Analytical Solutions for Traveling Pulses and Wave Trains in Neural Models: Excitable and Oscillatory Regimes

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Abstract We consider a piecewise linear approximation of the diffusive Morris-Lecar model of neuronal activity, the Tonnelier-Gerstner model. Exact analytical solutions for one-dimensional excitation waves are derived. The dynamics of traveling waves is related to two basic regimes of wave propagation: excitable and oscillatory cases. In the first case we describe mathematically the structure of a solitary pulse and in the second case—the form of a periodic sequence of pulses (a periodic wave train).

Keywords Reaction-diffusion equations · Piecewise linear models · Traveling waves

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1 Introduction

Mathematical modeling of the neuronal activity is a common problem in the theoretical biophysics and neuroscience. The neurosystems show the complex spatiotemporal behavior of extended objects from diverse interacting elements. To describe such a behavior, the formalism of reaction-diffusion wave processes is usually applied. The reaction-diffusion wave processes present self-sustaining wave

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processes when there appear excitation traveling waves with constant shape and speed of propagation in an excitable (active) nonlinear medium.

In the active medium may occur three fundamental types of the reaction-diffusion wave behavior: bistable, excitable, and oscillatory regimes. The bistable regime is characterized by the appearance of a switching wave (a front). The excitable regime exhibits a single pulse wave. In the oscillatory regime there exists a periodic pulse sequence (a wave train). All these types of nonlinear traveling waves (fronts, pulses, and wave trains) have been intensely studied. Such nonlinear waves arise, propagate, and interact each other and with boundary in active media and are described mathematically with related nonlinear equations in partial derivatives of the reaction-diffusion type. Diverse wave patterns may appear due to spontaneous formation of wave sources as well as dissipative structures under conditions when traveling waves do not collide but move away from each other. The developing pattern depends on conditions and may exist in different states characterized by the presence of one as well as several reaction-diffusion wave processes.

One of the well-known models of the reaction-diffusion systems includes two equations, with a cubic nonlinearity in the first one and a linear reaction term in the second one. The equations of this system were proposed by FitzHugh [1] and Nagumo et al. [2]. This FitzHugh-Nagumo (FHN) model is also referred to as the Bonhoeffer-van der Pol model. It was originally presented as a simplification of the Hodgkin-Huxley equations describing the propagation of an action potential along nerve fibers. The FHN model is described by the reaction-diffusion equations

$$\frac{\partial u}{\partial t} = u(1-u)(u-a) - v + D_u \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

$$\frac{\partial v}{\partial t} = \varepsilon(u-v) + D_v \frac{\partial^2 v}{\partial x^2}. \tag{2}$$

The positive parameters a and ε are the excitation threshold and the ratio of time scales. The constants $D_{u,v}$ are diffusion coefficients. The variable u represents the “activator” or potential variable. It corresponds to the potential across the membrane of the nerve fiber in the original application to the Hodgkin-Huxley model. The variable v represents the “inhibitor” or recovery variable. Depending on the parameter values, all three regimes are realized in such a system.

Rinzel and Keller [3] applied a piecewise linear approximation of the McKean-type [4] for the nonlinear reaction term in the first equation and deduced an example of an analytically solvable model qualitatively well reproducing the FitzHugh-Nagumo dynamics. They wrote

$$\frac{\partial u}{\partial t} = -u - v + H(u-a) + D_u \frac{\partial^2 u}{\partial x^2}, \tag{3}$$

$$\frac{\partial v}{\partial t} = \varepsilon(u-v) + D_v \frac{\partial^2 v}{\partial x^2}, \tag{4}$$

where $H(u - a)$ is the Heaviside step function. Such approach allows to solve the equations analytically and obtain the wave solutions for u and v. Tonnelier and Gerstner [5] considered the piecewise linear version of the FHN model with the nonlinear inhibitor function of a sigmoidal type as a simplification of the Morris-Lecar [6] equations related to neuron models and proposed the corresponding modification of the Rinzel-Keller model with the Heaviside step function in both equations. This model will be used in the present research. We wish to solve the reaction-diffusion equations analytically and find the exact solutions in the form of the traveling pulses and wave trains. To the best of our knowledge, before the present study, no fully analytical solutions for the traveling pulses and wave trains in the two-component reaction-diffusion systems with piecewise linear reaction functions in both equations were available.

