Chapter 7
Reaching the Unphysical Riemann Sheets. A Nonlinear Integral Equation to Calculate a PWA

Now, let us discuss how to proceed to calculate the T matrix of PWAs in an unphysical Riemann sheet (RS). In order to give a general discussion let us use a generic parameterization for a T matrix by explicitly isolating the RHC. Performing a DR of the inverse of the T matrix by employing Eq. 2.51 we have

$$T_L(s)^{-1} = \mathcal{N}_L(s)^{-1} + a(s_0) - \frac{s-s_0}{\pi}\int_{s_{\text{th}}} \frac{\rho(s')ds'}{(s'-s_0)(s'-s)} \,, \tag{7.1}$$

$$T_L(s) = \left[\mathcal{N}_L(s)^{-1} + g(s)\right]^{-1} \,. \tag{7.2}$$

where we have included a subtraction at s_0 because $\rho(s)$ tends to constant as $s \to \infty$. Here $\mathcal{N}_L(s)$ is a matrix that only has crossed-channel cut (although it could also have CDD poles). In the limit in which crossed cuts are neglected this function and the $N(s)$ function of the N/D method can be made to coincide. In addition, the dispersive integral plus the subtraction constant $a(s_0)$ (so that the result is independent of the subtraction point s_0) is denoted by $g(s)$. The matrices $g(s)$ and $a(s_0)$ are diagonal (recall that $\rho(s)$ is a diagonal matrix), whose matrix elements are explicitly,

$$g_i(s) = a_i(s_0) - \frac{s-s_0}{\pi}\int_{s_{\text{th};i}}^{\infty} \frac{\rho_i(s')ds'}{(s'-s_0)(s'-s)} \,. \tag{7.3}$$

Notice that if the only singularities of $T_L(s)_{ij}$ were a RHC, a LHC, and possible poles in between the two cuts, and if it were furthermore bounded in the complex s plane by some power of s for $s \to \infty$, we could then apply the Sugawara–Kanazawa theorem, Chap. 4. This is clear because from unitarity we have that $T_{ij} = (S_{ij} - 1)/\left(2i\rho_i^{1/2}\rho_j^{1/2}\right)$, $|S_{ij}| \le 1$, and we would expect for the case of finite-range interactions that S_{ij} tends to a definite limit for $s \to \infty + i\varepsilon$ (let us note that the Schwarz reflection principle is fulfilled by the PWA). We could then conclude from the application of the Sugawara–Kanazawa theorem that $T_{ij}(s)$ would tend to constant for $s \to \infty$, like $(S_{ij}(\infty + i\varepsilon) - 1)/(2i\rho_i(\infty + i\varepsilon)^{1/2}\rho_j(\infty + i\varepsilon)^{1/2})$ for

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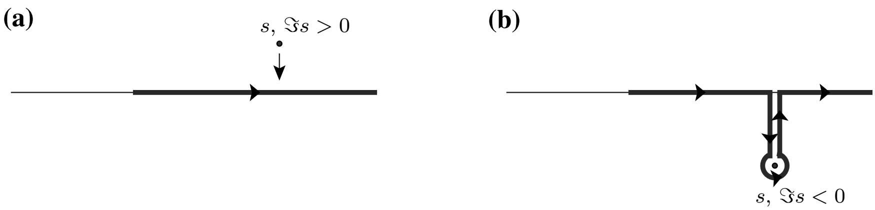

Fig. 7.1 Contour deformation (thick solid line) for reaching the second Riemann sheet of the function $g_i(s)$ by crossing the RHC from top (**a**) to bottom (**b**). The deformation of the integration contour results in order to avoid the pole singularity of the integrand in Eq. (7.3) at $s' = s$, for $s \in \mathbb{R}$ and $s > s_{\text{th};i}$. Subsequently the process continues to further avoid the crossing of the pole at s in the integrand with the deformed contour when moving deeper in the complex s plane. This figure could be seen also as a way to reach the first Riemann sheet from the second one by crossing again the RHC from the later (**a**) to the former (**b**). Of course, the RHC could also be crossed from bottom to top, with the deformed contour being the mirror image of the one pictured in (**b**)

$\Im s > 0$ and like its complex conjugate for $\Im s < 0$. Nonetheless, in practical applications we have to handle, at least at the effective level, with singular interactions for which the PWAs are not bounded in the complex s plane. For examples the interested reader might consult Ref. [7], where a formula is derived that allows to calculate the exact discontinuity of a PWA along the LHC both for regular and singular potentials. For the latter ones, the modulus of this discontinuity diverges stronger than any polynomial of s for $s \to -\infty$. Therefore, the Sugawara–Kanazawa theorem does not apply in this case and $T_{ij}(s)$ is divergent for $s \to \infty$, as the explicit calculation of the discontinuity along the LHC shows.

