e.g., with coherence, congruence), and for creativity (e.g., imagination). Furthermore, divergent thinking is a compound construct, consisting of various separate mental processes that cannot be isolated into the cognitive elements that turn ordinary thinking into creative thinking. This composite nature makes the construct nearly impossible to trace with today's neuroimaging tools, and there is therefore no theory that fully explains the brain activity of the creative process. In fact, one of the strongest findings in the literature is that creativity is not particularly associated with any single brain region, excluding the prefrontal cortex (Dietrich and Kanso 2010). Nevertheless, divergent thinking is still considered one of the more fruitful ways to study ideation and, thus, potential for creativity and problem solving.

The second investigative model posited by Haylock (1987) emphasizes the process of mathematical creativity and the importance of overcoming mental fixations. Creative thinking is related to flexibility of thought (Haylock 1997). This capacity to break from established mental sets is an important aspect of the creative process. Overcoming fixations as an aspect of mathematical creativity can be traced back to the writings of Hadamard and Poincaré and Gestalt psychology (Sriraman et al. 2013). The Gestaltists described the process of creative problem solving through four stages: (1) preparation, (2) incubation, (3) illumination, and (4) verification. Here, illumination occurs once the problem solver, either through conscious or unconscious work, is able to break from established mindsets and overcome certain fixations. A recent example of this approach can be found in Lithner's (2008) framework for creative and imitative reasoning, in which the author separates mathematical reasoning into two categories: (1) creative mathematically founded reasoning and (2) imitative reasoning. Creative mathematically founded reasoning is a sequence of arguments that are original, plausible, and based on mathematical properties. Imitative reasoning, on the other hand, is built on copying task solutions or through remembering an algorithm or answer. The key difference is seen in the reasoner's ability to break from established mindsets and come up with novel and plausible reasoning sequences.

## Conceptual Relationships

Although we are still learning about the relationship between mathematical creativity and mathematical giftedness, certain features of mathematical creativity are found throughout the literature. The characteristics of mathematical creativity and its relationship to other theoretical constructs have been investigated further in recent research. Several issues of *ZDM*, The International Journal of Mathematics Education, and books (e.g., Leikin and Sriraman 2017) have, for instance, been exclusively devoted to the concepts of mathematical creativity and mathematical giftedness. In this section, we mention some of these findings.

In one study, a team of researchers examined how different conceptualizations of giftedness were related to mathematical creativity (Lev and Leikin 2017). Hundred and eighty-four students (aged 16–18) were assigned to four different groups, determined by a combination of general intelligence (high IQ) and high performance in mathematics. A total of 665 students (aged 16–18) were given a multiple solution test in mathematics; 184 belonged to the research sample and 481 students served as a comparison group. Both general intelligence and high performance in mathematics were found to have a significant effect on mathematical creativity. However, the study also demonstrates a distinction between general intelligence and mathematical performance. High performance in mathematics appeared to be a prerequisite for mathematical creativity, but general intelligence offered an added effect in particular on originality of solutions to mathematical problems (Lev and Leikin 2017). Although the exact relationship between general intelligence and general creativity is still being investigated, the findings of Lev and Leikin (2017) resonate with findings in general psychology. Decades ago, creativity and intelligence were seen as distinct concepts. However, in recent years, most researchers have begun to see creativity and intelligence as related concepts. The contemporary view is that creativity and intelligence are closely linked. People who do better on typical intellectual tasks also do well on creativity tasks. Instead of talking about creativity and intelligence as separate things, they should be seen as "families of processes and functions that the mind can do" (Silvia 2015).

Several other findings in the literature are related to these processes and functions. Pitta et al. (2013) investigated the relationship between mathematical creativity and cognitive styles. A mathematical creativity test consisting of five tasks was given to 96 prospective primary school teachers and was assessed on the basis of fluency, flexibility, and originality. Cognitive style was measured with the Object-Spatial Imagery and Verbal Questionnaire (OSIVQ) with respect to three styles: spatial, object, and verbal. Using multiple regression, the authors conclude that spatial and object styles were significant predictors of mathematical creativity, while verbal style was not significant. Spatial cognitive style was positively related to mathematical creativity, while object cognitive style was negatively related to mathematical creativity. Furthermore, spatial cognitive style was positively related to fluency, flexibility, and originality, while object cognitive style was negatively related to originality and verbal cognitive style was negatively related to flexibility.

In another study, Pitta-Pantazi and Christou (2009) investigated the relationship of students' spatial and object visualization to their analytical, creative, and practical abilities in three dimensional geometry. The analysis conducted showed that object visualization was related to the students' creative abilities. Other researchers, working on general creativity research, agree that cognitive styles are related to creativity (see for instance Sternberg 2012). However, some researchers deny this relationship (Kirton 1989). The disagreement seems to stem from a difference in how cognitive styles and creative behavior have been defined and investigated. In a study of a theoretical model that attempts to explain mathematical creativity, Haavold (2016) demonstrated that intrinsic motivation in mathematics was a significant predictor of mathematical creativity. In other studies, positive affect in general (feelings, emotions, dispositions, and beliefs) has been associated with the creative process. A common theme in these studies is that affective states play a significant role in stimulating creative thinking and is a factor that can be influenced (Mann et al. 2017).

No clear picture of creativity and its relationship to other constructs emerges from these, and other, studies. However, it is clear that creativity as a concept is closely related to other cognitive and affective features of the mind, such as ability, intelligence, cognitive style, motivation. As Silvia (2015) argues, we should look at creativity and its related concepts as part of family of processes and functions of the mind, instead of deterministic predictors or requirements of creativity.

## Giftedness and Creativity in Psychology and Neuroscience

Within mathematics education, mathematical creativity is often claimed to be an ill-defined concept. However, within a bigger context of creativity research, there has been a steady progress on numerous fronts. Sriraman (2017) summarizes some of this progress into three postulates. The first postulate states that incubation facilitates creativity. A century ago, the school of Gestalt psychology put forward model of preparation-incubation-illumination and verification. Since then there have been numerous studies that have verified the importance of the "rest hypothesis" for facilitating creativity, a fresh brain in a new state of mind triggers illumination and the incubation phase gets rid of false leads. The second postulate states that intrinsic motivation is positively related to creativity. Creative results are often the product of a period of prolonged and sustained activity, which in turn is driven by the intrinsic motivation of the individual. The third postulate tells us that divergent thinking is not the sole marker of mathematical creativity. Most research into creativity has used divergent production as a marker of creativity. While divergent thinking is an important factor of creativity, too much divergent thinking could lead to an excess of novelty at the expense of usefulness. Convergent thinking has to work in tandem with divergent thinking, in order to align ideas with the rules, norms, and knowledge of the field in question.

In general, cognitive psychology studies have shown that creativity involves many cognitive processes, including defocused attention, cognitive control, flexibility, fluency, and working memory (Dietrich 2004). Recently, neuroscience has become an increasingly popular approach for studying creativity, and EEG and fMRI research lends neuroscientific support to the behavioral evidence from cognitive psychology. Although there are both benefits and drawbacks to using these techniques, neuroscience has already proven its worth in the study of creativity. For instance, neuroscience has helped debunk the myth that the right hemisphere of the brain is responsible for creative thought. Numerous studies have shown that a diffuse network of neurons across both hemispheres is involved in creative processes (Sawyer 2011). EEG-based studies have established that both creativity and tasks that require higher cognitive abilities are connected to variations in alpha power. In addition, fMRI-based studies have demonstrated that executive functions and creativity active the prefrontal and parietal regions of the neocortex (Cropley et al. 2017).