The method of the piecewise linear approximation has general applicability and is often the only way to study a variety of nonlinear problems analytically in an approximate fashion [7]. Piecewise linear models have been employed to use the translational invariance of equations as a speed selection mechanism [8, 9], to study the effect of transport memory [10–13] and the wave propagation in discrete [14–16] and inhomogeneous [17–19] media and to consider a forcing [20, 21]. Rinzel and Keller [3] described the pulses and the wave trains, they calculated the wave speeds and performed the stability analysis in a reaction-diffusion model with non-diffusing inhibitor. Koga [22] described wave solutions of the bistable double-diffusive piecewise linear model. He considered only the case where the activator (the first variable) diffuses faster than the inhibitor (the second variable). The linear stability analysis was performed both by Rinzel and Keller and by Koga. Ito and Ohta [23] derived a motionless localized solution and a propagating-pulse solution. The research was focused on the effect of the inhibitor diffusion. In most papers related to the piecewise linear reaction-diffusion equations, one-component [8, 10, 12, 20, 24, 25] and two-component [22, 23, 26–29] systems are investigated, i.e., such an analytical approach is important despite the existence of many numerical or seminumerical results.

2 Tonnelier-Gerstner Model

There are three wave phenomena related to traveling waves: wave formation, propagation, and interaction. Wave formation and propagation are simple processes, whereas wave interaction shows complex dynamics. In many reaction-diffusion systems, it leads to wave annihilation or wave reflection [30, 31]. These phenomena occur usually after a collision of a pair of counter-propagating waves or after a collision of a wave with no-flux boundaries of the medium. Recently [32] we have found more complex behavior at such collisions: wave reflection at a growing distance (*remote* reflection). The present research continues our preceding work and extend the investigations of traveling front dynamics [33] to the solitary pulses and the periodic wave trains. Here we develop the analytical description for the

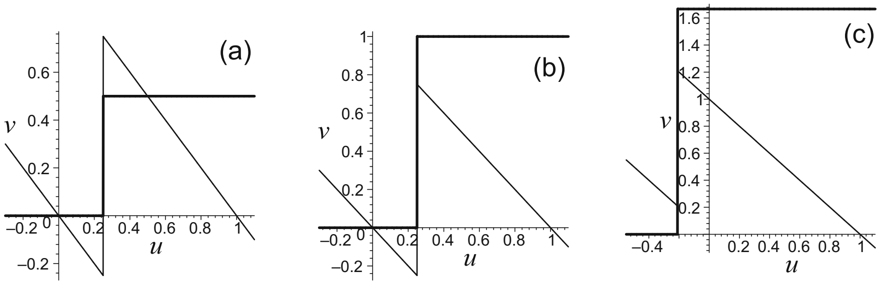

Fig. 1 Null-clines of the sigmoidal reaction-diffusion system with piecewise linear functions in (**a**) bistable, (**b**) excitable, and (**c**) oscillatory regimes. Null-cline related to the activator (the first variable) reaction function is shown by thin line, whereas to the inhibitor (the second variable) function by thick line

traveling waves in the piecewise linear reaction-diffusion system, the Tonnelier-Gerstner [5] model, which is referred also as the sigmoidal model. Wave solutions in this reaction-diffusion model depend on the intersection of null-clines, i.e., the curves plotting the equations of zero-valued reaction functions $f(u, v) = 0$ and $g(u, v) = 0$. The first regime (Fig. 1a) is bistable and the corresponding solution is a front wave (heteroclinic). The second one (Fig. 1b) is excitable and the corresponding solution is a pulse (homoclinic). In the last case (Fig. 1c), the system is in oscillatory regime and demonstrates the sequences of pulses or periodic wave trains.

Reaction-diffusion system considered here incorporates the Tonnelier-Gerstner kinetics [5] and a spatial coupling via diffusion that allows traveling wave propagation. The model is described by equations

$$\frac{\partial u}{\partial t} = -u - v + H(u - a) + D_u \frac{\partial^2 u}{\partial x^2}, \tag{5}$$

$$\frac{\partial v}{\partial t} = -\varepsilon v + \alpha H(u - a) + D_v \frac{\partial^2 v}{\partial x^2}, \tag{6}$$

where ε, α, a, and $D_{u,v}$ are positive constants.

