Chapter 16
Near-Threshold Scattering

In this section we consider the physics in a near-threshold region so that the nonrelativistic limit is appropriate. Furthermore, we assume that the LHC is relatively weak and/or far away. In the later case the LHC admits a Taylor expansion in the region of interest and its effects can be accounted for without explicitly including it. For the former case we assume that it can be neglected in good approximation because of its weakness. With this scenario in mind we use the general results deduced in Chap. 6, cf. Eq. (6.24), which are applicable when the cut associated with the LHC is not explicitly realized.

We consider the S-wave scattering, which is expected to be dominant since the energy is supposed to be near the threshold of the reaction of the two particles with masses m_1 and m_2. In such circumstances the general structure of a PWA corresponds to Eq. (6.24) with $L = 0$. The relativistic phase space in the integral along the RHC is $p(s)/8\pi\sqrt{s}$. We also introduce the kinetic energy E by its nonrelativistic expression, namely,

$$
\begin{aligned}
E &= \frac{p^2}{2\mu} ,\\
p &= \sqrt{2\mu E} ,
\end{aligned}
\tag{16.1}
$$

which is more appropriate for nonrelativistic dynamics, with the relation $\sqrt{s} = m_1 + m_2 + E + \mathcal{O}(p^4)$.

The unitary loop function, that corresponds to the integral along the RHC in Eq. (6.24), is already given in Eq. (8.3). Its series in powers of $p = |\mathbf{p}|$ around threshold gives rise to an expansion involving powers of p^2 [it corresponds to the expansion in real variable of its real part for physical values of $s > s_{\text{th}}$. This is given by the Cauchy's principal value of the integral in Eq. (8.3)] and odd powers of p [which stems from the expansion of its imaginary part $-p/8\pi\sqrt{s}$]. The first terms of this nonrelativistic series of $g(p)$ in powers of p read

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$$g(p) = \frac{a}{16\pi^2} + \frac{1}{8\pi^2(m_1+m_2)}(m_1 \log\frac{m_1}{\mu} + m_2 \log\frac{m_2}{\mu}) \\ - i\frac{p}{8\pi(m_1+m_2)} + \mathcal{O}(\frac{p^2}{\Sigma^2}) , \tag{16.2}$$

where Σ has dimension of mass and it is made out of the masses m_1 and m_2. For every $\mathcal{O}(p^m)$ term with $m \geq 1$ there is always a neat power of mass m_i, $i = 1, 2$, in the denominator, avoiding any relative enhancement from powers of the factor m_1/m_2 with $m_1 \gg m_2$. This is clear from the nonrelativistic reduction of the RHC integral in Eq. (8.3),

$$g(p) = \widetilde{a} - \frac{s}{\pi}\int_0^\infty \frac{dq^2}{\omega_1'(q)\omega_2'(q)} \frac{q}{8\pi[\omega_1'(q)+\omega_2'(q)]} \\ \times \frac{1}{[\omega_1'(q)+\omega_2'(q)-\omega_1(p)-\omega_2(p)][\omega_1'(q)+\omega_2'(q)+\omega_1(p)+\omega_2(p)]} \\ = \widetilde{a} + \frac{1}{8\pi(m_1+m_2)}\int_0^\infty \frac{q\,dq^2}{p^2-q^2} + \mathcal{O}(\frac{p^2}{\Sigma^2}) , \tag{16.3}$$

with $\omega_i(q)$ already introduced in Eq. (8.5).

In the following we adopt the more standard nonrelativistic normalization of the PWA $t(p^2)$, already introduced in Eq. (11.9) for $m_1 = m_2 = m$. Additionally, we denote by β the momentum-independent contribution on the rhs of Eq. (16.2) (times $8\pi(m_1 + m_2)$ because of the change in normalization). Namely,

$$\beta = \frac{a(m_1+m_2)}{2\pi} + \frac{1}{\pi}(m_1 \log\frac{m_1}{\mu} + m_2 \log\frac{m_2}{\mu}) . \tag{16.4}$$

Attending to Eq. (6.24) with $L = 0$, in addition to the subtraction constant we also have the sum over the CDD poles. Looking for relevant structures in the near-threshold region apart from the threshold branch-point singularity, we explore the consequences of including a CDD pole. In this way, we recast Eq. (6.24) as

$$t(E) = \left(\frac{\gamma}{E - M_{\rm CDD}} + \beta - ip(E)\right)^{-1} , \tag{16.5}$$

where we use the kinetic energy E as variable, cf. Eq. (16.1). In this equation γ is the residue of the CDD pole and $M_{\rm CDD}$ is its position in energy E.

