Computable Isomorphisms of Distributive Lattices

Nikolay Bazhenov[1,2](✉), Manat Mustafa[3], and Mars Yamaleev[4]

[1] Sobolev Institute of Mathematics,
4 Acad. Koptyug Ave., Novosibirsk 630090, Russia
[2] Novosibirsk State University,
2 Pirogova St., Novosibirsk 630090, Russia
bazhenov@math.nsc.ru
[3] Department of Mathematics, School of Science and Technology,
Nazarbayev University, 53 Qabanbaybatyr Avenue, Astana 010000, Kazakhstan
manat.mustafa@nu.edu.kz
[4] Kazan Federal University,
18 Kremlevskaya St., Kazan 420008, Russia
mars.yamaleev@kpfu.ru

Abstract. A standard tool for the classifying computability-theoretic complexity of equivalence relations is provided by computable reducibility. This gives rise to a rich degree-structure which has been extensively studied in the literature. In this paper, we show that equivalence relations, which are complete for computable reducibility in various levels of the hyperarithmetical hierarchy, arise in a natural way in computable structure theory. We prove that for any computable successor ordinal α, the relation of Δ^0_α isomorphism for computable distributive lattices is $\Sigma^0_{\alpha+2}$ complete. We obtain similar results for Heyting algebras, undirected graphs, and uniformly discrete metric spaces.

Keywords: Distributive lattice · Computable reducibility · Equivalence relation · Computable categoricity · Heyting algebra · Computable metric space

1 Introduction

We study computability-theoretic complexity of equivalence relations which arise in a natural way in computable structure theory. Our main working tool is *computable reducibility*.

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Definition 1.1. *Suppose that E and F are equivalence relations on the domain ω. The relation E is* computably reducible *to F (denoted by $E \leq_c F$) if there is a total computable function $f(x)$ such that for all $x, y \in \omega$, the following holds:*

$$(xEy) \ \Leftrightarrow \ (f(x)Ff(y)).$$

In what follows, we assume that every considered equivalence relation has domain ω.

The systematic study of c-degrees, i.e. degrees induced by computable reducibility, was initiated by Ershov [12,13]. His approach stems from the category-theoretic methods in the theory of numberings. In 1980s, the research in the area of c-degrees was concentrated on computably enumerable equivalence relations (or *ceers* for short): in particular, provable equivalence in formal systems was studied (see, e.g., [10,11,28]). Note that the acronym *ceer* was introduced in [17]. Recently, Andrews and Sorbi [1] provided a profound analysis of the structure of c-degrees of ceers. For the results and bibliographical references on ceers, the reader is referred to, e.g., the survey [2] and the articles [1,3,17].

Computable reducibility also proved to be useful for classifying equivalence relations having higher complexity than ceers. In particular, recent works [8,24] consider c-degrees of Δ^0_2 equivalence relations.

Definition 1.2. *Let Γ be a complexity class (e.g., Σ^0_1, d-Σ^0_1, Σ^0_2, or Π^1_1). An equivalence relation E is Γ* complete (for computable reducibility) *if $E \in \Gamma$ and for every equivalence relation $R \in \Gamma$, we have $R \leq_c E$.*

Examples of known Γ complete equivalence relations include:

- The relation of provable equivalence in Peano arithmetic is Σ^0_1 complete [11].
- 1-equivalence and m-equivalence on indices of c.e. sets are both Σ^0_3 complete [14].
- Turing equivalence on indices of c.e. sets is Σ^0_4 complete [21].
- For every $n \in \omega$, 1-equivalence on indices of $\emptyset^{(n+1)}$-c.e. sets is Σ^0_{n+4} complete [21].

Furthermore, in [21], it was proved that for any computable ordinal α, there is no $\Pi^0_{\alpha+2}$ complete equivalence relation.

Some of Γ complete equivalence relations have origins in computable structure theory: Given a class of structures K, one can treat the *isomorphism relation* on (the set of computable members of) the class K as an equivalence relation on ω (to be formally explained in Sect. 2.1). In [15], it was proved that for each of the following classes K, the isomorphism relation on K is Σ^1_1 complete for computable reducibility: trees, graphs, torsion-free abelian groups, abelian p-groups, linear orders, fields (of arbitrary characteristic), 2-step nilpotent groups.

Fokina, Friedman, and Nies [14] investigated the relation of *computable isomorphism* on a given class. In particular, they showed that for predecessor trees, equivalence structures, and Boolean algebras, the computable isomorphism relation is Σ^0_3 complete.