## Teaching for Creativity

Although creativity, like most cognitive processes have a genetic component, it can be developed and nurtured (see for instance Beghetto 2013). What can we do to stimulate creativity in school mathematics? In this section, we look at some techniques and methods for fostering creativity in the classroom. In general, a state of doubt is important for triggering the creative learning process (Beghetto and Schreiber 2017). Sternberg (2017) writes that creativity is a habit, and if we want to promote it, we need to treat it as a desirable practice. To develop the creativity habit, one needs opportunities to exercise creativity. This means that students must be willing to take sensible risks, to see conventional problems in new ways, and persist when others question one's creative approach to problems. According to Sternberg (2017), teaching for creative thinking means that students should be encouraged to create, invent, discover, predict, and imagine. However, this requires teachers to not only support and encourage creativity, but also role-model it and reward it. Closely related to doubt and the development of creativity as a habit is inquiry-oriented mathematics instruction that includes problem-solving and problem-posing tasks (Silver 1997; Leikin 2009). These types of activities can assist students to develop more creative approaches to mathematics. With problem solving and problem posing, teachers can increase their students' capacity with respect to the core dimensions of creativity, namely, fluency, flexibility, and novelty.

In recent years, several studies have looked at methods for stimulating mathematical creativity in the classroom. Beghetto and Schreiber (2017) propose abductive reasoning as an approach to stimulate creativity. Abductive reasoning starts with an observation and then seeks to find the most likely explanation. It represents a special form of creative reasoning that is triggered by states of genuine doubt, as it typically arises when we cannot explain an observed phenomenon. Through abductive reasoning, we resolve our doubt and this in turn is a key motivator in the creative learning process. An example of creating doubt, closely related to abductive reasoning, is found in a recent article in ZDM by Sriraman and Dickman (2017). Here, the authors advocate the use of mathematical pathologies as a means of fostering creativity in the classroom. Pathologies refers here to mathematical objects "cooked up" to "provide interesting examples of counterintuitive behavior." One example of "mathematical pathology" is provided in the context of cancellation as a means to simply fractions. By cancelling the 9 s in $\frac{19}{95}$, we get $\frac{1}{5}$. A correct result by erroneous methods. The natural follow-up question is whether there are other two-digit fractions with this property.

## Cross-References


- ▶ Giftedness and High Ability in Mathematics
- ▶ Mathematical Cognition: In Secondary Years [13–18] Part 1
- ▶ Mathematical Cognition: In Secondary Years [13–18] Part 2


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# Critical Mathematics Education

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**Keywords**

Mathematics education for social justice · Critical mathematics education · Ethnomathematics · Mathematics in action · Students' foregrounds · Mathemacy · Landscapes of investigation · Mathematization · Dialogic teaching and learning

## Definition

Critical mathematics education can be characterized in terms of concerns: to address social exclusion and suppression, to work for social justice, to open new possibilities for students, and to address critically mathematics in all its forms and application.

## Characteristics

### Critical Education

Inspired by the students' movement, a New Left, peace movements, feminism, and antiracism, critical education proliferated. A huge amount of literature became published, not least in Germany, and certainly the work of Paulo Freire was recognized as crucial for formulating radical educational approaches.

However, critical education was far from expressing any interest in mathematics. In fact, with reference to the Frankfurt School, mathematics was considered almost an obstruction to critical education. Thus, Habermas, Marcuse, and many others associated instrumental reason with, on the one hand, domination and, on the other hand, the rationality cultivated by natural science and mathematics. Mathematics appeared as the grammar of instrumental reason. How could one imagine any form of emancipatory interests being associated to this subject?

### Steps into Critical Mathematics Education

Although there were no well-defined theoretical frameworks to draw on, there were from the beginning of the 1970s many attempts in formulating a critical mathematics education. Let me mention some publications.

The book *Elementarmathematik: Lernen für die Praxis* (*Elementary mathematics: Learning for the praxis*) by Peter Damerow, Ulla Elwitz, Christine Keitel, and Jürgen Zimmer from 1974 was crucial for the development of critical mathematics education in a German context. In the article "Plädoyer für einen problemorientierten Mathematikunterrich in emanzipatorisher Absicht" ("Plea for a problem-oriented mathematics education with an emancipatory aim") from 1975, Dieter Volk emphasized that it is possible to establish mathematics education as a critical education. The book *Indlæring som social proces* (*Learning as a social process*) by Stieg Mellin-Olsen was published in 1977. It provided an opening of the political dimension of mathematics education, a dimension that was further explored in Mellin-Olsen (1987). *Indlæring som social proces* was crucial for the development of critical mathematics education in the Scandinavian context. An important overview of Mellin-Olsen's work is found in Kirfel and Lindén (2010). Dieter Volk's *Kritische Stichwörter zum Mathematikunterricht* (*Critical notions for mathematics education*) from 1979 provided a broad overview of what could be called the first wave in critical mathematics education, soon after followed, in Danish, Skovsmose (1980, 1981a, b).

Marilyn Frankenstein (1983) provided an important connection between critical approaches in mathematics education and the outlook of Freire, and in doing so, she was the first in English to formulate a critical mathematics education (see also Frankenstein 1989). Around 1990, together with Arthur Powell and several others, she formed the critical mathematics education group, emphasizing the importance of establishing a united concept of critique and mathematics (see Frankenstein 2012; Powell 2012). Skovsmose (1994) provided an interpretation of critical mathematics education and Skovsmose (2012) a historical perspective.

Critical mathematics education developed rapidly in different directions. As a consequence, the very notion of critical mathematics education came to refer to a broad range of approaches, such as mathematics education for social justice (see, e.g., Sriraman 2008; Penteado and Skovsmose 2009; Gutstein 2012), pedagogy of dialogue and conflict (Vithal 2003), responsive mathematics education (Greer et al. 2009), and, naturally, critical mathematics education (Skovsmose 2011). Many ethnomathematical studies also link closely with critical mathematics education (see, e.g., D'Ambrosio 2006; Knijnik 1996; Powell and Frankenstein 1997).

### Some Issues in Critical Mathematics Education

Critical mathematics education can be characterized in terms of concerns, and let me mention some related to mathematics, students, teachers, and society:

***Mathematics*** can be brought in action in technology, production, automatization, decision-making, management, economic transaction, daily routines, information procession, communication, security procedures, etc. In fact, mathematics in action plays a part in all spheres of life. It is a concern of a critical mathematics education to address mathematics in its very many different forms of applications and practices. There are no qualities, like objectivity and neutrality, that automatically can be associated to mathematics. Mathematics-based actions can have all kind of qualities, being risky, reliable, dangerous, suspicious, misleading, expensive, brutal, profit generating, etc. Mathematics-based action can serve any kind of interest. As with any form of action, also mathematics in action is in need of being carefully criticized. This applies to any form of mathematics: everyday mathematics, engineering mathematics, academic mathematics, and ethnomathematics.

***Students***. To a critical mathematics education, it is important to consider students' interests, expectations, hopes, aspirations, and motives. Thus, Frankenstein (2012) emphasizes the importance of respecting student knowledge. The notion of students' foregrounds has been suggested in order to conceptualize students' perspectives and interests (see Skovsmose 2014a). A foreground is defined through very many parameters having to do with economic conditions, social-economic processes of inclusion and exclusion, cultural values and traditions, public discourses, and racism. However, a foreground is, as well, defined through the person's experiences of possibilities and obstructions. It is a preoccupation of critical mathematics education to acknowledge the variety of students' foregrounds and to develop a mathematics education that might provide new possibilities for the students. The importance of recognizing students' interest has always been a concern of critical mathematics education.

***Teachers***. As it is important to consider the students' interests, it is important to consider the teachers' interests and working conditions as well. Taken more generally, educational systems are structured by the most complex sets of regulations, traditions, and restrictions, which one can refer to as the "logic of schooling." This "logic" reflects (if not represents) the economic order of today, and to a certain degree, it determines what can take place in the classroom. It forms the teachers' working conditions. It becomes important to consider the space of possibilities that might be left open by this logic. These considerations have to do with the micro–macro (classroom-society) analyses as in particular addressed by Paola Valero (see, e.g., Valero 2009). Naturally, these comments apply not only to the teachers' working conditions but also to the students' conditions for learning. While the concern about the students' interests has been part of critical mathematics education right from the beginning, a direct influence from the students' movements, the explicit concern about teaching conditions is a more recent development of critical mathematics education.