Despite the straightforward derivation of Eq. (16.5) by attending to basic analytical properties of PWAs, in this case, the presence of the RHC and of a pole in the inverse of the PWA, goes beyond an ERE, up to an including $\mathcal{O}(p^4)$, cf. Eq. (11.10). The reason is because the presence of a zero in $t(p^2)$ [or a pole in $1/t(p^2)$] sets a limit in the applicability of the ERE because at this point $p \cot\delta = \infty$ and it is singular. Thus, if this zero happens very close to threshold it makes the ERE to have a very small region of validity, which would typically invalidate it as an adequate approach to study the near-threshold scattering. One can work out straightforwardly the relationship between the parameters a, r_2, and v_2 in the ERE with γ, M_{CDD} and β in Eq. (16.5), it reads

$$\begin{aligned}\frac{1}{a} &= \frac{\gamma}{M_{\text{CDD}}} - \beta\,,\\ r &= -\frac{\gamma}{\mu M_{\text{CDD}}^2}\,,\\ v_2 &= -\frac{\gamma}{4\mu^2 M_{\text{CDD}}^3}\,.\end{aligned} \tag{16.6}$$

An important output of these relations [96] is that a near-threshold CDD pole, $M_{\text{CDD}} \to 0$, is characterized by giving rise to large values of r in absolute module (γ can take any sign). This is also the expected situation for v_2 and higher shape parameters v_i, $i \geq 4$. However, the value for the scattering length in the same limit $M_{\text{CDD}} \to 0$ would tend to zero as M_{CDD}/γ. Of course, the actual situation in which this limit takes place depends on the value of the residue of the CDD pole, the larger is γ the sooner this scenario takes place.

Another parameterization that is usually employed in the literature to describe near-threshold resonances is the so-called Flatté parameterization [97], that we denote as $t_F(E)$ and corresponds to

$$\begin{aligned}t_F(E) &= \frac{g^2/2}{M_F - i\frac{1}{2}\Gamma(E) - E}\,,\\ \Gamma(E) &= g^2 p(E)\,,\, E > 0\,,\\ \Gamma(E) &= i g^2 |p(E)|\,,\, E < 0\,,\end{aligned} \tag{16.7}$$

and $\Gamma(E) \geq 0$ for $E > 0$, which determines that $g^2 \geq 0$. The Flatté mass M_F is the value of the energy for which the real part of the denominator in $t_F(E)$ vanishes. The energy dependence of $\Gamma(E)$ is a characteristic aspect of a Flatté parameterization.

We notice here that $t_F(E)$ is a particular case of an ERE, with the denominator in Eq. (16.7) involving up to quadratic powers in p. The relationship between the ERE parameters a, r and those in the Flatté parameterization g^2, M_F is

$$\begin{aligned}a &= -\frac{g^2}{2M_F}\,,\\ r &= -\frac{2}{g^2\mu}\,.\end{aligned} \tag{16.8}$$

Thus, a Flatté parameterization can only give rise to negative values for the effective range, $r < 0$. The scattering length changes of sign with M_F and, for fixed coupling g^2, it is infinity for $M_F = 0$, in which case $t_F(0)$ becomes infinity too. There is indeed a qualitatively different behavior of the pole content of $t_F(E)$ depending on whether M_F is positive or negative. Solving the roots in p of the denominator of $t_F(E)$, we can write the latter as

$$t_F(E) = \frac{-\mu g^2}{(p(E) - p_1)(p(E) - p_2)} , \tag{16.9}$$

$$p_{1,2} = -i\frac{g^2\mu}{2}\left(1 \pm \sqrt{1 - \frac{8M_F}{g^4\mu}}\right) , \tag{16.10}$$

with the subscript 1(2) corresponding to the $+(-)$ sign in front of the square root. Thus, if $M_F \leq 0$ then $p_{1,2}$ are purely imaginary, but with opposite signs. In this way, $\Im p_1 < 0$ and it corresponds to a virtual state in the second RS, while $\Im p_2 > 0$ and it gives rise to a bound state (in the first RS). Furthermore, $|p_1| > |p_2|$ and the virtual state is deeper than the bound state, which is then closer to threshold. For $M_F = 0$ the bound state has zero binding energy. When M_F becomes positive and lies in the interval $0 < M_F < g^4\mu/8$, the second pole turns out as another virtual state closer to threshold than p_1.

