Chapter 5
On the Concept of Curve: Geometry and Algebra, from Mathematical Modernity to Mathematical Modernism

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Abstract We consider the concept of curve in the context of the transition from mathematical "modernity" to mathematical "modernism," the transition defined, the article argues, by the movement from the primacy of geometrical to the primacy of algebraic thinking. The article also explores the ontological and epistemological aspects of this transition and the connections between modernist mathematics and modernist physics, especially quantum theory, in this set of contexts.

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5.1 Prologue

The frescoes of the Chauvet-Pont-d'Arc Cave in southern France, painted roughly 32,000 years ago, and the subject of Werner Herzog's 3-D documentary, *The Cave of Forgotten Dreams* (2010), are remarkable not only because of the extraordinary richness and quality of their paintings, or how well they are preserved, or, closer to my subject here, their prehistorical images of curves, which are found in other, some earlier, cave paintings, but also and especially because these curves, delineating animal figures (there is only one human figure), are drawn on the intricately curved surfaces of the cave. This unfolding curved-surface imagery compelled Herzog to use 3-D technology for his film. The resulting cinematography, the temporal image of curves and curved surfaces, the curved image of time (Herzog's theme as well) is remarkable phenomenologically, aesthetically, and, for a geometer, mathematically. The access to the cave, discovered in 1994, is severely limited, and Herzog was lucky to get permission to enter and film in it. But one could imagine what the likes of Gauss, Lobachevsky, Riemann and Poincaré (who spent a lifetime thinking about curves on surfaces) would have thought if they had had a chance to see the cave,

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which seems, by its very existence, to challenge Euclid, nearly our contemporary on this 30,000-year-old time scale. Almost, but not quite! One needs to know and, first, to invent a great deal of mathematics to think of this challenge, and the mathematics as developed, along with science, philosophy, and art, by the ancient Greeks brings them and us together as contemporaries.

There is an immense intervening history between these paintings and us, a history that has erased beyond recovery most of the thinking that created them, and whatever comments one can make concerning this thinking are bound to be conjectural. It is unlikely that mathematics existed at the time, although this may depend on how one understands what mathematics is. What may be said with more confidence is that human thinking, thanks to the neurological structure of the human brain, had by then (50,000–100,000 years ago is a current rough estimate for when this structure emerged) a component that led to the rise of mathematical thinking and eventually to mathematics itself. Concerning the longer prehistory one can only invent evolutionary fables, akin to those concerning the origins of thinking, consciousness, language, logic, music, or art, which may be plausible and useful but are unlikely to ever be confirmed.[1] Art is one endeavor where we might be close to the cave painters in the Chauvet Cave and elsewhere. But then, is art possible without some mathematical thinking, or mathematics without some artistic thinking, or thinking in general without either, or without philosophical thinking? This is doubtful, as my argument in this article will suggest, without, however, making a definitive claim to that effect, which may not be possible given where our understanding (neurological, psychological, or philosophical) of the nature of thinking stands now.

5.2 From Mathematical Modernity to Mathematical Modernism

While, inevitably, invoking earlier developments, in particular ancient Greek mathematics, the history I address here begins with mathematics at the rise of *modernity*, especially the work of Fermat and Descartes, crucial to our concept of curve as well, but having a much greater significance for all subsequent mathematics. One can assess the character of mathematical thinking then more reliably because this character is close to and has decisively shaped the character of our mathematical thinking now. I then move to Riemann and, finally, to mathematical *modernism*, which emerged sometime around 1900, shaped by the preceding development of nineteenth-century mathematics, roughly, from K. F. Gauss on, with Riemann as the most crucial figure of this development and, in the present view, conceptually, already of modernism. Thus, even this shorter trajectory, curve, of the idea of curve

[1] G. Tomlinson's book on the prehistory of music, with the revealing title "A Million Years of Music," confirms this view, even if sometimes against its own grain [64].

will be sketched by way of a discrete set of points, which would hopefully allow the reader to envision or surmise the curve or curves (for there are, again, more than one of them) connecting the dots. This *uncircumventable circumstance* notwithstanding (the metaphor of a curve invades this sentence, too), I do aim to offer a thesis concerning the idea of curve in mathematics and a thesis concerning mathematics itself, as defined by mathematical modernism as fundamentally algebraic, and to offer a historical and conceptual argument supporting this thesis.

I shall more properly outline my key concepts in the next section, merely sketching them in preliminary terms here, beginning with modernity and modernism. Modernity is a well-established broad cultural category, defined by a set of cultural transformations, "revolutions," that extends, roughly, from the sixteenth century to our own time. The rise of modernity has been commonly associated with the concept of Renaissance, especially in dealing with its cultural aspects, such as philosophy, mathematics and science, and literature and art. We are more cautious in using the rubric of Renaissance now, and prefer to speak of the early modern period, implying a greater continuity with the preceding as well as subsequent history. This caution is justified. On the other hand, in the present context, invoking the Renaissance is not out of place either, as referring to the rebirth of the ways of thinking, not the least the mathematical thinking, of ancient Greece. The rise of modernity was in part shaped by new mathematics, such as analytic geometry, algebra, and calculus, and new, experimental-mathematical sciences of nature, beginning with the physics of Kepler, Descartes, Galileo and Newton, a set of developments often referred to as the Scientific Revolution.

By contrast, modernism is a well-established, even if not uniquely defined, denomination only when applied to literature and art, while a recent and infrequent denomination when applied to mathematics or science, a denomination, moreover, commonly borrowed from its use in literature and art, as by H. Mehrtens and J. Gray, my main references here [26, 44]. *Historically*, both phenomena, modernist literature or art and modernist mathematics or science, are commonly understood as belonging to the same period, roughly from the 1890s to the 1940s, but as in various ways extending to and, certainly, continuing to impact our own time. Both forms of modernism are considered to be defined by major transformations in their respective domains, and there is a consensus that significant changes in both domains did take place during that time. However, the complex and multifaceted nature of these transformations makes it difficult to conclusively ascertain their nature and causality. Some of these transformational effects had multiple causes, and conversely, some of these causes combine to produce single effects. It is hardly surprising, then, that, *conceptually*, the thinking concerning modernism in any field is diverse and, in each case, only partially reflects the nature of modernism in a given field or the relationships, on modernist lines, between different fields, even between modernist mathematics and modernist physics.

This is an unavoidable limitation, and it cannot be circumvented by the conception of mathematical modernism to be offered in this article, which intersects with other such conceptions, but, to the best of my knowledge, does not coincide with any of them. I can only argue that this type of mathematical thinking emerged during the historical period in question, but not that it exhausts what can be termed mathematical modernism, let alone capture the development of mathematics during that period. Not all of this mathematics was modernist by any definition I am familiar with. Indeed, it cannot be captured by any single definition, any more than mathematics in general. While I find the term mathematical modernism useful and historically justified, more important are key conceptual formations that, I argue, decisively, even if not uniquely, characterize the mathematics or science that emerged during that period. I am, however, ready to admit that these formations could be given other denominations. One cannot hope for a unique name here (any more than in general), which is a good thing, because new names open new trajectories of thought.

Mathematical modernism will be primarily understood here as mathematical thinking that gives mathematics a *fundamentally algebraic character*. By way of broad preliminary definitions, I understand algebra as the mathematical formalization of the relationships between symbols, arithmetic as the mathematical practice dealing specifically with numbers, geometry as the mathematical formalization of spatiality, especially (although not exclusively) in terms of measurement, and topology as the mathematization of the structure of spatial or spatial-like objects apart from measurements, specifically in terms of continuity and discreteness. The corresponding mathematical fields are algebra, number theory, geometry, and topology. Analysis deals with the questions of limit, and related concepts such as continuity (where it intersects with topology), differentiation, integration, measurement, and so forth. There are multiple intersections between these fields, and there are numerous subfields and fields, like arithmetic algebraic geometry, that branched off these basic fields.

Defining algebra as the mathematical formalization of the relationships between symbols makes it part of all mathematics, at least all modern mathematics. Ancient Greek geometry was grounded, at least expressly, in arithmetic, although one might detect elements of symbolism there as well, especially at later stages of its development, certainly by the time of Diaphantus, sometimes called "the father of algebra." Geometrical and topological mathematical objects always have algebraic components as part of their structure, while algebraic objects may, but need not have geometrical or topological components. Two other, narrower or field-specific, considerations of algebra are important for my argument as well. The first, standing at the origins of algebra as a mathematical discipline, is that of algebra as the study of algebraic (polynomial) equations, is important also because all equations are in effect forms of algebra, which includes equations associated with calculus and then differential equations, crucial to the history of mathematics from mathematical modernity to mathematical modernism. The second is that defined by algebraic structures, such as groups or associative algebras (groups, especially symmetry groups, are also crucial to geometry and topology). These two senses of algebra bring into the landscape of mathematical modernity and then modernism, and the transition from one to the other, the figure of Galois who was the first to connect these two senses of algebra, which he did in a radically revolutionary way (also by introducing the concept of group). Galois is, arguably, the most notable figure, next to Riemann, in this history and is an even earlier (proto)modernist than Riemann was, although the limits of this article will allow me to comment on Galois only in passing. All these aspects of algebra are part of the algebraization defining mathematical modernism and the concepts of curve that come with it.

What gives the present conception of mathematical modernism its bite is that it applies *fundamentally*, rather than merely operationally, across modernist mathematics: It defines not only fields, such as analysis or mathematical logic that, while not disciplinarily classified as algebra, are governed by structures that are algebraic, but also fields like geometry and topology, that, while having technical algebraic aspects, are conceptually and disciplinarily juxtaposed to algebra. According to the present view, it is not only a matter of having an algebraic component as part of the mathematical structure of their objects but also and primarily a matter of defining these objects algebraically. Without aiming thus to contain the nature of mathematics (which is impossible in any event), one might say that the following three elements are always found in mathematics: concepts, structures, and logic, each generally more rigorously formalized than when they are found elsewhere, especially when mathematics is not used, the way it is used in physics, for example. While, however, structures and logic always entail one or another form of algebra, this is not necessarily so in the case of concepts, which may be strictly geometrical or topological. My argument here is that mathematical modernism brings algebra into the architecture of mathematical concepts, including those found in geometry or topology, even though algebra in turn accommodates the disciplinary demands of these fields.

An emblematic case is algebraic topology (a revealing denomination in itself), a field important for my argument here on several accounts, especially given its significance for algebraic geometry and Grothendieck's work.[2] Algebraic topology does have an earlier history (extending from Leibniz and, more expressly, Euler) preceding the rise of the discipline as such with Riemann, Poincaré, and others. This history, however, is not comparable to that of geometry from the ancient Greeks on, until modernism. By contrast, from modernism on, geometry and topology developed equally and in interaction with each other, and with differential topology, a field that emerged along with algebraic topology. What makes algebraic topology a mathematical discipline is that one can associate algebraic structures (initially numbers, eventually groups and other abstract algebraic structures, such as rings) to the architecture of spatial objects that are invariant under continuous transformations, independently of their geometrical properties, such as those associated, directly or implicitly, with measurements. This makes topology topo-*logy* vs. geo-*metry*. By the same token, in retrospect, topology is almost inherently categorical. It relates, functorially, the objects of topological and algebraic categories, a form of algebraic

[2] I will be less concerned with general or point-set topology, which has a different and much longer history, extending, arguably, to the ancient Greek thinking, although my claim concerning the modernist algebraization of mathematics could still be made in this case. See A. Papadopoulos' contribution to this volume for an illuminating discussion of the topological aspects of Aristotle's philosophy, via Thom's engagement with Aristotle [48].

thinking that is one of the culminating conceptions of mathematical modernism, for example and in particular, in Grothendieck's algebraic geometry.

This is not to say that the spatial (geometrical or topological) character of mathematical objects defined in modernist geometry and topology in terms of algebra disappears. It remains important at least on two counts, both of which are, however, consistent with my argument. First, the algebra defining these objects has a special form that may be called "spatial algebra."[3] Spatial algebra arises from algebraic structures that mathematically define geometrical or topological objects and reflects their proximity to $\mathbb{R}^3$ and mathematical spatial objects there, that are close to our phenomenal intuition and the geometry and topology associated with this intuition. This proximity may be, and commonly is, left behind in rigorous mathematical definitions and treatments of such objects, beginning with $\mathbb{R}^3$ itself. The same type of algebra may also be used to define mathematical objects that are no longer available to our phenomenal intuition, apart from using the latter to create heuristic metaphorical images of such space. Among characteristic examples of such objects are a projective space (a set of lines through the origin of a vector space, such as $\mathbb{R}^2$ in the case of the projective plane, with projective curves defined algebraically, as algebraic varieties) and an infinite-dimensional Hilbert space (the points of which are typically square-integrable functions or infinite series, although a Euclidean space of any dimension is a Hilbert space, too). In sum, spatial algebra is an algebraization of spatiality that makes it rigorously mathematical, topologically or geometrically, as opposed to something that is phenomenally intuitive or is defined philosophically, even in the case of spatial objects in $\mathbb{R}^3$. As such, it also allows us to define spatial-algebraic objects across a broad mathematical spectrum, and by doing so to extend the fields of topology and geometry.

At the same time, and this is the second count on which mathematical objects defined by spatial algebra retain their connections to geometrical and topological thinking, analogies with $\mathbb{R}^3$ continue to remain useful and even indispensable. Such analogies may be rigorous (and specifically algebraic) or metaphorical, with both types sometimes used jointly. Thus, the analogues of the Pythagorean theorem or parallelogram law in Euclidean geometry, which holds in infinite-dimensional Hilbert spaces over either $\mathbb{R}$ or $\mathbb{C}$, are important, including in applications to physics, especially quantum theory, the mathematical formalism of which is based in Hilbert spaces (of both finite and infinite dimensions) over $\mathbb{C}$. More generally, our thinking concerning geometrical and topological objects is not entirely translatable into algebra. This was well understood by D. Hilbert in his axiomatization of Euclidean geometry, even though this axiomatization had a spatial-algebraic character, in the

[3]Finding a good term poses difficulties because such, perhaps more suitable, terms as "geometric algebra" and "algebraic geometry," are already in use for designating, respectively, the Clifford algebra over a vector space with a quadratic form and the study of algebraic varieties, defined as the solutions of systems of polynomial equations. This object and this field, however, equally exemplify the modernist algebraization of mathematics.

present sense, including in establishing an algebraic model (the field $\mathbb{C}$) of his system of axioms in order to prove its consistency [34]. According to D. Reed:

> [A]fter a chapter in which [Hilbert] provides himself with more tools like geometry and algebra [in this following Descartes], he goes on to demonstrate in a truly spectacular way:
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> (*) a "theory of plane area can be derived from the axioms" (but not a theory of volume);
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> (*) Desargues' theorem, which states that if two triangles are situated in a plane so that pairs of corresponding sides are parallel then the lines joining the corresponding sides pass through one and the same point or are parallel, expresses a criterion for a "plane" geometry to form part of "space" geometry; and
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> (*) Pascal's theorem, which states that if A, B, C, and A^1, B^1, C^1 are two sets of points on two intersecting lines and if AB^1 is parallel to BC^1 and AA^1 is parallel to CC^1 then BA^1 is parallel to CB^1, is dependent in a very specific way on the so-called Axiom of Archimedes.
>
> None of these statements can be given a simple unequivocal expression in the realm of algebra even though models from "analytic geometry" are used in the demonstrations. In other words, while algebra is useful as a tool in the demonstration of geometrical statements it is not useful in formulating the statements themselves. [59, pp. 33–34]

Reed is right in arguing for the significance of geometrical thinking and expression in mathematics. On the other hand, his claim concerning algebra as not being useful in formulating geometrical statements is an over-simplification, whether as a general claim or as reflecting Hilbert's thinking, even in Euclidean geometry, where our geometrical intuition is more applicable and where certain proofs, such as many of those supplied by the Elements could be geometrical [22].[4] Thus, as Hilbert was well aware, a more natural setting for Desargues' and Pascal's theorems is projective geometry, which these theorems helped to usher in, in a setting, however, that we cannot visualize and that is spatial-algebraic. In other words, making a symmetrical assessment, while (Euclidean) geometrical and topological intuitions are helpful and even irreducible, spatial algebra and, thus, algebra itself, at least since Fermat and Descartes, or Diaphantus, if not Euclid, is irreducible in turn even in topology and geometry.

One can get further insight into this situation by considering a related principle due to J. Tate, whose thinking bridged number theory and algebraic geometry in highly original and profound ways: "Think geometrically, prove algebraically." It was introduced in the book (co-authored with J. Silverman) on "the rational

[4]This may remain true in low-dimensional geometry or topology. I would argue, however, that spatial algebra is still irreducible there because one commonly converts topological operations into algebraic ones. This conversion in low dimensions was essential to the origin of algebraic topology. On the other hand, the recent development of low-dimensional topology, following, among others, W. Thurston's work, from the 1970s on, is a more geometrically oriented trend that, to some degree, counters the twentieth-century modernist algebraic trends and returns to Riemann's and Poincaré's topological thinking, but only to a degree, because the algebraic structures associated with these objects remain crucial. Some of the most powerful (modernist) algebraic tools of algebraic topology and algebraic geometry have been used and sometimes developed during this more geometrical stage of the field. These areas have important connections to quantum field theory and then string theory, as in E. Witten's work, which, especially in quantum field theory, are fundamentally algebraic, in part by virtue of their probabilistic nature.

points of elliptic curves," a context that is more expressly modernist as far as the algebraization of the geometrical is concerned and as such is more illuminating in the present context. The title-phrase combines algebra ("rational points") and geometry ("curves"), and implies that geometry, at least beyond that of $\mathbb{R}^3$ and even there, requires algebra to be mathematically rigorous. According to Silverman and Tate:

> It is also possible to look at polynomial equations and their solutions in rings and fields other than $\mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$. For example, one might look at polynomials with coefficients in the finite field $\mathbb{F}_p$ with p elements and ask for solutions whose coordinates are also in the field $\mathbb{F}_p$. You may worry about your geometric intuition in situations like this. How can one visualize points and curves and directions in A^2 when the points of A^2 are pairs (x, y) with $x, y \in \mathbb{F}_p$? There are two answers to this question. The first and most reassuring one is that you can continue to think of the usual Euclidean plane, i.e., $\mathbb{R}^2$, and most of your geometric intuition concerning points and curves will still be true when you switch to coordinates in $\mathbb{F}_p$. The second and more practical answer is that the affine and projective planes and affine and projective curves are defined algebraically in terms of ordered pairs (r, s) or homogeneous triples $[a, b, c]$ without any reference to geometry. So, in proving things one can work algebraically using coordinates, without worrying at all about geometrical intuitions. We might summarize this general philosophy as: *Think geometrically, prove algebraically*. [62, p. 277]

Affine and projective planes and curves, no longer available to our phenomenal intuition, can in principle be defined without any reference to ordinary language and concepts. The latter are more difficult and perhaps impossible to avoid in geometry, at least in the kind of intuitive geometry Silverman and Tate refer to, rather than what I call spatial algebra, which, I argue, ultimately defines (nearly) all geometry rigorously. Even in these more intuitively accessible cases, we still think algebraically, too, by using spatial algebra, if with the help of geometrical intuitions, except, as noted, possibly in dealing with low-dimensional topological and geometrical objects, where more immediately spatial (topological and geometrical) arguments could be used more rigorously. It is also true that a mathematician can develop and use intuition in dealing with discrete geometries as such, say, that of the Fano plane of order 2, which has the smallest number of points and lines (seven each). However, beyond the fact that they occur in the two-dimensional regular plane, the diagrammatic representations of even the Fano plane are still difficult to think of as other than spatial algebra, in this case, combinatorial in character. While useful and even indispensable, our Euclidean intuitions are limited even when we deal with algebraic curves in the Euclidean plane, let alone in considering something like a Riemann surface as a curve over $\mathbb{C}$, or curves and other objects of finite or projective geometries, abstract algebraic varieties, Hilbert spaces, the spaces of noncommutative geometry, or geometric groups, a great example of the extension of spatial algebra to conventionally algebraic objects.