General traveling wave ($\xi = x - ct$ is the traveling-frame coordinate and c is the wave speed) solution reads

$$u(\xi) = A_1 e^{\lambda_1 \xi} + A_2 e^{\lambda_2 \xi} + \frac{B_3}{\mu_3} e^{\lambda_3 \xi} + \frac{B_4}{\mu_4} e^{\lambda_4 \xi} + u^*, \tag{7}$$

$$v(\xi) = B_3 e^{\lambda_3 \xi} + B_4 e^{\lambda_4 \xi} + v^*, \tag{8}$$

where

$$\lambda_{1,2} = \frac{1}{2D_u}\left(-c \pm \sqrt{c^2 + 4D_u}\right), \tag{9}$$

$$\lambda_{3,4} = \frac{1}{2D_v}\left(-c \pm \sqrt{c^2 + 4D_v\varepsilon}\right) \tag{10}$$

are the eigenvalues of the characteristic equation,

$$\mu_{3,4} = c\left(1 - \frac{D_u}{D_v}\right)\lambda_{3,4} - \left(1 - \varepsilon\frac{D_u}{D_v}\right) \tag{11}$$

and u^*, v^* are constants.

We consider here two types of solutions: solitary pulses and sequences of pulses (periodic wave trains) [34].

2.1 Solitary Pulses

Solitary pulses occur in the excitable regime of the active media. Calculations show that there are two pulse waves at the fixed parameter values, the fast and slow waves, as obtained elsewhere [3], where it was found that the fast pulse is a stable solution, whereas the slow wave is unstable. The case of the pulse with oscillatory tails reproduces such a situation with non-oscillatory waves.

The pulse solution in this piecewise linear model consists of three segments, first of which vanishes as $\xi \to -\infty$ and the third too as $\xi \to +\infty$, i.e., the boundary conditions for the pulse solutions are as follows:

$$u_1(\xi \to -\infty) = 0, \quad u_3(\xi \to +\infty) = 0, \tag{12}$$

$$v_1(\xi \to -\infty) = 0, \quad v_3(\xi \to +\infty) = 0. \tag{13}$$

Since $D_{u,v}$ and ε are positive, $\lambda_{1,3} > 0$ and $\lambda_{2,4} < 0$, and the pulse solution has the form

$$u(\xi) = \begin{cases} A_{11}e^{\lambda_1\xi} + \dfrac{B_{31}}{\mu_3}e^{\lambda_3\xi}, & \xi \le \xi_0, \\ A_{12}e^{\lambda_1\xi} + A_{22}e^{\lambda_2\xi} + & \\ \dfrac{B_{32}}{\mu_3}e^{\lambda_3\xi} + \dfrac{B_{42}}{\mu_4}e^{\lambda_4\xi} + 1 - \alpha/\varepsilon, & \xi_0 \le \xi \le \xi_0^*, \\ A_{23}e^{\lambda_2\xi} + \dfrac{B_{43}}{\mu_4}e^{\lambda_4\xi}, & \xi \ge \xi_0^* \end{cases} \tag{14}$$

and

$$v(\xi) = \begin{cases} B_{31}e^{\lambda_3 \xi}, & \xi \leq \xi_0, \\ B_{32}e^{\lambda_3 \xi} + B_{42}e^{\lambda_4 \xi} + \alpha/\varepsilon, & \xi_0 \leq \xi \leq \xi_0^*, \\ B_{43}e^{\lambda_4 \xi}, & \xi \geq \xi_0^*. \end{cases} \tag{15}$$

From the continuity of the solutions and its derivative at the matching points $\xi_0 = 0$ and ξ_0^* we find matching conditions

$$\begin{gathered} u_1(\xi_0) = u_2(\xi_0), \quad u_2(\xi_0^*) = u_3(\xi_0^*), \\ \left.\frac{\mathrm{d}u_1(\xi)}{\mathrm{d}\xi}\right|_{\xi_0} = \left.\frac{\mathrm{d}u_2(\xi)}{\mathrm{d}\xi}\right|_{\xi_0}, \quad \left.\frac{\mathrm{d}u_2(\xi)}{\mathrm{d}\xi}\right|_{\xi_0^*} = \left.\frac{\mathrm{d}u_3(\xi)}{\mathrm{d}\xi}\right|_{\xi_0^*}, \\ v_1(\xi_0) = v_2(\xi_0), \quad v_2(\xi_0^*) = v_3(\xi_0^*), \\ \left.\frac{\mathrm{d}v_1(\xi)}{\mathrm{d}\xi}\right|_{\xi_0} = \left.\frac{\mathrm{d}v_2(\xi)}{\mathrm{d}\xi}\right|_{\xi_0}, \quad \left.\frac{\mathrm{d}v_2(\xi)}{\mathrm{d}\xi}\right|_{\xi_0^*} = \left.\frac{\mathrm{d}v_3(\xi)}{\mathrm{d}\xi}\right|_{\xi_0^*}, \\ u_1(\xi_0) = a, \quad u_3(\xi_0^*) = a. \end{gathered} \tag{16}$$