For $M_F > g^4\mu/8$ the two pole positions $p_{1,2}$ have the same negative imaginary part but real parts with opposite signs. These are poles corresponding to resonances, such that in the limit $M_F \to g^4\mu/8 + \varepsilon$ the real part tends to zero and we would end with a a double virtual-state pole [98]. This is also a limitation of the Flatté model, no higher than double poles can arise from this parameterization.

The resonance poles happens in complex conjugate positions in the complex E plane ($E = p^2/2\mu$), which is generally required because of the fulfillment by the PWAs of the Schwarz reflection principle. For this situation, we read from Eq. (16.7) that in the limit in which $\Gamma(M_F) \ll M_F$ the nearest pole position to the physical axis occurs in good approximation at

$$E_F = M_F - i\frac{\Gamma(M_F)}{2} , \tag{16.11}$$

where the equation $M_F - i\Gamma(E)/2 - E = 0$ is solved by iterating it once in $\Gamma(E)$. This is the situation corresponding to the narrow resonance case. The other pole at $E = E_F^*$ in the second RS is further away from the physical or first RS, because the physical values are obtained in the latter sheet by taking $E + i\varepsilon$ with $\varepsilon \to 0^+$. This upper part of the physical axis is connected smoothly with the negative vanishing imaginary part in the second RS, which is the region in which the approximate pole position of Eq. (16.11) lies. Nonetheless, the pole with positive complex imaginary part is connected with the values of the scattering amplitudes in the complex E plane below the real axis (where the scattering amplitude is the complex conjugate of the physical one).

The poles of $t_F(E)$ in the second RS of the complex E plane are $E_{1,2} = p_{1,2}^2/2\mu$, with $p_{1,2}$ given in Eq. (16.10). The corresponding expressions for $E_{1,2}$ are

$$E_{1,2} = M_F - \frac{g^4\mu}{4} \mp i\frac{g^4\mu}{4}\sqrt{\frac{8M_F}{g^4\mu} - 1}\ ,\ M_F > \frac{g^4\mu}{8}\ , \tag{16.12}$$

There is a variation of the real part of $E_{1,2}$ with respect to M_F due to the self-energy contribution $-g^4\mu/4$. On the one hand, from Eq. (16.12) we identify the resonance mass, M_R, as $\Re E_{1,2}$,

$$M_R = M_F - \frac{g^4\mu}{4}\ . \tag{16.13}$$

On the other hand, twice the modulus of the imaginary part of $E_{1,2}$ is identified with the width of the resonance, Γ, given then by

$$\Gamma = \frac{g^4\mu}{2}\sqrt{\frac{8M_F}{g^4\mu} - 1}\ . \tag{16.14}$$

Let us notice that, this expression for Γ is only equal to $g^2\sqrt{2\mu M_R}$, cf. Eq. (16.7), for the case in which $M_F \gg g^4\mu$. In this situation the expression for Γ in Eq. (16.11) also holds in good approximation. Thus, the narrow resonance limit actually requires that $M_F \gg g^4\mu$.

It is also interesting to workout the residues of $t_F(E)$ at the pole positions, either in the complex momentum or energy spaces,

$$\gamma_k^2 = -\lim_{p\to p_i}(p - p_i)t_F(p^2/2\mu)\ , \tag{16.15}$$

$$\gamma_E^2 = -\lim_{E\to E_P}(E - E_i)t_F(E)\ , \tag{16.16}$$

in order. Both types of residues are related by

$$\gamma_E^2 = \gamma_k^2\left.\frac{dE}{dp}\right|_{p_{1,2}} = g_k^2\frac{p_{1,2}}{\mu}\ . \tag{16.17}$$

Working out the residue γ_k^2 is straightforward from Eq. (16.9), with the result

$$\gamma_k^2 = \pm\frac{\mu g^2}{p_1 - p_2} = \pm\frac{1}{\sqrt{\frac{8M_F}{g^4\mu} - 1}}\ , \tag{16.18}$$

with $+(-)$ for the pole $p_1(p_2)$, in this order. Notice that in the narrow resonance case, $M_F \gg g^4\mu/8$, the coupling $\gamma_k^2 \to 0$. The opposite situation occurs for $M_F \to g^4\mu/8$ in which case the coupling diverges, because we end with a double virtual-state pole [98]. Let us recall that this is the starting point for having resonance poles, cf. Eq. (16.12).