Paul Dirac, recognized as the greatest algebraic virtuoso among the founding figures of quantum theory, was, nevertheless, reportedly fond of referring to geometrical thinking in quantum mechanics and quantum field theory, both mathematically based in Hilbert spaces, of both finite and infinite dimensions over $\mathbb{C}$, and the algebras of Hermitian operators there (e.g., [23]). It is difficult to surmise, especially just from reported statements, what Dirac, famous for his laconic style, exactly had in mind. If, however, one is to judge by his writings, they appear to suggest that at stake are the algebraic properties and relations, and methods of investigations they suggest, modeled on those found in geometrical objects, defined by algebraic structures, in short, spatial algebra, as just explained, working with which was part of Dirac's algebraic virtuosity. Indicatively, notwithstanding his insistence on the role of geometry in Dirac's quantum-theoretical thinking, O. Darigold's analysis of this thinking shows the significance of algebra there [16]. Thus, as he says, "roughly, Dirac's quantum mechanics could be said to be to ordinary mechanics what noncommutative geometry is to intuitive geometry" [16, p. 307]. Noncommutative geometry, however, the invention of which was in part inspired by quantum mechanics, is a form of spatial algebra ([13, p. 38]; [53, pp. 112–113]). One encounters similar appeals to geometrical thinking in referring to transfers of geometrical methods and techniques to spatially algebraic or just algebraic objects (thus making them spatially algebraic), such as in dealing with groups and group representations in quantum mechanics, initially developed in a more geometrical context beginning with S. Lie and F. Klein or in using the idea of metrics in geometric group theory.

In what sense, then, apart from being defined by spatial algebra, may such spaces be seen as spaces, in particular, as relates to our phenomenal intuition, including visualization? The subject is complex and it is far from sufficiently explored in cognitive psychology and related fields, an extensive research during recent decades notwithstanding, including as concerns cultural or technological (digital technology in particular) factors affecting our spatial thinking. It would be difficult to make any definitive claims here. It does appear, however, that, these factors notwithstanding, our three-dimensional phenomenal intuition is shared by us cognitively and even neurologically in shaping our sense of spatiality. Part of this sense appears to be Euclidean, insofar as it corresponds to what is embodied in $\mathbb{R}^3$ (again, a mathematical concept), keeping in mind that the idea of empty space, apart from bodies of one kind or another defining or framing it, is an extrapolation, because we cannot have such a conception phenomenally or, as Leibniz argued against Newton, physically. We can have a mathematical conception of space itself. To what degree our phenomenal spatiality is Euclidean remains an open question, for example, in dealing with the visual perception of extent and perspective (e.g., [21, 25, 63]). It is nearly certain, however, that, when we visualize spatial-algebraically defined objects or even more conventional geometrical spaces or geometries once the number of dimensions is more than three, we visualize only three- (or even two-) dimensional configurations and supplement them by algebraic structures and intuitions. R. Feynman instructively explained this process in describing visual intuition in thinking in quantum theory, as cited in S. Schweber [61, pp. 465–466]. Obviously, such anecdotal evidence is not sufficient for any definitive claim. It appears, however, to be in accord with the current neurological and cognitive-psychological research, as just mentioned, which suggests the dependence of our spatial intuition, including visualization, on two- and three-dimensional phenomenal intuition. This was arguably why Kant thought of this intuition, which he saw as that of Euclidean three-dimensional spatiality, as given to us a priori. That this intuition is entirely Euclidean or that it is given to us a priori, rather than developed by experience (we would now say, neurologically), may be and has been challenged. On the other hand, its three-dimensional character appears to be reasonably certain.

As Tate must have been aware, mathematical thinking concerning geometrical and topological objects cannot be reduced to our naïve Euclidean intuitions, even though it may not be possible or desirable to exclude them. Silverman and Tate's example from differential calculus (given to further illustrate their philosophy of thinking geometrically but proving algebraically), that of finding a tangent line to a curve, confirms this point [62, pp. 377–378]. The invention of calculus, an essentially algebraic form of mathematics, was not so much about proving algebraically, as the standard of proof then was geometry. Newton, was compelled to present his mechanics in terms of geometry rather than calculus in his 1687 *Principia*, in part, as he explained, to assure a geometrical demonstration of his findings, also in the direct sense of showing something by means of phenomenal visualization, rather than in terms of the algebra of calculus [45]. Calculus was about *thinking* algebraically, as was especially manifested in Leibniz's version, rather than about rigorous proofs.

Calculus was a decisive development in understanding the geometry of curves as continuous objects, a major rethinking of the nature of curve and curvature, with, artistically and culturally, deep connections to the Baroque, the style, or more accurately, the mode of thought, defined by the ideas of curve and inflection, with Leibniz being the defining philosopher of the Baroque as well as, in his case correlatively, the coinventor of calculus [17, 51]. "Inflection is the ideal genetic element of the variable curve," G. Deleuze says in *The Fold: Leibniz and the Baroque* [17, p. 15]. Baroque thinking was also thinking in terms of infinite variations of curves, reminding one of moduli spaces of curves, yet another of Riemann's major discoveries. For the moment, one might argue that it is not in fact possible to understand the concept of continuous curve mathematically apart from calculus or some form of proto-calculus (as in Archimedes, for example), and the subject, accordingly, should have been given more consideration here. However, even in enabling this understanding, calculus was a new form of algebra, as is, again, especially manifested in Leibniz's version of it, but found in Newton as well. Fermat, the founding father of the study of elliptic curves (which led him to his famous "last" theorem), played a key role in this history, too, even if he fell short of inventing calculus. Mathematical modernism, I argue, is ultimately defined by thinking in terms of algebra rather than in terms of continuity, even in thinking of continuity itself, for example, and in particular, in considering differentiable objects, differentiable manifolds, as we define them, following Riemann. The field known as "differential algebra" (introduced in the 1950s) is another confirmation of this modernist view in one of its later incarnations, and, as it may be argued to have a Leibnizian genealogy, the connection, via algebra, between modern and modernist mathematics. An earlier modernist example of this connection was "symbolic differentiation" for Hilbert space operators (infinite-dimensional matrices) in quantum mechanics by M. Born and P. Jordan [9, 52, p. 121].

My conception of modernist mathematics as an algebraization of mathematics, even in the case of topology and geometry, is an extension of Tate's principle. This extension retains its second part but modifies the first: "Think both Intuitively Geometrically and Spatially-Geometrically: Prove Algebraically." In this form the principle could sometimes apply in algebra as well, which has benefited from the introduction of spatial-algebraic objects from Fermat and Descartes to Grothendieck and beyond, for example, in the case of geometric group theory, the study of which was founded by M. Gromov, one of the more intuitive contemporary geometers, on this type of principle. But then proving something is thinking, too, as Tate would surely admit.

The history of mathematical thinking concerning curves or straight lines (a special class of curves) is part of the origin, if not a unique origin, of this philosophy "Think both Intuitively Geometrically and Spatially-Geometrically: Prove Algebraically," which, in modernity, begins with Fermat and Descartes. Their thinking and work, which overshadow Silverman and Tate's passage just cited, bridge modern and modernist mathematics and physics, from the birth of modern mathematics. Silverman and Tate's statement that "in proving things one can work algebraically using coordinates, without worrying at all about geometrical intuitions" could have been made by Descartes, and it was one of his main points in his analytic geometry. The concept of an elliptic curve, especially when considered in its overall conceptual architecture, presented in their book, is strictly modernist, as is in fact is all algebraic geometry. This concept has other modernist dimensions, for example, by virtue of its Riemannian genealogy as (a) belonging to the theory of functions of a complex variable; (b) as, for each such a curve, both a two-dimensional topological (real) manifold and a one-dimensional complex manifold, to Riemann's theory of manifolds, central to the history of modernist geometry, and (c) as, topologically, a torus, a figure at the origin of topology, as a mathematical discipline. Both (b) and (c) manifest the modernist algebraization of geometry and topology, via spatial algebra, expressly, but it is found in (a), too, even if in a more oblique and subtler way. It might be added that a major part of Grothendieck's work in algebraic geometry, his theory of étale cohomology, discussed below, originates in Riemann's ideas of a covering space over a Riemann surface, one of Riemann's several great inventions. Algebraic curves, beginning with elliptic curves (the simplest abelian varieties) were the objects for which étale cohomology groups were established first, by an elegant calculation, exemplary of the mathematical technologies to which modernism gave rise [4, 5]).

Although their manifestation in Silverman and Tate's passage cited here is particularly notable because of echoing Descartes, these historical connections to the rise of modern algebra and analytic geometry (which algebraic geometry brings to its, for now, ultimate form) are not surprising. As all conceptions or undertakings, no matter how innovative, the modernist algebraization of mathematics has a history. While more prominent in the nineteenth century, the history of algebraization, at the very least, again, by way of practice, although it has, especially with Descartes, deep philosophical roots as well, begins with the mathematics at the outset of modernity, such as that of Fermat and Descartes.[5] Geometry was then still more dominant than algebra, and it had continued to be dominant for quite a while, even though this dominance diminished with the shift, often noted, of interest from geometry to algebra and number theory from around the time of Gauss, a key figure in this shift. Gauss' work was also central to the development of geometry during the same period and a major influence on Riemann's thinking concerning geometry, which, however, only testify to the rising significance of the relationships between algebra and geometry during the period leading to modernism. In any event, the possibility of making geometry algebraic (in either sense, that of algebraic geometry and the present one) entered mathematical thinking with Fermat and Descartes.

In some respects, the view of mathematics as fundamentally algebraic returns to *a* Pythagorean view of mathematics, which is not the same as *the* Pythagoreans' view, which was more arithmetical, although arithmetic is a form of algebra in its broad modern sense (the mathematical field-specificity of arithmetic or number theory, say, from Gauss on, is a separate issue). Geometry was of course a key part of Pythagorean mathematics. For one thing, it appears that these were the Pythagoreans (who exactly, is conjectural) who discovered the existence of incommensurable magnitudes by considering the diagonal of the square and thus in geometry, in effect by means of what I call spatial algebra, or proto-spatial algebra. (The "irrationals" in our algebraic language and, with it, our sense, are borrowed from Latin, and not Greek.) This discovery led to the crisis of ancient Greek mathematics. According to Heath's commentary: The "discovery of incommensurability must have necessitated a great recasting of the whole fabric of elementary geometry, pending the discovery of the general theory of proportion applicable to incommensurable as well as to commensurable magnitudes" ("Introductory Note," [22, v. 3, p. 1]; [53, pp. 416–417]). Thus, the *history* of mathematical modernism defined by algebra is very old, possibly as old as mathematics itself. On the other hand, thus combining, as history often does, the continuous and the discontinuous along different lines, the specific form this definition takes in modernism is a break with the past.

At stake, thus, is the rethinking of the very nature and practice of mathematics by making algebra a fundamental part of it, including topology and geometry, even in the cases of mathematicians whose thinking has a strong geometrical or topological orientation, such as Riemann, who figures centrally in this history. Riemann is, arguably, a unique case of a modernist combination of geometrical, topological, and algebraic thinking, further combined with real and complex analysis, and with number theory (the ζ-function and the distribution of primes) added to the mix, even though, as might be expected, various aspects of this Riemannian synthesis are found in the work on his predecessors, such as Gauss, Cauchy, Abel, and Dirichlet. I am referring not only to the multifaceted character of Riemann's work and his contributions to the interrelationships of these diverse fields in his work, but also and primarily to the significance of these interrelationships for modernism, which could,

[5]Descartes' *La Géométrie* was originally published as an appendix to his *Discourse on Method*, and it was part of a vast philosophical agenda that encompassed mathematics [18].

nevertheless, still be defined, *in these relationships*, in terms of the algebraization of mathematics. This situation makes Riemann's position in the history of modernism more complex, especially because of strong geometrical and topological dimensions of his thinking that resist algebraization, without, as I shall argue, diminishing his importance in this history but instead reflecting the complexity of this history and of modernism itself.

I shall further argue that this algebraization was often accompanied or even codefined by three additional, often interrelated, features, which are, along with the algebraization of the mathematics used, equally found in modernist physics. It is possible to define modernism in mathematics and physics by the presence of all four features. Doing so, however, would narrow modernism too much, as against seeing it in terms of modernist algebraization of mathematics, possibly accompanied by some or all of these additional features. These features are as follows.

The first feature, which gradually emerged throughout the nineteenth century, with Gauss, Abel, and Galois, as notable early examples, was a movement toward the independence and self-determination of mathematics as a field, especially its independence from physics and, with it, from the representation of natural objects. This feature has been seen as central to mathematical modernism by commentators who used the rubric and even defined it accordingly by Merthens and, following him, Gray [26, 44]. As will be seen presently, however, modernist mathematics, either in this or the present (algebraically oriented) definition, acquired a new, nonrepresentational, role in physics with quantum theory. This feature was closely related to the development of algebra, beginning with Gauss, Abel, and Galois; new, sometimes related, areas of analysis, such as the theory of elliptic functions; and then projective and finite geometries, in part following Riemann's work. Riemann's own thinking, as that of his teacher Gauss, retained close connections to physics, testifying to the complex nature of this history. As I explain below, this independence is also related to the independence of mathematics' from ordinary language and concepts, with which algebra could dispense more easily than geometry. This independence becomes crucial for modernist physics as well, especially quantum theory, which is essentially algebraic in character, in contrast to more geometrically oriented classical physics and relativity.

The second feature, discussed most explicitly in the final section of this article, is the role of technological thinking, in this case in considering mathematical technology in mathematics itself and in physics (where the use of mathematics is technological), in contrast to the dominance of ontological or realist thinking, defined by claims concerning *how* what exists or is claimed to exist actually exists. The "ontological" and "realist" are not always seen as the same, but their shared aspects allow these terms to be used interchangeably in the present context. On the other hand, the nouns "ontology" and "reality" will be used differently, because, as I shall explain presently, "reality" may be defined as disallowing ontology or realism.

The third feature, the emphasis on which, arguably, distinguishes most the present understanding of mathematical modernism from other concepts of mathematical modernism, is a radical form of epistemology linked to and in part enabled by the combination of other modernist trends: the modernist algebraization of mathematics, a movement toward the independence of mathematics from physics, and, especially, a shift from more ontological to more technological thinking, in the case of quantum theory (in the present interpretation) to the point of abandoning or even precluding ontological thinking altogether. As will be seen, ontological thinking (in this case concerning the ontology of thought rather than matter) retains a greater role in mathematics itself. In physics this epistemology, again, extends to the point of placing the ultimate constitution of reality (referring, roughly, to what exists) beyond a representation or even beyond conception, and thus beyond ontology, referring, as just noted, to such a representation or at least conception of the constitution of reality or existence, rather than merely to the fact that something exists. In this view, quantum objects or something in nature that compels us to think of quantum objects is assumed to exist, while no representation or even conception of what they ultimately are or how they exist is possible. That does not preclude thinking and knowledge in quantum theory or elsewhere, along with and in part enabled by surface-level ontologies (physical, mathematical, conceptual, and so forth) which enable this thinking and knowledge. On the other hand, any knowledge or even conception concerning and thus any ontology of the ultimate nature of reality is precluded. Thinking and knowledge would concern certain surface levels of reality, surface ontologies. Indeed, the unknowable or even unthinkable ultimate nature of reality is inferred from the effects it has at these surface levels. Importantly, however, this conception of reality as that which is beyond thought is still the product of thought. This conception is, moreover, interpretive in nature and the justification for assuming it is practical, and applicable only insofar as things stand now rather than is theoretically guaranteed to be true.

This epistemology, too, can be traced to Riemann's Habilitation lecture of 1854, "The Hypotheses That Lie at the Foundations of Geometry" [60], one of the founding works of mathematical modernism, as Riemann's view of the foundations of geometry is a radical reconceptualization of mathematics, pursued, correlatively, in his other works as well, such as that on the concept of a Riemann surface. In his Habilitation lecture, Riemann uses a remarkable phrase "a reality underlying space" [60, p. 33]. This phrase implies, on Kantian lines, that this reality may not be spatial in the sense of our usual phenomenal sense of spatiality: it could be discrete, for example. I am not contending that Riemann saw this reality as beyond representation (discrete or continuous, flat or curved, or three- or more-dimensional, all of which possibilities he entertained), let alone conception, any more than did Kant, a key figure in this history. While, in defining his epistemology by distinguishing noumena or things-in-themselves, as objects, and phenomena or representations appearing in our thought, Kant places things-in-themselves beyond representation or knowledge, he allows that a conception of them could be formed and, if logical, accepted for practical reasons, and even in principle be true, although this truth cannot be guaranteed [35, p. 115]. In its most radical form, the modernist epistemology, as defined in this article, in principle excludes that such as a conception can be formed, keeping in mind the qualifications just noted to the effect that this conception of reality as that which is beyond conception is still human and is only practically justified.

Both Riemann and then Einstein appear to have thought that an adequate mathematical representation of the ultimate nature of physical reality, a conception ideally close to the truth of nature, is, in principle, possible, as deduced from our experience and knowledge. This, for example, would allow one to conclude, against Kant's view, the geometry of the space is not Euclidean and its physics is not Newtonian, although of all people Kant might have been more open to this view than others, given new mathematics and science. A similar, more mathematically grounded, view was found in Heisenberg's later works. Heisenberg argued there that "the 'thing-in-itself' is for the atomic physicist, if he uses this concept at all, finally a mathematical structure; but this structure is—contrary to Kant—indirectly deduced from experience" [33, p. 91]. Kant's view was more complex and more open. It is impossible to know what Kant would have thought if he'd had been confronted with quantum physics, or, again, non-Euclidean geometry or relativity. If anything, his epistemology is closer to the one advocated here than just about any modern philosopher, apart from Nietzsche.