There are 10 equations for 10 unknowns: 4 constants A, 4 constants B, the coordinate ξ_0^* of the second matching point and the speed c; the first matching point ξ_0 may be chosen arbitrarily, usually as zero, due to the translational invariance of the model equations.

An example of traveling solitary pulses at different values of the excitation threshold a is shown in Fig. 2. We see that the pulse wave profile consists of two parts: front and back. The activator u and the inhibitor v waves have different front and back parts. The front part of activator is always monotonic, whereas the back part has a well. For the inhibitor, the front and back parts are both monotonic. When the speed of the pulse wave takes positive value, the pulse propagates from left to right, when the speed is negative, the wave travels from right to left.

2.2 *Periodic Wave Trains*

Periodic sequences of pulses can be found in the same active media where the solitary pulses occur. Contrary to the pulse case, the number of wave trains may change depending on parameter values. When the period of the wave train is varied, there appear one or several wave trains with different speeds. The diagrams for the wave speed vs. period are called dispersion relations. For wave trains with a standard

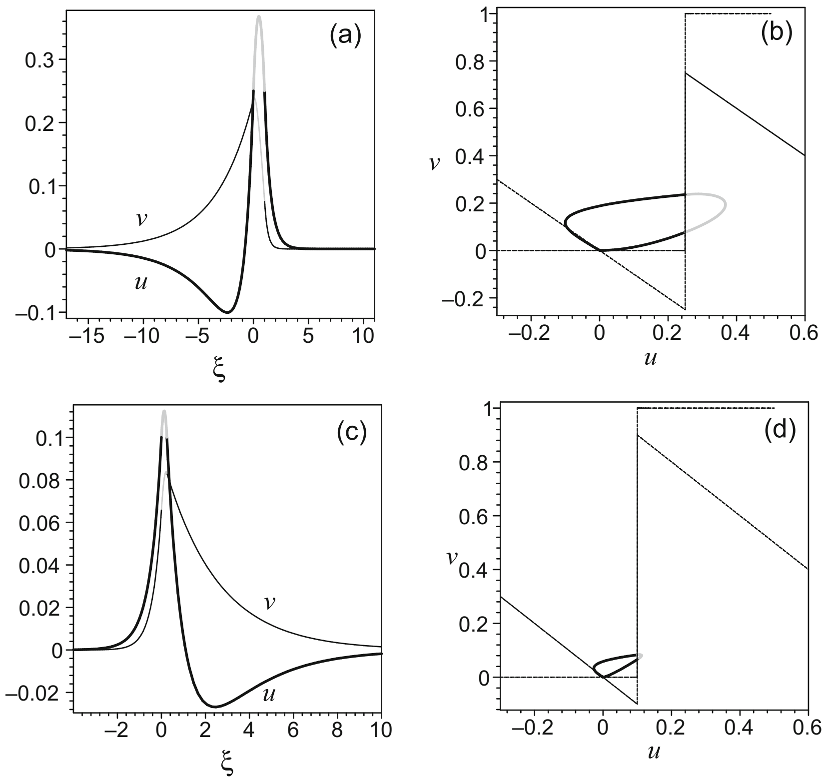

Fig. 2 An example of traveling pulses at $D_u = 0.5$, $D_v = 0.1$, $\varepsilon = 0.1$ and $\alpha = 0.1$: (**a, c**) $u = u(\xi)$ (thick line) and $v = v(\xi)$ (thin line) profiles and (**b, d**) $u - v$ diagrams. Pulse propagates with (**a, b**) positive ($c \approx 0.311$ at $a = 0.25$) and with (**c, d**) negative ($c \approx -0.201$ at $a = 0.1$) speeds. The second (intermediate) piece of each wave is marked by gray color

shape, the dispersion relation curves are monotonic, whereas for wave trains with oscillations in profile the dispersion relations are anomalous.

