Chapter 9
$SU(3)$ Analysis of the Subtraction Constants in $g_i(s)$

In the $SU(3)$ limit the masses of all the hadrons belonging to a given $SU(3)$ multiplet have the same value. By direct product of monoparticle states of particles belonging to different $SU(3)$ multiplets we have multiparticle scattering states. In particular, for the two-body interaction process involving such particles we would have only to distinguish between the common masses of the particles in the $SU(3)$ representation involved. For instance, for the lightest pseudoscalar–pseudoscalar scattering we would have only one mass since all the particles belong to the same octet $SU(3)$ representation. Other two-body states of interest for our purposes is the one made by a baryon and one of the lightest pseudoscalars, all of them belonging to octet representations.

It is clear that because of the Wigner–Eckart theorem the matrix element of $SU(3)$ operators transforming within a given $SU(3)$ multiplet between states belonging to definite representations are independent of the hypercharge and third component of isospin, that characterize the different states and operators in a given irreducible representation [46]. The T matrix is an $SU(3)$ singlet and therefore the scattering matrix is diagonal in a basis of states with definite transformation properties under $SU(3)$. Denoting these states by $|R,\lambda\rangle$, with R corresponding to the $SU(3)$ irreducible representation and λ including the other quantum numbers needed to distinguish between states within R (e.g., third component of isospin and hypercharge). The momenta and spin indices are not indicated in the following since they do not play any active role in the next considerations. We then have for the T matrix,

$$\begin{aligned}\langle R',\lambda'|T|R,\lambda\rangle &= T_R\delta_{RR'} \\ &= \frac{1}{\mathcal{N}_R^{-1}+g_R(s)}\delta_{RR'}\,. \end{aligned} \tag{9.1}$$

Here we have used the general parameterization of Eq. (7.2) with the unitarity function $g_R(s)$ containing the subtraction constant a_R.

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Let us further denote by $|i\rangle$ the physical states in the charged basis and the Clebsch–Gordan coefficients connecting both bases by $\langle i, R\lambda\rangle$. These real coefficients satisfy the orthogonality relations

$$\sum_i \langle i, R\lambda\rangle\langle i, R'\gamma\rangle = \delta_{RR'}\delta_{\lambda\gamma}\;, \qquad (9.2)$$
$$\sum_{R,\lambda} \langle i, R\lambda\rangle\langle j, R\lambda\rangle = \delta_{ij}\;.$$

Notice that since $\langle i, R\lambda\rangle = \langle i|R,\lambda\rangle$ and is real, then it also follows that $\langle i, R\lambda\rangle = \langle R,\lambda|i\rangle$.

In the physical basis the T-matrix elements $T_{ij}(s)$ also obeys Eq. (7.2), with $\mathcal{N}(s)$ calculated in the charged basis and the functions $g_i(s)$ involving the subtraction constants a_i. Let us show that all the a_R and a_i have the same value in the $SU(3)$ limit, as derived in Ref. [47]. For that we proceed with the change of basis of a singlet $SU(3)$ matrix A, from the physical basis to the $SU(3)$ one. Then,

$$\sum_{ij} \langle i, R\lambda\rangle A_{ij}\langle j, R'\gamma\rangle = A_R\delta_{RR'}\delta_{\lambda\gamma}\;. \qquad (9.3)$$

For instance, this is case for the QCD Hamiltonian, and therefore for the T matrix, as well as for the unitarity loop function $g_i(s)$, Eq. (8.5). The latter function, contrary to the T matrix, is also diagonal in the physical basis. This is a key distinctive feature that allows us to perform the following manipulations. By inverting Eq. (9.3) with $g(s)$ instead of A, we have that

$$g_i(s)\delta_{ij} = \sum_{R,\lambda} \langle i, R\lambda\rangle g_R(s)\langle j, R\lambda\rangle\;. \qquad (9.4)$$

Next, we multiply by $\langle j, R'\gamma\rangle$ and sum over j,

$$g_i(s)\langle i, R'\gamma\rangle = \sum_{R,\lambda}\sum_j \langle i, R\lambda\rangle g_R(s)\underbrace{\langle j, R\lambda\rangle\langle j, R'\gamma\rangle}_{\delta_{RR'}\delta_{\lambda\gamma}} = \langle i, R'\gamma\rangle g_{R'}(s)\;. \qquad (9.5)$$

From this relation it is sufficient to take physical states with components in different irreducible representation to conclude that for any state $|i\rangle$ in the charged basis and for any irreducible $SU(3)$ representation R involved in the decomposition of the charged basis in $SU(3)$ multiplets, one has

$$g_i(s) = g_R(s) = g(s)\;. \qquad (9.6)$$

As a result it follows the equality in the $SU(3)$ limit of all the subtraction constants for the two-particles states $|AB\rangle$, with A and B belonging to the irreducible $SU(3)$ representations R_A and R_B, in order.

This result is manifestly evident when every subtraction constant $a_i(\mu)$ is given by its natural value, Eq. (8.8), because then the masses m_1 and m_2 (as well as the three-momentum cut-off Λ) are common to all the two-particle states in the $SU(3)$ limit.