The Eq. (7.2) gives $T_L(s)$ in the first Riemann sheet. In order to reach resonance poles we should consider the T matrix in unphysical Riemann sheets as well. This is accomplished by performing the analytical continuation of the different matrix elements of the diagonal dispersive integral in Eq. (7.1). The function $g_i(s)$ has a branch-point singularity at the i_{th} threshold $s_{\text{th};i}$ and a cut starting from this point that we take along the positive real s axis, that is, a standard RHC or unitarity cut. Now, in order to reach the second Riemann sheet of $g_i(s)$ one should cross the RHC and proceed by analytical continuation to the second Riemann sheet. This analytical continuation can be accomplished by deforming the integration contour [2] as depicted in Fig. 7.1.

We then have to add to $g_i(s)$ the result of the integration along the closed integration contour around s. Thus, if we denote by $g_{II;i}(s)$ the $g_i(s)$ function in the second Riemann sheet we have the relation

$$g_{II;i}(s) = g_i(s) - 2i\rho_{II;i}(s) = g_i(s) + 2i\rho_{I;i}(s) \,, \tag{7.4}$$

where the function $\rho_{I;i}(s)$ in the complex s plane is

$$\rho_{I;i}(s) = \frac{1}{16\pi}\sqrt{\frac{\lambda(s, m_1^2, m_2^2)}{s^2}}\,, \tag{7.5}$$

with the square root $\sqrt{z}$ taken in its first Riemann sheet and defined as having a RHC, that is, with $\arg z \in [0, 2\pi[$. Notice, that the minus sign in the term after the first equal sign in Eq. (7.4) is due to the fact that $\rho_{II;i}(s)$ is the same function as $\rho_{I;i}(s)$ but defined in its second RS (the procedure of analytically continuing an integral by deforming its integration contour requires using the integrand analytically continued to its corresponding Riemann sheet).

The Eq. (7.4) also shows that this is a two-sheet cut, because by crossing again the RHC we would have to add $+2i\rho_{I;i}(s)$, because of the addition of the circle to the integration contour, but this time added to $g_{II;i}(s) = g_i(s) + 2i\rho_{II;i}(s)$ (the square-root function $\rho_{I;i}(s)$ in Eq. (7.4) is also analytically continued to its second Riemann sheet). Then, the extra terms cancel and we come back again to $g_i(s)$ in the first Riemann sheet. This analysis shows that the RHC is a two-sheet cut and because of this the different Riemann sheets can be characterized as the Riemann sheets of the square root present in the definition of the CM three-momentum p,

$$p(s) = \pm\sqrt{\frac{\lambda(s, m_1^2, m_2^2}{4s}}\,. \tag{7.6}$$

Our convention to nominate all the possible 2^n RS for a scattering process with n channels is the following. The physical or first Riemann sheet (RS) corresponds to take the plus sign in all the channels, $(+, +, \ldots)$, the second RS to take the minus sign in the first channel, $(-, +, \ldots)$, the third RS to $(+, -, +, \ldots)$, the fourth RS to $(-, -, +, \ldots)$, etc. Thus, before we flip the sign of the m_{th} channel we have 2^{m-1} RSs.

We now discuss a DR for $\mathcal{N}_L(s)$ following the derivation of Ref. [8]. This representation also provides a nonlinear IE for $\mathcal{N}_L(s)$. To simplify the discussion we consider an uncoupled PWA taken as a function of the CM three-momentum squared, p^2. This is done so as to avoid the circular cuts for unequal mass scattering, §1.1 of Chap. 8 in Ref. [5], so that $T_L(p^2)$ has only a LHC and a RHC. The procedure discussed could be generalized straightforwardly to coupled PWAs.