In any event, neither Riemann nor Einstein thought that the ultimate constitution of physical reality could be beyond conception altogether. This is the position adopted here in view of quantum mechanics and following Heisenberg (in his early work, as opposed to his later thinking) and N. Bohr, although neither might have assumed that quantum objects and behavior are beyond conception rather than only beyond representation and knowledge.[6] As I explain below, however, Heisenberg's and Bohr's positions are still different from that of Kant concerning phenomena vs. things-in-themselves, in this case, defining phenomena as what is observed in measuring instruments and objects as quantum objects, which cannot be observed, as effects they have on measuring instruments by interacting with the latter. Bohr, it could be noted in passing, was influenced in his interpretation of quantum mechanics in terms of what he called complementarity (a mutually exclusive nature of certain experiments we can perform and, correlatively, certain concepts we can use) by the concept of a Riemann surface as a way of dealing with multivalued functions of a complex variable [50, pp. 235–238].

Heisenberg's and Bohr's epistemology arises in part in view of the algebraic rather than, as in classical physics or relativity, geometrical, relationship between the mathematics of a physical theory and physical reality in its ultimate constitution, assumed by theory. This algebraic relationship between a (mathematical) physical theory and physical reality was no longer representational, because, in Bohr's words, "In contrast to ordinary mechanics, the new quantum mechanics does not deal with a [geometrical] space-time description of the motion of atomic particles," while, nevertheless, providing probabilistic or statistical predictions that are fully in accord with the available experimental evidence [8, v. 1, p. 48]. Eventually, Bohr argued, more radically, that "in quantum mechanics, we are not dealing with an arbitrary renunciation of a more detailed analysis of atomic [quantum] phenomena, but with a recognition that such an analysis is *in principle excluded*" [8, v. 2, p. 62]. The

[6]For a detail discussion of the subject, see the companion article by the present author [56].

true meaning of this statement is brought out by Bohr's view of the (irreducible) difference, following but ultimately reaching beyond Kant, between quantum *phenomena*, defined by what is observed in measuring instruments, and quantum *objects*, responsible for these phenomena, as effects of the interactions between quantum objects and measuring instruments, effects manifested in the measuring instruments. Bohr's statement, then, means that there could be no analysis that would allow us to represent, physically or mathematically, quantum objects and behavior, although Bohr might not have thought that they are beyond conception, which is an *interpretation* adopted here. It is, again, important that at stake here are interpretations, those of the type, indeed two types, just defined, amidst still other interpretations (some of which are realist) of quantum theory, and not the ultimate truth of nature, which we do not know and may never know or even conceive of, concerning which this article makes no claims.

It is worth noting that probability theory is fundamentally algebraic, as is, accordingly, its use in physics or elsewhere. Indeed, that probability theory is defined by the role of events, either in the real world or some model world, makes it akin to physics and its use of mathematics, in the case of quantum theory, in nonrealist interpretations, in effect making the latter a form of probability theory. The origin of probability theory, in the work of G. Cardano, B. Pascal and Fermat (who thus makes yet another appearance in the history of algebra) coincides with the emergence of algebra, as part of the rise of modernity. As I. Hacking persuasively argued in explaining why the theory emerged in the seventeenth century rather than earlier, some form of algebra was necessary for probability theory [28]. Quantum mechanics, however, at least, again, in nonrealist interpretations, reshaped, the relationships between the algebra of probability and the algebra of theoretical physics, as against previous uses of probability, for example, in classical statistical physics. There the relationships between them is underlain by a geometrical picture of the behavior of the individual constituents of the systems considered, assumed to follow the (causal) laws of classical mechanics. By contrast, as became apparent beginning with M. Planck's discovery of quantum phenomena in 1900, even elementary individual quantum objects and the events they give rise to had to be treated probabilistically. One needed to find a new theory to make correct probabilistic or statistical predictions concerning them. Heisenberg was able to accomplish this task with quantum mechanics, which only predicted the probabilities of what was observed in measuring instruments, considered as quantum phenomena, without representing the behavior of *quantum objects*, even the elementary ones, an imperative that had previously defined fundamental physics, including relativity.[7] This mathematics, never previously used in physics, was

[7]That, again, does not exclude either realist or causal interpretations of quantum mechanics or alternative theories of this behavior that are realist or causal. The so-called Bohmian mechanics is one example of such an alternative theory. Unlike quantum mechanics, however, Bohmian mechanics expressly violates the requirement of locality, which entered physics with relativity theory and which dictates that the instantaneous transmission of physical influences between spatially separated systems is forbidden.

essentially (Heisenberg did not initially use these terms) that of infinite-dimensional Hilbert spaces over $\mathbb{C}$, a modernist concept.

As I shall discuss later in this article, related epistemological considerations are relevant in considering modernist mathematics itself, as became apparent beginning at least with G. Cantor's set theory and became more pronounced in subsequent developments, such as those leading to K. Gödel's incompleteness theorems and P. Cohen's proof of the undecidability of Cantor's continuum hypothesis. In mathematics, moreover, it may not be possible to speak of the ultimate nature of reality, however inconceivable, as existing independently of thought, in the way one is able to do in quantum physics. There one might more readily assume the ultimate reality of matter that exists independently of us, say, as something that has existed before we were here and will continue to exist when we are no longer here, even if any conception concerning this reality or the impossibility of forming such a conception is a product of our thought and thus can only exist insofar as we exist.[8] On the other hand, while one might easily accept what we think of as real in our thought, assuming the existence of a single nonmaterial reality existing independently of our individual thinking is a more complicated matter. This is not to say that this type of assumption has not been made in mathematics, philosophy, or art, from Parmenides and Plato to the mathematical Platonism of the twentieth century (which was important to the project of the independence of mathematics and to mathematical modernism), with numerous Platonisms, whether so named or not, between them or after mathematical Platonism. Not many of them, certainly not twentieth-century mathematical Platonism, are the same as Plato's own Platonism.

The concept of curve, as it emerged in modernist mathematics, is, I argue here, exemplary, in some respects even uniquely exemplary, of the modernist situation outlined here, beginning with the modernist extension of the view of Riemann surfaces as curves over $\mathbb{C}$, which is only possible if one thinks of them spatial-algebraically. The mathematics of complex numbers was, especially, again, in and following Gauss' work, itself a crucial part of the history that eventually brought modern mathematics to modernist mathematics. This mathematics, too, is traceable to the origin of modern algebra, in considering the roots of polynomial equations, essentially related to the algebra of curves. However, the view that something (topologically) two-dimensional is a curve is essentially modernist. But then, as noted, in modernist mathematics, even something topologically zero-dimensional may be a curve, a situation anticipated by Riemann as well in considering discrete manifolds in his Habilitation lecture.

Between his work on Riemann surfaces, which are both (differentiable) manifolds and, while topologically two-dimensional, curves, and his ideas concerning the foundations of geometry, which properly grounded non-Euclidean, *curved*,

[8] The so-called many-worlds interpretation of quantum mechanics, which aimed to resolve some of the paradoxes of the theory in a realist and causal way, does not affect this point, because this kind of material reality is still retained within each world involved, and there are no connections between these worlds.

geometries, Riemann becomes a key figure for our understanding of the idea of curve, as he is for many developments of modernist mathematics. Riemann's thinking figures in quantum theory, too, by virtue of his introduction of the idea of an infinite-dimensional manifold, of which Hilbert spaces are examples. Riemann is the highest point of the arc, a curve, from Fermat and Descartes to our own time, via A. Weil, Grothendieck, and their followers, making the work of each of these figures, in Nietzsche's phrase, the mathematical "philosophy of the future," a subtitle of his 1888 *Beyond Good and Evil: A Prelude to a Philosophy of the Future* [46]. As most of Nietzsche's works, it belongs to the time around the rise of modernism, which it influenced in philosophy, and literature and art, or even in mathematics, as in the case of F. Hausdorff [26, p. 222], and physics, as, likely, in the case of Bohr [54, p. 116].

5.3 Fundamentals of Mathematical Modernism

I would like now to establish more firmly the key concepts that ground my view of mathematical modernism, as sketched in the Introduction, beginning with modernity and modernism. "Modernity" is customarily seen, and will be seen here, as a broad cultural category. It refers to the period of Western culture extending from about the sixteenth century to our own time: we are still modern, although during the last 50 years or so, modernity entered a new stage, sometimes known as postmodernity, defined by the rise of digital information technology.[9] Modernity is defined by several interrelated transformations, sometimes known as revolutions, although each took a while. Among them are scientific (defined by the new cosmological thinking, beginning with the Copernican heliocentric view of the Solar system, and the introduction by Descartes, Galileo, and others, of the mathematical-experimental science of nature); industrial or, more broadly, technological (defined by the transition to the primary role of machines in industrial production and beyond); philosophical-psychological (defined by the rise of the concept of the individual human self, beginning with Descartes' concept of the Cogito); economic (defined by the rise of capitalism); and political (defined by the rise of Western democracies).

One might add to this standard list, in which Descartes figures prominently already, the mathematical revolution, which is rarely expressly discussed as such, although it figures in discussions of the rise of modernity as part of the scientific revolution and, occasionally, because of the invention of calculus and then probability theory, both seen as defined by a modern way of thinking. The rise of algebra was, however, equally important in this revolution and conceptually fundamental because algebra was also crucial to the discoveries and developments of calculus and probability theory, in which calculus came to play a major role as

[9] Thus, postmodernity was also epistemologically shaped by certain developments in mathematics and science, most of which are modernist in the present sense (e.g., [37]).

well. Algebra was the defining aspect of modern mathematics and physics, although geometry remained dominant for a quite while in both and has never, including in modernism, entirely lost its independence and importance. Thus, while the laws of classical mechanics, embodied in its equations, are algebraic (all equations are), they are grounded in a geometrical picture of the world, including the curved motion of classical bodies, such as, paradigmatically, planets moving around the Sun, although analytic geometry or algebraic laws of classical mechanics added algebra to this geometry. This type of modern geometrical thinking will continue to define physics, including Einstein's relativity (although it does have modernist aspects as well), until quantum mechanics and its modernist algebraic approach, introduced by Heisenberg.

As I noted at the outset, in contrast to modernity, "modernism" has been primarily used as an aesthetic category, referring to certain developments in literature and art in the first half of the twentieth century, from roughly the 1900s on, represented by such figures as Stéphane Mallarmé, W. B. Yeats, Ezra Pound, James Joyce, Franz Kafka, Reiner M. Rilke, Virginia Woolf, and Jorge Luis Borges in literature; Pablo Picasso, Wassily Kandinsky, and Paul Klee, in art; and Arnold Schoenberg and Igor Stravinsky in music. On occasion, it has been applied to the philosophy of, roughly, the same period, such as that of Nietzsche, Bergson, Husserl, and Heidegger. Gray considers Husserl in the context of the foundations of mathematics and mentions Nietzsche because of Hausdorff's interest in him [26, p. 222], but he does not discuss modernism in philosophy. The denomination has rarely been used in considering mathematics and physics, or science, as opposed to "modern," used frequently, but with different periodizations. In mathematics, "modern" tends to refer to the mathematics that had emerged in the nineteenth century, with the likes of Gauss, Abel, Cauchy, and Galois, and then developed into the twentieth century, thus overlapping with modernist mathematics in the present definition. In fact, the term "modern algebra" was introduced, referring essentially to abstract algebra (presented axiomatically), as late as 1930 by van der Waerden in his influential book under this title, based on the lectures given by Emil Artin and E. Noether [65]. In physics it refers to all mathematical-experimental physics, from Galileo and Descartes on, which is fitting because this physics emerged along with and shaped the rise of modernity as a cultural formation, as just explained, making it fundamentally scientific. After the discovery of relativity and quantum theory, the term "classical physics" was adopted for the preceding physics, still considered modern, by virtue of its mathematical-experimental character. The present article, by contrast, uses the designation modern for the mathematics emerging at the same time. If modernity is scientific, it is also because it is mathematical. As Heidegger argued in commenting on Galileo and Descartes, "modern science is experimental because of its mathematical project" [29, p. 93]. Thus, it was the concept of the second-degree curve that supported and even defined the experimental basis of physics and astronomy, in Kepler, Galileo, and Descartes, who gave these mathematics "coordinates," the concept central to all modern and then modernist physics.

Using of the term “modernism” in considering, historically and conceptually, mathematics and science is, as noted, still quite infrequent. Two most prominent examples, mentioned from the outset of this article, are H. Mehrtens’ 1990 *Moderne Sprache, Mathematik: Eine Geschichte des Streits um die Grundlagen der Disziplin und des Subjekts formaler Systeme* [44] and, in part following Mehrtens’ book (but also departing from it in several key respects), Gray’s 2007 *Plato’s Ghost: The Modernist Transformation of Mathematics* [26]. Gray’s conception of modernism covers developments in topology, set theory, abstract algebra, mathematical logic, and foundations of geometry that had reached their modernist stage around 1900, focusing most on geometry, with Hilbert’s Foundations of Geometry as his conceptual center in this regard, and on logical, especially set-theoretical, foundations of mathematics, where Hilbert again, figures centrally. In sum, for Gray, the most representative figure of mathematical modernism is Hilbert. By contrast, in the present view of modernism (defined differently), it is Riemann, while still by and large respecting the chronology of modernism adopted by Gray, a chronology contemporaneous with the rise of literary or artistic modernism.[10]

Gray briefly comments on literary and artistic modernism, and his title comes, not coincidentally, from that of a poem by Yeats, one of the major modernist poets. Gray only minimally considers these connections (e.g., [26, p. 185]). He prefers to focus on mathematics and the philosophy of mathematics. He could in my view

[10]The modernist aspects of Riemann’s work, equally in Gray’s definition of modernism, pose difficulties for Gray, because Riemann preceded modernism by several decades [26, p. 5]. It is not a problem for the present argument, firstly, because the present view of modernism is different, and, secondly, because modernism is seen here as more continuous with modern mathematics from Fermat and Descartes on, a longer history in which Riemann’s work is a decisive juncture. This continuity is recognized by Gray, but it seems to worry him because it disturbs the stricter chronology he considers. The present view emphasizes, in part following G. W. F. Hegel, the conceptual over the chronological, even in historical considerations. Gray, in addition, appears to see the axiomatic, not central for Riemann (in contrast to Hilbert), rather than the conceptual, as more characteristic of modernism. In the present view, modernism is more about concepts and their history than about the chronology of events or developments, such as those associated with the spreading of modernist thinking or practices. This chronology cannot of course be disregarded, but a concept or a form of practice in a given field can precede a chronologically defined state of this field, with which this concept or practice would be in accord. This accord is not an “anticipation” but a determinate quality of a concept or a form of practice. Riemann’s concepts and practice are modernist, in the present (or, with some differences, Gray’s) definition, and a similar claim could be made, helped by his revolutionary algebraic thinking, concerning Galois. The degree or even the existence of such an accord, or to what degree this accord reflects the understanding of this concept by its inventor, is a matter of interpretation, which could be contested. Riemann’s thinking has complexities when it comes to the role of algebra there because of the topological and geometrical aspects of his thinking, which often take the center stage, while algebra, when still present, appears in a supporting role. This is, however, only so in a more narrow or technical sense, as opposed to the broader sense assumed here as defining modernism. Riemann’s work, as noted, is defined by the joint workings of geometry, topology, algebra, and analysis in his mathematics, added by philosophical and physical, aspects of his thinking. Hilbert made major contributions in all these areas as well (apart from topology), but one does find the same type of fusion of different fields dealing with a given subject that one finds in Riemann, as in the case of Riemann surfaces.

have given more attention to physics, especially quantum theory, which he by and large bypasses. (Gray does comment on relativity.) Unfortunately, especially given the role of the image of the curve in modernist art, such as that of Klee (e.g., [17, pp. 14–15]), I cannot address the connections between modernist mathematics or sciences and modernist literature and art in detail either. I would, however, argue, more strongly than Gray, for the validity of these connections in sharing some of the key conceptual features. This view follows Bohr, who said, in speaking of quantum theory: "We are not dealing here with more or less vague analogies, but with an investigation of the conditions for the proper use of our conceptual means shared by different fields" [8, v. 2, p. 2]. It is not merely a matter of traffic, for example, metaphorical, between fields, but of parallel situations in each that justifies the use of the term modernism in considering them. Indeed, this article explores this type of parallel between modernist mathematics and quantum theory, mathematically equally defined by the role of algebra in them.[11] That does not of course mean that the specificity of each field, such as that of mathematics vs. physics, or that of either vs. that of literature and art (or that of literature vs. that of art) is dissolved even in considering such parallel situations, let alone in general. Such parallels often give new dimensions to this specificity, for example, as I argue, in the case of modernist mathematics and modernist physics in bringing out the fundamentally algebraic character of both.

Although Gray's concept of mathematical modernism is different from the one adopted here, there are relationships between them. These relationships are complex and considering them in detail would be difficult. While Gray offers a discussion of modernist algebra (which would of course be impossible to avoid), he does not address, except occasionally and mostly by implication, the modernist algebraization of mathematics, including geometry and topology. In fact, some key developments in modernist algebra, too, are not given by Gray the attention they deserve, such as Noether's work in algebra, one of the great examples of mathematical modernism, central to more abstract developments of algebraic topology (as in H. Hopf's work) and a bridge between R. Dedekind and Grothendieck, helpfully discussed by C. McLarty [42]. Gray also largely bypasses epistemological considerations central to the present analysis. Gray acknowledges the connections in mathematical modernism in the case of relativity [26, p. 324, n. 28]. But he misses nonrealist thinking found in quantum theory, which connects physical reality in its ultimate constitution with mathematics without recourse to realism. In fairness, related epistemological aspects of modernism are suggested by Gray in the context

[11]The nature of these connections and, in part correlatively, the effectiveness of using the term modernism, specifically by Mehrtens and Gray, have been questioned, for example, by S. Feferman [24] and L. Corry [15]. While both articles (that of Feferman is a review of Gray's book) make valid points, I don't find them especially convincing on either count, in part because their engagement with modernist art is extremely limited and because neither considers the epistemological dimensions of modernism, which are, in my view, important in addressing these connections. For an instructive counter argument to Mehrtens, challenging his historical claims, specifically those concerning F. Klein, see [6].

of Cantor's set theory and logical foundations of mathematics, especially Cantor's continuum hypothesis and Gödel's theorems. There is no discussion of quantum theory either.

On the other hand, the project of the *independence* and *self-determination* of mathematics is central to Gray, as it was to Mehrtens (whose views I shall put aside). This trend had, as I said, been gradually emerging throughout the nineteenth century. This independence is especially manifested as an independence from physics or, more generally, from considering mathematical objects as idealizations from natural objects, the type of idealization that was central to physics and its use of mathematics from Descartes and Galileo on. As I argue here, however, quantum theory, by its algebraic nature, also established new, nonrepresentational, relationships between mathematics and physics, and thus mathematics and nature, by using modernist mathematics. It was, echoing the literary parallel, the end of realism and the beginning of modernism, and not only echoing because similar relationships between representation and reality emerged in literary or artistic modernism. Quantum mechanics did not diminish faith in the classical ideal. Einstein or E. Schrödinger, the coinventor of quantum mechanics (and Einstein, too, made momentous contributions to quantum theory), never relinquished the hope that this ideal would be eventually restored to fundamental physics. Their uncompromising positions have served as inspirations for many others who share this hope, in fact a majority among physicists and philosophers alike. Einstein won this *philosophical* part of his debate with Bohr. *Physics* is a different matter. The question, which was the main question in the Bohr-Einstein debate, is whether nature would allow us a return to realism. While Einstein thought that it should, Bohr thought that it *might not*, which is not the same as it never will. As our fundamental theories are manifestly incomplete, especially given that of quantum field theory, our best theory of the fundamental forces of nature (electromagnetism, the weak force, and the strong forces) apart from gravity, and general relativity, our best theory of gravity, are inconsistent with each other, the question remains open, and the debate concerning it continues with undiminished intensity.

