Chapter 17
An Example of Application of Analyticity in the Nuclear Medium: The Nuclear Energy Density

In this section, following Ref. [8], we apply DRs to calculate the nuclear-matter energy density, $\mathcal{E}$, as a function of the Fermi momenta for the protons and neutrons, ξ_p and ξ_n, respectively. Ref. [8] evaluates the contributions to the energy density of the nuclear medium up to and including NLO in the in-medium chiral counting developed in Ref. [109]. The different contributions are represented in Fig. 17.1. Without entering in the details of this in-medium chiral power counting, for what we refer to the latter reference, we focus our attention here to the contributions that generally stem from the iteration in the nuclear medium of the two-nucleon interactions, represented by the diagrams (c.1) and (c.2) in Fig. 17.1. The former corresponds to the direct NN interactions (Hartree diagrams) and the later to the crossed ones because of the Fermi statistics (Fock diagrams).

The contribution from the sum over the kinetic energies of the nucleons is given by the diagram (a) of Fig. 17.1, it is denoted by $\mathcal{E}_1$ and its expression is

$$\mathcal{E}_1 = \frac{3}{10m}\left(\rho_p \xi_p^2 + \rho_n \xi_n^2\right) , \tag{17.1}$$

where ρ_p and ρ_n are the proton and neutron densities. The latter read in terms of the corresponding Fermi momentum ξ_i,

$$\rho_i = 2\int \frac{d^3k}{(2\pi)^3}\theta(\xi_i - |\mathbf{k}|) = \frac{\xi_i^3}{3\pi^2} , \tag{17.2}$$

with $i = 1(2)$ for the proton(neutron). The magnitude of $\mathcal{E}_1$ is suppressed with respect to its chiral order because it is divided by the relatively large nucleon mass. It is a contribution of recoil nature.

For the analysis of the rest of contributions in Fig. 17.1 we need to discuss the nucleon propagator in the nuclear medium with four-momentum k, $G_0(k)$. It can be written as [8, 110]

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$$G_0(k) = \left(\frac{1+\tau_3}{2}\theta(\xi_p - |\mathbf{k}|) + \frac{1-\tau_3}{2}\theta(\xi_n - |\mathbf{k}|)\right)\frac{1}{k^0 - E(\mathbf{k}) - i\epsilon} + \left(\frac{1+\tau_3}{2}\theta(|\mathbf{k}| - \xi_p) + \frac{1-\tau_3}{2}\theta(|\mathbf{k}| - \xi_n)\right)\frac{1}{k^0 - E(\mathbf{k}) + i\epsilon} . \quad (17.3)$$

In this expression $E(\mathbf{k})$ is the nucleon energy, $E(\mathbf{k}) = \sqrt{m^2 + \mathbf{k}^2}$ (we take the isospin limit for vacuum dynamics), and the τ_i are the Pauli matrices. We can also rewrite equivalently the nucleon propagator in Eq. (17.3) by doing the transformation $1/(x - i\varepsilon) = 1/(x + i\varepsilon) + 2i\pi\delta(x)$, with $x \to k^0 - E(\mathbf{k})$. It then reads,

$$G_0(k) = \frac{1}{k^0 - E(\mathbf{k}) + i\epsilon} + i(2\pi)\delta(k^0 - E(\mathbf{k}))\left(\frac{1+\tau_3}{2}\theta(|\mathbf{k}| - \xi_p) + \frac{1-\tau_3}{2}\theta(|\mathbf{k}| - \xi_n)\right) . \quad (17.4)$$

The first term is the free part of the propagator and the second one is the in-medium one. The latter one is also indicated as an in-medium insertion of a baryon propagator, or simply as an in-medium insertion. In Feynman diagrams an in-medium part of the nucleon propagator is depicted by a thick line, the free part by a line with a slash, and the full in-medium propagator is drawn by a plain line. The one-baryon propagator in Eqs. (17.3) and (17.4) is given in a matrix notation, while its components are denoted by $G_0(k)_i$.