There is also another interesting limit that corresponds to the case in which the real part of $E_{1,2}$ starts becoming positive. According to Eq. (16.12), this occurs for $g^4\mu/4 < M_F < \infty$ and then $1 \geq |\gamma_k|^2 \geq 0$. Indeed, one can develop a probabilistic interpretation for $|\gamma_k|^2$ when the real part of the pole position in energy of the resonance is larger than zero. According to this interpretation, $|\gamma_k|^2$ is the weight of the two-body continuum component in the composition of the resonance [96, 99].

As noted above, the effective range r for a Flatté parameterization must be negative, cf. Eq. (16.8). In general terms, a PWA $t(p^2)$ from an ERE up to and including p^2, cf. Eq. (11.9), and denoted by $t_r(E)$, is given by

$$t_r(E) = \frac{1}{-\frac{1}{a} + \frac{1}{2}rp(E)^2 - ip(E)} \; . \tag{16.19}$$

Given the quadratic nature of the denominator in p we also have two poles corresponding to the values

$$p_{1,2} = \frac{1}{r}\left(i \mp \sqrt{\frac{2r}{a} - 1}\right) \; . \tag{16.20}$$

We have resonance poles for

$$r/a > 1/2 \text{ and } r < 0 \; . \tag{16.21}$$

Notice that the imaginary part for a resonance pole should be negative as it lies on the second RS, and this is why we have required that $r < 0$. Let us also indicate that, if the requirements in Eq. (16.21) are applied to the expressions for a and r of a Flatté parameterization in Eq. (16.8), we have the constraint $M_F > g^2\mu/8$, which we already derived as necessary so as to end with resonance poles. It is also interesting to work out the residues γ_k^2 of $t_r(E)$ for the poles in Eq. (16.20). The corresponding expression is

$$\gamma_k^2 = \frac{1}{rp_{1,2} - i} = \mp \frac{1}{\sqrt{\frac{2r}{a} - 1}} \; . \tag{16.22}$$

The requirement for $0 \leq |\gamma_k^2| \leq 1$ implies that

$$\frac{r}{a} \geq 1 \; . \tag{16.23}$$

Again, if we consider this constraint in terms of the values of a and r as corresponding to the Flatté parameterization we then have the condition $M_F \geq g^2\mu/4$, which is also needed so that the real part of the resonance energy is positive. Indeed, from Eq. (16.20) the pole energy $E_{1,2} = p_{1,2}^2/2\mu$ is given by

$$E_{1,2} = \frac{p_{1,2}^2}{2\mu} = \frac{1}{ar\mu}\left(1 - \frac{a}{r}\right) \mp i\,\frac{1}{r^2\mu}\sqrt{\frac{2r}{a} - 1}\,. \tag{16.24}$$

It is then concluded form the previous expression that $r/a > 1$ so that $\Re E_{1,2} \geq 0$, and then the criterion for the probabilistic interpretation of $|\gamma_k|^2$, as developed in Refs. [96, 99], can be applied. From Eq. (16.24) we also have for the width of the resonance (identified as twice the modulus of the imaginary part of $E_{1,2}$),

$$\Gamma = \frac{2}{r^2\mu}\sqrt{\frac{2r}{a} - 1}\,. \tag{16.25}$$

From Eq. (16.22) $|\gamma_k|^2$ decreases as r/a increases. At this point it is worth connecting with the values of a and r given by a near-threshold CDD pole, worked out in Eq. (16.6). From this equation it follows that as $M_{\rm CDD} \to 0$ one is driven towards values of a and r for which $|r/a|$ increases and, therefore, $|\gamma_k|^2$ decreases. This fact is interpreted as that the resonance when $M_{\rm CDD} \to 0$ becomes purely elementary, in the sense that the weight in the resonance state of the two-body asymptotic states, whose scattering is described by $t_r(E)$, tends to vanish. Precisely, this is connected with the standard interpretation for a CDD pole which is typically associated with the need to introduce explicitly in the equations the exchange of an explicit bare resonance. In particular, notice that a bare resonance is characterized by two basic parameters, its mass and coupling to a given state. Similarly, a CDD pole implies two free parameters, its mass and residue. According to Eq. (16.24) in the limit $M_{\rm CDD} \to 0$ we have for the resonance poles positions

$$E_{1,2} \xrightarrow[M_{\rm CDD}\to 0]{} -\frac{M_{\rm CDD}^3}{\lambda^2} \mp i\,\frac{(-M_{\rm CDD})^{7/2}\sqrt{2\mu}}{\lambda^2} \tag{16.26}$$