In this paper, we study the relation of Δ^0_α *isomorphism*, denoted by $\cong_{\Delta^0_\alpha}$, where α is a non-zero computable ordinal. Δ^0_α isomorphisms and the closely related notion of Δ^0_α-*categoricity* have been extensively studied in the literature (see, e.g., [6,16] for a survey of results).

Following the approach of [14], our paper shows that the relation $\cong_{\Delta^0_\alpha}$ fits well in the setting of computable reducibility. The outline of the paper is as follows. Section 2 contains the necessary preliminaries. In Sect. 3, we prove our main result: For every computable successor ordinal α, the relation $\cong_{\Delta^0_\alpha}$ on computable distributive lattices is $\Sigma^0_{\alpha+2}$ complete for computable reducibility.

In Sect. 4, we prove consequences of the main theorem: similar results are obtained for Heyting algebras, undirected graphs, and uniformly discrete metric spaces. We also give a partial result for Boolean algebras with distinguished subalgebra. Section 5 discusses some open problems.

2 Preliminaries

We consider only computable languages. For any considered countable structure $\mathcal{S}$, its domain is contained in the set of natural numbers. By $D(\mathcal{S})$ we denote the atomic diagram of $\mathcal{S}$.

For a language L, *infinitary formulas* of L are formulas of the logic $L_{\omega_1,\omega}$. For a countable ordinal α, infinitary Σ_α and Π_α formulas are defined in a standard way (see, e.g., [6, Chap. 6]).

2.1 Isomorphism Relation

Suppose that L is a computable language. For a computable L-structure $\mathcal{S}$, its *computable index* is a number e such that the characteristic function $\chi_{D(\mathcal{S})}$ of the atomic diagram $D(\mathcal{S})$ is equal to φ_e, where $\{\varphi_e\}_{e\in\omega}$ is the standard enumeration of all unary partial computable functions.

For $e \in \omega$, by $\mathcal{M}_e$ we denote the structure with computable index e. Suppose that K is a class of L-structures. The *index set* of the class K is the set

$$I(K) = \{e : \mathcal{M}_e \in K\}.$$

Let $\sim$ be an equivalence relation on (computable members of) the class K. Then we will identify $\sim$ with the following equivalence relation $\sim_\#$ on the set of natural numbers:

$$(i \sim_\# j) \;\Leftrightarrow\; (i = j) \vee (i, j \in I(K) \,\&\, \mathcal{M}_i \sim \mathcal{M}_j).$$

Therefore, one can consider the relations of isomorphism and Δ^0_α isomorphism in the setting of computable reducibility.

Lemma 2.1. *Let K be a class of structures, and α be a computable non-zero ordinal. If the index set $I(K)$ is $\Sigma^0_{\alpha+2}$, then the relation of Δ^0_α isomorphism on computable members of K is also $\Sigma^0_{\alpha+2}$.*

Proof. Essentially follows from [19, Proposition 4.10]. □

It is not hard to establish the following result (e.g., compare [19, Proposition 4.1]).

Lemma 2.2. *For each of the following classes K (in an appropriate language, to be discussed in the corresponding sections), the index set $I(K)$ is Π^0_2:*

(a) distributive lattices,
(b) Heyting algebras,
(c) undirected graphs,
(d) Boolean algebras with distinguished subalgebra.

Lemmas 2.1 and 2.2 together show that on each of the classes K considered above, the relation $\cong_{\Delta^0_\alpha}$ is $\Sigma^0_{\alpha+2}$. Hence, in order to prove our results, it is sufficient to establish the $\Sigma^0_{\alpha+2}$ hardness of the relation $\cong_{\Delta^0_\alpha}$: Given an arbitrary $\Sigma^0_{\alpha+2}$ equivalence relation E, we produce a uniformly computable sequence $\{\mathcal{S}_n\}_{n\in\omega}$ of structures from K such that:

$$(mEn) \;\Leftrightarrow\; (\mathcal{S}_m \cong_{\Delta^0_\alpha} \mathcal{S}_n).$$

We leave the discussion of metric spaces until Sect. 4.3.

2.2 Hyperarithmetical Equivalence Relations

In order to obtain our results on the relation of Δ^0_α isomorphism, we will work with some special hyperarithmetical equivalence relations. Note that the exposition in this subsection mirrors the corresponding recursion-theoretical results from [14].

Consider an oracle $X \subseteq \omega$. For $e \in \omega$, by W^X_e we denote the X-c.e. set that has index e in the standard numbering of all X-c.e. sets.