The contribution (b) in Fig. 17.1, $\mathcal{E}_2$, arises from the nucleon self-energy due to a pion loop. It entails only one in-medium insertion, because a contribution with two in-medium insertions is already accounted for by the diagram (c.2) [due to the isovector nature of the pion–nucleon coupling there is no one-pion loop (c.1)-like diagram for this case]. We denote by Σ_f^π the nucleon self-energy in vacuum by a pion loop, which expression reads [8, 111]

$$\Sigma_f^\pi(k) = \frac{3g_A^2 b}{32\pi^2 f_\pi^2}\left[-\omega + \sqrt{b}\left(i\log\frac{\omega + i\sqrt{b}}{-\omega + i\sqrt{b}} + \pi\right)\right] - \frac{3g_A^2 m_\pi^3}{32\pi f_\pi^2} , \quad (17.5)$$

with $g_A \simeq 1.26$ the axial coupling of the nucleon related by chiral symmetry (partially conserved axial-vector current) to the pion–nucleon coupling constant. In the previous equation $\omega = k^0$ is the nucleon energy once its rest mass is discounted and $b = m_\pi^2 - \omega^2 - i\varepsilon$. The last term in Eq. (17.5) is subtracted because the self-energy is zero for the $\omega = 0$, which corresponds to the vacuum nucleon mass at rest. Since in the diagram (b) of Fig. 17.1 the nucleon energy is a kinetic one, with $\xi_i \ll m$, it follows that this diagram is indeed a small contribution to the total energy density in the medium.

For evaluating the contributions (c.1) and (c.2) of Fig. 17.1 we need the in-medium NN interactions, that are depicted by the iteration of the zig-zag lines. For an in-medium NN PWA we use Eq. (7.2) in terms of $\mathcal{N}$, that only has LHC, and the two-nucleon unitary function, that in the nuclear medium corresponds to $L_{10}^{I_3}$ instead

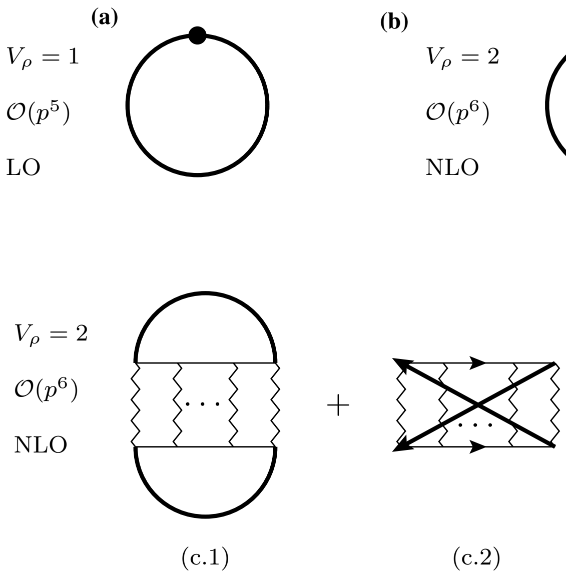

Fig. 17.1 Set of diagrams for the evaluation of the energy per baryon in nuclear matter up to an including two-nucleon interactions in the nuclear medium. In-medium insertions are represented in the figure by thick solid lines, and the thin ones correspond to the full baryon propagator $G_0(k)$, cf. Eq. (17.4). The diagram (**a**) is the kinetic energy, (**b**) represents the nucleon self-energy due to a pion loop [it involves one in-medium and one free baryon propagator (solid line with a dash), so as not to double count with the diagrams in (**c**)]. Finally, diagrams (c.1) (Hartree) and (c.2) (Fock diagrams) are the contributions due to the direct and exchange two-nucleon interactions, in order, with at least two in-medium interactions in the baryon propagators. Its evaluation [8], by making use of a partial-wave expansion and the analytical properties of the PWAs in the nuclear medium, is the main point of the present section

of $g(s)$. Contrarily to the vacuum case, the in-medium unitarity loop function function also depends on the total CM three-momentum of the two nucleons. In terms of the four-momenta k_1 and k_2 of the two nucleons we introduce the four-momenta a and p defined as

$$\begin{aligned} a &= \frac{1}{2}(k_1 + k_2) \;, \\ p &= \frac{1}{2}(k_1 - k_2) \;. \end{aligned} \tag{17.6}$$

We also use below the quantity

$$A = 2ma^0 - \mathbf{a}^2 \;. \tag{17.7}$$

The two-nucleon unitarity function depends also on the total charge of the two nucleons because the different values that the Fermi momenta of protons and nucleons could have. This is indicated by the superscript I_3 in $L_{10}^{I_3}$ which corresponds to the total third component of the isospin of the NN system. The explicit expression for $L_{10}^{I_3}$ is [8]

$$L_{10}^{I_3} = i\int \frac{d^4k}{(2\pi)^4}\left[\frac{\theta(\xi_1 - |\mathbf{a}-\mathbf{k}|)}{a^0 - k^0 - E(\mathbf{a}-\mathbf{k}) - i\epsilon} + \frac{\theta(|\mathbf{a}-\mathbf{k}| - \xi_1)}{a^0 - k^0 - E(\mathbf{a}-\mathbf{k}) + i\epsilon}\right] \tag{17.8}$$
$$\times\left[\frac{\theta(\xi_2 - |\mathbf{a}+\mathbf{k}|)}{a^0 + k^0 - E(\mathbf{a}+\mathbf{k}) - i\epsilon} + \frac{\theta(|\mathbf{a}+\mathbf{k}| - \xi_2)}{a^0 + k^0 - E(\mathbf{a}+\mathbf{k}) + i\epsilon}\right].$$