***Society*** can be changed. This is the most general claim made in politics. It is the explicit claim of any activism. And it is as well a concern of critical mathematics education. Following Freire's formulations, Gutstein (2006) emphasizes that one can develop a mathematics education which makes it possible for students to come to read and write the world: "read it," in the sense that it becomes possible to interpret the world filled with numbers, diagrams, figures, and mathematics, and "write it," in the sense that it becomes possible to make changes. However, a warning has been formulated: one cannot talk about making sociopolitical changes without acknowledging the conditions for making changes (see, e.g., Pais 2012). Thus, the logic of schooling could obstruct many aspirations of critical mathematics education. Anyway, I find that it makes good sense to articulate a mathematics education for social justice, not least in a most unjust society.

## Some Notions in Critical Mathematics Education

Notions such as social justice, mathemacy, dialogue, and uncertainty together with many others are important for formulating concerns of critical mathematics education. In fact we have to consider ourselves with clusters of notions of which I highlight only a few:

***Social justice***. Critical mathematics education includes a concern for addressing any form of suppression and exploitation. As already indicated, there is no guarantee that an educational approach might in fact be successful in bringing about any justice. Still, working for social justice is a principal concern of critical mathematics education. Naturally, it needs to be recognized that "social justice" is an open concept, the meaning of which can be explored in many different directions. Addressing *equity* also represents concerns of critical mathematics education, and the discussion of social justice and equity brings us to address processes of *inclusion* and *exclusion*. Social exclusion can take the most brutal forms being based on violent discourses integrating racism, sexism, and hostility toward "foreigners" or "immigrants." Such discourses might label groups of people as being "disposable," "a burden," or "nonproductive," given the economic order of today. It is a concern of critical mathematics education to address any form of social exclusion. As an example, I can refer to Martin (2009). However, social inclusion might also represent a questionable process: it could mean an inclusion into the capitalist mode of production and consumption. So critical mathematics education needs to address inclusion–exclusion as contested processes. However, many forms of inclusion–exclusion have until now not been discussed profoundly in mathematics education: the conditions of blind students, deaf students, and students with different handicaps – in other words, students with particular rights. However, such issues are now being addressed in the research environment created by the Lulu Healy and Miriam Goody Penteado in Brazil. Such initiatives bring new dimensions to critical mathematics education.

***Mathemacy*** is closely related to literacy, as formulated by Freire, being a competence in reading and writing the world. Thus, D'Ambrosio (1998) has presented a "New Trivium for the Era of Technology" in terms of literacy, matheracy, and technoracy. Anna Chronaki (2010) provided a multifaceted interpretation of mathemacy, and in this way, it is emphasized that this concept needs to be reworked, reinterpreted, and redeveloped in a never-ending process. Different other notions have, however, been used as well for these complex competences, including *mathematical literacy* and *mathematical agency*. Eva Jablonka (2003) provides a clarifying presentation of mathematical literacy, showing how this very notion plays a part in different discourses, including some which hardly represent critical mathematics education. The notion of mathematical agency helps to emphasize the importance of developing a capacity not only with respect to understanding and reflection but also with respect to acting.

***Dialogue***. Not least due to the inspiration from Freire, the notion of dialogue has played an important role in the formulation of critical mathematics education. Dialogic teaching and learning has been presented as one way of developing broader critical competences related to mathematics. Dialogic teaching and learning concerns forms of interaction in the classroom. It can be seen as an attempt to break at least some features of the logic of schooling. Dialogic teaching and learning can be seen as a way of establishing conditions for establishing mathemacy (or mathematical literacy, or mathematical agency). *Problem-based learning* and *project work* can also be seen as way of framing a dialogic teaching and learning.

***Uncertainty***. Critique cannot be any dogmatic exercise, in the sense that it can be based on any well-defined foundation. One cannot take as given any particular theoretical basis for critical mathematics education; it is always in need of critique (see, e.g., Ernest 2010, and Skovsmose 2014b). In particular one cannot assume any specific interpretation of social justice, mathemacy, inclusion–exclusion, dialogue, critique, etc. They are all contested concepts. We have to do with concepts under construction.

## Critical Mathematics Education for the Future

The open nature of critical mathematics education is further emphasized by the fact that forms of exploitations, suppressions, environmental problems, and critical situations in general are continuously changing. Critique cannot develop according to any preset program.

For recent developments of critical mathematics education, see, for instance, Alrø, Ravn, and Valero (Eds.) (2010), Wager, A. A. and Stinson, D. W. (Eds.) (2012), Skovsmose and Greer (Eds.) (2012), and Ernest, Sriraman, and Ernest (Eds.) (2015). In Portuguese, one also finds important new contributions to critical mathematics education. Denival Biotto Filho (2015) addresses students in precarious situations and in particular their foregrounds. Raquel Milani (2015) and Ana Carolina Faustino (in progress) explore further the notion of dialogue, while Renato Marcone (2015) addresses the notion of inclusion–exclusion, emphasizing that we do not have to do with a straightforward good-bad duality. Inclusion could also mean an inclusion into the most questionable social practices.

Critical mathematics education is an ongoing endeavor. And naturally we have to remember that as well the very notion of critical mathematics education is contested. There are very many different educational endeavors that address critical issues in mathematics education that do not explicitly refer to critical mathematics education. And this is exactly as it should be as the concerns of critical mathematics cannot be limited by choice of terminology.

## Cross-References

► Critical Thinking in Mathematics Education
► Dialogic Teaching and Learning in Mathematics Education
► Mathematical Literacy
► Mathematization as Social Process

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C

# Critical Thinking in Mathematics Education

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**Keywords**
Logical thinking · Argumentation · Deductive reasoning · Mathematical problem solving · Critique · Mathematical literacy · Critical judgment · Goals of mathematics education

## Definition

Mainstream educational psychologists view critical thinking (CT) as the strategic use of a set of reasoning skills for developing a form of reflective thinking that ultimately optimizes itself, including a commitment to using its outcomes as a basis for decision-making and problem solving. In such descriptions, CT is established as a general methodological standard for making judgments and decisions. Accordingly, some authors also include a sense for fairness and the assessment of practical consequences of decisions as characteristics (e.g., Paul and Elder 2001). This conception assumes rational, autonomous subjects who share a common frame of reference for representation of facts and ideas, for their communication, as well as for appropriate (morally "good") action. Important is the difference as to what extent a critical examination of the criteria for CT is included in the definition: If education for CT is conceptualized as instilling a belief in a more or less fixed and shared system of skills and criteria for judgment and associated values, then it seems to contradict its very goal. If, on the other hand, education for CT aims at overcoming potentially limiting frames of reference, then it needs to allow for transcending the very criteria assumed for legitimate "critical" judgment. The dimension of not following rules and developing a fantasy for alternatives connects CT with creativity and change. In Asian traditions derived from the Mãdhyamika Buddhist philosophy, critical deconstruction is a method of examining possible alternative standpoints on an issue, which might amount to finding self-contradictions in all of them (Fenner 1994). When combined with meditation, the deconstruction provides for the student a path toward spiritual insight as it amounts to a freeing from any form of dogmatism. This position coincides with some postmodern critiques of purely intellectual perspectives that lack contact with experience and is echoed in some European traditions of skepticism (Garfield 1990). Hence, paradoxical deconstruction appears more radical than CT as it includes overcoming the methods and frames of reference of previous thinking and of purely intellectual plausibility.

## Introduction

The role assigned to CT in mathematics education includes CT as a by-product of mathematics learning, as an explicit goal of mathematics education, as a condition for mathematical problem solving, as well as critical engagement with issues of social, political, and environmental relevance by means of mathematical modeling and statistics. Such engagement can include a critique of the very role mathematics plays in these contexts. In the mathematics education literature, explicit reference to CT as defined in educational psychology or philosophy is not very widespread, but general mathematical problem-solving and mathematical reasoning are commonly associated with critical thinking, even though such association remains under-theorized. On the other hand, the notion of critique, rather than CT, is employed in the mathematics education literature in various programs related to critical mathematics education. In these programs, the adjective "critical" is used to modify "mathematics (education)" rather than "thinking."