By contrast, mathematical realism and, especially, mathematical Platonism (a modernist development, which is, as I said, only superficially related to Plato's thought) has been important for the project of the independence of mathematics. This project had been developing as part of *modern* mathematics, but by 1900, with the rise of mathematical modernism, it reached the stage of breaking with connections representing or idealizing natural objects in all areas of mathematics, notably in geometry, making it "profoundly counterintuitive." "This realization," Gray contends, "marks a break with all philosophy of mathematics that present mathematical objects as idealizations from natural ones: it is characteristic of modernism" [26, p. 20].

The history of realization is much longer and, to some degree or in some of its aspects, it began even with the emergence of mathematics itself, including geometry, but it was certainly quite advanced by 1800 or thereabout, with non-Euclidean geometry as part of it (e.g., [27]). Indeed, this history may also be seen as that of divorcing mathematical concepts from our general phenomenal intuition as well, culminating in modernism and more characteristic of it (than a break with the view of mathematical objects as idealization from natural ones, although there are connections between these two breaks). In this divorce, algebra, including spatial algebra, has, I argue, played a major role. This divorce was stressed by H. Weyl, himself a major figure of mathematical modernism. Weyl made his point in his 1918 book, *The Continuum* [69], following closely his 1913 book on Riemann surfaces [73], and followed even more closely by his 1918 classic on the mathematics of relativity, *Space Time Matter* [68].[12] All three books were linked by their shared modernist problematic, as defined here, keeping in mind that Weyl's own position was realist, which was, however, common among mathematical modernists. The idea of curve, too, was equally crucial in all three contexts—Riemann surfaces, the continuum, and, again, via Riemann, Einstein's general relativity. According to Weyl: "the conceptual world of mathematics is so foreign to what the intuitive continuum presents to us that the demand for coincidence between the two must be dismissed as absurd" [69, p. 108]. "Coincidence" is not the same as "relations," which, as noted above, are unavoidable, at least insofar as it is difficult to think of continuity, spatially, apart from one or another phenomenal intuition of it. Even algebra involves general phenomenal intuition, even a spatial one, for example, in considering matrices as arrangements of symbols, which was crucial to Heisenberg's discovery of quantum mechanics, in the course of which he reinvented matrix algebra through so arranging certain mathematical elements involved [52, pp. 30–31]. On the other hand, it is entirely possible to define a given continuum, such as that of a line or curve, algebraically. This situation emerged with Cantor's introduction of set theory, and the multiplicity of infinities, the infinity of infinities, there, and his continuum hypothesis, and then the discovery of esoteric objects, such as Peano's curve, and related developments leading to Gödel's incompleteness theorem, and finally Cohen's proof of the undecidability of the continuum hypothesis, which brought new, ultimately irresolvable, complexity to the idea of continuum.[13] In sum, we do not, and even cannot, know how a continuous line, straight or curved (which does not matter topologically), is spatially constituted by its points, but we have algebra to address this question, and have a proof that the answer is rigorously undecidable. I shall further address the philosophical underpinning of this situation in the final section of this article. This history and related modernist developments, such as the concept of dimension (which Cantor's rethinking of the concepts of continuum required) is extensively considered by Gray, confirming Weyl's point, made as part of his own important contribution to this history in *The Continuum*.

Weyl's point concerning the conceptual world of mathematics as unavailable to our general phenomenal intuition or, by implication, ordinary language, exceeds the question of the continuum, and pertains to most of modernist mathematics

[12] Weyl's classic book had undergone several editions, some of them with significant revisions. I cite here the last edition.

[13] Intriguingly, Cohen ultimately thought that the hypothesis was likely to be false [12, p. 151].

or physics, such as relativity and quantum theory, which was even able to take advantage of this divorce between mathematics and our general phenomenal intuition, or a representation of natural objects. Weyl added to his statement cited above: “Nevertheless, those abstract schemata supplied us by mathematics must underlie the exact sciences of domains of objects in which continua play a role” [69, p. 108]. This comment was undoubtedly made with Einstein’s relativity in mind, as Weyl’s next book, *Space Time Matter*, was already in the works [68]. While his careful formulation implies the representational role of such schemata, it has a modernist twist given that those abstract schemata are, mathematically, divorced to one degree or another from our common phenomenal intuition, which entails the use of algebra, in the case of Einstein’s general relativity, based in Riemannian geometry of differentiable manifolds as a form of spatial algebra.

This type of divorce, I argue, has been equally at work in modernist mathematics and modernist physics, and, while prepared by previous developments, such as analytical mechanics and Maxwell’s electromagnetism, it reaches its modernist stage with relativity, beginning with special relativity, and especially quantum mechanics. Both used modernist mathematics, respectively, that of Riemannian geometry (Minkowski’s spacetime of special relativity is a pseudo-Riemannian manifold) and that of infinite-dimensional Hilbert spaces, equally mathematically divorced, as abstract continuous schemata, from our general phenomenal intuition. Relativity still did this in a realist way, as the break from our phenomenal intuition does not entail a divorce from realism or ontology, because the latter could be mathematical. In fact, this ontology has been mathematical in all modern physics from Descartes and Galileo on, even when it is supplemented by or, in classical mechanics, originates in our phenomenal intuition. In addition, as noted above, one could still use one’s phenomenal, including geometrical, intuition heuristically, to help our thinking, or, to return to Tate’s principle, one could still think geometrically, as well as spatial-algebraically, while proving things or (which is not the same) making more rigorous arguments and calculations in physics, algebraically. Still, relativity entailed a radical departure from classical physics. For one thing, as Weyl was of course aware, the relativistic law of addition of velocities (defined by the Lorentz transformation) in special relativity, $s = (v + u)/(1 + \frac{1}{c^2}vu)$, for collinear motion ($c$ is the speed of light in a vacuum), runs contrary to any intuitive (geometrical) representation of motion that we can have. This concept of motion is, thus, no longer a mathematical refinement of a daily concept of motion in the way the classical concept of motion is. Relativity was the first physical theory that defeated our ability to form a phenomenal conception of an elementary physical process. But it still allowed for a mathematical and conceptual representation of physical reality.

Quantum mechanics, by contrast, only used mathematics for providing probabilistic predictions concerning the outcomes of quantum experiments, quantum events, without providing a representation or even conception of the processes responsible for these events, in which case geometrical intuitions are of no help to us at all. At most we can have spatial algebra. This gives an entirely new role to abstract continuous schemata, such as those of Hilbert spaces, in physics, that of predicting, probabilistically, the outcome of irreducibly discrete events. These predictions are, moreover, made possible by rules *added* to the formalism rather than being part of it, such as Born's rule, which relates, essentially by using complex conjugation, complex quantities of formalism to real numbers corresponding to the probabilities of quantum events. Basically, one takes the square moduli of the eigenvalues of the operators associated with quantum variables, such as position, momentum, or energy (or equivalently, multiply these eigenvalues by their complex conjugates), which gives one real numbers, corresponding, once suitably normalized, to the probabilities of observed events, associated with the corresponding measurements. The standard rule for adding the probabilities of alternative outcomes is changed to adding the corresponding amplitudes and deriving the final probability by squaring the modulus of the sum. The algebra of probabilities changes!

The modernist situation just outlined bears significantly on the question of language in quantum theory and in mathematics itself. It is helpful to briefly consider first this role in mathematics, beginning with geometry, where ordinary language or ordinary concepts have always played a greater role than in algebra, beginning with Euclid's *Elements* and its very first definition: "A point is that which has no part" [22, v. 1, p. 15]. At the same time, there is a movement, enabled by and enabling mathematics, away from ordinary language and concepts, because what makes "points," or "lines," part of geometry as mathematics are not their definition but the relationships between and among them in Euclidean geometry, a fact on which Hilbert capitalized two millenia later. Descartes' "geometry," as presented in *La Géométrie*, offers an important contrast to both Euclid and Hilbert alike. It is not axiomatic but "problematic," as well as, correlatively, algebraic. It primarily deals with problems (some of which may be theorems in the usual sense), thus nearly erasing Euclid's distinction, which is difficult to sustain, between problems and theorems. There is an affinity with Riemann in this regard. But there is also a major difference. Riemann thinks in terms of concepts [55]. Descartes thinks in terms of equations, and points and lines are understood accordingly. Modern algebraic geometry will eventually bring Descartes and Riemann together, with Grothendieck's work as the culmination of this history.

While not axiomatic, Descartes' thinking suggests the possibility of a different, more algebraic, axiomatization, which was part of the project of mathematical modernism (both in the present and in Gray's definition), as manifested in Hilbert's *Foundations of Geometry*, first published in 1899. Hilbert's often cited earlier remark, apparently made in 1891, offers an intriguing angle: "One must be able to say at all times—instead of points, straight lines, and planes—tables, chairs, and beer mugs" [71, p. 635]. What Hilbert exactly had in mind is not entirely certain and has been interpreted in a variety of ways. Without attempting to give it a definitive interpretation, my reading would be as follows, in accordance with Weyl's point concerning the conceptual world of mathematics as foreign to that of our general phenomenal intuition. Hilbert uses his example only to indicate that both sets are that of connected entities, and that one should properly speak of neither "points, straight lines, and planes," as geometry did from Euclid on, nor "tables, chairs, and beer mugs," nor anything else referred to by means of ordinary language, but instead use algebraic symbols and algebraic relationships between them, without referring to any objects in the world represented by ordinary language.[14] Ordinary language, however, still plays an important role and, arguably, cannot be entirely dispensed with. While we can replace points, straight lines, and planes, and relationships between them with symbols, it may be difficult for our thought, at least our unconscious thought, but perhaps even our conscious thought, to replace them with tables, chairs, and beer mugs. As explained earlier, a decade later, Hilbert's *Foundations of Geometry* tells us as much. Still, with modernism, mathematics and physics break with ordinary language or thinking more deliberately and radically, as expressed in Weyl's 1918 remark just considered. In physics, this break, although gradually emerging earlier as well, becomes pronounced with quantum mechanics, where it became complete in considering quantum objects and behavior, in nonrealist interpretations.

According to Heisenberg, "it is not surprising that our language [or concepts] should be incapable of describing the processes occurring within atoms, for . . . it was invented to describe the experiences of daily life, and these consist only of processes involving exceedingly large numbers of atoms. It is very difficult to modify our language so that it will be able to describe these atomic processes, for words can only describe things of which we can form mental pictures, and this ability, too, is a result of daily experience" [32, p. 11]. Words can do more, including make the statement that tells us that words cannot describe the processes occurring within atoms, which, however, does not undermine Heisenberg's main point. It follows that, while classical physics, at least classical mechanics, may rely on, and was born from, a mathematical refinement of our daily phenomenal intuition, concepts, and language, atomic physics can no longer do so. However, as Heisenberg realized in his discovery of quantum mechanics, it can still use mathematics. As just discussed in considering Weyl's argument, relativity and the preceding quantum theory, or even some developments of classical physics have, with the help of mathematics, already broken, at least in part, with our daily intuition and concepts. Heisenberg clearly realized this. As he said, following the passage just cited: "Fortunately, mathematics is not subject to this limitation, and it has been possible to invent a mathematical scheme—the quantum theory [e.g., quantum mechanics]—which seems entirely adequate for the treatment of atomic processes" [32, p. 11]. Mathematics allows one to circumvent the limits of our phenomenal representational intuition, also involving visualization, sometimes used, including by Bohr, to translate the German word for intuition, *Anschaulichkeit*. "Visualization" and its avatars are often invoked by Bohr, by way of this translation, in considering quantum objects and behavior, as being beyond our capacity to phenomenally represent them (e.g., [8, v. 1, pp. 51, 98–100, 108, v. 2, p. 59]). Ultimately, Bohr came to see quantum objects and behavior as being beyond any representation, if not conception (a view adopted here), including a mathematical

[14] I borrow the juxtaposition between Hilbert's remark and Euclid's definition of a point from G. E. Martin [40, p. 140], who, however, only states this juxtaposition without interpreting it.

one, a view adopted by Heisenberg at the time of the comments just cited. By contrast, as I noted, in his later thinking Heisenberg appears to be more open to the possibility of a mathematical representation of the ultimate structure of matter, still, however, in the absence of a physical representation of it, a form of strictly mathematical realism or Platonism. (e.g., [33, pp. 91, 147–166]).

That Heisenberg *found* a mathematical scheme that could predict the data in question was as fortunate as that mathematics is free of this limitation, for, as just noted and as Heisenberg must have realized, this freedom is also at work in relativity or even classical physics, beginning at least with Lagrange's and Hamilton's analytical mechanics. It is true that matrix algebra was introduced in mathematics before Heisenberg, who was, again, unaware of it and reinvented it, although the unbounded infinite matrices that he used were not previously studied in mathematics and were given a proper mathematical treatment by M. Born and P. Jordan later [9]. But, even if Heisenberg had been familiar with it, his scheme would still have needed to be invented as a mathematical model dealing with quantum phenomena. Heisenberg discovered that this was possible to do in terms of probabilistic or statistical predictions in the absence of any representation or even conception of quantum objects and their behavior. Indeed, mathematics now becomes primary in an even more fundamental sense than in its previous use in physics. This is because, given that we have no help from physical concepts, mathematics is our only means to develop the formalism we need. Quantum physics does contain an irreducible nonmathematical remainder because no mathematics can apply to quantum objects and behavior. But then, nothing else, physics or philosophy, for example, could apply either. Heisenberg's key physical intuition was that there could be no physical intuition that could apply to quantum objects and processes, while one could use mathematics to predict the outcomes of experiments, thus redefining the relationships between mathematics and physics.

This redefinition was grounded in the primacy of algebra, moreover, not only as against classical physics and relativity but also as against the preceding quantum theory, specifically, Bohr's 1913 atomic theory, initially, as that of the hydrogen atom. The theory retained a geometrical, orbital, representation of electrons' motion in so-called stationary states, even though it renounced any mechanical conception of transitions between such states. It had its Keplerian, "Harmonia-Mundi," appeal (Bohr's orbits were elliptical, too) in defining the ultimate microscopic constitution of nature. Developed by Bohr and others to apply to more complex atoms, the theory had major successes over the next decade. However, it ran into formidable problems and proved to be inadequate as a fundamental theory of atomic constitution. To rectify the situation Heisenberg made an extraordinary move, unanticipated at the time, because nearly everyone was expecting a return to a more geometrical picture partially abandoned by Bohr's theory. Against these expectations, in Heisenberg's scheme there were no orbits anymore but only states of quantum objects, states, moreover, never accessible as such and hence not available to a theoretical representation, but only manifested in their effects on measuring instruments. This, however, still allowed his theory to predict the probabilities of what can be observed in quantum experiments, which became the core of Heisenberg's approach.

Even before his paper announcing his discovery was published [31], Heisenberg explained this "more suitable" concept as follows: "What I really like in this scheme is that one can really reduce *all interactions between atoms and the external world ... to transition probabilities*" between quantum measurements (Heisenberg, Letter to Kronig, 5 June 1925; cited in [43, v. 2, p. 242]; emphasis added). It was Heisenberg's renunciation of any geometrical representation of quantum objects and behavior, thus replacing the geometry of curves with the algebra of probabilities, that led him to the discovery of quantum mechanics.

As he says at the outset of his paper: "in quantum theory it has not been possible to associate the electron with a point in space, *considered as a function of time*, by means of observable quantities. However, even in quantum theory it is possible to ascribe to an electron the emission of radiation" [31, p. 263; emphasis added]. The effect of such an emission could be observed in a measuring instrument and its occurrence can be assigned probability or (if the experiment is repeated many time) statistics. My emphasis reflects the fact that, in principle, a measurement could associate an electron with a point in space, but not by linking this association to a function of time representing the continuous motion of this electron, in the way it is possible in classical mechanics. If one adopts a nonrealist interpretation, one cannot assign any properties to quantum objects themselves, not even single out such properties, such as that of having a position, rather than only certain joint ones, which are precluded by the uncertainty relations. One could only assign physical properties to the measuring instruments involved. On the other hand, Heisenberg's approach put into question the privileged *position* that the position variable had previously occupied in physics. Heisenberg described his next task as follows, which shows the genealogy of his derivation in Bohr's atomic theory:

> In order to characterize this radiation we first need the frequencies which appear as functions of two variables. In quantum theory these functions are in the form:
>
> $$\nu(n, n-\alpha) = 1/h\{W(n) - W(n-\alpha)\}$$
>
> and in classical theory in the form
>
> $$\nu(n, \alpha) = \alpha\nu(n) = \alpha/h(dW/dn)$$
>
> [31, p. 263]

This difference leads to a difference between classical and quantum theories as regards the combination relations for frequencies, which, in the quantum case, correspond to the Rydberg-Ritz combination rules, again, reflecting, in Heisenberg's words, "the discrepancy between the calculated orbital frequency of the electrons and the frequency of the emitted radiation." However, "in order to complete the description of radiation [in correspondence, by the mathematical correspondence principle, with the classical Fourier representation of motion] it is necessary to have not only frequencies but also the amplitudes" [31, p. 263]. On the one hand, then, by the correspondence principle, the new, quantum-mechanical equations must formally contain amplitudes, as well as frequencies. On the other hand, these amplitudes could no longer serve their classical physical function (as part of a continuous representation of motion) and are instead related to discrete transitions between stationary states. (Nor ultimately do frequencies because of the non-classical character of the Rydberg-Ritz combination rules.) In Heisenberg's theory and in quantum theory since then, these "amplitudes" are no longer amplitudes of physical motions, which makes the name "amplitude" itself an artificial, symbolic term. Linear superposition in quantum mechanics is of a fundamentally different nature from any superposition found in the classical wave theory. In nonrealist interpretations, this superposition is not even physical: it is only mathematical. In classical physics this mathematics represents physical processes; in quantum mechanics it does not. Amplitudes are instead linked to the probabilities of transitions between stationary states: they are what we now call probability amplitudes. The corresponding probabilities are derived, from Heisenberg's matrices, by a form of Born's rule for this limited case. (Technically, one needs to use the probability density functions, but this does not affect the main point in question.) One can literally see here a conversion of the classical continuous geometrical picture of oscillation or wave propagation, as defined by frequencies and amplitudes, into the algebra of probabilities of transitions between discrete quantum events.