From Eq. (7.2) we have that $\Im\mathcal{N}(p^2)$ satisfies along the LHC (we omit the subscript L to shorten the writing),

$$\Im T(p^2) = \Im\frac{1}{\mathcal{N}(p^2)^{-1} + g(p^2)} = -\frac{\Im\mathcal{N}(p^2)^{-1}}{|\mathcal{N}(p^2)^{-1} + g(p^2)|^2} = \Im\mathcal{N}(p^2)\frac{|T(p^2)|^2}{|\mathcal{N}(p^2)|^2}\,. \tag{7.7}$$

Therefore,

$$\Im\mathcal{N}(p^2) = \frac{|\mathcal{N}(p^2)|^2}{|T(p^2)|^2}\Im T(p^2) \tag{7.8}$$
$$= \left|1 + g(p^2)\mathcal{N}(p^2)\right|^2 \Delta(p^2)\ ,\ p^2 < p^2_{\text{Left}}\ .$$

Here, we have introduced the function $\Delta(p^2)$ defined as

$$\Delta(p^2) = \Im T(p^2)\ ,\ p^2 < p^2_{\text{Left}}\ , \tag{7.9}$$

where p^2_{Left} is the upper bound of the LHC. Assuming that $\mathcal{N}(p^2)/p^{2n}$ vanishes for $p^2 \to \infty$ we can write an n-times subtracted DR for $\mathcal{N}(p^2)$,

$$\mathcal{N}(p^2) = \sum_{m=0}^{n-1} a_m p^{2m} + \frac{p^{2n}}{\pi}\int_{-\infty}^{p^2_{\text{Left}}} \frac{\left|1 + g(q^2)\mathcal{N}(q^2)\right|^2 \Delta(q^2)dq^2}{q^{2n}(q^2 - p^2)}\ . \tag{7.10}$$

This is a nonlinear IE which input is the knowledge of $\Delta(p^2)$ along the LHC. In terms of this DR we can write for $T(p^2)$,

$$T(p^2) = \left[\left(\sum_{m=0}^{n-1} a_m p^{2m} + \frac{p^{2n}}{\pi}\int_{-\infty}^{p^2_{\text{Left}}} \frac{\left|1 + g(q^2)\mathcal{N}(q^2)\right|^2 \Delta(q^2)dq^2}{q^{2n}(q^2 - p^2)}\right)^{-1} + g(p^2)\right]^{-1}\ . \tag{7.11}$$

In this form the subtraction constants a_m can be determined in terms of physical parameters of the T matrix, e.g., by fitting phase shifts, reproducing the effective range expansion (ERE) shape parameters, etc.

The Ref. [8] also shows that $T(p^2)$ is independent of the subtraction constant in $g(s)$. We reproduce here the arguments given in this reference and, as there, we take only one subtraction constant in $\mathcal{N}(p^2)$, which is enough for illustrating the point. We perform a DR for $T^{-1}(p^2)$ taking into account the RHC and LHC with an integration contour that consists of a circle at infinity that engulfs the two mentioned cuts. We use that the $\Im T(p^2)^{-1}$ along the RHC is $-\rho(p^2)$, Eq. 2.51. Then, one has

$$T^{-1}(p^2) = \beta - \frac{p^2}{\pi}\int_0^{\infty} \frac{\rho(q^2)dq^2}{q^2(q^2 - p^2)} + \frac{p^2}{\pi}\int_{-\infty}^{p^2_{\text{Left}}} \frac{\Delta(q^2)dq^2}{|T(q^2)|^2 q^2(q^2 - p^2)} + R(p^2)\ , \tag{7.12}$$

where $R(p^2)$ is a rational function taking care of the possible zeroes of $T(p^2)$ and that it does not play an active role in the considerations that follow. It is clear from the previous equation that there is only one free parameter (subtraction constant) to be determined, β, even though we could split it in two constants and add one of them to the integral over the RHC. The sum of this constant plus the RHC integral is the unitarity function $g(s)$. Thus, the inclusion of a subtraction constant in $g(p^2)$ appears just as a matter of convenience.