### Critical Thinking and Mathematical Reasoning

Mathematical argumentation features prominently as an example of disciplined reasoning based on clear and concise language, questioning of assumptions, and appreciation of logical inference for deriving conclusions. These features of mathematical reasoning have been contrasted with intuition, associative reasoning, justification by example, or induction from observation. While the latter are also important aspects of mathematical inquiry, a focus on logic is directed toward extinguishing subjective elements from judgments, and it is the essence of deductive reasoning. Underpinned by the values of rationalism and objectivity, reasoning with an emphasis on logical inference is opposed to intuition and epiphany as a source of knowledge and viewed as the counterinsurance against blind habit, dogmatism, and opportunism.

The enhancement of students' general reasoning capacity has for quite some time been seen as a by-product of engagement with mathematics. Francis Bacon (1605), for example, wrote that it would "remedy and cure many defects in the wit and faculties intellectual. For if the wit be too dull, they [the mathematics] sharpen it; if too wandering, they fix it; if too inherent in the sense, they abstract it" (VIII (2)). Even though this promotion of mathematics education is based on its alleged value for developing generic thinking or reasoning skills, these skills are in fact not called "critical thinking." Historically, the notion of critique was tied to the tradition of historic, esthetic, and rhetoric interpretation and evaluation of texts. Only through the expansion of the function of critique toward general enlightenment, critique became a generic figure of thinking, arguing, and reasoning. This more general notion, however, transcends what is usually associated with accuracy and rigor in mathematical reasoning. Accordingly, CT in mathematics education not only is conceptualized as evaluating rigor in definitions and logical consistency of arguments but also includes attention to informal logic and heuristics, to the point of identifying problem-solving skills with CT (e.g., O'Daffer and Thomquist 1993). Applebaum and Leikin (2007), for example, see the faculty of recognizing contradictory information and inconsistent data in mathematics tasks as a demonstration of CT. However, as most notions of CT include an awareness of the subject doing it, neither a mere application of logical inference nor successful application of mathematical problem-solving skills would reasonably be labeled as CT. But as a consequence of often identifying CT with general mathematical reasoning processes embedded in mathematical problem solving, there is a large overlap of literature on mathematical reasoning, problem solving, and CT.

There is agreement that CT does not automatically emerge as a by-product of any mathematics curriculum but only with a pedagogy that draws on students' contributions and affords processes of reasoning and questioning when students collectively engage in intellectually challenging tasks. Fawcett (1938), for example, suggested that teachers (in geometry instruction) should make use of students' disposition for critical thinking and that this capacity can be harnessed and cultivated by an appropriate choice of pedagogy. Reflective thinking practices could be enacted when drawing the students' attention to the need for clear definition of key terms in statements, for examination of alleged evidence, for exposition of assumptions behind their beliefs, and for evaluation of arguments and conclusions. Fawcett's teaching experiments included the critical examination of everyday notions. A more recent example of a pedagogical approach with a focus on argumentation is the organization of a "scientific debate" in the mathematics classroom (Legrand 2001), where students in an open discussion defend their own ideas about a conjecture, which may be prepared by the teacher or emerge spontaneously during class work. Notably, in these examples CT in mathematics education is developed as a social activity.

While cultivating some form of discipline-transcending CT has long been promoted by mathematics educators, explicit reference to CT is not very common in official mathematics curriculum documents internationally. For example, "critical thinking" is not mentioned in the US Common Core Standards for Mathematics (Common Core State Standards Initiative 2010). However, in older recommendations from the US National Council of Teachers of Mathematics, mention of "critical thinking" is made in relation to creating a classroom atmosphere that fosters it (NCTM 1989). A comparative analysis of associations made between mathematics education and CT in international curriculum documents remains a research desideratum.

Notions of CT in mathematics education with a focus on argumentation and reasoning skills have in common that the critical competence they promote is directed toward claims, statements, hypotheses, or theories ("texts") but do include neither a critique of the social realities, in which these texts are produced, nor a critique of the categories, in which these texts describe realities. As it is about learning how to think, but not what to think about, this notion of CT can be taken to implicate a form of thinking without emotional or moral commitment. However, the perspective includes the idea that the same principles that guide critical scientific inquiry could also guide successful problem solving in social and moral matters and this would lead to improvement of society, an idea that was, for example, shared by Dewey (Stallman 2003). Education for CT is then by its nature emancipatory.

## Critical Thinking and Applications of Mathematics

For those who see dogmatic adherence to the standards of hypothetical-deductive reasoning as limiting, the enculturation of students into a form of CT derived from these standards alone cannot be emancipatory. Such a view is based on a critique of Enlightenment's scientific image of the world. The critique provided by the philosophers of the Frankfurt School is taken up in various projects of critical mathematics education and critical mathematical literacy. This critique is based on the argument that useful things are conflated with calculable things and thus formal reasoning based on quantification, which is made possible through the use of mathematics, is purely instrumental reasoning. Mathematics educators have pointed out that reliance on mathematical models implicates a particular worldview and mathematics education should widen its perspective and take critically into account ethical and social dimensions (e.g., Steiner 1988). In order to cultivate CT in the mathematics classroom, reflection not only of methodological standards of mathematical models but also of the nature of these standards themselves, as well as of the larger social contexts within which mathematical models are used, has been suggested (e.g., Skovsmose 1989; Keitel et al. 1993; Jablonka 1997; Appelbaum and Davila 2009; Fish and Persaud 2012). Such a view is based on acknowledging the interested nature of any application of mathematics. This is not to dismiss rational inquiry; it rather aims at expanding rationality beyond instrumentality through inclusion of moral and political thought. Such an expansion is seen as necessary by those who see purely formally defined CT as ultimately self-destructive and hence not emancipatory.

### Limitations of Developing CT Through Mathematics Education

The take-up of poststructuralist and psychoanalytic theories by mathematics educators has afforded contributions that hold CT up for scrutiny. Based on the postmodern acknowledgment that all forms of reasoning are only legitimized through the power of some groups in society and in line with critics who see applied mathematics as the essence of instrumental reason, an enculturation of students into a form of CT embedded in mathematical reasoning must be seen as disempowering. As it excludes imagination, fantasy, emotion, and the particular and metaphoric content of problems, this form of CT is seen as antithetical to political thinking or social commitment (Walkerdine 1988; Pimm 1990; Walshaw 2003; Ernest 2010; see also Straehler-Pohl et al. 2017). Hence, the point has been made that mathematics education, if conceptualized as enculturation into dispassionate reason and analysis, limits critique rather than affording it and might lead to political apathy.

### Further Unresolved Issues

Engaging students in collaborative CT and reasoning in mathematics classrooms assumes some kind of an ideal democratic classroom environment, in which students are communicating freely. However, classrooms can hardly be seen as ideal speech communities. Depending on their backgrounds and educational biographies, students will not be equally able to express their thoughts and not all will be guaranteed an audience. Further, the teacher usually has the authority to phrase the questions for discussion and, as a representative of the institution, has the obligation to assess students' contributions. Thus, even if a will to cultivate some form of critical reasoning in the mathematics classroom might be shared among mathematics educators, more attention to the social, cultural, and institutional conditions under which this is supposed to take place needs to be provided by those who frame CT as an offshoot of mathematical reasoning. Further, taxonomies of CT skills, phrased as metacognitive activities, run the risk of suggesting to treat these explicitly as learning objectives, including the assessment of the extent to which individual students use them. Such a didactical reification of CT into measurable learning outcomes implicates a form of dogmatism and contradicts the very notion of CT.

The antithetical character of the views of what it means to be critical held by those who see CT as a mere habit of thought that can be cultivated through mathematical problem solving, on the one hand, and mathematics educators inspired by critical theory and critical pedagogy, on the other hand, needs further exploration.

Attempts to describe universal elements of critical reasoning, which are neither domain nor context specific, reflect the idea of rationality itself, the standards of which are viewed by many as best modeled by mathematical and scientific inquiry. The extent to which this conception of rationality is culturally biased and implicitly devalues other "rationalities" has been discussed by mathematics educators, but the implications for mathematics education remain under-theorized.