Algebra, in part as the spatial algebra of Hilbert spaces, was the mathematical technology of predictions concerning the outcomes of quantum experiments, eventually, with quantum field theory, in high energy (relativistic) quantum regimes, in the absence of mathematical ontology of the ultimate reality, defined by the quantum constitution of nature, an ontology found in relativity or classical physics before it. Quantum electrodynamics is the best experimentally confirmed physical theory ever. It was the triumph of "the Heisenberg [algebraic] method," as Einstein characterized it in 1936, while still skeptical about its future, a decade of major successes of quantum mechanics notwithstanding. Even apart from the fact that Einstein's unwavering discontent with quantum mechanics and his debate with Bohr concerning it were a decade long by then as well, Einstein's assessment of Heisenberg's algebraic method was hardly unexpected given Einstein's preference for realism and geometry. As he said: "[P]erhaps the success of the Heisenberg method points to a purely algebraic method of description of nature, that is, to the elimination of continuous functions from physics. Then, however, we must give up, in principle, the space-time continuum [at the ultimate level of reality]. It is not unimaginable that human ingenuity will some day find methods which will make it possible to proceed along such a path. At present however, such a program looks like an attempt to breathe in empty space" [20, p. 378]. For some, by contrast, beginning with Bohr, Heisenberg's method was more like breathing fresh mountain air. The theory has been extraordinarily successful and remains our standard theory of quantum phenomena in both low and high-energy quantum regimes, governed by quantum mechanics and quantum field theory respectively.

A few qualifications are in order, however. First of all, one must keep in mind the complexity of this algebra, which involves objects that are not, in general, discontinuous, although certain key elements involved are no longer continuous functions, such as those used in classical physics, and are replaced by Hilbert-space operators (over $\mathbb{C}$). Some continuous functions are retained, because the Hilbert spaces involved are those of such functions, considered as infinite-dimensional vectors in dealing with continuous variables such as position and momentum, keeping in mind that these variables themselves are represented by operators. These functions, vectors, are those of complex (rather than, as in classical physics, real) variables and the vector spaces that they comprise, or associated objects such as operator algebras, have special properties, such as, most crucially, noncommutativity. These vectors, of which Schrödinger's wave function is the most famous example, play an essential role in calculating (via Born's rule) the probabilities of the outcomes of quantum experiments. In fact, given that they deal with Hilbert spaces, quantum mechanics and quantum field theory involve mathematical objects whose continuity is denser than that of regular continua such as the (real number) spacetime continuum of classical physics or relativity. In contrast to these theories, however, the continuous and differential mathematics used in quantum theory, along with the discontinuous algebraic one, relates, in terms of probabilistic predictions, to the physical discontinuity defining quantum phenomena, which are discrete in relation to each other, while, at least in nonrealist interpretations, quantum objects and their behavior are not given any physical or mathematical representation or even conceptions—continuous or discontinuous.

Thus, as Bohr was the first to fully realize, Heisenberg's algebraic method brings about a radical change of our understanding of the nature of physical reality, an understanding ultimately depriving us not of reality but of realism, which was, for Einstein, the most unpalatable implication of Heisenberg's method. In saying that "we must give up, in principle, the space-time continuum," Einstein must have had in mind the spacetime continuum in representing, by means of the corresponding theory, the ultimate reality considered, and possibly in attributing the spacetime continuum to this reality, something, defining his geometrical philosophy of physics (embodied in general relativity), that Einstein was extremely reluctant to give up. The idea that this reality may ultimately be discrete had been around for quite a while by then. In particular, it was, as noted, proposed by Riemann as early as 1854, speaking of "the reality underlying space" [60, p. 33]. It was Riemann's concept of continuous (actually, differentiable) manifolds and Riemannian geometry this concept defined that grounded Einstein's general relativity and his view of the ultimate nature of physical reality as the spacetime continuum, threatened by quantum theory. The idea of the discrete nature of ultimate reality has acquired new currency in view of quantum mechanics and quantum field theory, as advocated by, among others, Heisenberg in the 1930s, and is still around. In the present view, the ultimate nature of physical reality is beyond representation and even conception (neither Bohr nor Heisenberg might, again, have been ready to go that far) and, as such, may not be seen as either continuous or discontinuous. Discreteness only pertains to quantum phenomena, observed in measuring instruments, while continuity has no physical significance at all. It is only a feature of the formalism of quantum mechanics, which at the same time relates to discrete phenomena by predicting the probabilities or statistics of their occurrence.

While Kant's philosophy may be seen as an important precursor to this epistemology, beginning with the difference between objects and phenomena as its basis, in its stronger form, which places the ultimate nature of reality beyond conception (rather than only beyond representation or knowledge), this epistemology is manifestly more radical than that of Kant. This because, as I explained, in Kant's epistemology, noumena or objects as things-in-themselves are, while unknowable, still in principle conceivable and that conception might even be true, even though there is no guarantee that it is true [35, p. 115]. Even if Bohr adopted a weaker view, which only precludes a representation of quantum objects and behavior, it is still more radical than that of Kant, because, while a conception of quantum objects and behavior is in principle possible, it cannot be unambiguously used in considering quantum phenomena, at least as things stand now. I am not saying that the stronger view is physically necessary, but only that it is interpretively possible. There does not appear to be experimental data that would compel one to prefer either view. These views are, however, different philosophically because they reflect different limits that nature allows our thought in reaching its ultimate constitution. "As things stand now" is an important qualification, equally applicable to the strong view adopted here, even though it might appear otherwise, given that this view precludes any conception of the ultimate reality not only now but also ever, by placing it beyond thought altogether. This qualification still applies because a return to realism is possible, either on experimental or theoretical grounds even for those who hold this view. This return may take place because quantum theory, as currently constituted, may be replaced by an alternative theory that allows for or requires a realist interpretation, or because either the weak or the strong nonrealist view in question may become obsolete, even for those who hold this view, with quantum theory in place in its present form. It is also possible, however, that this view, in either the weak or strong version, will remain part of our future fundamental theories.

It is reasonable to assume that something "happens" or "changes," for example, that an electron changes its quantum state in an atom, say, from one energy level to another, between observations that then register this change. But, if one adopted the present interpretation, one could do so only if one keeps in mind the provisional nature of such words as "happen," "change," or "atom," which are ultimately inapplicable in this case, as are any other words or concepts. Quantum objects are defined by their capacity to create certain specific effects observed in measuring instruments and changes in what is so observed from one measurement to the other, changes described in language with the help of mathematics, without allowing one to represent or even conceive of what they are or how they change. According to Heisenberg:

> There is no description of what happens to the system between the initial observation and the next measurement. . . . The demand to "describe what happens" in the quantum-theoretical process between two successive observations is a contradiction in adjecto, since the word "describe" [or "represent"] refers to the use of classical concepts, while these concepts cannot be applied in the space between the observations; they can only be applied at the points of observation. [33, pp. 47, 145]

The same, it follows, must apply to the word "happen" or any word we use, and we must use words and concepts associated with them, even when we try to restrict ourselves to mathematics as much as possible. There can be no physics without language, but quantum physics imposes new limitations on using it. Heisenberg adds later in the book: "But the problem of language is really serious. We wish to speak in some way about the structure of the atoms and not only about 'facts'—the latter being, for instance, the black spots on a photographic plate or the water droplets in a cloud chamber. But we cannot speak about the atoms in ordinary language" [33, pp. 178–179]. Nor, by the same token, can we use, in referring to the atoms, ordinary concepts, from which our language is not dissociable, or for that matter philosophical or physical concepts. Heisenberg's statements still leave space for the possibility of representing "the structure of atoms" and thus the ultimate constitution of matter mathematically, without providing a physical description of this constitution. Indeed, as I said, this was the position adopted by Heisenberg by the time of these statements [33, pp. 91, 147–166]. At the time of his discovery of quantum mechanics, he saw the quantum-mechanical formalism strictly as the means of providing probabilistic predictions of the outcomes of quantum experiments. Physically, it was only assumed that "it [was] possible to ascribe to an electron the emission of radiation [a photon] [the effect of which could be observed in a measuring instrument]," without providing any physical mechanism for this emission [31, p. 263].

Language remains unavoidable and helpful in mathematics and physics alike. In physics, this significance of language is more immediate, as Bohr, again, observed on many occasions. Thus, he said: "[W]e must recognize above all that, even when the phenomena transcend the scope of classical physical theories, the account of the experimental arrangement and the recording of observations must be given in plain language, suitably supplemented by technical physical terminology. This is a clear logical demand, since the very word 'experiment' refers to a situation where we can tell others what we have done and what we have learned" [8, v. 2, p. 72; emphasis added]. This also ensures the objective and (objectively) verifiable nature of our measurements or predictions, just as in classical physics. The fundamental difference in this regard between classical and quantum physics is that in quantum physics, we deal with objects, quantum objects, which cannot be observed or represented, in contradistinction to quantum phenomena, defined by what is observed in measuring instruments as the impact of unobservable quantum objects. This difference in principle exists in classical mechanics as well, just as it does in our observations of the world, as was realized by Kant, who introduced his epistemology in the wake of Newton, whose mechanics was crucial to Kant, along with and correlatively to Euclidean geometry. There, however, as Bohr noted on the same occasion, the interference of observation "may be neglected," which is no longer the case in quantum physics [8, v. 2, p. 72].[15] Thus, paradigmatically, we can

[15]Classical statistical physics introduces certain complications here, which are, however not essential because the behavior of individual constituents of the systems considered there is

observe how planets move along the curves of their orbits, without our observational process having any effect. Not so in quantum mechanics. Nobody has ever observed, at least thus far, an electron or photon as such, in motion or at rest, to the degree that either concept ultimately applies to them, or any quantum objects, qua quantum objects, no matter how large (and some could be quite large). It is only possible to observe traces, such as spots on photographic plates, left by their interactions with measuring instruments. Hence, Bohr invokes "the essential ambiguity involved in a reference to physical attributes of [quantum] objects when dealing with phenomena where no sharp distinction can be made between the behavior of the objects themselves and their interaction with the measuring instruments" [8, v. 2, p. 61]. It follows that any meaningful ("unambiguous") representations or even conception of quantum objects and their independent behavior is "*in principle* excluded" [8, v. 2, p. 62]. On the other hand, each such trace in measuring instruments or a specific configuration of such traces can be treated as a permanent record, which can be discussed, communicated, and so forth. In this sense, such traces or our predictions concerning them are, again, as objective as they are in classical physics or relativity, except that quantum records are only verifiable as probabilistic or statistical records in all quantum physics, which is only the case in classical statistical physics. Classical mechanics or relativity give ideally exact predictions, which are not possible in quantum mechanics, because identically prepared quantum experiments in general lead to different outcomes. Only the statistics of multiple identically prepared experiments are repeatable. It would be difficult, if not impossible, to do science without being able to reproduce at least the statistical data and thus to verify the prediction of a given theory, which is possible in quantum physics.

Bohr's qualification, "plain language, suitably supplemented by technical physical terminology," introduces an additional subtlety, which extends to the mathematics of quantum theory and to mathematics itself. In the latter case, however, Bohr's formulation may be reversed to "technical terminology, suitably supplemented by plain language," although it may be a matter of balance, especially when philosophical considerations are involved. Thus, Riemann's Habilitation lecture famously contains only one real formula, which did not prevent it from decisively shaping the subsequent history of geometry, dominated, especially from modernism on, by technical, sometimes nearly impenetrably technical, algebraic treatments.

Consider his defining concept, that of manifold—*Mannigfaltigkeit*. Riemann's German is important. Although the term "*Mannigfaltigkeit*" was not uncommon in German philosophical literature, including in Leibniz and Kant, it is worth noting that the German word for the Trinity is "*Dreifaltigkeit*," thus, etymologically, suggesting a kind of "three-folded-ness," which could not have been missed by Riemann, or, for that matter, Leibniz and Kant. It is the "folded-ness" that is of the main significance here in shaping Riemann's concept philosophically. English

governed by the deterministic laws of classical mechanics. In quantum mechanics, even elementary individual objects (the so-called elementary particles) can only be handled probabilistically, and in the present view, their behavior is beyond representation or even conception.

“manifold” picks it up, as does French “*multiplicité*” [*pli*] which was initially used to translate Riemann’s term, but is no longer, being replaced by *variété* (English “variety” is used for algebraic varieties), perhaps because, unlike German *Mannigfaltigkeit* and English *manifold*, it also refers to multiplicity in general.[16] Different general (or philosophical) concepts implied by terminological fluctuations of these terms do shape their mathematical choices and uses. These concepts add important dimensions to our understanding of these choices or their intellectual and cultural significance in a given case, such as that of Riemann’s concept of *Mannigfaltigkeit* [55]. On the other hand, a mathematical definition of a manifold allows us to dispense with these connections or, again, from its connection to intuitive geometrical thinking, and also extend this concept in mathematics or in physics. This is something Riemann’s lecture gives us as well, even though some of this mathematics is still expressed verbally, which would be quite uncommon now and has been uncommon for quite a while, uncommon but not entirely absent.

Thus, one finds this type of approach in Poincaré’s work, as in parts of his series of papers on the curves defined by differentials published in the 1880s and related work [57], which also led him to the so-called qualitative theory of differential equations.[17] Poincaré’s strategy in these papers was also novel (and exhibited a contrast to or even a reversal of algebraic modernist trends emerging at the time) in that, in Gray’s words, it was to consider “the solutions as curves, not as functions, and to consider the global behavior of these curves” [26, p. 254]. Gray adds: “Two kinds of topological thinking entered this early work: the algebraic topological ideas of the genus of a surface and the recognition that many surfaces are characterized by their genus alone; and the point-set topological idea of everywhere dense and perfect sets, which though not original with Poincaré, are put to novel uses” [26, p. 254]. Among many remarkable outcomes of this thinking was Poincaré’s analysis of curves and flows on a torus, an elliptic curve, if considered over $\mathbb{C}$. Poincaré’s work is a chapter in its own right in the modernist history of the concept of curve. His “conventionalism” in physics is also important for the history of modernist

[16]See [3, pp. 523–524] on Grothendieck’s use of the term “multiplicity,” which is, on the one hand, specific (close to what is now called “orbifold”), and on the other hand, is clearly chosen to convey the multiple, plural nature of the objects considered. This is also true concerning Riemann’s concept of manifold. I would argue that Riemann and Grothendieck share thinking in terms of multiplicities as their primary mathematical philosophy, a modernist trend that is especially pronounced in their thinking. As will be seen, this philosophy, manifested already in Grothendieck’s early work in functional analysis, drives his use of sheaves and category theory (both concepts of the multiple), and then his concept of topos. Nothing is ever single. Everything is always positioned in relation to a multiplicity, is “sociological,” and is defined and studied as such, which is itself a trend characteristic of modernism.

[17]While, the concept of “qualitative” is of much interest in the context of this article, it would require a separate treatment. I might note, however, that, while the qualitative could be juxtaposed to the quantitative, it has more complex relationships with the algebraic, which is not the same as the quantitative, just as the geometrical is not the same the qualitative. Still the genus of a surface, which is a number and thus is quantitative, is important in a qualitative approach to its topology or geometry. See note 4 above.

epistemology, from relativity to quantum mechanics, even though his position was ultimately realist. I cannot unfortunately address either subject within my scope.

Poincaré, however, joins Fermat, Descartes, Gauss, Riemann, and Hilbert in reminding us that curves are still curves, and while they may and even must be replaced by or rather translated into algebra, their geometry never quite leaves our thought and our work and exposition of mathematics. Riemann makes an extraordinary use of this situation in his work, again, nearly unique, even next to other great figures just mentioned, in mixing geometry, topology, analysis, and algebra with each other and all of them with philosophical concepts, general phenomenal intuition, and the power of language, in turn intermixed as well. It is, I might add, not a matter of inventing evocative metaphors, but rather of using these multiple, manifold, means for creating new mathematical and physical (and sometimes philosophical) concepts, such as that of *Mannigfaltigkeit* [55, pp. 341–342].

I close this section with a more general point central for my argument concerning the modernist transformation of mathematics and, in part via this mathematics, modernist physics, into essential algebraic mathematical theories, keeping in mind other components of this transformation to which this qualification equally pertains. As discussed from the outset, we still depend on and are helped by a more conventional geometrical or topological thinking in modernist, fundamentally algebraic, thinking; general phenomenal intuition; ordinary language and concepts, or other general aspects of human thinking or cognition, such as narrative, for example.[18] On the other hand, these aspects of our thinking may also become limitations in mathematics and physics alike, and, as Heisenberg argued, in quantum mechanics, modernist mathematics frees us from these limitations, or at least gives us more freedom from them. Technically, so does all mathematics, geometry and topology included, vis-à-vis other components just listed, but algebra and, with it, modernist mathematics extends this freedom. Making a curve an algebraic equation, as in Fermat and Descartes; extending the concept of curve in mathematics to include (topologically speaking) surfaces by making them curves over $\mathbb{C}$, as (at least in

[18] On narrative in mathematics, see [19]. Of particular interest in the present context, as part of the history leading to the modernist algebraization of mathematics, is B. Mazur's contribution there, which offers a discussion of L. Kronecker's "dream, vision, and mathematics" in "Visions, Dreams, and Mathematics" [41]. It might be added that Kronecker's "dream, vision, and mathematics," also decisively shaped those full-fledged modernist ideas of Weil. It may also be connected to Grothendieck's work. See the article by A'Campo et al for a suggestion concerning this possibility, as part of a much broader network, opened by Grothendieck's work on Galois theory ("the absolute Galois group"), which confirms Galois' work as a key juncture of the trajectory leading from modernity to modernism in mathematics [2, p. 405, also n. 12]. These themes could be conceptually linked to quantum field theory, via M. Kontsevitch's work on the "Cosmic Galois Group" (Cartier 2001), noted below (note 19). The article by A'Campo et al is also notable for a remarkable narrative trajectory of Grothendieck's work it traces. This confirms the role of narrative as part of mathematics itself and the philosophy of mathematics rather than only of the history of mathematics, a key theme of Mazur's and other articles in [19]. The present author's contribution to this volume deals with the epistemology of narrative, along the lines of this article [53].

effect) in Riemann, or even more so making a curve, or even just a point, a topos in Grothendieck; and using infinite-dimensional Hilbert spaces as the predictive mathematical technology of quantum mechanics (in the absence of a representation of quantum objects and behavior, which would depend on physical concepts) in Heisenberg, are all examples of taking advantage of this freedom. This freedom may not be complete, but it makes possible pursuits of previously insurmountable tasks.

5.4 Curves from Modernity to Modernism: Three Cases

5.4.1 Curves as Algebra: Descartes/Fermat/Diophantus

I shall now discuss three cases shaping the idea of curve, and geometry and mathematics in general from modernity to modernism. For the reasons explained in the Introduction, I leave aside most of the relevant earlier history and begin with Fermat and Descartes. Then I move to Riemann, and finally, to Weil and Grothendieck, with Riemann still as a key background figure.