Suppose that A and B are subsets of ω. We say that A is *1-X-reducible* to B, denoted by $A \leq^X_1 B$, if there is a total X-computable, injective function $f(x)$ such that for every $x \in \omega$, we have $x \in A$ iff $f(x) \in B$. As usual, we write $A \equiv^X_1 B$ if $A \leq^X_1 B$ and $B \leq^X_1 A$.

The sets A and B are *X-computably isomorphic* if there is an X-computable permutation σ of the set of natural numbers such that $\sigma(A) = B$. The following lemma is a relativization of Myhill Isomorphism Theorem [23].

Lemma 2.3. *Sets A and B are X-computably isomorphic iff $A \equiv^X_1 B$.*

Now one can consider a relativized version of [14, Theorem 1]:

Theorem 2.1 (essentially [14]). *For any $\Sigma^0_3(X)$ equivalence relation E, there is a total computable function $g(x)$ such that:*

(a) If (yEz), then $W^X_{g(y)} \equiv^X_1 W^X_{g(z)}$.
(b) If $\neg(yEz)$, then $W^X_{g(y)} \not\leq_T W^X_{g(z)} \oplus X$ and $W^X_{g(z)} \not\leq_T W^X_{g(y)} \oplus X$.

Proof (sketch). Proceed with a straightforward relativization of [14, Theorem 1]. Note that this gives only an X*-computable* function $g_0(x)$ with the desired properties. Nevertheless, there is a *computable* function $g(x)$ such that $W^X_{g(e)} = W^X_{g_0(e)}$ for all e. Indeed, the set $\{\langle k, e\rangle : k \in W^X_{g_0(e)}\}$ is c.e. in X and hence, the function g can be recovered by using s-m-n Theorem (see, e.g., Exercise 1.20 in [26, Chap. III] for more details). □

Suppose that α is a computable non-zero ordinal. For convenience, we use the following notation:

$$\emptyset_{(\alpha)} := \begin{cases} \emptyset^{(\alpha-1)}, & \text{if } \alpha < \omega, \\ \emptyset^{(\alpha)}, & \text{if } \alpha \geq \omega. \end{cases}$$

Notice that for every α, we have $\Sigma^0_\alpha = \Sigma^0_1(\emptyset_{(\alpha)})$ and $\Delta^0_\alpha = \Delta^0_1(\emptyset_{(\alpha)})$. The theorem above implies the following.

Corollary 2.1. *Let α be a computable non-zero ordinal. Then the relation $\equiv_1^{\emptyset_{(\alpha)}}$ on the indices of $\emptyset_{(\alpha)}$-c.e. sets is $\Sigma^0_{\alpha+2}$ complete for computable reducibility.*

2.3 Pairs of Computable Structures

Our proofs heavily rely on the technique of pairs of computable structures developed by Ash and Knight [5,6]. Here we give necessary preliminaries on the technique.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are L-structures. We say that $\mathcal{B} \leq_\alpha \mathcal{A}$ if every infinitary Π_α sentence true in $\mathcal{B}$ is also true in $\mathcal{A}$.

Let α be a computable ordinal. A family $K = \{\mathcal{A}_i : i \in I\}$ of L-structures is α*-friendly* if the structures $\mathcal{A}_i$ are uniformly computable in $i \in I$, and the relations

$$B_\beta = \{(i, \bar{a}, j, \bar{b}) : i, j \in I,\, \bar{a} \in \mathcal{A}_i,\, \bar{b} \in \mathcal{A}_j,\, (\mathcal{A}_i, \bar{a}) \leq_\beta (\mathcal{A}_j, \bar{b})\}$$

are computably enumerable, uniformly in $\beta < \alpha$.

Theorem 2.2 ([5, Theorem 3.1]). *Suppose that α is a non-zero computable ordinal, $\mathcal{A}$ and $\mathcal{B}$ are L-structures. If $\mathcal{B} \leq_\alpha \mathcal{A}$ and the family $\{\mathcal{A}, \mathcal{B}\}$ is α-friendly, then for any Σ^0_α set X, there is a uniformly computable sequence of L-structures $\{\mathcal{C}_n\}_{n\in\omega}$ such that*

$$\mathcal{C}_n \cong \begin{cases} \mathcal{A}, & \text{if } n \notin X; \\ \mathcal{B}, & \text{if } n \in X. \end{cases}$$

Theorem 2.2 and the description of the relations $\leq_\alpha$ for countable well-orders [4,6] together imply the following:

Proposition 2.1. *Let β be a computable ordinal.*

(i) For any $\Sigma^0_{2\beta+1}$ set S, there is a uniformly computable sequence of linear orders $\{\mathcal{C}_n\}_{n\in\omega}$ such that

$$\mathcal{C}_n \cong \begin{cases} \omega^\beta, & \text{if } n \notin S;\\ \omega^\beta \cdot 2, & \text{if } n \in S. \end{cases}$$

(ii) For any $\Sigma^0_{2\beta+2}$ set S, there is a uniformly computable sequence of linear orders $\{\mathcal{C}_n\}_{n\in\omega}$ such that

$$\mathcal{C}_n \cong \begin{cases} \omega^{\beta+1}, & \text{if } n \notin S;\\ \omega^{\beta+1} + \omega^\beta, & \text{if } n \in S. \end{cases}$$

A sketch of the proof of Proposition 2.1 can be found, e.g., in [9, Theorem 4].