Performing explicitly the integration over k^0 we have for this loop function

$$L_{10}^{I_3} = m\int \frac{d^3k}{(2\pi)^3}\left[\frac{\theta(|\mathbf{a}-\mathbf{k}| - \xi_1)\theta(|\mathbf{a}+\mathbf{k}| - \xi_2)}{A - \mathbf{k}^2 + i\epsilon} - \frac{\theta(\xi_1 - |\mathbf{a}-\mathbf{k}|)\theta(\xi_2 - |\mathbf{a}+\mathbf{k}|)}{A - \mathbf{k}^2 - i\epsilon}\right], \tag{17.9}$$

in which the first term between square brackets is the free particle-free particle part (in the following we drop the adjective "free" as usual in the literature) and the last one is the so-called hole–hole part (because it involves two insertions of Fermi seas due to the Heaviside functions in the numerator). The integration over $\mathbf{k}$ can also be performed algebraically and the explicit expressions can be found in the Appendix C of Ref. [8]. It is clear from Eq. (17.9) the dependence of $L_{10}^{I_3}$ on I_3 and the CM variables contained in a and A. In this respect, notice that in the CM frame and for on-shell k_1 and k_2, it follows from Eq. (17.7) that $A = \mathbf{p}^2$. The Eq. (17.9) also establishes the appearance of the RHC when the real part of any of its denominators vanishes. The resulting imaginary part has the same sign from both contributions because of the minus sign in front of the hole–hole term. At LO in the in-medium chiral counting of Ref. [109] the matrix $\mathcal{N}$ is the same as the one already determined in vacuum. In general its characteristic facet at any order in the chiral expansion, as expressed above, is that it has no RHC, being the latter contained entirely in $L_{10}^{I_3}$. Employing the notation of Ref. [8] we denote the former by $\mathcal{N}_{JI}(\bar{\ell}, \ell, S)$ and similarly for the PWA, $T_{JI}^{I_3}(\bar{\ell}, \ell, S)$. Here, J is the total angular momentum, S the total spin, $\bar{\ell}$ the final orbital angular momentum and ℓ the initial one, always referred to the initial/final NN systems (the meaning of the different labels is in harmony with the notation introduced in Chap. 2). After this preamble, the in-medium expression equivalent to Eq. (7.2) is

$$T_{JI}^{I_3}(\bar{\ell}, \ell, S; \mathbf{p}^2, \mathbf{a}^2, A) = \left[\mathcal{N}_{JI}^{I_3}(\bar{\ell}, \ell, S; \mathbf{p}^2, \mathbf{a}^2, A)^{-1} + L_{10}^{I_3}(\mathbf{a}^2, A)\right]^{-1}. \tag{17.10}$$

It is worth clarifying that the total isospin I of a NN state is a good quantum number because the $L_{10}^{I_3}$ function is symmetric under the exchange of the two particles, cf. Eq. (17.8). This is a general rule because the $I_3 = 0$ operators are symmetric under the exchange $p \leftrightarrow n$, and therefore the symmetric properties under the transposition of the two particles in the $I_3 = 0$ NN state are not altered by the iterative interacting process.

Let us come back to evaluate the diagrams (c.1) and (c.2) in Fig. 17.1. Its sum is denoted by $\mathcal{E}_3$ and it is given by

$$\mathcal{E}_3 = \frac{1}{2} \sum_{\sigma_1,\sigma_2} \sum_{\alpha_1,\alpha_2} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} e^{ik_1^0\eta} e^{ik_2^0\eta} G_0(k_1)_{\alpha_1} G_0(k_2)_{\alpha_2} \times T^{\sigma_1\sigma_2}_{\alpha_1\alpha_2}(\mathbf{p}, \mathbf{a}, A) \ . \tag{17.11}$$