## Cross-References


- ▶ Argumentation in Mathematics Education
- ▶ Authority and Mathematics Education
- ▶ Critical Mathematics Education
- ▶ Dialogic Teaching and Learning in Mathematics Education
- ▶ Didactic Contract in Mathematics Education
- ▶ Logic in Mathematics Education
- ▶ Mathematical Modelling and Applications in Education
- ▶ Mathematical Proof, Argumentation, and Reasoning
- ▶ Mathematization as Social Process
- ▶ Metacognition
- ▶ Problem-Solving in Mathematics Education
- ▶ Questioning in Mathematics Education


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# Cultural Diversity in Mathematics Education

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Cultural identity and learning · Vygotsky · Home and school mathematics · Immigrant students · Minority students · Cultural discontinuity · Sociocultural approaches

## Introduction

Cultural diversity in mathematics education is a widely used expression to discuss questions around why students from different cultural, ethnic, social, economic, and linguistic groups perform differently in their school mathematics. These questions are not new in cultural perspectives to mathematics education developed since the late 1980s (Bishop 1988) and in cultural approaches to mathematical cognition (Cole 1996). However, until recently issues of cultural diversity were considered to be out there in other non-Western cultures or to be issues of marginalized and poor groups in society. Globalization changed this perspective. With changes in communication, technologies, and unprecedented levels of migration, cultures have become increasingly complex, connected, and heterogeneous. One of the major impacts on education has been a substantial change in the cultural and ethnic composition of the school population.

Schools and classrooms become places where teachers, students, and parents are exposed to and have to respond to many types of cultural differences. For many these differences are resources enriching the learning opportunities and environments. For many others, diversity is experienced as a problem, which is reflected in school achievement (Secada 1995). The issues cultural diversity poses to education have many facets and have been approached from different perspectives in social sciences (De Haan and Elbers 2008). Conceptions of culture and the role of culture in psychological development inform these perspectives. Examining culture as a way of life of specific cultural groups has contributed to the understanding of cultural discontinuities between schools and the home background of the students. In this perspective, the emphasis has been on the shared cultural practices of the group. A more recent perspective focuses on more dynamic aspects of culture, i.e., on the way a person experiences participation in multiple practices, and the production of new cultural knowledge, meaning, and identities. Mathematics education research draws on these perspectives but also considers issues that are specific to mathematics learning (Cobb and Hodge 2002; Nasir and Cobb 2007; de Abreu 2008; Gorgorió and de Abreu 2009).

Here the focus is on the development of ideas that examine mathematics as a form of cultural knowledge (Bishop 1988; Asher 2008) and learning as a socioculturally mediated process (Vygotsky 1978). These ideas offer a critique to approaches that locate the sources of diversity in the autonomous individual mind. More importantly, sociocultural approaches have contributed to rethinking cultural diversity as “relational” and “multilayered” phenomena, which can be studied from different angles (Cobb and Hodge 2002; De Haan and Elbers 2008). Empirical research following these approaches has evolved from an examination of diversity between cultural groups, i.e., the nature of mathematical knowledge specific to cultural practices, to an examination of the person as a participant in specific sociocultural practices.

## Diversity and Uses of Cultural Mathematical Tools

A driving force for researching the impact of cultural diversity in mathematics education has been to understand why certain cultural groups experience difficulties in school mathematics. In the culture-free view of mathematics, poor performance in school mathematics was explained in terms of deficits, namely, cognitive deficits that could be the result of cultural deficits. However, since the 1980s, this view has become untenable. Researchers exploring the difficulties non-Western children, such as the Kpelle children in Liberia, experienced with Western-like mathematics introduced with schooling (Cole 1996) realized that their difficulties could not be explained by cognitive deficits or cultural deficits. They discovered that differences in mathematical thinking could be linked to the tools used as mediators. Thus, for instance, the performance in a mathematical task, such as estimating length, was linked to the use of a specific cultural measuring system. With the advance of cultural research and the view of mathematics and cognition as cultural phenomena, alternative explanations of poor performance in school mathematics have been put forward in terms of cultural differences.

Drawing on the insights from examining the mathematics of particular cultural groups research moved to explore cultural differences within societies, which is still the major focus of current research on cultural diversity in mathematics education. A classic example of this research is the "street mathematics" investigations in Brazil by Nunes et al. (1993). In a series of studies that started with street children, Nunes and her colleagues examined differences between school mathematics and out-of-school mathematics. Their findings added support to the notion that mathematical thinking was mediated by cultural tools, such as oral and written arithmetic. The within society studies also highlighted the situated nature of mathematical cognition. Depending on the context of the practice, the same person may draw on different cultural tools; they can call on an oral method to solve a shopping problem and a written method to solve a school problem.

How cultural tools mediate mathematical thinking and learning continues to be a key aspect in investigations in culturally diverse classrooms. Research with minority and immigrant students in different countries shows that the students learned often to use different forms of mathematics at home and at school (Bishop 2002; Gorgorió et al. 2002; de Abreu 2008). Similarly, research with parents shows that they refer often to differences in their methods and the ones their children are being taught in school. To sum up, research shows that students from culturally diverse backgrounds are exposed often to different cultural tools in different contexts of mathematical practices. It also suggests that many students experience cultural discontinuities in their transitions between contexts of mathematical practices. A cultural discontinuity perspective offers only a partial account of the impact of diversity, however. The fact that students from similar home cultural groups perform differently at school requires research to consider other aspects of diversity. A fruitful way of continuing to explore the different impacts of diversity in school mathematical learning focuses on how the person as a participant in mathematical practices makes sense of their experiences. The person here can be, for example, an immigrant student in a mathematics classroom, a parent that supports their children with their school homework, and a teacher that is confronted with students from cultural backgrounds they are not familiar with. Here the focus turns to culture as being reconstructed in contexts of practices, and issues of identity and social representations are foregrounded.

## Diversity and Cultural and Mathematical Identities

Many studies with immigrant and minority students have now illustrated that they become aware of the differences between their home culture and their school practices (Bishop 2002; see also ▶ "Immigrant Students in Mathematics Education"). Accounts from parents of their experiences of supporting their children's school mathematics at home (e.g., homework) also illustrate the salience of differences between home and school mathematics. These could be experienced in terms of (a) the content of school mathematics and in the strategies used for calculations, (b) the methods of teaching and the tools used in teaching (e.g., methods for learning times tables, use of calculators), (c) the language in which they learned and felt confident doing mathematics, and (d) the parents' and the children's school mathematical identities. Though all the dimensions are important, this research shows that identities take a priority in the way the parents organize their practices to support their children. The societal and institutional valorization of mathematical practices plays a role on this process (de Abreu 2008).

Recent studies also show that students talk about differences in relation to how they perceive their home cultural identities as intersecting with their school mathematical learning. Studies with students from minority ethnic backgrounds in England whose parents had been schooled in other countries show that differences between school mathematical practices at home and at school have implications on their mathematical identities. For example, some students report trying to separate home and school, i.e., to use the "home way" at home and the "school way" at school. The reason provided for the separation is that they do not feel that the home ways are valued at school. Other students simply claim that their parents do not know or that their knowledge is old fashioned. In both cases, the construction of a positive school mathematical identity involves suppressing the home mathematical identity (Crafter and de Abreu 2010). Identities, as socially constructed, can then be conceptualized as powerful mediators in the way diversities are being constructed in the context of school practices. Indeed, studies examining other types of diversity, such as gender, have also implied similar processes (Boaler 2007).