and we see that the width vanishes faster than the mass (the real part of $E_{1,2}$) by an extra factor $(-M_{\rm CDD})^{1/2}$. This pole is then characterized by a small mass but even a much smaller width, so that the narrow resonance limit holds. Indeed, the decoupling limit of a bare resonance from the two-body continuum requires a zero in the PWA in order to remove the bare pole of the resonance from $t(E)$. This shows in simple terms that the weak coupling limit of a resonance and the presence of a CDD pole are related. We also mention that in order to fulfill the requirements in Eq. (16.21) it is necessary to have negative $M_{\rm CDD}$ and positive γ in the limit $M_{\rm CDD} \to 0$.

The situation in Eq. (16.26) is opposite to the one when $\gamma_k^2 \to 1$, which according to Eq. (16.22) happens for $r/a \to 1$. In such a case, we infer from Eq. (16.24) that the mass of the resonance vanishes in this limit and the energy becomes purely imaginary and finite. Therefore, a resonance that is purely composite of the asymptotic two-body state whose interaction is given by $t_r(E)$ is characterized by having a width much larger than its mass.

We have shown that the parameterization in Eq. (16.5) is more general than an ERE up to an including $\mathcal{O}(p^4)$, because the former accounts for the possibility of near-threshold zeroes while the latter loses its meaning for energies beyond the zero and its convergence is much worse for lower energies. If the parameterization based on the ERE is restricted to terms up to $\mathcal{O}(p^2)$ in the p^2 expansion one then has a PWA, Eq. (16.19), that is more general than the one obtained by applying the Flatté parameterization, Eq. (16.7), because the later can only give rise to negative values of r.

Another type of parameterization that one usually finds in the literature for describing near-threshold scattering stems from the use of a dynamical model based on solving a LS equation with a potential that also includes the exchange of an explicit bare resonance. This is a definite model that exemplifies the connection between the exchange of a bare resonance and the appearance of a CDD pole in the PWA, as commented above. We qualify the scattering due to an energy-independent potential $V(\mathbf{p}, \mathbf{p}')$ as direct scattering between the two-body states in the continuum. On top of it, the exchange of a bare state is also considered, so that the total potential in the continuum, $V_T(\mathbf{p}, \mathbf{p}', E)$, is given by

$$V_T(\mathbf{p}, \mathbf{p}', E) = V(\mathbf{p}, \mathbf{p}') + \frac{f(\mathbf{p})f(\mathbf{p}')}{E - E_0} . \tag{16.27}$$

Here E_0 is the bare mass of the discrete state that is exchanged. The real function $f(\mathbf{p})$ is the bare coupling of this state to the two-body states. The scattering amplitude is given by solving the LS equation in momentum space, cf. Eq. (2.65),

$$T(\mathbf{p}, \mathbf{p}', E) = V_T(\mathbf{p}, \mathbf{p}', E) + \int \frac{d^3q}{(2\pi)^3} \frac{V_T(\mathbf{p}, \mathbf{q}, E)T(\mathbf{q}, \mathbf{p}', E)}{q^2/(2\mu) - E - i\varepsilon} . \tag{16.28}$$

The solution of this IE is clear and intuitive by employing a graphical method. First consider those diagrams without the exchange of any bare-state propagator. This is represented in the panel (a) of Fig. 16.1, where the point vertices, each with four lines attached, indicate the insertion of a factor of $V(\mathbf{q}, \mathbf{q}')$. In turn, the circles joining vertices correspond to the loops with two propagators associated with the two-body intermediate states in the continuum. The panel (a) of Fig. 16.1 represents the iteration of $V(\mathbf{p}, \mathbf{p}')$ that gives rise to the direct-scattering amplitude $T_V(\mathbf{p}, \mathbf{p}', E)$, that results by solving the LS equation of the pure potential problem,

$$T_V(\mathbf{p}, \mathbf{p}', E) = V(\mathbf{p}, \mathbf{p}') + \int \frac{d^3q}{(2\pi)^3} \frac{V(\mathbf{p}, \mathbf{q})T_V(\mathbf{q}, \mathbf{p}', E)}{q^2/(2\mu) - E - i\varepsilon} . \tag{16.29}$$