Fermat's work is both remarkable and seminal historically, also in influencing Descartes' work and the development of calculus, and, of course, especially in view of his famous, "last," theorem, the study of algebraic and specifically elliptic curves. The deeper mathematical nature of elliptic curves was ultimately revealed by unexpected connections, via the Taniyama-Shimura conjecture (now the "modularity theorem") and related developments, to Fermat's last theorem, which enabled Wiles' proof of the theorem, as a consequence of the modularity theorem for semistable elliptic curves, which he proved as well. These connections could not of course have been anticipated by Fermat. On the other hand, his ideas concerning elliptic curves remain relevant, and are a powerful manifestation of the algebraization and number-theorization of the geometrical ideas then emerging. Thus, according to Weil, first commenting on Fermat's last theorem and Fermat's famous remark "that he discovered a truly remarkable proof for [it] 'which this margin is too narrow to hold,' " and then on Fermat's study of elliptic curves:

> How could he have guessed that he was writing for eternity? We know his proof for biquadrates . . . ; he may well have constructed a proof for cubes, similar to the one which Euler discovered in 1753 . . . ; he frequently repeated those two statements . . . , but never the more general one. For a brief moment perhaps, and perhaps in his younger days . . . , he must have deluded himself into thinking that he had the principle of a general proof; what he had in mind on that day can never be known.
>
> On the other hand, what we possess of his methods for dealing with curves of genus 1 is remarkably coherent; it is still the foundation for the modern theory of such curves. It naturally falls into two parts; the first one, directly inspired by Diophantus, may conveniently be termed a method of ascent, in contrast with the descent which is rightly regarded as Fermat's own. Our information about the latter, while leaving no doubt about its general features, is quite scanty in comparison with Fermat's testimony about the former . . . .

> and the abundant (and indeed superabundant) material collected by Billy in the *Inventum Novum*.
>
> In modern terms, the "ascent" is nothing else than a method of deriving new solutions for the equations of a curve of genus 1. What was new here was of course not the principle of the method: it has been applied quite systematically by Diophantus... and, as such, referred to, by Fermat as well as by Billy, as "*methodus vulgaris*." The novelty consisted in the vastly extended use which Fermat made of it, giving him at least a partial equivalent of what we would obtain by the systematic use of the group theoretical properties of the rational points on a standard cubic. Obviously Fermat was quite proud of it; writing for himself on the margins of his Diophantus, he calls it "*nostro invention*," and again, writing to Billy: "it has astonished the greatest experts." [67, pp. 104–105]

It still does, which was Weil's point as well. The epoch of algebraization and spatial algebraization of elliptic curves and of mathematical curves in general had commenced, with the arc from Fermat to Wiles through many points of modern and then modernist algebraic geometry. Weil's own work was one these points, even a trajectory of its own, leading him to his rethinking of algebraic geometry, which had a momentous impact on Grothendieck, who, however, also radically transformed it in turn. It would, again, be more accurate to speak of a network of trajectories, manifested in Wiles' proof, which brings together so many of them. It is hard, however, to abandon the metaphor of a curve when dealing with the history of the idea of curve itself.

Descartes took full advantage of this algebraization and gave it its modern coordinate form, still very much in use, thus, as I said, making his project of analytic geometry an intimation of modernist thinking in mathematics and physics at the heart of modernity. This project has its history, too, as part of the history of algebra, especially the concept of equation that, as we just saw, emerged in ancient Greek mathematics, especially with Diophantus (around the third century CE), whose ideas were, again, central to Fermat. Analytic geometry, however, by expressly making geometry algebra, gave mathematics its, in effect, independence of physics and of material nature, thus, along with the work of Descartes' contemporary fellow algebraists, again, in particular Fermat, initiating mathematical modernism within modernity.

In the simplest possible terms, analytic geometry did so because the equation corresponding to a curve, say, $X^2 - 1 = 0$ for the corresponding parabola, could be studied as an algebraic object, independently of its geometrical representation or its connection to physics, which eventually enabled us to define curves even over finite fields and thus as discrete objects, as considered above. A curve becomes, in its composition, defined by its equation, divested from its representational geometrical counterpart. It no longer geometrically idealizes the reality exterior to it. It only represents itself, is its own ontology, akin to a *line* of poetry. The equation, algebra, is the poetry of the curve, confirming and amplifying a separation of a mathematical curve from any curve found in the world, which defines all mathematics. When we say in mathematics, "consider a curve X," we separate it from every curved object in the world, in the way poetry separates its words and ideas from those denoted by ordinary language and the world they represent, as A. Badiou noted in commenting on Mallarmé's theory of poetry, based in this separation [7, p. 47]. This poetry of algebra can define a discrete curve, or can make a curve a surface, or a surface a curve, give it an even more complex spatial algebraic architecture, or, to continue with my artistic metaphor, an ever-more complex composition, such as that of a moduli space, the Teichmüller space (also the Teichmüller *curve*), Grothendieck's or Hilbert's scheme and representable-functors, ... we are as yet far from exhausting the limits of this "poetry" of Riemann surfaces/curves (e.g., [1]).

5.4.2 Curves as Surfaces, Surfaces as Curves: Riemann/Riemann/Riemann

The idea of a Riemann surface is one of Riemann's (many) great contributions to modern and eventually modernist mathematics. In Papadopoulos' cogent account:

> In his doctoral dissertation, Riemann introduced Riemann surfaces as ramified coverings of the complex plane or of the Riemann sphere. He further developed his ideas on this topic in his paper on Abelian functions. This work was motivated in particular by problems posed by multi-valued functions $w(z)$ of a complex variable z defined by algebraic equations of the form
>
> $$f(w, z) = 0,$$
>
> where f is a two-variable polynomial in w and z.
>
> Cauchy, long before Riemann, dealt with such functions by performing what he called "cuts" in the complex plane, in order to obtain surfaces (the complement of the cuts) on which the various determinations of the multi-valued functions are defined. Instead, Riemann assigned to a multi-valued function a surface which is a ramified covering of the plane and which becomes a domain of definition of the function such that this function, defined on this new domain, becomes single-valued (or "uniform"). Riemann's theory also applies to transcendental functions. He also considered ramified coverings of surfaces that are not the plane. [47, p. 240]

The idea of a Riemann surface gains much additional depth and richness when considered along with, and in terms of, Riemann's concept of manifold, his other great invention, introduced, around the same time, in his Habilitation lecture [60]. Riemann did not do so himself, although he undoubtedly realized that Riemann surfaces were manifolds, and they have likely been part of the genealogy of the concept of manifold. Riemann's surfaces were first expressly defined as manifolds by Weyl in *The Concept of a Riemann Surface* [73]. Understanding the concept of a Riemann surface as a complex curve is helped by this perspective. It is an intriguing question whether Riemann himself thought of them as curves, but it would not be surprising if he had. Weyl undoubtedly did, although the point does not figure significantly in his book, focused on the "surface" nature of Riemann surfaces, defined, however, in spatial-algebraic terms. This may be surprising. But then, Weyl was not an algebraic geometer. The work of É. Picard, a key figure in the history of algebraic geometry would be more exemplary in considering this aspect of Riemann's concept [47, 49]. However, that a Riemann surface (with which a family of algebraic curves could be associated) is a manifold is crucial for "making it" both a surface and a curve.

Weyl's argumentation leading him to his definition is an application of a principle very much akin to Tate's "Think geometrically, prove algebraically" or its extension here, "Think both Intuitively Geometrically and Spatially-Geometrically: Prove Algebraically." It is also a manifestation of the spirit of Riemann's thinking in which, as noted earlier, geometry and algebra, indeed geometry, topology, algebra, and analysis, come together in a complex mixture of the rigorous and the intuitive, algebraic and spatial-algebraic, mathematical and physical, mathematical and philosophical, and so forth. His work on his ζ-function and number theory could be brought into this mix as well. According to Weyl:

> It was pointed out . . . that one's intuitive grasp of an analytic form [an analytic function to which a countable number of irregular elements have been added] is greatly enhanced if one represents each element of the form by a point on a surface F in space in such a way that the representative points cover F simply and so that every analytic chain of elements of the form becomes a continuous curve on F. To be sure, from a purely objective point of view, the problem of finding a surface to represent the analytic form in this visual way may be rejected as nonpertinent; for in essence, three-dimensional space has nothing to do with analytic forms, and one appeals to it not on logical-mathematical grounds, but because it is closely associated with our sense perception. To satisfy our desire for pictures and analogies in this fashion by forcing inessential representation of objects instead of taking them as they are could be called an anthropomorphism contrary to scientific principles. [73, p. 16]

I note in passing a criticism, apparent here, of the logicist philosophy of mathematics, which theorized mathematics as an extension of logic and, championed by, among others, Bertrand Russell, was in vogue at the time. This is, however, a separate subject. Weyl will now proceed, again, in the spirit of Riemann, to his definition of a two-dimensional manifold and eventually Riemann's surface, *intrinsically*, rather than in relation to its ambient three-dimensional space. Riemann was building on Gauss' ideas concerning the curvature of a surface and his, as he called it, "*theorema egregium*," which states that the curvature of a surface, which he defined as well, was intrinsic to the surface. It is also this concept and the corresponding spatial algebra that enables one to define a Riemann surface as a curve over C. This intrinsic and abstract, spatial-algebraic, view of a Riemann surface was often forgotten by Riemann's followers, especially at earlier stages of the history of using Riemann's concept. According to Papadopoulos, who in part follows Klein's assessment:

> Riemann not only considered Riemann surfaces as associated with individual multi-valued functions or with meromorphic functions in general, but he also considered them as objects in themselves, on which function theory can be developed in the same way as the classical theory of functions is developed on the complex plane. Riemann's existence theorem for meromorphic functions with specified singularities on a Riemann surface is also an important factor in this setting of abstract Riemann surfaces. Riemann conceived the idea of an abstract Riemann surface, but his immediate followers did not. During several decades after Riemann, mathematicians (analysts and geometers) perceived Riemann surfaces as objects embedded in three-space, with self-intersections, instead of thinking of them abstractly. They tried to build branched covers by gluing together pieces of the complex

> plane cut along some families of curves, to obtain surfaces with self-intersections embedded in three-space. [47, p. 242]

According to Weyl (whom Papadopoulos cites): "Thus, the concept 'two-dimensional manifold' or 'surface' will not be associated with points in three-dimensional space; rather it will be a much more general abstract idea," in effect a spatial-algebraic one in the present definition, and thus is modernist. Weyl's position concerning the nature of mathematical reality is a different matter. As is clear from his philosophical writings (e.g., [72]), Weyl was ultimately a realist (albeit not a Platonist) in mathematics and physics alike, his major contribution to quantum mechanics notwithstanding, contributions also dealing with the role of group theory there, yet another modernist trend in mathematics and physics alike [70]. This aspect of the situation is, however, secondary for the moment, although one might still ask whether if considering a given Riemann surface as either a (topologically) real two-dimensional surface or a curve over C, deal with the same mathematical object. Weyl continued as follows:

> If any set of objects (which will play the role of points) is given and a continuous coherence between them, similar to that in the plane, is defined we shall speak of a two-dimensional manifold. Since all ideas of continuity may be reduced to the concept of neighborhood, two things are necessary to specify a two-dimensional manifold:
>
> (1) to state what entities are the "points" of the manifold;
> (2) to define the concept of "neighborhood." [73, pp. 16–17]

One hears here an echo, deliberate or not, of Hilbert's "tables, chairs, and beer mugs," for "points, straight lines, and planes," mentioned above. In the present view, this means one should define entities, such as points, lines, neighborhoods, by using algebraic symbols and algebraic relationships between them, without referring to any objects in the world represented by ordinary language, even if still using this language, as the concept of a Riemann surface as a curve and then its avatars such as Gromov's concept of a pseudoholomorphic curve (a smooth map from a Riemann surface into an almost complex manifold) exemplify. Its connections to our phenomenal sense of surface are primarily, if not entirely, intuitive, when it comes to the idea of continuity, for example, as defined by Weyl here, in terms of the concept of neighborhood. In any event, a Riemann surface is certainly not a curve in any phenomenal sense. As defined by Weyl, in a pretty much standard way, the concept of manifold is a spatial-algebraic one in the present definition. Weyl's more technical definition, again, pretty much standard, given next, and then his analysis of Riemann surfaces only amplified this point.

This multifaceted nature of Riemann surfaces equally and often jointly defined the history of complex analysis, the main initial motivation for Riemann's introduction of the concept of a Riemann surface, and the history of algebraic curves, both building on this concept, and other developments, for example, in abstract algebra and number theory, including Riemann's work on the ζ-function and the distribution of primes.[19] All these developments were unfolding towards modernism during the period between Riemann and Weyl, whose book initiated the (modernist) treatment of the concept of a Riemann surface that defines our understanding of the concept. This history explains my triple subtitle, "Riemann/Riemann/Riemann"—the Riemann of the concept of manifold, the Riemann of the concept of Riemann surfaces, the Riemann of complex analysis. A few more Riemanns could be added. This multiple and entangled history shaped algebraic geometry, eventually leading to the work of A. Weil, Grothendieck, M. Artin, J. Tate, and others, ultimately extending mathematical modernism to our own time.

5.4.3 Curves as Discrete Manifolds: Grothendieck/Weil/Riemann

One of the great examples of this extension is the concept of algebraic curve or algebraic variety in general over a finite field and the study of such objects by the standard tools of algebraic topology, in particular homotopy and cohomology theories, which have previously proven to be effective tools, technologies, for the study of complex algebraic varieties.

The origin of this project goes back to Weil, a key figure of the later stage of mathematical modernism, especially in bringing together algebra, geometry, and number theory, in which he was a true heir of Fermat (and he probably saw himself as one), as well as of Kronecker (in this case, Weil certainly saw himself as one). Riemann is still a key figure in the history leading to Weil's work in algebraic geometry, first of all, again, in view of his concepts of a Riemann surface and a covering space, but also the least by virtue of introducing the concept of a discrete manifold in his Habilitation lecture. (Riemann, thus, was instrumental in the history of both discrete and infinite-dimensional spaces of modernism.) G. Fano, one of the founders of finite geometry, belonged to the Italian school of geometry (1880s–1930s), contemporary with and an important part of the history of mathematical

[19]For an extensive historical account of the history of complex function theory, only mentioned in passing here, see [10], which considers at length most key developments conjoining geometry and complex analysis, from Cauchy to Riemann and then of Riemann's work [10, pp. 189–213, 259–342]. Intriguingly, the algebra of quantum field theory found the way to use Riemann's algebraic work, his work and his hypothesis concerning the ζ-function (one of the greatest, if not the greatest, of yet unsolved problems of mathematics). The ζ-function plays an important role in certain versions of higher-level quantum field theory. See P. Cartier's discussion, which introduces an intriguing idea of the "Cosmic Galois group" [11] and A. Connes and M. Marcoli's book [14], which explores the role of Riemann's differential geometry in this context. The latter is a long and technical work in noncommutative geometry, which uses Grothendieck's motive cohomology theory, but see p. 10 for an important definition of "the Riemann-Hilbert correspondence." This is yet another testimony to the fact that much of modernism in mathematics and even in physics takes place along the trajectory or again, a network of trajectories between Riemann and Grothendieck. See note 16.

modernism (Fano was a student of F. Enriques), strongly influenced by Riemann's thought. Representatives of the Italian school (quite a few of them, even counting only major figures) made major contributions to many areas of geometry, especially algebraic geometry, which formed an important part of the very rich and complex (modernist) history, leading to Weil's work under discussion. Weil suggested that a cohomology theory for algebraic varieties over finite fields, now known as Weil cohomologies, could be developed, by analogy with the corresponding theories for complex algebraic varieties or topological manifolds in general. Weil's motivation was a set of conjectures (these go back to Gauss), known as the Weil conjectures, concerning the so-called local ζ-functions, which are the generating functions derived from counting numbers of points on algebraic varieties over finite fields. These conjectures, Weil thought, could be attacked by means of a proper cohomology theory, although he did not propose such a theory himself.

In order to be able to do so, one needed, first, a proper topology, which was nontrivial because the objects in question are topologically discrete. A more "native" topology that could be algebraically defined by them, known as Zariski's topology, did not work, because it had too few open sets. The decisive ideas came from Grothendieck, helped by the sheaf-cohomology theory and category theory, known as "cohomological algebra," by then the standard technology of algebraic topology. Using these tools, a hallmark of Grothendieck's thinking throughout his career, and his previous concepts, such as that of "scheme," eventually led him to topos theory, arguably the culminating example of spatial algebra, and étale cohomology, as a viable candidate for Weil's cohomology, which it had quickly proven to be. By using it, Grothendieck (with Artin and J.-L. Verdier) and P. Deligne (his student) were able to prove Weil's conjectures, and then Deligne, who previously proved the Riemann hypothesis conjecture (considered the most difficult one), found and proved a generalization of Weil's conjectures. Grothendieck's key, extraordinary, insight, also extending what I call here spatial algebra in a radically new direction, was to generalize, in terms of category theory, the concept of "open set," beyond a subset of the algebraic variety, which was possible because the concept of sheaf and of the cohomology of sheaves could be defined by any category, rather than only that of open sets of a given space. Étale cohomology is defined by this type of replacement, specifically by using the category of étale mappings of an algebraic variety, which become "open subsets" of the finite unbranched covering spaces of the variety, a vast and radical generalization of Riemann's concept of a covering space. Grothendieck was also building on some ideas of J.-P. Serre. Part of the origin of this generalization was the fact that the fundamental group of a topological space, say, again, a Riemann surface, could be defined in two ways: it can either be defined more geometrically, as a group of the sets of equivalence classes of the sets of all loops at a given point, with the equivalence relation given by homotopy (itself an example of the history of the idea of curve); or it can be defined even more algebraically, as a group of transpositions of covering spaces. In this second, algebraic, definition, the fundamental group is analogous to the Galois group of the algebraic closure of a field. Serre was the first to consider for finite fields, importantly for Grothendieck's work on étale cohomology, a concept that, thus, has its genealogy in both Galois' and Riemann's thought. (The connection has been established in the case of Riemann surfaces long before then [e.g., [73, p. 58]].) Grothendieck's concept of étale mappings gives a sufficient number of additional open sets to define adequate cohomology groups for some coefficients, for algebraic varieties over finite fields. In the case of complex varieties, one recovers the standard cohomology groups (with coefficients in any constructible sheaf).

Some of the most elegant calculations concern algebraic curves over algebraically closed fields, beginning with elliptic ones [4, 5]. These calculations are also important because they are the initial step in calculating étale cohomology groups for other algebraic varieties by using the standard means of algebraic topology, such as spectral sequences of a fibration. My main point at the moment is that (spatial) algebra makes algebraic curves over such finite fields fully mathematically analogous to standard algebraic curves, beginning, again, with elliptic curves, as studied by Fermat, again, a major inspiration for Weil.