Studies with immigrant students with a history of success in their school mathematical learning in their home country are also particularly interesting to illustrate the intersection of identities. Firstly, the difficulties of these students cannot be easily attributed to the individual mathematical ability as they have a personal history of being "good mathematics students." Secondly, in this case the cultural diversity is already internalized as part of the student's previous schooling. These students' positive school mathematical identities get disrupted when they receive low grades in the host country school mathematics. Suddenly, the students' common representation that mathematics is just about numbers and formulae and that these are the same everywhere is challenged. It is revealing that young people from different immigrant backgrounds and going to school in different countries report similar experiences (e.g., Portuguese students in England; Ecuadorian students in Catalonia, Spain). This can be interpreted as evidence that when a student joins a mathematical classroom in a new cultural context, their participation is mediated by representations of what counts as mathematical knowledge. These examples illustrate a culture-free view of mathematics that is still predominant in many educational systems but that could be detrimental to immigrant students' academic mathematical careers. Having shown that issues of diversity are very salient in the experiences of students and their parents, the next section briefly examines teachers' representations.

## Diversity and Teachers' Social Representations of Cultural Differences

In many schools, teachers, who have trained to teach monolingual and monocultural students from their own culture, teach students who may speak a different language and come from cultures they are not familiar with. However, in communities with a stronger tradition of receiving immigrants, some teachers themselves have already had to negotiate the practices of the home and school culture. This complex situation may add insight into the ways that cultural differences and identities come to be constructed as significant for the school mathematical learning. An examination of studies carried out in culturally diverse schools in Europe reveals two views in the way teachers make sense of the cultural and ethnic background on their students' mathematical learning (de Abreu and Cline 2007; Gorgorió and de Abreu 2009). One view stresses "playing down differences" and the other "accepting differences." The view of playing down cultural differences draws upon representations of mathematics as a culture-free subject (that it is the same around the world). This view can also draw on a representation of the child's ability as the key determinant factor in their mathematical learning. The universal construction of children takes priority over their ethnic and cultural backgrounds. Treating everyone as equal based on their merits is also used as a justification for not taking into account cultural differences. The lack of recognition of the cultural nature of mathematical practices may restrict opportunities for students to openly negotiate the differences at school. This way, diversity may become a problem instead of a resource. The alternative positioning of accepting cultural differences represents a minority voice outside the consensus that mathematics is a culture-free subject and that ability is the main factor in the mathematical learning.

## Conclusion

Diversity in mathematics education includes complex and multilayered phenomena that can be explored from different perspectives. Drawing on sociocultural psychology, empirical research on uses and learning of mathematics in different cultural practices offered key insights on understandings of cultural diversity considering (i) mathematical tools (the specific forms of mathematical knowledge associated with cultural groups and sociocultural practices), (ii) identities (the ways differences are experienced by the students and the impact on how they construct themselves as participants in these practices), and (iii) social representations (the images and understandings that enable people to make sense of mathematical practices, such as images of learners and the learning process and views of mathematical knowledge). These understandings emerged from looking at diversity from complementary perspectives. One perspective focuses on the discontinuities between the cultural practices, and the other on how discontinuity is experienced by the person as a participant in school mathematical practices. This second perspective is more recent and is key for the development of approaches where diversity becomes a resource. The extent to which approaches that stress the importance of cultural identities can be used as resources for change from culture-free to culturally sensitive practices in mathematics education is a question for further research. The fact that the views of cultural identities as mediators of school mathematical learning are still marginalized can be seen as a consequence of the dominant cultural practices and representations. For example, this can include practices in teacher training, where little attention is given to preparing teachers to understand the cultural nature of (mathematical) learning and human development (see also, ► "Immigrant Students in Mathematics Education"). Secondly, implicit conceptions of the social and emotional development of the child at school draw on representations of childhood which often do not take into account the cultural diversity of current societies.

## Cross-References

► Ethnomathematics
► Immigrant Students in Mathematics Education
► Situated Cognition in Mathematics Education
► Theories of Learning Mathematics

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# Cultural Influences in Mathematics Education

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Access · Belonging · Barriers · Context · Culture · Equity

## Definition

Extensive educational scholarship investigates why different demographic groups of students are less successful than others; much of that scholarship has focused on characteristics of the students themselves, for example, their motivation, affect, attitudes, preparation, and ability. In this chapter, we turn from characteristics of students to identify some ways that cultural and societal contexts surrounding schooling and mathematics affect opportunities and performance for groups of people who have traditionally been underrepresented in mathematics. Understanding these contexts and the constraints they impose on some students is crucial for the development of strategies to create more accessible and equitable learning environments.

> Miners used to take a canary into the mines to signal whether or not the air was safe to breath. If the canary thrived, the atmosphere was safe. If the canary became sick or died, the atmosphere was toxic. Members of oppressed groups – people of color, poor and working classes, women, gays, bisexuals, and lesbians – are like the canary: They signal when the atmosphere is not healthy. . .. Trying to "fix" the canary or blaming the toxic atmosphere on the canary makes the atmosphere no less toxic to everyone in it. (Weber 2001, p. 22)

## Introduction

School mathematics can serve as a barrier or a catalyst for further educational and career opportunities. A substantial body of research has explored the reasons for the differences in the achievement, attitudes, learning styles, strategy use, and persistence between girls and boys and among students of different races, ethnicities, social classes, and language proficiencies (e.g., Leder 1992; OECD 2015; Tate 1997). Although gaps have gotten narrower, differences among groups remain, as do important differences among countries (Else-Quest et al. 2010; Lubienski and Ganley 2017). Ironically, the work of many researchers has had the paradoxical effect of creating a discourse that females and students of color cannot do math (Boaler and Sengupta-Irving 2006; Fennema 2000; O'Connor and Joffe 2014). This *deficit model* stereotypes some groups of students as defective and in need of repair, and the goal becomes developing interventions to fix the students who are less successful. As a result, when students do not succeed or persist in mathematics, the reason is framed as a problem with the students themselves, rather than as the result of broader social or cultural issues (e.g., Sheldon et al. 2016).

While research in mathematics education identifies some features and behaviors of students – for example, ability, persistence, and affect – that can affect success, it has also become clear that success in school mathematics is influenced by far more than characteristics of the students themselves (Herzig 2004a, b; Lubienski and Ganley 2017). Some scholars have looked beyond characteristics of students to describe political, economic, social, and cultural contexts in which education is situated and how those contexts affect who succeeds (Apple 1992; Else-Quest et al. 2010; Gutiérrez 2013; Martin et al. 2017; Tate 1997).

In this essay, we examine social and cultural barriers, both within and surrounding mathematics, that affect who succeeds in mathematics, including (1) features of mathematics as it is represented in classrooms and (2) the way the broader society perceives mathematics, mathematical ability, and the students who succeed in math.

## Features of Mathematics

Mathematics is often perceived, by both teachers and students, as a set of manipulations that lead to predetermined results or, at a more advanced level, as sequence of deductive proofs of clearly stated theorems. This abstraction of mathematics has little or no explicit connection to other mathematical ideas, ideas outside of mathematics, or the mathematical "big picture" (Herzig 2002, 2004b; Stage and Maple 1996). Some feminist scholars have challenged the predominance of abstraction in mathematics, arguing that abstraction in mathematics is a consequence of modern industrial society, which is based on the idea of separating things into manageable pieces, distinct from their context (Johnston 1995). This abstraction of mathematics denies the social nature of mathematics. In an abstract context like the one that is common in Western school mathematics, a quest for certain types of understanding can actually interfere with success, as when students look to understand, for example, What does this have to do with the world? With my world? With my life? (Johnston 1995). Of course, intuition, creativity, insight, and even trial-and-error give rise to important mathematics as well, and give meaning to the results (Burton 1999; Herzig 2002). Applications of mathematics are often included merely as demonstrations rather than as the meaning of mathematics itself. Also omitted are the political, economic, social, and personal contexts and applications, and the esthetics of mathematics that have inspired mathematicians, musicians, and visual artists (Montano 2014).

## Perceptions of Mathematical Success

Building students' sense of belongingness and engagement with mathematics has been proposed as a critical feature of an equitable education (Allexsaht-Snider and Hart 2001; Darragh 2013; Herzig 2002; Ladson-Billings 1997; Tate 1995). Allexsaht-Snider and Hart (2001) define belonging as "the extent to which each student senses that she or he belongs as an important and active participant" in mathematics (p. 97). A similar construct has been proposed at the post-secondary level, with several authors arguing belonging in the communities of practice of mathematics is important for student success and persistence (Herzig 2002, 2004a; Solomon 2007).