2.4 Distributive Lattices

Consider a language $L_{BL} := \{\vee, \wedge; 0, 1\}$. Recall that a lattice is *bounded* if it has the least element 0 and the greatest element 1. In this paper, we consider only bounded lattices. Thus, we treat lattices as L_{BL}-structures. The reader is referred to [20] for the background on lattice theory.

A partial order $\leq$ in a lattice $\mathcal{A}$ is recovered in a standard lattice-theoretical way: $x \leq y$ if and only if $x \vee y = y$. For elements $a, b \in \mathcal{A}$, by $[a; b]$ we denote the interval $\{c \in \mathcal{A} : a \leq c \leq b\}$.

Suppose that $\{\mathcal{A}_n\}_{n\in\omega}$ is a sequence of distributive lattices. The *direct sum* of the sequence $\{\mathcal{A}_n\}_{n\in\omega}$ (denoted by $\sum_{n\in\omega} \mathcal{A}_n$) is the substructure of the product $\prod_{n\in\omega} \mathcal{A}_n$ on the domain

$$\left\{ f \in \prod_{n\in\omega} \mathcal{A}_n : (\exists c \in \{0,1\}) \exists m (\forall k \geq m)(f(k) = c^{\mathcal{A}_k}) \right\}.$$

It is not hard to show that $\sum_{n\in\omega} \mathcal{A}_n$ is a distributive lattice. Furthermore, if the sequence $\{\mathcal{A}_n\}_{n\in\omega}$ is computable, then one can build a computable copy of the sum $\sum_{n\in\omega} \mathcal{A}_n$, in a standard way (see, e.g., [9, § 2.1] for details). Hence, in this case, we will identify the direct sum with its standard computable presentation.

If $a_i \in \mathcal{A}_i$, $i \leq n$, and $a_n \neq 0^{\mathcal{A}_n}$, then $(a_0, a_1, \dots, a_n, \bot_{n+1})$ denotes the element $(a_0, a_1, \dots, a_n, 0, 0, 0, \dots)$ from $\sum_{n\in\omega} \mathcal{A}_n$. If $a_n \neq 1^{\mathcal{A}_n}$, then by $(a_0, a_1, \dots, a_n, \top_{n+1})$ we denote the element $(a_0, a_1, \dots, a_n, 1, 1, 1, \dots)$.

If $\mathcal{L}$ is a linear order with the least and the greatest elements, then (as per usual) $\mathcal{L}$ can be treated as bounded distributive lattice $\mathcal{D}(\mathcal{L})$.

3 Δ^0_α Isomorphism for Distributive Lattices

Theorem 3.1. *Suppose that α is a computable successor ordinal. The relation of Δ^0_α isomorphism of computable distributive lattices is a complete $\Sigma^0_{\alpha+2}$ equivalence relation under computable reducibility.*

Proof. Here we give a detailed proof for the case when α is odd, i.e. $\alpha = 2\beta + 1$. At the end of the proof, we will briefly comment on how to deal with even α.

Suppose that E is a $\Sigma^0_{\alpha+2}$ equivalence relation on ω. Then by Corollary 2.1, there is a computable function $g(x)$ with the following property: for any $m, n \in \omega$,

$$(mEn) \;\Leftrightarrow\; W^{\emptyset^{(\alpha)}}_{g(m)} \equiv^{\emptyset^{(\alpha)}}_1 W^{\emptyset^{(\alpha)}}_{g(n)}. \tag{1}$$

Since $\alpha = 2\beta + 1$, the first part of Proposition 2.1 gives a computable sequence $\{\mathcal{L}_{n,k}\}_{n,k\in\omega}$ of linear orders such that

$$\mathcal{L}_{n,k} \cong \begin{cases} \omega^\beta, & \text{if } k \notin W^{\emptyset^{(\alpha)}}_{g(n)}; \\ \omega^\beta \cdot 2, & \text{if } k \in W^{\emptyset^{(\alpha)}}_{g(n)}. \end{cases} \tag{2}$$