Now, the category of étale mapping is a topos, a concept that is, for now, the most abstract form of what I call here spatial algebra. Although, as became apparent later, étale cohomologies could be defined for most practical uses in simpler settings, the concept of topos remains crucial, especially in the present context, because it can be seen as the concept of a covering space over a Riemann surface converted into the (spatial-algebraic) concept of topology of the surface itself, and then generalized to any algebraic variety. The concept of topos also came to play a major role in mathematical logic, a major development of mathematical modernism, thus bringing it together with the modernist problematic considered here. The subject cannot, however, be addressed here, except by noting that mathematical logic is already an example of modernist algebraization of mathematics, with radical epistemological implications concerning the nature of mathematical reality, or the impossibility of such a concept. On the other hand, Grothendieck's use of his topoi in algebraic geometry is essentially ontological rather than logical, although his overall philosophical position concerning the nature of mathematical reality remains somewhat unclear, for example, whether it conforms or not to mathematical Platonism, and the subject will be put aside here as well. In any event, it does not appear that Grothendieck was ever thinking of his topoi or in general in terms of breaking with the ontological view of mathematics. My main focus here is the mathematical technologies that the concept of topos, whatever its ontological status, enables, such as étale cohomology. Such technologies may suggest a possible break with the possibility of the ultimate ontological description of mathematical reality, again, assuming that any ultimate reality, say, again, of the type considered in physics, is even possible in mathematics.

It would not be possible here to present topos theory in its proper abstractness and rigor, prohibitive even for those trained in the field of algebraic geometry. The essential philosophical nature of the concept, briefly indicated above, may, however, be sketched in somewhat greater detail, as an example of both a rich mathematical concept in its own terms and of the modernist problematic in question here. First, very informally, consider the following way of endowing a space with a structure, generalizing the definition of topological space in terms of open subsets, as mentioned above. One begins with an arbitrarily chosen space, X, potentially any given space, which may initially be left unspecified in terms of its properties and structure. What would be specified are the relationships between spaces applicable to X, such as mapping or covering one or a portion of one, by another. This structure is the arrow structure $Y \rightarrow X$ of category theory, where X is the space under consideration and the arrow designates the relationship(s) in question. One can also generalize the notion of neighborhood or of an open subspace of (the topology of) a topological space in this way, by defining it as a relation between a given point and space (a generalized neighborhood or open subspace) associated with it. This procedure enables one to specify a given space not in terms of its intrinsic structure (e.g., a set of points with relations among them) but "sociologically," throughout its relationships with other spaces of the same category, say, that of algebraic varieties over a finite field of characteristic p [38, p. 7]. Some among such spaces may play a special role in defining the initial space, X, and algebraic structures, such as homotopy and cohomology, as Riemann in effect realized in the case of covering spaces over Riemann surfaces, which, as I explained, was one of the inspirations for Grothendieck's concept of topos and more specifically of an étale topos.

To make this a bit more rigorous (albeit still quite informal), I shall briefly sketch the key ideas of category theory. It was introduced as part of cohomology theory in algebraic topology in 1940 and, as I said, later extensively used by Grothendieck in his approach to cohomological algebra and algebraic geometry, eventually leading him to the concept of topos.[20] Category theory considers multiplicities (which need not be sets) of mathematical objects conforming to a given concept, such as the category of differential manifolds or that of algebraic varieties, and the arrows or morphisms, the mappings between these objects that preserve this structure. Studying morphisms allows one to learn about the individual objects involved, often to learn more than we would by considering them only or primarily individually. In a certain sense, in his Habilitation lecture, Riemann already thinks categorically. He does not start with a Euclidean space. Instead, the latter is just one specifiable object of a large categorical multiplicity, here that of the category of differential or, more narrowly, Riemannian manifolds, an object marked by a particularly simple way we can measure the distance between any two points. Categories themselves may be viewed as such objects, and in this case one speaks of "functors" rather than "morphisms." Topology relates topological or geometrical objects, such as manifolds, to algebraic ones, especially, as in the case of homotopy and cohomology groups, introduced by Poincaré. Thus, in contrast to geometry (which relates its spaces to algebraic aspects of measurement), topology, almost by its nature, deals

[20]One of his important, but rarely considered, contributions is his work on Teichmüller space, the genealogy of which originates in Riemann's moduli problem, powerfully recast by Grothendieck in his framework. Especially pertinent in the present context is the idea of a "Teichmüller curve" and then Grothendieck's recasting of it, another manifestly modernist incarnation of the idea of curve, via Riemann. Conversely, the theory provided an important case for Grothendieck to use his new technology. Étale cohomology came next. This is yet another modernist trajectory extending from Riemann and Grothendieck. For an excellent account, see A'Campo et al. [1].

with functors between categories of topological objects, such as manifolds, and categories of algebraic objects, such as groups.

A topos in Grothendieck's sense is a category of spaces and arrows over a given space, used especially for the purpose of allowing one to define richer algebraic structures associated with this space, as explained above. There are additional conditions such categories must satisfy, but this is not essential at the moment. To give one of the simplest examples, for any topological space S, the category of sheaves on S is a topos. The concept of topos is, however, very general and extends far beyond spatial mathematical objects (thus, the category of finite sets is a topos); indeed, it replaces the latter with a more algebraic structure of categorical and topos-theoretical relationships between objects. On the other hand, it derives from the properties of and (arrow-like) categorical relationships between properly topological objects. The conditions, mentioned above, that categories that form topoi must satisfy have to do with these connections.

Beyond enabling the establishing of a new cohomology theory for algebraic varieties, as considered above, topos theory allows for such esoteric constructions as nontrivial or nonpunctual single-point "spaces" or, conversely, spaces (topoi) without points (first constructed by Deligne), sometimes slyly referred to by mathematicians as "pointless topology." Philosophically, this notion is far from pointless, especially if considered within the overall topos-theoretical framework. In particular, it amplifies a Riemannian idea that "space," defined by its relation to other spaces, is a more primary object than a "point" or, again, a "set of points." Space becomes a Leibnizean, "monadological" concept, insofar as points in such a space (when it has points) may themselves be seen as a kind of monad, thus also giving a nontrivial structure to single-point spaces. These monads are certain elemental but structured entities, spaces, rather than structureless entities (classical points), or at least as entities defined by (spatial) structures associated with and defining them [1]. Naturally, my appeal to monads is qualified and metaphorical. Leibniz's monads are elemental souls, the atoms of soul-ness, as it were. One might, however, also say, getting a bit more mileage from the metaphor, that the space thus associated with a given point is the soul of this point, which defines its nature. In other words, not all points are alike insofar as the mathematical (and possibly philosophical) nature of a given point may depend on the nature or structure of the space or topos to which it belongs or with which it is associated in the way just described. This approach gives a much richer architecture to spaces with multiple points, and one might see (with caution) such spaces as analogous to Leibniz's universe composed by monads. It also allows for different (mathematical) universes associated with a given space, possibly a single-point one, in which case a monad and a universe would coincide. Grothendieck's topoi are possible universes, possible worlds, or com-possible worlds in Leibniz's sense, without assuming, like Leibniz (in dealing with the physical world), the existence of only one of them, the best possible.

One might also think of this ontology as an assembly of surface ontologies (Grothendieck's concept of topos is, again, ontological, rather than logical, as in his logical followers), in the absence of any ultimate ontology, or even, as against physics, any ultimate mathematical or otherwise mental reality, thus connecting on modernist lines the multiple and the unthinkable. Topoi are multiple universes, defined ontologically, in the absence of a single ultimate reality underlying them; they are investigated by means of technologies such as cohomologies or homotopies (which can be defined for them as well). I shall not consider Grothendieck's topos as such from this perspective, which, again, does not appear to be Grothendieck's own. Instead, I shall discuss next the ontological and epistemological architecture of modernism mathematics more generally in relation to the concept of technology, conceived broadly so as to include the means by which mathematics studied itself; and I shall briefly comment on topos theory in this context. Quantum theory will, yet again, serve as a convenient bridge, in this case as much because of the differences as the similarities between physics and mathematics.[21]

5.5 Mathematical Modernism Between Ontologies and Technologies

While, roughly speaking, technology is a means of doing something, enabling us to get "from here to there," as it were, the concept of mathematical technology that I adopt extends more specifically the concept of "experimental technology" in modern, post-Galilean, physics, defined, as explained, by its jointly experimental and mathematical character. I note, first, that experimental technology is a broader concept than that of measuring instruments, with which it is most commonly associated in physics. It would, for example, involve devices that make it possible to use the measuring instruments, a point that, as will be seen, bears on the concept of technology in mathematics. Thus, the experimental technology of quantum physics, from Geissler tubes and Ruhmkorrf coils of the nineteenth century to the Large Hadron Collider of our time, enables us to understand how nature works at the ultimate level of its constitution. In the present interpretation, this technology allows us to know the effects this constitution produces on measuring devices (described, along with these effects themselves, by classical physics), without allowing us to represent or even conceive of the character of this constitution. The character of these effects is, however, sufficient for creating theories, defined by their *mathematical technologies*, such as quantum mechanics and quantum field theory, that can predict these effects. Thus, quantum physics is only about the relationships between mathematical and experimental technologies used, vis-à-vis classical physics or relativity, or mathematics itself. All mathematics used in quantum physics is technology; in mathematics, or in classical physics or relativity, some mathematics is also used ontologically. Quantum objects themselves are not technology; they are a form of reality that technology helps us to discover, understand, work with, and so forth, but in this case, at least in the present

[21]The discussion to follow is partly adopted from [54, pp. 265–274]. My argument here is essentially different, however.

interpretation, without assuming or even precluding any ontological representation of this reality. Quantum objects can of course become part of technology, beginning with the quantum parts of measuring instruments through which the latter interact with quantum objects, or as parts of devices we use elsewhere, such as lasers, electronic equipment, MRI machines, and so forth.

In the wake of Heisenberg's discovery and Born and Jordan's work in 1925 [9, 31], Bohr commented as follows:

> In contrast to ordinary mechanics, *the new quantum mechanics does not deal with a space-time description of the motion of atomic particles*. It operates with manifolds of quantities which replace the harmonic oscillating components of the motion and symbolize the possibilities of transitions between stationary states [manifested in measuring instruments]. These quantities satisfy certain relations which take the place of the mechanical equations of motion and the quantization rules [of the preceding quantum theory]. . . .
>
> It will interest mathematical circles that the mathematical instruments created by the higher algebra play an essential part in *the rational formulation* of the new quantum mechanics. Thus, the general proofs of the conservation theorems in Heisenberg's theory carried out by Born and Jordan are based on the use of the theory of matrices, which go back to Cayley and were developed especially by Hermite. It is to be hoped that a new era of mutual stimulation of mechanics and mathematics has commenced. To the physicists it will at first seem deplorable that in atomic problems we have apparently met with such a limitation of our usual means of visualization. This regret will, however, have to give way to thankfulness that mathematics in this field, too, presents us with the tools to prepare the way for further progress. [8, v. 1, pp. 48, 51; emphasis added]

Bohr's appeal to "the *rational* formulation of the new quantum mechanics" merits a brief digression, especially in conjunction with his several invocations of the "irrationality" inherent in quantum mechanics, a point often misunderstood. The "irrationality" invoked here and elsewhere in Bohr's writings is not any "irrationality" of quantum mechanics, which Bohr, again, sees as a "rational" theory [8, v. 1, p. 48]. Bohr's invocation of "irrationality" is based on an analogy with irrational numbers, reinforced perhaps by the apparently irreducible role of complex numbers and specifically the square root of -1, i (an irrational magnitude in the literal sense because it cannot be presented as a ratio of two integers) in quantum mechanics, or quantum field theory. It is part of the history of the relationships between algebra (initially arithmetic) and geometry, from the ancient Greeks on. As noted earlier, the ancient Greeks, who discovered the (real) irrationals, could not find an arithmetical, as opposed to geometrical, form of representing them. The Greek terms were "alogon" and "areton," which may be translated as "incommensurable" and "incomprehensible," the latter especially fitting in referring to quantum objects and processes. The problem was only resolved, by essentially modernist mathematical means (algebra played a major role), in the nineteenth century, after more than 2000 years of effort, with Dedekind and others, albeit, in view of the undecidability of Cantor's continuum hypothesis, perhaps only resolved as ultimately unresolvable. It remains to be seen whether quantum mechanical "irrationality" will ever be resolved by discovering a way to mathematically or otherwise represent quantum objects and processes. As thing stand now, quantum mechanics is a rational theory of something that is irrational in the sense of being inaccessible to a rational representation or even to thinking itself. In other words, at stake is a replacement of a *rational representational* theory, classical mechanics, with a *rational probabilistically or statistically predictive* theory. This replacement is the rational quantum mechanics introduced by Heisenberg, a reversal of what happened in the crisis of the incommensurable in ancient Greek mathematics, which compelled it to move from arithmetic to geometry.

Heisenberg's thinking revolutionized the practice of theoretical physics and redefined experimental physics or reflected what the practice of experimental physics had in effect become in dealing with quantum phenomena. The practice of experimental physics no longer consists of tracking what happens or what would have happened independently of our experimental technology, but in creating new configurations of this technology, which allows us to observe effects of quantum objects and behavior manifested in this technology.[22] This practice reflects the fact that what happens is unavoidably defined by what kinds of experiments we perform, and how we affect quantum objects, rather than by tracking their independent behavior, although their independent behavior does contribute to what happens. The practice of theoretical physics no longer consists in offering an idealized mathematical representation of quantum objects and their behavior, but in developing mathematical technology that is able to predict, in general (in accordance with what obtains in experiments) probabilistically, the outcomes of always discrete quantum events, observed in the corresponding configurations of experimental technology.

Taking advantage of and bringing together two meanings of the word "experiment" (as a test and as an attempt at an innovative creation), one might say that the practice of quantum physics is the first practice of physics that is both, jointly, fundamentally experimental and fundamentally mathematical. That need not mean that this practice has no history; quite the contrary, creative experimentation has always been crucial to mathematics and science, as the work of all key figures discussed in this article demonstrates. Galileo and Newton, are two great examples in classical physics: they were experimentalists, both in the conventional sense (also inventors of new experimental technologies, new telescopes in particular) and, in their experimental and theoretical thinking alike, in the sense under discussion at the moment. Nevertheless, this experimentation acquires a new form with quantum mechanics and then extends to higher level quantum theories, and, as just explained, a new understanding of the nature of experimental physics. The practice of quantum physics is *fundamentally* experimental because, as just explained, we no longer track, as we do in classical physics or relativity, the independent behavior of the systems considered, and thus track what happens in any event, by however ingenious

[22] I qualify by "unavoidably" because we can sometimes define by an experiment what will happen in classical physics, say, by rolling a ball on a smooth surface, as Galileo did in considering inertia. In this case, however, we can then observe the ensuing process without affecting it. This is not so in quantum physics, because any new observation essentially interferes with the quantum object under investigation and defines a new experiment and a new course of events. Only some observations do in classical physics.

experiments. We *define* what will happen in the experiments we perform, by how we experiment with nature by means of our experimental technology.

By the same token, quantum physics is fundamentally mathematical, because its mathematical formalism is equally not in the service of tracking, by way of a mathematical representation, what would have happened anyhow, which tracking would shape the formalism accordingly, but is in the service of predictions required by experiments. Indeed, quantum theory experiments with mathematics itself, more so and more fundamentally than does classical physics or relativity. This is because quantum theorists invent, in the way Heisenberg did, effective mathematical schemes of whatever kind and however far they may be from our general phenomenal intuition, rather than proceeding by refining mathematically our phenomenal representations of nature, which limits us in classical physics or even (to some degree) in relativity. One's choice of a mathematical scheme becomes relatively arbitrary insofar as one need not provide any representational physical justification for it, but only need to justify this scheme by its capacity to make correct predictions for the data in question. It is true that in Heisenberg's original work the formalism of quantum mechanics extended (via the correspondence principle) from the representationally justified formalism of classical mechanics. Heisenberg and then other founders of the theory (such as Born and Jordan, or Dirac) borrowed the equations of classical mechanics. However, they replaced the variables used in these equations with Hilbert-space operators, thus using modernist mathematics, which was no longer justified by their representational capacity but, in Heisenberg's words, by "the agreement of their predictions with the experiment" [32, p. 108]. One's mathematical experimentation may, thus, be physically motivated, but it is not determined by representational considerations, the freedom from which also liberates one's mathematical creativity. Rather than with the equations of classical mechanics, one could have started directly with Hilbert-spaces and derived the necessary formalism by certain postulates, as was done by von Neumann in his classical book, admittedly, with quantum mechanics already in place [66]. Other versions of the formalism, such as the C*-algebra version and, more recently, the category-theory version are products of this type of mathematical experimentation. It is true that all these versions have thus far been essentially mathematically equivalent, and in particular, the role of complex numbers appears to be unavoidable. It is difficult, however, to be entirely certain that this will remain the case in the future, even if no change is necessary because of new experimental data. The invention of quantum theory was essentially modernist in its epistemology and its spirit of creative experimentation (which it shared with contemporary modernist literature and art) alike, as well as in its use of modernist mathematics. Heisenberg was the Kandinsky of physics.

One could indeed think of the technological functioning of mathematics even in mathematics itself: certain mathematical instruments, such as homotopy or cohomology groups, are technologies akin to measuring instruments in physics, with the role of reality taken in each case by the corresponding topological space. According to J.-P. Marquis, who borrows his conception of mathematical technology from quantum physics "they provide information about the corresponding topological space. . . . [T]hey are epistemologically radically from . . . transformation [symmetry] groups of a space. They do not act on anything. The purpose of these geometric devices is to classify spaces by their different *homotopy [or cohomology] types*." By contrast, fibrations, for example, important for using homotopy and cohomology groups, including, as noted, in étale cohomologies, are not "measuring instruments," but rather "devices that make it possible to apply measuring instruments [such as cohomology and homotopy groups] and other devices" [39, p. 259]. The key point here is that the invention and use of mathematical technologies is crucial for mathematics, in modernism from Riemann's concept of the genus of a Riemann surface to Grothendieck's invention of étale cohomology, with the whole history of algebraic topology between them, keeping in mind that any technology can and is eventually likely to become obsolete, as Marquis notes [39, p. 259]. In quantum theory, all mathematics used is technology (vs. classical physics or relativity where it can also be used ontologically) and it can become obsolete, too, as that of classical physics became in quantum physics. Ontologies can become obsolete, too, such as, at least for some, that of set theory, replaced by category theory, which redefines, for example, the concept of topological space. On the other hand, the concept of physical reality is unlikely to go away any time soon. (The name may change, and "matter" has sometimes been used instead.) Could the same be said about some form of mathematical reality? I would like to offer a view that suggests that it is possible to answer this question in the negative, thus fundamentally differentiating mathematical and physical reality.