The way that mathematics students are perceived outside the classroom also affects students' involvement and sense of belonging in mathematics (Campbell 1995; Damarin 2000). Noddings (1996) argued that

> There seems to be something about [mathematics] or the way it is taught that attracts a significant number of young people with underdeveloped social skills.... If this impression of students who excel at math is inaccurate, researchers ought to produce evidence to dispel the notion, and teachers should help students to reject it. If it is true, math researchers and teachers should work even harder to make the "math crowd" more socially adept. Because that group so often tends to be exclusive, girls and minority youngsters may wonder whether they could ever be a part of it. But when the group is examined from a social perspective, many talented

> young people may question whether they *want* to be a part of it. (p. 611; italics in original)

As Noddings (1996) argued, mathematics educators need to find ways to make the social world of mathematics – its culture – more accessible to a broader range of people, and the world outside of mathematics needs to change its perception of those who succeed within it. Only then can more students, including females and people of color, find a way come to feel that they truly belong in some part of the mathematics world.

Damarin (2000) compared people with mathematical ability to "marked categories" such as women, people of color, criminals, people of disability, and people who identify as LGBTQ, and identified these characteristics:

1. Members of marked categories are ridiculed and maligned, and descriptions of marked categories are used to harass, tease, and discipline members of the larger society.
2. Members of marked categories are portrayed as incompetent in dealing with daily life.
3. In institutions designed to meet the needs of all, the needs of members of marked categories are deferred to the needs of the members of unmarked categories.
4. Members of marked categories are feared as powerful even as they are marked as powerless.
5. Explicit or social marking serves to define communities of the marked.
6. Membership in multiple marked categories places individuals in the margins of each marked community.
7. The study of a marked category leads to the construction and study of the complementary class of people.
8. The unmarked category is generally larger than the marked category; even when this is not the case, the marked category is not recognized as the majority (Damarin 2000, pp. 72–74).

Damarin then presents an analysis of discourses surrounding mathematical ability and concludes:

> From leading journals of pubic intellectual discussion, from the analyses of sociologists of science, from the work of (genetic) scientists themselves, from the pages of daily papers, and from practices of students and adults within the wall[s] of our schools, there emerges and coalesces a discourse of mathematics ability as marking a form of deviance and the mathematically able as a category marked by the signs of this deviance. (p. 78)

Given the common perceptions of mathematics students as being white, male, childless, and socially inept, having few interests outside of mathematics, students who explicitly do *not* fit this description might conclude that they do not *wish* to fit in. Thus belonging in mathematics might not be an entirely good thing, as it "marks" a student as deviant and as socially inept. Herzig (2004b) found that some female graduate students described ways that they worked to distance themselves from some of these common constructions of ineptness and social deviance, which, paradoxically, led them to resist belonging in mathematics.

Damarin (2000) argued that membership in the deviant category provides the "deviant" with a community with which to affiliate: Being identified and marked as mathematically able encourages mathematics students to form a community among themselves – if there are enough of them and if they have the social facility needed. Unfortunately, females are members of (at least) two marked categories, and the double marking is not merely additive: That is, females are constructed as deviant as females separately within each marked category in which they are placed. Within mathematics, they are marked as females, but among females, their mathematical ability defines them as deviant. In particular, given common stereotypes of mathematics as a male domain, mathematical women are marked among mathematicians as not actually being mathematicians. For women of color, the marking is three-fold and even more complex, making women of color "deviant" within each of the communities to which they belong.

Researchers have described the phenomenon of *stereotype threat* (Steele and Aronson 1995), in which student achievement tends to mimic stereotypes (Hill et al. 2010; Nguyen and Ryan 2008). For example, female students who are reminded before a test of the stereotype that females perform more poorly than males, perform worse than those who did not receive a reminder (Spencer et al. 1999). In their review of research on gender in mathematics, Lubienski and Ganley (2017) cite conflicting evidence of the effect of stereotype threat on gender differences in mathematics, but theorize that more nuanced research may reveal ways in which stereotype threat affect specific populations or in specific contexts.

## Summary

Educational scholarship has made great strides in understanding why mathematics has generally attracted certain types of students. Rather than studying what is different about women and minorities – groups that have typically been viewed as unsuccessful in mathematics – some scholarship now acknowledges and investigates cultural and societal contexts affecting the opportunities and performance for groups of people who have traditionally been underrepresented in mathematics. In addition, the literature has shown that students are most engaged in an educational environment that fosters belonging, which can be difficult for some students. The stereotypical views of mathematics students can make it particularly challenging for women and minorities to succeed. The mathematically capable may not wish to be socially or culturally marked as such due to common preconceived notions of mathematics students. However, by understanding the cultural and societal issues in mathematics learning, researchers and educators can begin to implement policies and strategies to create more accessible and equitable learning environments and atmospheres.

## Cross-References

- ▶ Ability Grouping in Mathematics Classrooms
- ▶ Cultural Diversity in Mathematics Education
- ▶ Engagement with Mathematics
- ▶ Epistemological Obstacles in Mathematics Education
- ▶ Equity and Access in Mathematics Education
- ▶ Gender in Mathematics Education
- ▶ Mathematics Learner Identity

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# Curriculum Resources and Textbooks in Mathematics Education

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**Keywords**
Curriculum resources · Digital curriculum resources · ICT · Internet · Professional development · Teacher knowledge · Teacher design · Textbooks · Use of curriculum resources

Typically, curriculum resources including textbooks are seen to reside at the interface between policy and practice (e.g., Valverde et al. 2002), as they translate policy (the intended curriculum) into practice (the enacted curriculum). More recently mathematics teachers increasingly rely on digital resources to prepare their lessons and to design their mathematics curriculum, and students use such resources in class and to complement their courses. These materials are said to become key tools for teachers; as in many countries (e.g., France, the Netherlands, the United Kingdom, the United States), teachers are increasingly encouraged to (re) design the curriculum in planning their instruction.

In the next section we define curriculum resources; in particular we distinguish digital curriculum resources from educational technology. In the subsequent section, we discuss the design and "use" of mathematics curriculum resources by teachers (and students). In the last section, we develop further perspectives.

## Definition of Curriculum Resources

We define mathematics curriculum resources as all the material resources that are developed and used by teachers and students in their interaction with mathematics in/for teaching and learning, inside and outside the classroom. Hence, curriculum resources would include the following:

Text resources (e.g., textbooks, teacher curricular guidelines, websites, worksheets, syllabi, tests)
Other material resources (e.g., manipulatives, calculators)
Digital-/ICT-based curriculum resources (e.g., interactive e-textbooks)

Leaning on work by Pepin et al. (2017a), we distinguish digital curriculum resources including e-textbooks, from instructional technology (e.g., digital geometry software), in the sense that:

> It is the attention to sequencing—of grade-, or age-level learning topics, or of content associated with a particular course of study (e.g., algebra)—so as to cover (all or part of) a curriculum specification, which differentiates Digital Curriculum Resources from other types of digital instructional tools or educational software programmes. . . . Of course, Digital Curriculum Resources make use of these other types of tool and software: indeed, what differentiates them from pre-digital curriculum programmes is that they are made accessible on electronic devices and that they often incorporate the dynamic features of digital technologies. (p. 647)

Seen this way, it makes the study of curriculum resources, whether digital or non-digital, and student and teacher interaction with such resources, a crucial ingredient of teacher education and professional development.

There are other "nonmaterial" resources used by teachers to design their curriculum, for example, social resources (e.g., direct and/or web-based conversations with colleagues) and cognitive resources (e.g., conceptual frames that are used, for example, in professional development sessions to develop particular competencies). These two further categories are not addressed in this text.

## Design and "Use" of Mathematics Curriculum Resources

In this section we provide a condensed overview of the relevant issues and literature organized under two headings: (1) research about the design and the quality of curriculum resources and (2) research about the use of and interaction with resources, including their adaptation and transformation by users, in particular teachers.