For a natural number n, we define a computable distributive lattice $\mathcal{S}_n$ as follows:

$$\mathcal{S}_n := \sum_{k\in\omega} \mathcal{D}(\mathcal{L}_{n,k} + 1).$$

Now it is sufficient to prove the following fact: For every $m, n \in \omega$,

$$(mEn) \;\Leftrightarrow\; (\mathcal{S}_m \text{ and } \mathcal{S}_n \text{ are } \Delta^0_\alpha\text{-computably isomorphic}).$$

For $n, k \in \omega$, consider the element $e_{n,k} := (0, 0, \dots, 0, c_{n,k}, \bot_{k+1})$ from $\mathcal{S}_n$, where $c_{n,k}$ is the greatest element in the order $(\mathcal{L}_{n,k} + 1)$. Clearly, the sequence $\{e_{n,k}\}_{n,k\in\omega}$ is uniformly computable.

We define auxiliary finitary formulas

$$\begin{aligned} Lin(x) &:= \forall y \forall z[(y \le x)\&(z \le x) \to (y \le z) \vee (z \le y)], \\ MaxLin(x) &:= Lin(x)\&\forall y[(x \le y)\&Lin(y) \to (y = x)]. \end{aligned}$$

The $\forall\exists$-formula $MaxLin(x)$ says that an element x is maximal such that the interval $[0; x]$ is linearly ordered. It is not hard to show that $MaxLin(\mathcal{S}_n) = \{e_{n,k} : k \in \omega\}$, see [9, Lemma 3] for details. Since the sequence $\{e_{n,k}\}_{n,k\in\omega}$ is computable, one may assume that the sets $MaxLin(\mathcal{S}_n)$ are computable, uniformly in n.

Lemma 3.1. *If $\mathcal{S}_m$ and $\mathcal{S}_n$ are Δ^0_α-computably isomorphic, then m and n are E-equivalent.*

Proof. Let F be a Δ^0_α isomorphism from $\mathcal{S}_m$ onto $\mathcal{S}_n$. Note that the map $F_1 := F \restriction MaxLin(\mathcal{S}_m)$ is a Δ^0_α bijection from $MaxLin(\mathcal{S}_m)$ onto $MaxLin(\mathcal{S}_n)$. Define a map $\sigma\colon \omega \to \omega$ as follows:

$$\sigma(i) = j, \text{ if } F(e_{m,i}) = e_{n,j}.$$

It is easy to see that σ is well-defined. Moreover, σ is a Δ^0_α permutation of ω.

For every $i \in \omega$, the intervals $[0; e_{m,i}]_{\mathcal{S}_m}$ and $[0; e_{n,\sigma(i)}]_{\mathcal{S}_n}$ are isomorphic. Thus, for any i, the following conditions are equivalent:

$$i \in W^{\emptyset^{(\alpha)}}_{g(m)} \;\Leftrightarrow\; \mathcal{L}_{m,i} \cong \omega^\beta \cdot 2 \;\Leftrightarrow\; \mathcal{L}_{n,\sigma(i)} \cong \omega^\beta \cdot 2 \;\Leftrightarrow\; \sigma(i) \in W^{\emptyset^{(\alpha)}}_{g(n)}.$$

Therefore, the permutation σ witnesses that the sets $W^{\emptyset^{(\alpha)}}_{g(m)}$ and $W^{\emptyset^{(\alpha)}}_{g(n)}$ are $\emptyset^{(\alpha)}$-computably isomorphic. Equation (1) implies that the numbers m and n are E-equivalent. □

Lemma 3.2. *If* (mEn)*, then the lattices* $\mathcal{S}_m$ *and* $\mathcal{S}_n$ *are* Δ^0_α*-computably isomorphic.*

Proof. Assume that m and n are E-equivalent. By Eq. (1), there is a Δ^0_α permutation σ such that $\sigma(W^{\emptyset^{(\alpha)}}_{g(m)}) = W^{\emptyset^{(\alpha)}}_{g(n)}$. Therefore, for every $i \in \omega$, the orders $\mathcal{L}_{m,i}$ and $\mathcal{L}_{n,\sigma(i)}$ are isomorphic. Recall that every $\mathcal{L}_{m,i}$ is isomorphic either to ω^β, or to $\omega^\beta \cdot 2$.

In [9, p. 609] (see also Proposition 2 in [9]), the following fact was proved: There is an effective procedure which, given computable indices of linear orders $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M}$ and $\mathcal{N}$ are both isomorphic to some $\mathcal{A} \in \{\omega^\beta, \omega^\beta \cdot 2\}$, computes a $\Delta^0_{2\beta+1}$ index of an isomorphism F from $\mathcal{M}$ onto $\mathcal{N}$.