In this equation $\eta \to 0^+$ at the end of the calculation. It is introduced so as to enforce that at least two in-medium insertions get involved in the calculation [8, 110]. The NN scattering amplitude from the initial state $|k_1, k_2, \sigma_1\sigma_2, \alpha_1\alpha_2\rangle_S$, cf. Eq. (2.55), to the same final one is indicated in the previous equation by $T^{\sigma_1\sigma_2}_{\alpha_1\alpha_2}(\mathbf{p}, \mathbf{a}, A)$. The two states are the same because one has to take the trace of the scattering amplitudes when calculating the self-interactions of the system giving rise to $\mathcal{E}_3$. As in Chap. 2, the labels σ_i and α_i refer to the third components of the spin and isospin of the ith nucleon, in order. Since Eq. (17.11) is already a NLO contribution we can use for its evaluation the LO NN PWAs amplitudes, by employing Eq. (17.10) with $\mathcal{N}$ calculated as in vacuum. In such a case, $\mathcal{N}_{JI}(\bar{\ell}, \ell, S)$ is a function only of $\mathbf{p}^2$, the momentum transfer squared, cf. Eqs. (17.6), and (17.10) becomes

$$T^{I_3}_{JI}(\bar{\ell}, \ell, S; \mathbf{p}^2, \mathbf{a}^2, A) = \left[\mathcal{N}^{I_3}_{JI}(\bar{\ell}, \ell, S; \mathbf{p}^2)^{-1} + L^{I_3}_{10}(\mathbf{a}^2, A)\right]^{-1} \ . \tag{17.12}$$

Once the integration variables k_1 and k_2 are changed by A, $\mathbf{a}$ and p in Eq. (17.11), it is then possible to perform straightforwardly the integration over p^0. Notice that the only dependence on p^0 in the integrand of Eq. (17.11) is in the propagators $G_0(k_i)_{\alpha_i}$. It results,

$$\int \frac{dp^0}{2\pi} G_0(a+p)_{\alpha_1} G_0(a-p)_{\alpha_2} = -i\left[\frac{\theta(|\mathbf{a}+\mathbf{p}| - \xi_{\alpha_1})\theta(|\mathbf{a}-\mathbf{p}| - \xi_{\alpha_2})}{2a^0 - E(\mathbf{a}+\mathbf{p}) - E(\mathbf{a}-\mathbf{p}) + i\epsilon} - \frac{\theta(\xi_{\alpha_1} - |\mathbf{a}+\mathbf{p}|)\theta(\xi_{\alpha_2} - |\mathbf{a}-\mathbf{p}|)}{2a^0 - E(\mathbf{a}+\mathbf{p}) - E(\mathbf{a}-\mathbf{p}) - i\epsilon}\right] \ . \tag{17.13}$$

For convenience we also introduce the splitting of the particle–particle contribution in the form

$$\begin{aligned}\theta(|\mathbf{a}+\mathbf{p}| - \xi_{\alpha_1})\theta(|\mathbf{a}-\mathbf{p}| - \xi_{\alpha_2}) &= [1 - \theta(\xi_{\alpha_1} - |\mathbf{a}+\mathbf{p}|)][1 - \theta(\xi_{\alpha_2} - |\mathbf{a}-\mathbf{p}|)] \\ &= 1 - \theta(\xi_{\alpha_1} - |\mathbf{a}+\mathbf{p}|) - \theta(\xi_{\alpha_2} - |\mathbf{a}-\mathbf{p}|) + \theta(\xi_{\alpha_1} - |\mathbf{a}+\mathbf{p}|)\theta(\xi_{\alpha_2} - |\mathbf{a}-\mathbf{p}|) \ .\end{aligned} \tag{17.14}$$

It follows from Eqs. (17.13) and (17.14) that the result of the integration in p^0 of Eq. (17.11) can be written as

$$\mathcal{E}_3 = -4i \sum_{\sigma_1,\sigma_2} \sum_{\alpha_1,\alpha_2} \int \frac{d^3a}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \frac{dA}{2\pi} e^{i(A+\mathbf{a}^2)\eta} T^{\sigma_1\sigma_2}_{\alpha_1\alpha_2}(\mathbf{p}, \mathbf{a}, A) \Bigg[\frac{1}{A - \mathbf{p}^2 + i\epsilon} - \frac{\theta(\xi_{\alpha_1} - |\mathbf{a}+\mathbf{p}|) + \theta(\xi_{\alpha_2} - |\mathbf{a}-\mathbf{p}|)}{A - \mathbf{p}^2 + i\epsilon} - 2\pi i \delta(A - \mathbf{p}^2)\theta(\xi_{\alpha_1} - |\mathbf{a}+\mathbf{p}|)\theta(\xi_{\alpha_2} - |\mathbf{a}-\mathbf{p}|) \Bigg] . \tag{17.15}$$

Here we have expressed all the explicit denominators having $+i\epsilon$ by employing the trick explained just before Eq. (17.4).