First, I note that, in parallel with the experimental and mathematical technology used in quantum physics, the mathematical technology in mathematics may not only be used to help us to represent mathematical reality (although it may be used in this way, too) but also to enable us to experiment with this reality, without representing it. In mathematics, moreover, where all our ontologies and technologies are mental (although they can be embodied and communicated materially), one need not assume the independent reality, shared or not, of the type assumed, as material reality, in physics, beyond representation or even conception as the ultimate character of this reality may be assumed to be, as it is in quantum theory in nonrealist interpretations. Technically, it follows that, if this reality is beyond representation or even conception, it is not possible to rigorously claim that this reality as such is single any more than multiple. The "sameness" of this reality is itself an effect ascertainable by our measuring instruments, which, however, compel us to assume that we deal with the same types of quantum objects (electrons, photons, and so forth) and their composites, regardless where and when we perform our experiments. In any event, mathematical realities are always multiple, a circumstance of which, as we have seen, Grothendieck's topos theory takes advantage, as it also experiments with and even creates them. In fact, as I shall now suggest, mathematical realities always belong to individual human thinking, although they may be related to each other. Indeed, they always are so related, to one degree or another, which prevents them from being purely subjective. On the other hand, they can acquire a great degree of objectivity because they can be (re)constructed, for example in checking mathematical proofs. This situation is parallel to that of quantum physics, too, where, as explained earlier, this objectivity is provided by quantum phenomena, observed in measuring instrument, while, however, the ultimate material reality is still assumed.

The nature of mathematical reality, representable or not, has been debated since Plato, whose ghost still overshadows this debate in modernist mathematics, as Gray rightly suggests, and the differences between his and the present view of modernism do not affect this point [26]. Mathematical Platonism assumes the existence of mathematical reality, whether representable by our mathematical concept or not (there are different views on this point within mathematical Platonism), as independent of our thinking. I shall not enter these debates, including those concerning mathematical Platonism, apart from noting that the questions of the domain, "location," of this reality have, in my view, never been adequately answered. It is difficult to think of that which is not material and yet is outside human thought, unless it is divine, which, however, is not a common assumption among those who subscribe to mathematical Platonism. In any event, I only assume here the reality of human thought, thus, generally, different for each of us, as the only domain in relation to which one can speak of the reality of mathematical objects and concepts, in juxtaposition to the material reality of nature in physics. It is possible to assume that the Platonist mathematical reality is a potentiality—the same, even if multiply branched, potentiality—in principle realizable by human thought. I shall comment on this possibility presently.

What could be claimed to exist, ontologically, in our thought without much controversy are mathematical specifications, from strict definitions to partial and indirect characterizations (implying more complete or direct future specifications). Such specifications would involve concepts, structures, logical propositions, or still others elements, which could be geometrical or topological, as well as algebraic. In this respect, there is no difference between geometrical (or topological) and algebraic specifications. All such specifications can, at least in principle, be expressed and presented in language, verbally or in writing, visually, digitally for example, in other words technologically. While algebra helps our mathematical writing (to paraphrase Tate, "think geometrically, write algebraically!"), digital technology helps our geometrical specifications and expression. The computer-generated images of chaos theory are most famous, but low-dimensional topology and geometry have been similarly helped by digital technology.

The question, then, is whether anything else exists in our thought beyond such specifications and the local ontologies they define, at least in our conscious thought, because we can unconsciously think of other properties of a given object or field, which either may eventually be made conscious or possibly never become conscious and thus known. This qualification does not, however, change the nature of the question, because one could either claim that the unconscious could still only contain such specifications or that the object or broader reality in question, as different or exceeding such specifications, somehow exists there. In other words, essentially the same alternative remains in place. On the other hand, some unconscious specifications may never become conscious. It is quite possible that some mathematics, even very great mathematics, or for that matter poetry, never left the unconscious, and there are accounts by mathematicians or poets of dreaming of mathematics or poetry which could not be remembered as what it actually was, although sometimes it can. Only our memory of dreams is conscious, but never dreams themselves. By the same token, whatever is in our unconscious can at some point enter our consciousness: if the Chauvet Cave is the cave of dreams, it is because the consciousness of those who painted them realized their previously unconscious thinking, even though some parts of these painting were, undoubtedly, still manifestations of the unconscious. It is beyond doubt also that our unconscious does a great deal of mathematical thinking, as it does most of our thinking in general, and some of it may never be realized by our conscious thought. Sometimes, not uncommonly in the logical foundations of mathematics (Hilbert held this view), the consistency of a given definition of a mathematical entity is identified with the existence of the corresponding object, a form of mathematical Platonism, if this existence is assumed to be possible outside human thought. Either way, this view poses difficulties given Gödel's incompleteness theorems or even Cantor's set theory and its paradoxes. Our mathematical specifications must of course be logically consistent.

My assumption here is that nothing mathematical actually exists in thought beyond what can be thus specified, perhaps, again, in one's unconscious. This differentiates the situation from that of quantum physics. While quantum phenomena or quantum theories are specified in the same sense (and quantum theories are mathematical in the first place), one assumes, by a decision of thought concerning one's interpretation of quantum phenomena, the existence of the ultimate physical reality, which is beyond representation or even conception and thus specification. I leave aside for the moment whether something nonmathematical can exist in thought apart from any specification, although the position I take here compels me to answer in the negative in this case as well. In the present view, only physical matter in its ultimate constitution exists in this way. This assumption has been challenged as well, with Plato as the most famous ancient case and Bishop Berkeley as the most famous modern case, and is occasionally revived, as a possibility, in the context of quantum theory, but it is still a common assumption. As just noted, it is quite possible that there are (mentally) real things that exist in our unconscious that will never become conscious. It is equally possible, however, that they will enter our consciousness at one point or another. The ultimate constitution of nature, in this interpretation of quantum physics, is not assumed to ever become available, as things stand now. This does not of course preclude that such specifications cannot be made more complete or modified by new concepts, structures, and logical propositions, which would change the objects or concept in question, as say, a Riemann surface, as it developed during over, by now, a long period. However, in the present view, it is no longer possible to see such changes as referring to the same mathematical object (which can, again, be a broad and multiple entity), approached by our evolving concepts. Instead, they create new objects or concepts. Thus, new classes of Riemann surfaces are created by each modification of the concept. There are no Riemann surfaces as such, existing by themselves and in themselves, at any given point of time; there is only what we can think or say about them at a given point of time. By contrast, as things stand now, nothing can be in principle improved in our understanding of the ultimate constitution of matter. We can only improve our understanding and, by using mathematics and new experimental technology, our predictions of the effects on this constitution manifested in measuring instruments.

The present view of mathematical reality has thus a constructivist flavor, in part following Kant's view of mathematics as the synthetic construction of mathematical proposition and concepts by thought [35].[23] According to Badiou, "mathematics is a thought," part of the ontology of thought, and for Badiou this ontology is mathematical, an argument he makes via modernist mathematics, from Cantor's set theory to Grothendieck's topos theory [7, p. 45]. Without addressing Badiou's argument itself (different from the one offered here), I take his thesis literally in the sense that mathematics is only what can be thought, created by thought and then expressed, communicated, and so forth, thus also in accord with the Greek meaning of *máthẽma* as that which can be known and learned, or taught.

It is of course not uncommon to encounter a situation in which a mathematical entity (again, possibly a large and multiple one) that cannot be given, now or possibly ever, an adequate mathematical specification, and is only specified partially mathematically or more fully otherwise. It may, for example, be specified as a phenomenal object or set of objects by means of philosophical concepts, but that can nevertheless be consistently related to, indirectly, and by means of a more properly specified mathematical concept or set of concepts. The latter concepts may, then, function as mathematical technologies which enable one to work with and, to the degree possible, understand this entity, as fibrations or homotopy and cohomology group allow us to understand better and more properly specify the corresponding topological spaces. These technologies are crucial and, while found in all mathematics, their persistent use, in part, against, relying on ontology, is characteristic of mathematical modernism, because of the persistence of the situations of the type just described. In the present view, however, any such entity can only be seen as existing or real if it is sufficiently specified in some way: in terms of phenomenal intuition or philosophical concepts, perhaps partially supplemented by mathematical concepts or structures. It cannot be assigned reality beyond such a specification.

In fact, as we know, in view of Gödel's incompleteness theorems, mathematics, at least if it is rich enough to contain arithmetic, cannot completely represent itself: it cannot mathematically formalize all of its concepts, propositions, or structures, and ultimately itself so as to guarantee its consistency. But it does not necessarily follow that the corresponding unspecifiable reality exists, although one can make this assumption, as Gödel ultimately did on Platonist lines, claiming that there is, at least for now, no human means, mathematical or other, to specify this reality. It only follows that it is impossible to prove that all possible specifications, within

[23] I do not refer by this statement to the trend known as "constructivism" in the foundational philosophy of mathematics, from intuitionism on, relevant as it may be, in part given Kant's influence. I use the term "constructivist" more generally.

any such system, are consistent. (Gödel's theorems do allow that the system can in fact be proven to be inconsistent.) What may be inconceivable is why this is the case, the reality that is responsible for it, which is, however, not a mathematical or even meta-mathematical question, any more than the question why our interaction with nature by means of quantum physics enables us to make correct probabilistic predictions. These questions belong to the biological and specifically neurological nature of our thought, although they may not ultimately be answerable by biology or neuroscience either.

One could speak in considering such as yet unspecified mathematical objects or concepts in terms of a hypothetical potentiality, defined by the assumption that a mathematical object or concept of a certain type could or should exist. Such potentialities are, moreover, only partially, probabilistically, determined by what is sometimes called plausible reasoning, very important in mathematical thinking, as rightly argued by Polya [58]. There are different and possibly incompatible way in which this potentially may be become reality. Consider, paradigmatically, thinking of the equation $X^2 + 1 = 0$ and complex numbers. While this equation (which may be safely assumed to exist in our thought as a mathematical entity) had no real solution, one could have and some had envisioned that it should have a solution and that a mathematical entity or a multiple of such entities, a new type of number, should exist. This hypothesis came to be realized, also literally, insofar, as complex numbers eventually became a mathematical reality. In the present view, however, they were not a mathematical reality before they were correspondingly specified in somebody's thought, say, by the time of Gauss, who was crucial in allowing complex numbers to become a mathematical reality, each time one thinks of them, but in the present view, not otherwise.

The present view, thus, precludes the assumption of an independent mathematical reality. This assumption, again, commonly defines reality in physics, even if this reality is assumed to be beyond representation or conception, as in quantum theory, thus, consistently with the present view of physical reality, as opposed to mathematical Platonism or other positions that claim the existence of mathematical reality independent of human thought, which is in conflict with the present view of mathematical reality. In sum, in the present view, in mathematics all reality is constructed, and this construction may, ontologically, involve multiple "mathematics," as Grothendieck's topos-theoretical ontology shows. This multiplicity is also a consequence of Gödel's undecidability, as exemplified by Cantor's continuum hypothesis, mentioned above. This hypothesis was crucial not only for the question of continuity but also for the question of Cantor's hierarchical order of infinities (the infinity of which was one of his discoveries) and thus for the whole edifice of Cantor's set theory. The hypothesis was proven undecidable by Cohen. It follows, however, that one can extend classical arithmetic in two ways by considering Cantor's hypothesis as either true or false, that is, by assuming either that there is no such intermediate infinity or that there is. This allows one, by decisions of thought, to extend arithmetic into mutually incompatible systems that one can construct, ultimately infinitely many such systems, because each on them will contain at least one undecidable proposition. It is, as noted, in principle possible to assume that all such possible constructions form a single, if multiply branching, potentiality, ultimately realizable in principle. I shall not assess this view except by noting that even if one adopted it strictly in this form, one would still only allow for a vast constructible mathematical potentiality rather than independent mathematical reality. Would this view be *in practice* equivalent to mathematical Platonism? While it may be in practice, the difference in principle would remain important, both in general and because that it would be impossible to assume that, being infinite, this potentiality could ever be realized. In practice, mathematics, again, creates new mathematical realities and, with them, new mathematical potentialities all the time, quite apart from any undecidable propositions. This process will only end when mathematics is no longer with us, and one day it might not be, although curves are likely to remain with us as long as we are around. By contrast, in quantum physics in nonrealist interpretations, the ultimate reality is assumed to exist as unconstructible or (as this view is still constructivist), *constructed* as unconstructible. But this unconstructible physical reality may be related to by means of constructed mathematical realities, such as that of Hilbert-spaces mathematics, again, meaning by a Hilbert space what we can think about or use and objectively share, rather than an independently existent mathematical object.

Thus, along with all realism in physics, the present view radically breaks with all Platonism in mathematics, especially with mathematical Platonism, but arguably with any form of Platonism hitherto. As I said, not all Platonism in mathematics is mathematical Platonism: that of Plato is not. Some forms of realism in physics are, again, forms of Platonism, too, as are, for example, some versions (known as ontological) of the so-called structural realism, according to which mathematical structures are the only reality [36]. As I indicated, Heisenberg, in his later thinking was inclined to this type of view, as against the time of his creation of quantum mechanics [33, pp. 91, 147–166].

While, however, breaking with Platonism, even Plato's own, the modernist thinking considered here in mathematics and physics does retain something, perhaps the most important thing, from Plato—from the *spirit* of Plato—rather than the *ghost* of Plato, intimately linked as these two words, spirit and ghost, are. This thinking retains the essential role of the movement of thought, something as crucial to Plato as to mathematical modernism, however anti-Platonist the latter may become. Heisenberg (whose father was a classicist) was reading Plato's *Timaeus* in the course of his discovery of quantum mechanics, in which he in effect reinvented Hilbert spaces over $\mathbb{C}$, a double, physical and mathematical, modernism [43, v. 2, pp. 11–14]. Some of Plato's thinking, led Heisenberg to his invention of a new mathematical technology in physics, under radically non-Platonist, epistemological assumptions. (Heisenberg, again, adopted a more Platonist view in his later thinking.) That this technology already existed in mathematics does not diminish the significance of this mathematical invention, especially given that Heisenberg used infinite unbounded matrices, never considered previously. The work of the mathematical figures considered here, from Fermat and Descartes to Riemann and from Riemann to Grothendieck and beyond, to split for a moment (but only for a moment) modernity and modernism, was shaped by the spirit of the movement of thought, the spirit that connects modernity and modernism, in mathematics and science, as it does in philosophy and art.

5.6 Conclusion

I close on a philosophical and artistic note by citing Heidegger's conclusion in "The Question Concerning Technology":

> There was a time when it was not technology alone that bore the name techne. Once that revealing that brings forth truth into the splendor of radiant appearing also was called techne.
>
> Once there was a time when the bringing forth of the true into the beautiful was called techne. And the poiesis of the fine arts also was called techne.
>
> In Greece, at the outset of the destining of the West, the arts soared to the supreme height of the revealing granted them.... And art was simply called techne. It was a single, manifold revealing. It was ..., promos, i.e., yielding to the holding-sway and the safekeeping of truth.
>
> The arts were not derived from the artistic. Art works were not enjoyed aesthetically. Art was not a sector of cultural activity.
>
> What, then, was art—perhaps only for that brief but magnificent time? Why did art bear the modest name techne? Because it was a revealing that brought forth and hither, and therefore belonged within poiesis. It was finally that revealing which holds complete sway in all the fine arts, in poetry, and in everything poetical that obtained poiesis as its proper name....
>
> Whether art may be granted this highest possibility of its essence in the midst of the extreme danger [of modern technology], no one can tell. Yet we can be astounded. Before what? Before this other possibility: that the frenziedness of technology may entrench itself everywhere to such an extent that someday, throughout everything technological, the essence of technology may come to presence in the coming-to-pass of truth.
>
> Because the essence of technology is nothing technological, essential reflection upon technology and decisive confrontation with it must happen in a realm that is, on the one hand, akin to the essence of technology and, on the other, fundamentally different from it.
>
> Such a realm is art. But certainly only if reflection on art, for its part, does not shut its eyes to the constellation of truth after which we are questioning.
>
> Thus questioning, we bear witness to the crisis that in our sheer preoccupation with technology we do not yet experience the coming to presence of technology, that in our sheer aesthetic-mindedness we no longer guard and preserve the coming to presence of art. Yet the more questioningly we ponder the essence of technology, the more mysterious the essence of art becomes. [30, pp. 34–35]

I would argue that modernist mathematics, in its more expressly technological aspects and in general, and physics, where in quantum theory all mathematics used is a technology, are *techne* in a sense *close* to that Heidegger wants to give this term here. The reason that I see them as close rather than the same is that Heidegger would allow that the ultimate reality could be accessed by what he saw as the true thought, which he saw as artistic or poetic thought in the sense of this passage. In contrast to some (including some modernist) poetry and art, he sees modernist technology (in its conventional sense) and modernist mathematics and science, including, one might plausibly surmise, as it is understood here, as a form of forgetting rather than approaching techne as art found in ancient Greek thinking.

It is not even clear that he would grant this to the ancient Greek mathematics, much as he admired ancient Greek, especially pre-Socratic, thought, in philosophy and poetry, and he sees the forgetting of the thought in question as beginning with Socrates and Plato. Admittedly, Heidegger's position is complex, especially insofar as how artistic thought can do this remains "mysterious," the mystery that appears to be deepened by our attempts to understand the essence of our, modern and modernist, technology. Nevertheless, Heidegger allows at least the possibility of *thinking* [*Denken*] (his preferred term) this truth, even if not representing it. I would contend, however, that modernist epistemology, even when, in its most radical form, it places the ultimate nature of reality beyond thought itself in physics or rejects the existence of such a (single) ultimate reality in mathematics altogether, does not preclude thought from reaching "the supreme height of the revealing granted them," albeit "creation" might be a better word than "revealing," if there is no ultimate reality that can be revealed. Even if it exists, as in physics, it still cannot be revealed, and in mathematics, again, everything is created, constructed. Coming together of techne and truth is still possible under these conditions and is perhaps not possible otherwise, regardless of one's aspirations for how far our thought can reach. We cannot dispense with truth. What changes are the relationships between truth and reality, and both concepts themselves, while realism and the corresponding concepts of truth still apply and are indispensable at surface levels. *Techne* and truth do come together under these conditions.

This, I have argued here, is precisely what happens in the thought of Riemann, Hilbert, Weyl, Weil, and Grothendieck, and those who followed them in mathematics, or their predecessors, from Fermat and Descartes, or the thought of those who used mathematics in physics, from Kepler and Galileo to Einstein and Heisenberg, and beyond, the Platonist or realist aspirations of many, even most, of these figures notwithstanding. Their thought continues, in mathematics or physics, not the least when it comes to the idea of curve, even when a curve is a surface, the project of the painters of curves of the Chauvet Cave, the cave of dreams, no longer forgotten. The discovery of the cave gave these dreams back to us, and these dreams are about much more than curves, just as modernist art, such as that of Klee, or the modernist mathematics of curves are so much more.

Perhaps, however, our history has kept these dreams alive all along by keeping alive the creative nature of our thought, dreams that we began dreaming well before the frescoes of the Chauvet Cave were painted. Some form of mathematical thinking, just as some form of artistic or philosophical thinking, must have always been part of our history as thinking beings and our dreams, in either sense. The history that at some point gave (we may never know how!) our brain the capacity to have these dreams is immeasurably longer, ultimately as long as the history of life or even the Universe itself, in which, at some point, life has emerged.

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