Recall that $\alpha = 2\beta + 1$. Hence, using the fact above, one can produce a uniform sequence of Δ^0_α isomorphisms $\{F_i\}_{i\in\omega}$ such that F_i maps $\mathcal{L}_{m,i}$ onto $\mathcal{L}_{n,\sigma(i)}$.

Now one can arrange a Δ^0_α isomorphism G from $\mathcal{S}_m$ onto $\mathcal{S}_n$ in a pretty straightforward way. A typical example looks like follows: Consider an element $a = (p_0, p_1, p_2, \top_3)$ from $\mathcal{S}_m$, where $0 \le p_i < e_{m,i}$. Then

$$G(a) := F_0(p_0) \vee F_1(p_1) \vee F_2(p_2) \vee b,$$

where the j^{th} coordinate of the element b (inside $\mathcal{S}_n$) is equal to

$$\begin{cases} 0, & \text{if } j \in \{\sigma(0), \sigma(1), \sigma(2)\}, \\ e_{n,j}, & \text{otherwise.} \end{cases}$$

Lemma 3.2 is proved. □

The proof of Theorem 3.1 for the case $\alpha = 2\beta + 2$ is essentially the same, modulo the following key modification: one needs to use the ordinals $\omega^{\beta+1}$ and $\omega^{\beta+1} + \omega^\beta$ in place of ω^β and $\omega^\beta \cdot 2$, respectively. More details on this case can be recovered from the discussion in [9, p. 610]. Theorem 3.1 is proved. □

4 Consequences of the Main Result

The (method of the) proof of Theorem 3.1 can be applied to obtain similar results for other familiar classes of structures.

4.1 Heyting Algebras

Heyting algebras are treated as structures in the language $L_{HA} = \{\vee, \wedge, \to; 0, 1\}$. An L_{HA}-structure $\mathcal{H}$ is a *Heyting algebra* if the $\{\vee, \wedge; 0, 1\}$-reduct of $\mathcal{H}$ is a bounded distributive lattice, and $\mathcal{H}$ satisfies the following three axioms:

(a) $\forall x \forall y [x \wedge (x \to y) = x \wedge y]$;
(b) $\forall x \forall y \forall z [x \wedge (y \to z) = x \wedge ((x \wedge y) \to (x \wedge z))]$;
(c) $\forall x \forall y \forall z [z \wedge ((x \wedge y) \to x) = z]$.

If $\mathcal{L}$ is a linear order with the least and the greatest elements, then it can be treated as Heyting algebra by introducing the operation:

$$x \to y := \begin{cases} 1, \text{ if } x \le y; \\ y, \text{ if } x > y. \end{cases}$$

Therefore, essentially the same proof as for Theorem 3.1 provides us with the following result:

Corollary 4.1. *Let α be a computable successor ordinal. The relation of Δ^0_α isomorphism of computable Heyting algebras is a $\Sigma^0_{\alpha+2}$ complete equivalence relation.*

More computability-theoretical results on Heyting algebras can be found in [7,9,27].

4.2 Undirected Graphs

Consider a linear order $\mathcal{L}$ on the domain $\{a_i : i \in \omega\}$. Assume that $\mathcal{L}$ has no greatest element. We define an undirected graph $G(\mathcal{L})$ as follows:

- $dom(G(\mathcal{L})) = dom(\mathcal{L}) \cup \{b_{i,j}, c_{i,j} : i < j\} \cup \{d, e, f\}$.
- We put (undirected) edges (d,e), (e,f), (f,d), $(a_i, b_{i,j})$, $(b_{i,j}, c_{i,j})$, $(c_{i,j}, a_j)$ for every $i < j$.
- Suppose that $i < j$. If $a_i <_{\mathcal{L}} a_j$, then add the edge $(c_{i,j}, d)$. Otherwise, put the edge $(b_{i,j}, d)$.

It is not hard to see that the set $dom(\mathcal{L})$ and the ordering $\le_{\mathcal{L}}$ are definable by both $\exists$- and $\forall$-formulas inside $G(\mathcal{L})$.