The next step is to perform the integration over A, which actually implies to compute

$$\int_{-\infty}^{\infty} \frac{dA}{2\pi} \frac{e^{iA\eta}}{A - \mathbf{p}^2 + i\epsilon} T^{\sigma_1\sigma_2}_{\alpha_1\alpha_2}(\mathbf{p}, \mathbf{a}, A) , \tag{17.16}$$

since the last term in the integrand of Eq. (17.15) is proportional to $\delta(A - \mathbf{p}^2)$ and the integral in A is then trivial. We proceed with Eq. (17.16) by enclosing the integration contour in A with a semicircle at infinity in the half complex A plane with positive imaginary part, by taking advantage of the factor $e^{iA\eta}$ with $\eta \to 0^+$. As it is evident from Eq. (17.9), the particle–particle contribution gives rise to a cut in A with a slightly negative imaginary part, so that it is not within the domain that results after closing the integration contour. Similarly the denominator in Eq. (17.16) gives rise to a pole singularity in A with also a negative imaginary part. However, the hole–hole part in $L^{I_3}_{10}(\mathbf{a}, A)$ generates a cut in A than runs slightly above the real axis with a positive imaginary part. This cut is of finite extent because of the Heaviside functions in the hole–hole part and extends from $A_1(|\mathbf{a}|)$ up to $A_2(|\mathbf{a}|)$ as depicted in Fig. 17.2 by the dashed line (explicit expressions for these limits are given in Eq. (C.19) of Ref. [8].)[1]

In order to go on and perform the integration in A we proceeds as follows. We consider two closed contours in the form stated above, but one of them runs above the hole–hole cut and the other below it. The former integration contour is denoted by $\mathcal{C}_{I'}$, the latter by $\mathcal{C}_I$, and both are represented in Fig. 17.2. We have the following preliminary results,

[1] These limits depend on I_3, although this is not explicitly written, since no ambiguity arises once the partial-wave expansion of the T matrix is performed below.

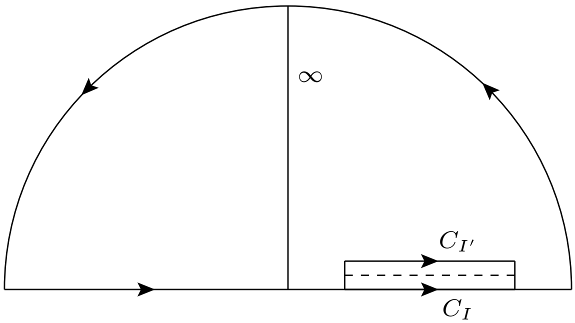

Fig. 17.2 Integration contours used to evaluate the A integration in Eq. (17.16). The dashed line is the RHC due to the hole–hole contribution in $L_{10}^{I_3}(\mathbf{a}^2, A)$

$$\int_{-\infty}^{\infty} \frac{dA}{2\pi} \frac{e^{iA\eta}}{A - \mathbf{p}^2 + i\epsilon} T_{\alpha_1\alpha_2}^{\sigma_1\sigma_2}(\mathbf{p}, \mathbf{a}, A) = \oint_{C_I} \frac{dA}{2\pi} \frac{e^{iA\eta}}{A - \mathbf{p}^2 + i\epsilon} T_{\alpha_1\alpha_2}^{\sigma_1\sigma_2}(\mathbf{p}, \mathbf{a}, A) ,$$
$$\oint_{C_{I'}} \frac{dA}{2\pi} \frac{e^{iA\eta}}{A - \mathbf{p}^2 + i\epsilon} T_{\alpha_1\alpha_2}^{\sigma_1\sigma_2}(\mathbf{p}, \mathbf{a}, A) = 0 . \tag{17.17}$$

Therefore, the subtraction of the two integrals gives

$$\oint_{C_I} \frac{dA}{2\pi} \frac{e^{iA\eta}}{A - \mathbf{p}^2 + i\epsilon} - \oint_{C_{I'}} \frac{dA}{2\pi} \frac{e^{iA\eta}}{A - \mathbf{p}^2 + i\epsilon}$$
$$= \int_{A_1(|\mathbf{a}|)}^{A_2(|\mathbf{a}|)} \frac{dA}{2\pi} \frac{T_{\alpha_1\alpha_2}^{\sigma_1\sigma_2}(\mathbf{p}, \mathbf{a}, A) - T_{\alpha_1\alpha_2}^{\sigma_1\sigma_2}(\mathbf{p}, \mathbf{a}, A + 2i\epsilon)}{A - \mathbf{p}^2 + i\epsilon} . \tag{17.18}$$

Notice that this result is also a consequence of deforming the integration contour C_I for avoiding the cut.