The periodic wave train is a two-piece solution of the form

$$u(\xi) = \begin{cases} u_1(\xi), & \xi_0^0 \le \xi \le \xi_0, \\ u_2(\xi), & \xi_0 \le \xi \le \xi_0^* \end{cases} \tag{17}$$

for u variable and

$$v(\xi) = \begin{cases} v_1(\xi), & \xi_0^0 \le \xi \le \xi_0, \\ v_2(\xi), & \xi_0 \le \xi \le \xi_0^* \end{cases} \tag{18}$$

for v variable, where

$$u_1(\xi) = A_{11}e^{\lambda_1\xi} + A_{21}e^{\lambda_2\xi} + \frac{B_{31}}{\mu_3}e^{\lambda_3\xi} + \frac{B_{41}}{\mu_4}e^{\lambda_4\xi}, \tag{19}$$

$$u_2(\xi) = A_{12}e^{\lambda_1\xi} + A_{22}e^{\lambda_2\xi} + \frac{B_{32}}{\mu_3}e^{\lambda_3\xi} + \frac{B_{42}}{\mu_4}e^{\lambda_4\xi} + u^*, \tag{20}$$

$$v_1(\xi) = B_{31}e^{\lambda_3\xi} + B_{41}e^{\lambda_4\xi}, \tag{21}$$

$$v_2(\xi) = B_{32}e^{\lambda_3\xi} + B_{42}e^{\lambda_4\xi} + v^* \tag{22}$$

with $u^* = 1 - \alpha/\varepsilon$ and $v^* = \alpha/\varepsilon$.

The trajectory of the periodic wave train on the $u - v$ plane is a closed curve, so that there are two matching points. The curve starts from point $u = a$ at $\xi = \xi_0^0$, passes this point at $\xi = \xi_0$, and ends in this point at $\xi = \xi_0^*$. At the matching point we have the conditions of continuity for functions, their derivatives, and an equation $u(\xi_0^0) = u(\xi_0) = u(\xi_0^*) = a$ of the fixed (matching) boundary. The value $L = \xi_0^* - \xi_0^0$ corresponds to the period of wave and is the new external parameter for the solutions. Thus, the matching conditions read

$$\begin{gathered}
u_1(\xi_0) = u_2(\xi_0), \quad u_2(\xi_0^*) = u_1(\xi_0^0), \\
\left.\frac{du_1(\xi)}{d\xi}\right|_{\xi_0} = \left.\frac{du_2(\xi)}{d\xi}\right|_{\xi_0}, \quad \left.\frac{du_2(\xi)}{d\xi}\right|_{\xi_0^*} = \left.\frac{du_1(\xi)}{d\xi}\right|_{\xi_0^0}, \\
v_1(\xi_0) = v_2(\xi_0), \quad v_2(\xi_0^*) = v_1(\xi_0^0), \\
\left.\frac{dv_1(\xi)}{d\xi}\right|_{\xi_0} = \left.\frac{dv_2(\xi)}{d\xi}\right|_{\xi_0}, \quad \left.\frac{dv_2(\xi)}{d\xi}\right|_{\xi_0^*} = \left.\frac{dv_1(\xi)}{d\xi}\right|_{\xi_0^0}, \\
u_1(\xi_0) = a, \quad u_1(\xi_0^0) = a.
\end{gathered} \tag{23}$$

Here is again 10 matching equations, but there is a new varied parameter: the period L of wave train.

An example of periodic wave trains at different values of the period is shown in Fig. 3. We see that the difference between wave trains with positive and negative speeds reflects in both u and v profiles [Fig. 3a, c]: the wave train is steeper in the direction of wave propagation. There is no difference between wave trains with positive and negative speeds in the $u - v$ diagrams [Fig. 3b, d]. The only difference in the size of the closed trajectory in the $u - v$ diagrams reflects the difference in the absolute value of the wave speed. The situation with the direction of wave propagation remains the same as for the solitary pulses: when the speed of the wave