### Design and Quality of Curriculum Resources

In terms of "text/paper" curriculum resources and textbooks, Fan et al. (2013) have developed a framework for classifying the literature in textbook research. They identified four categories, among them "textbook analysis and comparison" (p. 635). This category makes up 34% of empirical studies on mathematics textbooks in their survey ($n = 100$). According to this survey, textbook analyses (and comparisons) can be subdivided into five categories, i.e., studies focusing on (1) how different mathematics content or topic areas have been treated in textbooks; (2) cognition and pedagogy; (3) gender, equity, and values; (4) comparison of different textbooks internationally; and (5) methodological matters and frameworks for textbook analysis.

Leaning on the literature, we can distinguish three primary frameworks to inform the analysis of digital curriculum resources. The first is the Digital Typology created by Choppin et al. (2014), in which they outlined three categories for the analyses of digital curriculum resources: students' learning experiences, curriculum use and adaptation, and assessment systems. In the second framework, Choppin and Borys (2017) analyze digital curriculum resources in terms of four perspectives (private sector perspective, designer perspective, policy perspective, and user perspective) that inform the design, development, and dissemination of curriculum resources. In the third framework, Pepin et al. (2016) distinguish between three types of e-textbooks (according to their model of development and their functionality): integrative e-textbook, evolving or "living" e-textbook, and the interactive e-textbook.

All these studies, more or less explicitly, raise the issue of the quality of curriculum resources and in turn can be reinterpreted as contributions to quality studies (e.g., Gueudet et al. 2013). The issue of quality and evaluation is particularly developed in studies concerning digital resources, as the profusion of online resources has created a need for quality criteria. Moreover, it has become evident that quality and design issues are interrelated. Digital means lead to the development of new design modes and to new possibilities of teacher collaborative work around the design of curriculum resources. Research on curriculum resources needs to address questions, such as who are the designers and in which ways does the designer/group of designers impact on the quality of resources?

### The "Use" of Curriculum Resources

In this section, we address issues related to the "use" of resources, which include the interactions of teachers and students with resources.

We consider here the interactions between teachers or students and resources from the perspective of mediated activity. This leads to the consideration of a two-way process: (1) the resource's features influence the subject's activity and learning (for teachers, this can lead to policy choices, drawing on resources as a means for teacher education); at the same time, (2) the subject shapes his/her resources, according to his/her knowledge and beliefs. In short, the "use" of curriculum resources is recognized as a two-way interactive process (as acknowledged, e.g., in the Documentational Approach of Didactics).

Davis and Krajcik (2005) have coined the term "educational curriculum materials," emphasizing the importance of educative curriculum materials for teacher learning (in their case in science education). This is also acknowledged in mathematics education, although there is scarce research on this topic (e.g., Pepin 2018).

Considering the shaping of resources by teachers (or students), the ways teachers (or students) use, adapt, or transform the resources depend to a large extent on their knowledge and beliefs (see, e.g., Gueudet et al. 2012, or Pepin et al. 2013, or Remillard et al. 2009). The ways students "use," for example, a calculator is said to depend on their knowledge about the calculator and its affordances but also on their knowledge of the mathematics. The same holds true for textbooks: in order to find support for solving an exercise, some students will read the course materials, whereas others will search for worked examples. Similarly, two teachers will use the same textbook differently. A teacher can focus on the worksheets, or the provision of exercises, while another will consider the same book as curriculum guide. The notion of "implementation fidelity" is often used to denote that teachers align their lesson design with the textbook. At the same time, studying how the same teacher enacts the same (e.g., algebra) content of one textbook in two same grade classrooms, notable differences can be found. Thus, it can be said that curriculum resources offer personal possibilities for adaptations, and teachers have always adapted and transformed resources: selecting, changing, cutting, and rephrasing.

However, the main difference with digital resources, such as e-textbooks, is that these adaptations are technically anticipated and supported with specific technical means (Pepin et al. 2016). Considering teacher interaction with digital curriculum resources, Pepin et al. (2017b) defined mathematics *teacher design capacity* as consisting of three main aspects: (1) a clear goal orientation of the design (e.g., in terms of aims and content of learning), (2) a set of design principles/heuristics (e.g., a set of robust but flexible guidelines about how to address the design task), and (3) reflection-in-action type of understandings (e.g., the ability to collect information and adapt the initial design to circumstances during instruction). They developed this model for mathematics teacher design capacity when interacting with digital (and non-digital) curriculum resources.

In terms of interaction with digital curriculum resources, most teachers have now access to a profusion of freely available educational resources. However, teachers often find it difficult to analyze and choose from the profusion of materials available to fit their educational goals and classroom contexts. Pepin et al. (2017a) identify a number of practices/uses of digital curriculum resources, both by students and by teachers. There are at least three features that make it beneficial for teachers to work with digital curriculum resources: (1) their flexibility in terms of adaptation and redesign, for personal lesson preparation as well as collective design work with colleagues, at a distance or working together in professional development sessions; (2) the possibilities for personalization and differentiation, so as to attend to students' individual needs, for example, in providing particular tasks/activities or individual feedback on tasks; and (3) the many assessment features that allow "easy" access to different aspects of student learning.

In terms of student interaction with digital curriculum resources, we note that the interactive features of digital curriculum resources seem to be most useful with formative assessment practices, which help students (as well as teachers) to "feed forward" that is to drive the next learning (instructional) steps (e.g., Pepin et al. 2017a). At the same time, Ruthven (2018) points out that the general adaptivity of such digital resources is one of the biggest advantages, in particular with respect to personalized (diagnostic) assessment. Indeed, the adaptivity feature appears crucial for finding new pathways and sequencing of problems by students and in terms of assessment for leaving room for misunderstandings and amendments.

## Future Research Perspectives

Viewing curriculum resources as essential tools for teachers to accomplish their goals has been accepted for a long time. However, the vision of the teacher-tool relationship has changed and needs to be explored in more depth. Moreover, considering the evolution of resources available for teachers and students, this opens up new directions for research. It leads in particular (1) to view the teacher as a designer of his/her resources. Based on the interpretation of teaching as design, and teachers as designers, existing research emphasizes the vital interaction between the individuals/teachers and the tools/resources to accomplish their goals, an accomplishment inextricably linked to the use of cultural, social, and physical tools. This not only questions our conceptualization of "curriculum resources," but it also opens the door for many new avenues of researching mathematics curriculum resources and their interaction with the "learner," may it be the teacher or the student.

Linked to this, (2) it questions the nature of curriculum resources that are to be "teacher-educative." What kind(s) of curriculum resources does a group of teachers need for learning to take place? What is their nature, what are the criteria for educative curriculum resources? National policies for the design and use of curriculum resources are starting to take these evolutions into account, in particular by collecting users' comments on websites (e.g., dedicated websites for particular textbooks).

Furthermore, analyzing the quality of available resources, contributing to the design of resources (to be used by students and teachers), and proposing teacher development programs drawing on collaborative resource design and educative resources are important issues, which need to be addressed by research in mathematics education.

## Cross-References

- ▶ Communities of Inquiry in Mathematics Teacher Education
- ▶ Communities of Practice in Mathematics Education
- ▶ Cultural Diversity in Mathematics Education
- ▶ Design Research in Mathematics Education
- ▶ Didactic Transposition in Mathematics Education
- ▶ Documentational Approach to Didactics
- ▶ Information and Communication Technology (ICT) Affordances in Mathematics Education
- ▶ Instrumentation in Mathematics Education
- ▶ Learning Environments in Mathematics Education
- ▶ Manipulatives in Mathematics Education
- ▶ Mathematics Teacher Education Organization, Curriculum, and Outcomes
- ▶ Mathematics Teachers and Curricula
- ▶ Professional Learning Communities in Mathematics Education
- ▶ Teaching Practices in Digital Environments
- ▶ Technology and Curricula in Mathematics Education
- ▶ Technology Design in Mathematics Education

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