An interesting result in Ref. [8] is the derivation of the partial-wave expansion of the NN scattering amplitude in the nuclear medium, despite its dependence on $\mathbf{a}$. This is a generalization of the results in Chap. 2. The scattering amplitude $T_{\alpha_1\alpha_2}^{\sigma_1\sigma_2}(\mathbf{p}, \mathbf{a}, A)$ in terms of the in-medium NN PWAs, by making use of Eq. (A.8) of Ref. [8], reads

$$T_{\alpha_1\alpha_2}^{\sigma_1\sigma_2} = 4\pi \sum (\sigma_1\sigma_2 s_3|s_1 s_2 S)^2 (m' s_3 \mu|\ell' S J)(m s_3 \mu|\ell S J)(\alpha_1\alpha_2 i_3|\tau_1\tau_2 I)^2 \tag{17.19}$$
$$\times Y_{\ell'}^{m'}(\hat{\mathbf{p}}') Y_{\ell}^{m}(\hat{\mathbf{p}})^* \chi(S\ell' I)\chi(S\ell I) T_{JI}^{I_3}(\ell', \ell, S) .$$

For NN scattering $\tau_1 = \tau_2 = s_1 = s_2 = 1/2$ and the symbol $\chi(S\ell I)$ arises because Fermi statistics and it is

$$\chi(S\ell I) = \frac{1 - (-1)^{\ell+S+I}}{\sqrt{2}} = \begin{cases} \sqrt{2} & \ell + S + I = \text{odd} , \\ 0 & \ell + S + I = \text{even} . \end{cases} \tag{17.20}$$

This factor accounts for the unitary normalization introduced in Chap. 2.

The sum over the isospin and spin indices is straightforward,

$$\sum_{\alpha_1\alpha_2}(\alpha_1\alpha_2 i_3|\tau_1\tau_2 I)^2 = 1 \,, \tag{17.21}$$
$$\sum_{\sigma_1\sigma_2}(\sigma_1\sigma_2 s_3|s_1 s_2 S)^2 = 1 \,.$$

We continue next with the sum over s_3 and the third components of orbital angular momentum, which can also be summed in a close form as

$$\sum_{m',m,s_3}(m's_3\mu|\ell' SJ)(ms_3\mu|\ell SJ)Y_{\ell'}^{m'}(\hat{\mathbf{p}})Y_\ell^m(\hat{\mathbf{p}})^* = \delta_{\ell'\ell}\frac{2J+1}{4\pi} \,. \tag{17.22}$$

To arrive to this result we have used the following symmetry property of the Clebsch–Gordan coefficients [6],

$$(m_1 m_2 m_3|j_1 j_2 j_3) = (-1)^{m_2+j_2}\left(\frac{2j_3+1}{2j_1+1}\right)^{1/2}(-m_2 m_3 m_1|j_2 j_3 j_1) \,. \tag{17.23}$$

This property allows us to write the sum of Clebsch–Gordan coefficients in Eq. (17.22) as

$$\sum_{s_3,\mu}(m's_3\mu|\ell' SJ)(ms_3\mu|\ell SJ) = \frac{2J+1}{\sqrt{(2\ell+1)(2\ell'+1)}}\sum_{s_3,\mu}(-s_3\mu m'|S\ell' J)$$
$$\times\,(-s_3\mu m|S\ell J) = \frac{2J+1}{2\ell+1}\delta_{\ell'\ell}\delta_{m'm'} \,. \tag{17.24}$$

Notice that we could have included a sum over μ already in Eq. (17.22), because μ is fixed by the properties of the Clebsch–Gordan coefficients. Finally, the result in Eq. (17.22) follows by employing the addition theorem of the spherical harmonics

$$\frac{1}{2\ell+1}\sum_m |Y_\ell^m(\hat{\mathbf{p}})|^2 = \frac{1}{4\pi} \,. \tag{17.25}$$

Thus, the sum over the PWAs for calculating $\mathcal{E}_3$ simplifies to

$$\sum_{\alpha_1,\alpha_2}\sum_{\sigma_1,\sigma_2} T_{\alpha_1\alpha_2}^{\sigma_1\sigma_2}(\mathbf{p},\mathbf{a},A) = \sum_{I,I_3,J,\ell,S}(2J+1)\chi(S\ell I)^2\, T_{JI}^{I_3}(\ell,\ell,S;\mathbf{p}^2,\mathbf{a}^2,A) \,. \tag{17.26}$$