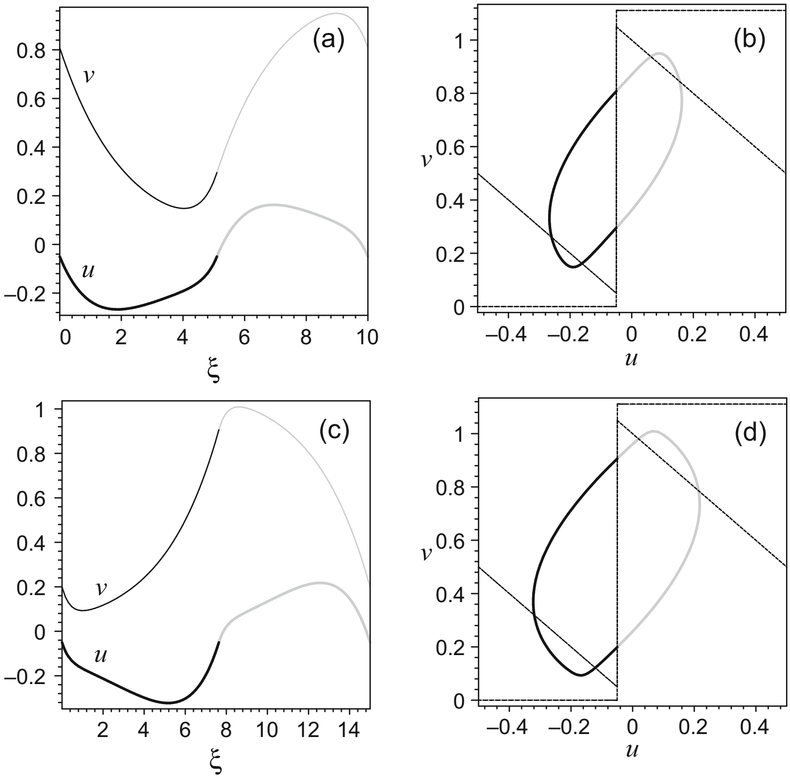

Fig. 3 An example of periodic wave trains at $D_u = 1$, $D_v = 1$, $a = -0.05$, $\varepsilon = 0.9$ and $\alpha = 1$: (**a, c**) $u = u(\xi)$ (thick line) and $v = v(\xi)$ (thin line) profiles and (**b, d**) $u - v$ diagrams. Wave train propagates with (**a, b**) negative ($c \approx -1.41$ at $L = 10$) and with (**c, d**) positive ($c \approx 2.109$ at $L = 15$) speeds. The second piece is marked by gray color

train has positive value, the wave train propagates from left to right, when the speed is negative, the direction of the wave propagation is opposite.

3 Morris-Lecar Model

Most models of excitation wave dynamics in mathematical neuroscience correspond to the Hodgkin-Huxley mechanism. The FHN model is a two-variable simplification of the Hodgkin-Huxley system, where the membrane potential and the recovery variable reflect the dynamics of transmembrane currents. The modification of the FHN model to more realistic systems with nonlinear inhibitor functions is the Tonnelier-Gerstner [5] caricature of the Morris-Lecar [6] model. Caricatures of nonlinear reaction functions by the Heaviside functions have much in their favor.

The piecewise linear approximation used in this paper provides insights into the most basic properties of traveling waves and can in many contexts be considered as an adequate approximation for the more complicated nonlinear reaction functions in most real models of the neural activity.

The analytical description for traveling waves may be developed for more general piecewise linear reaction-diffusion models of the activator-inhibitor type with nonlinear inhibitor [35] or in the piecewise linear approximation for the Morris-Lecar [6] model. Such a system is constructed from three pieces and is described by equations

$$\frac{\partial u}{\partial t} = f(u, v) + \frac{\partial^2 u}{\partial x^2}, \tag{24}$$

$$\frac{\partial v}{\partial t} = \varepsilon g(u, v) + \frac{\partial^2 v}{\partial x^2}, \tag{25}$$

where the reaction functions are

$$f(u, v) = \begin{cases} -\alpha_1 u - v, & u \leq a, \\ \alpha_2 u - v - \rho_2 & a < u < b, \\ -\alpha_3 u - v + \rho_3, & u \geq b \end{cases} \tag{26}$$

and

$$g(u, v) = \begin{cases} \beta_1 u - v - \sigma_1, & u \leq a, \\ \beta_2 u - v - \sigma_2 & a < u < b, \\ \beta_3 u - v + \sigma_3, & u \geq b. \end{cases} \tag{27}$$

The constants are

$$\rho_2 = a(\alpha_1 + \alpha_2) > 0, \tag{28}$$

$$\rho_3 = b(\alpha_2 + \alpha_3) - \rho_2 \tag{29}$$

and

$$\sigma_1 = 0, \tag{30}$$

$$\sigma_2 = a(\beta_2 - \beta_1) > 0 \tag{31}$$

in the excitable regime,

$$\sigma_1 > a(\alpha_1 + \beta_1), \tag{32}$$

$$\sigma_2 = a(\beta_2 - \beta_1) + \sigma_1 \tag{33}$$

in the oscillatory regime and

$$\sigma_1 = 0, \tag{34}$$

$$\sigma_2 = b^*(\beta_2 - \beta_1) > 0 \tag{35}$$

in the bistable regime;

$$\sigma_3 = b(\beta_2 - \beta_3) - \sigma_2 \tag{36}$$

is the same in all three cases.