The transformation $\mathcal{L} \mapsto G(\mathcal{L})$ allows us to obtain the following:

Proposition 4.1. *Let α be a computable successor ordinal. The relation of Δ^0_α isomorphism of computable undirected graphs is a complete $\Sigma^0_{\alpha+2}$ equivalence relation under computable reducibility.*

Proof (sketch). We follow the lines of Theorem 3.1, and after obtaining the sequence $\{\mathcal{L}_{n,k}\}_{n,k\in\omega}$, we introduce a uniformly computable sequence of undirected graphs $\{\mathcal{G}_n\}_{n\in\omega}$ which is constructed as follows. Put into $\mathcal{G}_n$ the graphs $G(\mathcal{L}_{n,k})$, $k \in \omega$, on disjoint domains, i.e. $dom(G(\mathcal{L}_{n,k})) \cap dom(G(\mathcal{L}_{n,i})) = \emptyset$ for $k \ne i$. Suppose that $e_{n,k}$ is the element which "plays role" of the node e in the graph $G(\mathcal{L}_{n,k})$. Introduce a fresh cycle of size five, fix a node v_0 inside the cycle, and add an edge between every $e_{n,k}$ and v_0.

It is not difficult to prove that $\mathcal{G}_m$ and $\mathcal{G}_n$ are Δ^0_α isomorphic if and only if $\mathcal{S}_m \cong_{\Delta^0_\alpha} \mathcal{S}_n$. $\square$

4.3 Metric Spaces

Consider a Polish metric space (M, d). Assume that $(q_i)_{i \in \omega}$ is a dense sequence in M without repetitions. A structure $\mathcal{M} = (M, d, (q_i)_{i\in\omega})$ is a *computable metric space* if the value $d(q_i, q_j)$ is a computable real, uniformly in i and j. The elements q_i are called *special points*. For the background on computable metric spaces, the reader is referred to [29].

Fix a (standard) effective enumeration $\{\psi_e\}_{e\in\omega}$ of all partial computable functions acting from ω^3 into the set $\{q \in \mathbb{Q} : q \geq 0\}$.

We say that a number $e \in \omega$ is a *computable index* of a computable metric space $\mathcal{M} = (M, d, (q_i)_{i\in\omega})$ if the function ψ_e is total and for all $i, j, t \in \omega$, the following holds:

$$|d(q_i, q_j) - \psi_e(i, j, t)| \leq 2^{-t}.$$

The notion of computable index allows us to introduce *index sets* in the same way as in Sect. 2.1 (for more details, we refer the reader to [22,25]). Thus, one can treat the relation of surjective isometry on computable metric spaces as an equivalence relation on ω.

Recall that a computable metric space is *discrete* if every its point is isolated. Note that in such a space, every point is special. A computable metric space $\mathcal{M}$ is *uniformly discrete* if there is a real $\varepsilon > 0$ such that for any points $a \neq b$ from $\mathcal{M}$, we have $d(a, b) \geq \varepsilon$. It is easy to see that any uniformly discrete space is discrete.

Corollary 4.2. *Let α be a computable successor ordinal. The relation of Δ^0_α surjective isometry of computable, uniformly discrete metric spaces is a $\Sigma^0_{\alpha+2}$ complete equivalence relation.*

Proof. Note that the property "e is a computable index of a metric space" is equivalent to a Π^0_2 description (see, e.g., [22, p. 322]). A computable index e encodes a uniformly discrete space if and only if the following holds:

$$(\exists \varepsilon \in \mathbb{Q})[(\varepsilon > 0) \,\&\, \forall i \forall j (i \neq j \rightarrow \exists t(\psi_e(i, j, t) \geq \varepsilon + 2^{-t}))].$$

This is a Σ^0_3 description, hence the index set of uniformly discrete metric spaces is Σ^0_3. By (an analogue of) Lemma 2.1, we obtain that Δ^0_α surjective isometry for computable, uniformly discrete spaces is a $\Sigma^0_{\alpha+2}$ relation.

Given a countable undirected graph G on the domain $\{a_i : i \in \omega\}$, we introduce a discrete metric space $\mathcal{M}(G)$ as follows. The domain of $\mathcal{M}(G)$ is equal to $dom(G)$, and for every $i \neq j$, we set

$$d(a_i, a_j) = \begin{cases} 1, & \text{if } G \models Edge(a_i, a_j), \\ 3/2, & \text{if } G \models \neg Edge(a_i, a_j). \end{cases}$$

It is easy to see that there is a Δ^0_α surjective isometry from $\mathcal{M}(G)$ onto $\mathcal{M}(H)$ iff $G \cong_{\Delta^0_\alpha} H$. Thus, the desired result follows from Proposition 4.1. □

4.4 Boolean Algebras with Distinguished Subalgebra

Consider a language $L_{BA} = \{\vee, \wedge, \overline{\cdot}; 0, 1\}$. A *Boolean algebra with a distinguished subalgebra* is a structure $\mathcal{S}$ in the language $L_{BA} \cup \{U\}$ such that:

- the L_{BA}-reduct of $\mathcal{S}$ (denoted by $\mathcal{S}_{BA}$) is a Boolean algebra, and
- the unary predicate U distinguishes a subalgebra of $\mathcal{S}_{BA}$.