In the rest of this section we suppress the arguments (ℓ, ℓ, S) in $T_{JI}^{I_3}$ and $\mathcal{N}_{JI}^{I_3}$ for brevity in the writing. We now perform the difference between PWAs needed to implement Eq. (17.18),

$$
\begin{aligned}
&T_{JI}^{I_3}(\mathbf{p}^2,\mathbf{a}^2,A)-T_{JI}^{I_3}(\mathbf{p}^2,\mathbf{a}^2,A+i2\epsilon)=\left[N_{JI}^{I_3}(\mathbf{p}^2)+L_{10}^{I_3}(\mathbf{a}^2,A)\right]^{-1}\\
&-\left[N_{JI}^{I_3}(\mathbf{p}^2)+L_{10}^{I_3}(\mathbf{a}^2,A+i2\epsilon)\right]^{-1}=\left[N_{JI}^{I_3}(\mathbf{p}^2)+L_{10}^{I_3}(\mathbf{a}^2,A)\right]^{-1}\\
&\times\left[L_{10}^{I_3}(\mathbf{a}^2,A+i2\epsilon)-L_{10}^{I_3}(\mathbf{a}^2,A)\right]\left[N_{JI}^{I_3}(\mathbf{p}^2)+L_{10}^{I_3}(\mathbf{a}^2,A+i2\epsilon)\right]^{-1}\ . \qquad (17.27)
\end{aligned}
$$

It follows from Eq. (17.9) that the difference $L_{10}^{I_3}(\mathbf{a}^2,A+i2\epsilon)-L_{10}^{I_3}(\mathbf{a}^2,A)$ is due entirely to the hole–hole part and it gives

$$
\begin{aligned}
&L_{10}^{I_3}(\mathbf{a}^2,A+i2\epsilon)-L_{10}^{I_3}(\mathbf{a}^2,A)=-m\int\frac{d^3q}{(2\pi)^3}\theta(\xi_{\alpha_1}-|\mathbf{a}+\mathbf{q}|)\theta(\xi_{\alpha_2}-|\mathbf{a}-\mathbf{q}|)\\
&\times\left(\frac{1}{A-\mathbf{q}^2+i\epsilon}-\frac{1}{A-\mathbf{q}^2-i\epsilon}\right)\\
&=i2\pi m\int\frac{d^3q}{(2\pi)^3}\theta(\xi_{\alpha_1}-|\mathbf{a}+\mathbf{q}|)\theta(\xi_{\alpha_2}-|\mathbf{a}-\mathbf{q}|)\delta(A-\mathbf{q}^2)\ . \qquad (17.28)
\end{aligned}
$$

Now, we come back to Eq. (17.15) and from Eqs. (17.26), (17.27) and (17.28) it follows that after performing the integration in A, cf. Eq. (17.18), we can write Eq. (17.15) as

$$
\begin{aligned}
&\mathcal{E}_3=-4\sum_{I,I_3,J,\ell,S}(2J+1)\chi(S\ell I)^2\int\frac{d^3a}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\theta(\xi_{\alpha_1}-|\mathbf{a}+\mathbf{q}|)\theta(\xi_{\alpha_2}-|\mathbf{a}-\mathbf{q}|)\\
&\times\Bigg(T_{JI}^{I_3}(\mathbf{q}^2,\mathbf{a}^2,\mathbf{q}^2)+m\int\frac{d^3p}{(2\pi)^3}\frac{1-\theta(\xi_{\alpha_1}-|\mathbf{a}+\mathbf{p}|)-\theta(\xi_{\alpha_2}-|\mathbf{a}-\mathbf{p}|)}{\mathbf{p}^2-\mathbf{q}^2-i\epsilon}\\
&\times\left[\mathcal{N}_{JI}^{I_3}(\mathbf{p}^2)^{-1}+L_{10}^{I_3}(\mathbf{a}^2,\mathbf{q}^2)\right]^{-1}\left[\mathcal{N}_{JI}^{I_3}(\mathbf{p}^2)^{-1}+L_{10}^{I_3}(\mathbf{a}^2,\mathbf{q}^2)^*\right]^{-1}\Bigg)_{(\ell,\ell,S)}\ . \qquad (17.29)
\end{aligned}
$$

This is our final equation for $\mathcal{E}_3$. In this equation we have dropped the exponent $e^{i\mathbf{a}^2\eta}$ since the integration in $|\mathbf{a}|$ is bounded because of the product of the Heaviside functions $\theta(\xi_{\alpha_1}-|\mathbf{a}+\mathbf{q}|)\theta(\xi_{\alpha_2}-|\mathbf{a}-\mathbf{q}|)$. Related to this factor, we have also written that $L_{10}^{I_3}(\mathbf{a}^2,\mathbf{q}^2+i2\epsilon)=L_{10}^{I_3}(\mathbf{a}^2,\mathbf{q}^2)^*$, because only the hole–hole part in this functions enters, cf. Eq. (17.9). The rest of sums and integrations in Eq. (17.29) are performed numerically in Ref. [8].