General traveling wave solution reads

$$u(\xi) = \sum_n A_n e^{\lambda_n \xi} + u^*, \tag{37}$$

$$v(\xi) = \sum_m B_m e^{\lambda_m \xi} + v^*, \tag{38}$$

where the eigenvalues are

$$\lambda_n = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + \frac{\varepsilon + \alpha_n}{2} \pm \sqrt{\frac{(\varepsilon + \alpha_n)^2}{4} - \varepsilon(\alpha_n + \beta_n)}}. \tag{39}$$

The specific feature is the case when the eigenvalues are complex, i.e., when

$$\frac{(\varepsilon + \alpha_n)^2}{4} - \varepsilon(\alpha_n + \beta_n) < 0. \tag{40}$$

In this case the traveling waves have cosine and sine terms and demonstrate the oscillations in the wave profile.

The analytical solutions for the specific types of the waves (fronts, pulses, and wave trains) in the piecewise linear Morris-Lecar model will be presented in detail elsewhere.

4 Discussion

The analytical approach to the solution of the reaction-diffusion equations that we consider here is much simpler than the standard solutions of the related nonlinear systems using numerical simulations. Moreover, such method allows to perform a linear stability analysis of the traveling waves. Exact results can be derived for the growth rates of disturbances. The advantage of the presented approach is that it can also be extended to the reaction-diffusion systems with inclusion of the perturbative effects. The perturbative factors, such as external fields (described by advection terms in equations), can produce a formation of the complex spatiotemporal waves and patterns in the reaction-diffusion systems. Such perturbations lead to a wave transition when the traveling wave changes the propagation direction, i.e., wave propagation can be effectively controlled by application of these external fields.

Another effect that can be treated perturbatively is an external forcing. The forcing can be prescribed a priori, i.e., as a modulation of excitability. There exist different types of the forcing: controlling by initial conditions, global feedback, periodic forcing, and traveling-wave modulation. The last type of forcing presents a perturbation on excitable medium through a moving mask (external potential) so that the type of induced pattern or wave depends on the width and velocity of the mask. The simplest situation occurs when the mask speed is equal to the velocity of propagating excitation waves. The source of such forcing is connected with the current position of the waves so that the origin of that modulation of an excitable medium may be found in that medium inside, i.e., this is a type of “automodulation" like the autocatalysis in chemical reactions.

The generalization of the problem using the above-described perturbative factors has not been explored in detail here, but we expect qualitatively similar behavior.

5 Conclusion

At the moment there exist two basic approaches to mathematical modelling of spatiotemporal phenomena in neuroscience: axiomatic and dynamic. The aim of the axiomatic approach involved the qualitative characterization of the reaction-diffusion wave evolution in neuronal systems, such as solitary pulse waves in nerve tissues. Moreover, the axiomatic approach does not require any additional information on the kinetics of operating processes, which allows solving a problem in its general formulation. However, there is an essential disadvantage of this approach that it is difficult to observe complex phenomena and to attain quantitative fit to experimental data.

The dynamic approach postulates that an excitable medium may be adequately described using the evolution equations in partial derivatives, the nonlinear reaction-diffusion equations. Such nonlinear equations are very complicated for analytical calculations. At present no exact solutions for traveling waves in general form have been found. Alternatively, approximate calculation methods are used, such as a kinematic approach. In the framework of this approach one can mathematically describe many processes and structures in active media.

In conclusion, we would like to emphasize that our presented results are expected from corresponding previous works [33, 34], of course. It is appropriate at this point to recall that a complete analytical derivation of solitary pulses and periodic wave trains in a two-variable reaction-diffusion system with piecewise linear activator and inhibitor functions has not been performed before. Another important point is that we now have the machinery at our disposal to generalize to the case with an added external field and forcing without the need to enter into the numerical details immediately.

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