Here we obtain a partial result on the relation of Δ^0_α isomorphism for this class of structures.

If $\mathcal{L}$ is a linear order with the least element, then $Int(\mathcal{L})$ denotes the corresponding *interval Boolean algebra*. The background on computable Boolean algebras can be found in [18].

Proposition 4.2. *Let β be a computable ordinal. The relation of $\Delta^0_{2\beta+1}$ isomorphism of computable Boolean algebras with distinguished subalgebra is a complete $\Sigma^0_{2\beta+3}$ equivalence relation under computable reducibility.*

Proof (sketch). Let $\alpha = 2\beta + 1$. As in Theorem 3.1, given a $\Sigma^0_{\alpha+2}$ equivalence relation E, we choose a computable function $g(x)$ which satisfies Eq. (1).

It is well-known that the transformation $\mathcal{L} \mapsto Int(\mathcal{L})$ is uniformly effective, i.e. given a computable index of a linear order $\mathcal{L}$ (with the least element), one can effectively find a computable index for the algebra $Int(\mathcal{L})$. Thus, using the sequence from Eq. (2), one can build a uniformly computable sequence of Boolean algebras

$$\mathcal{B}_{n,k} \cong \begin{cases} Int(\omega^\beta), & \text{if } k \notin W^{\emptyset^{(\alpha)}}_{g(n)}, \\ Int(\omega^\beta \cdot 2), & \text{if } k \in W^{\emptyset^{(\alpha)}}_{g(n)}. \end{cases}$$

Let $e_{n,k}$ be the greatest element in $\mathcal{B}_{n,k}$.

For a natural number n, we define the Boolean algebra $\mathcal{C}_n := \sum_{k\in\omega} \mathcal{B}_{n,k}$. Inside $\mathcal{C}_n$, we use a unary predicate U_n to distinguish the subalgebra generated by the elements $c_{n,k} := (0, 0, \dots, 0, e_{n,k}, \bot_{k+1})$, $k \in \omega$.

After that, one can show that

$$(mEn) \text{ iff } (\mathcal{C}_m, U_m) \text{ and } (\mathcal{C}_n, U_n) \text{ are } \Delta^0_\alpha\text{-computably isomorphic.}$$

First, note that the set $\{c_{n,k} : k \in \omega\}$ is precisely the set of atoms of the subalgebra U_n. This observation allows us to prove an analogue of Lemma 3.1.

In order to obtain an analogue of Lemma 3.2, we need the following fact: There is an effective procedure which, given computable indices of Boolean algebras $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \cong \mathcal{N} \cong \mathcal{A} \in \{Int(\omega^\beta), Int(\omega^\beta \cdot 2)\}$, computes a $\Delta^0_{2\beta+1}$ index of an isomorphism F from $\mathcal{M}$ onto $\mathcal{N}$. This is an easy consequence of the proofs of [6, Theorem 17.8] and [9, Proposition 2]. ◻

Note that in this setting, the proof of Theorem 3.1 for the case $\alpha = 2\beta + 2$ cannot be re-used in a direct way. Indeed, it is easy to see that the interval algebras $Int(\omega^{\beta+1})$ and $Int(\omega^{\beta+1} + \omega^\beta)$ are isomorphic, and hence, we cannot use these structures for encoding a $\Sigma^0_{\alpha+2}$ equivalence relation E.

5 Further Discussion

Note that in all our results, we consider only successor ordinals α. Therefore, the following is left open:

Question 5.1. Suppose that α is a computable limit ordinal. Is the relation of Δ^0_α isomorphism of computable structures $\Sigma^0_{\alpha+2}$ complete for computable reducibility?

Recall that in [14], it was shown that computable isomorphism of Boolean algebras is Σ^0_3 complete. We established $\Sigma^0_{\alpha+2}$ completeness of Δ^0_α isomorphism for Heyting algebras (Corollary 4.1). Since every Boolean algebra can be treated as Heyting algebra under the operation $x \to y := \overline{x} \vee y$, it is natural to ask the following:

Question 5.2. Suppose that α is a computable ordinal such that $\alpha \geq 2$. Is the relation of Δ^0_α isomorphism of computable Boolean algebras $\Sigma^0_{\alpha+2}$ complete for computable reducibility?

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