It is also of pedagogical interest to show explicitly following Ref. [8] that $\Im\mathcal{E}_3=0$, as it must be because $\mathcal{E}$ is a real quantity. The imaginary part of the second term between the round brackets in Eq. (17.29) stems only from the denominator of $1/(\mathbf{p}^2-\mathbf{q}^2-i\epsilon)\to i\pi\delta(\mathbf{p}^2-\mathbf{q}^2)$. We then have

$$
\begin{aligned}
\Im\mathcal{E}_3 = & -4 \sum_{I,I_3,J,\ell,S} (2J+1)\chi(S\ell I)^2 \int \frac{d^3a}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\theta(\xi_{\alpha_1}-|\mathbf{a}+\mathbf{q}|)\theta(\xi_{\alpha_2}-|\mathbf{a}-\mathbf{q}|) \\
& \times \Bigg(\Im T_{JI}^{I_3}(\mathbf{q}^2,\mathbf{a}^2,\mathbf{q}^2) + m\int \frac{d^3p}{(2\pi)^3}\big[1-\theta_{\alpha_1}(\xi_{\alpha_1}-|\mathbf{a}+\mathbf{p}|)-\theta_{\alpha_2}(\xi_{\alpha_2}-|\mathbf{a}-\mathbf{p}|)\big] \\
& \times \pi\delta(\mathbf{p}^2-\mathbf{q}^2)T_{JI}^{I_3}(\mathbf{p}^2,\mathbf{a}^2,\mathbf{q}^2)T_{JI}^{I_3}(\mathbf{p}^2,\mathbf{a}^2,\mathbf{q}^2)^*\Bigg)_{(\ell,\ell,S)} . \qquad (17.30)
\end{aligned}
$$

It is clear from Eq. (17.12) (with $A=\mathbf{q}^2$) that the imaginary part of $T_{JI}^{I_3}(\mathbf{q}^2,\mathbf{a}^2,\mathbf{q}^2)$ arises from the one of $L_{10}^{I_3}(\mathbf{a}^2,\mathbf{q}^2)$, which in turn is only due to the hole–hole part because of the product of the two Heaviside functions on the rhs of Eq. (17.30). Substituted the expression for $\Im T_{JI}^{I_3}$ into the previous equation one finds

$$
\begin{aligned}
\Im\mathcal{E}_3 = & -4 \sum_{I,I_3,\ell,S} (2J+1)\chi(S\ell I)^2 \int \frac{d^3a}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\theta(\xi_{\alpha_1}-|\mathbf{a}+\mathbf{q}|)\theta(\xi_{\alpha_2}-|\mathbf{a}-\mathbf{q}|) \\
& \times \int \frac{d^3p}{(2\pi)^3} m\pi\delta(\mathbf{p}^2-\mathbf{q}^2)\big[1-\theta(\xi_{\alpha_1}-|\mathbf{a}+\mathbf{p}|)-\theta(\xi_{\alpha_2}-|\mathbf{a}-\mathbf{p}|) \\
& + \theta(\xi_{\alpha_1}-|\mathbf{a}+\mathbf{p}|)\theta(\xi_{\alpha_2}-|\mathbf{a}-\mathbf{p}|)\big]T_{JI}^{I_3}T_{JI}^{I_3\,*}\Big|_{(\ell,\ell,S)} = 0\,. \qquad (17.31)
\end{aligned}
$$

To conclude that this expression is zero, we have taken into account from Eq. (17.14) that the function between square brackets in the previous equation is only the particle–particle part, given by $\theta(|\mathbf{a}+\mathbf{p}|-\xi_{\alpha_1})\theta(|\mathbf{a}-\mathbf{p}|-\xi_{\alpha_2})$. But since $|\mathbf{p}|=|\mathbf{q}|$ there is no way that this product of step functions can be satisfied because $\mathbf{a}$ and $\mathbf{q}$ are already constrained to satisfy the product of the two Heaviside functions in the first line of Eq. (17.31). Thus, this equation is zero.