Next, we consider those contributions containing at least the exchange of one bare state, which is represented pictorially by a double line. When iterating these contributions we have as intermediate states both two particles in the continuum and extra bare-state exchanges. In this way, we have the standard Dyson resummation for the bare-state propagator, giving rise to the dressed one, as represented in the panel (b) of Fig. 16.1. In addition, we also have the dressing of the bare coupling of the exchanged state to the continuum by the direct scattering of the latter, as represented

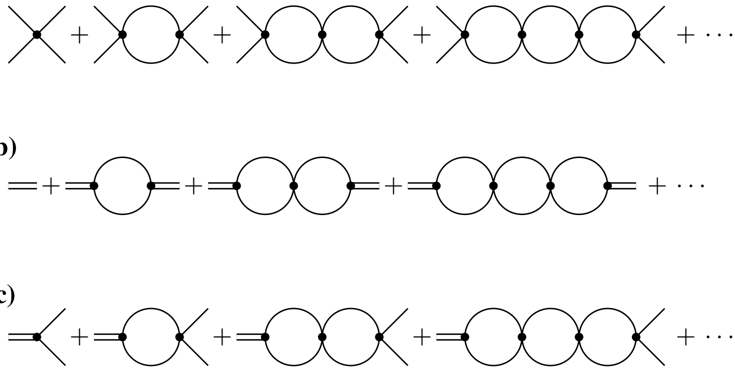

Fig. 16.1 Diagrammatic representation of the solution for $T(\mathbf{p}, \mathbf{p}', E)$ in Eq. (16.28). The diagrams in **a** represents the iteration of $V(\mathbf{p}, \mathbf{p}')$ without any bare-state exchange, which generates $T_V(\mathbf{p}, \mathbf{p}', E)$, Eq. (16.29). The panel **b** represents the self-energy for getting the dressed propagator. Those diagrams in panel **c** correspond to the dressing of the bare coupling due to the self-interactions (or final-state interactions) between the two-body states in the continuum

in the panel (c) of Fig. 16.1. Thus, the set of diagrams in the panels (b) and (c) of Fig. 16.1 gives rise finally to the exchange of a particle with dressed propagator and couplings, in the form

$$R(\mathbf{p}, \mathbf{p}', E) = \frac{\Theta(\mathbf{p}, E)\Theta(\mathbf{p}', E)}{E - E_0 + G(E)} , \tag{16.30}$$

where $\Theta(\mathbf{p}, E)$ represents the dressed coupling and $1/[E - E_0 + G(E)]$ the dressed propagator. We then conclude that the scattering amplitude $T(\mathbf{p}, \mathbf{p}', E)$ must be given by the sum of T_V and R,

$$T(\mathbf{p}, \mathbf{p}', E) = T_V(\mathbf{p}, \mathbf{p}', E) + \frac{\Theta(\mathbf{p}, E)\Theta(\mathbf{p}', E)}{E - E_0 + G(E)} . \tag{16.31}$$

First, we are going to show directly that indeed Eq. (16.31) is a solution of the LS equation in Eq. (16.28), for appropriate functions $\Theta(\mathbf{p}, E)$ and $G(E)$. Next, we give a more general derivation of the solution for the LS equation in terms of the solution of another LS equation with one less discrete intermediate state. In the present example for the total potential in Eq. (16.27), this is the scattering amplitude $T_V(\mathbf{p}, \mathbf{p}', E)$, which satisfies the LS equation of Eq. (16.29) without the intermediate bare state.

By inserting the tentative solution of Eq. (16.31) into Eq. (16.28), and taking into account that $T_V(\mathbf{p}, \mathbf{p}', E)$ fulfills Eq. (16.29), we are then left with the following IE for $R(\mathbf{p}, \mathbf{p}', E)$,

$$\frac{\Theta(\mathbf{p},E)\Theta(\mathbf{p}',E)}{E-E_0+G(E)} = \frac{f(\mathbf{p})f(\mathbf{p}')}{E-E_0} + \int \frac{d^3q}{(2\pi)^3}\frac{1}{q^2/(2\mu)-E-i\varepsilon}\Big[V(\mathbf{p},\mathbf{q})\frac{\Theta(\mathbf{q},E)\Theta(\mathbf{p}',E)}{E-E_0+G(E)} + \frac{f(\mathbf{p})f(\mathbf{q})}{E-E_0}T_V(\mathbf{q},\mathbf{p}',E) + \frac{f(\mathbf{p})f(\mathbf{q})}{E-E_0}\frac{\Theta(\mathbf{q},E)\Theta(\mathbf{p}',E)}{E-E_0+G(E)}\Big]. \quad (16.32)$$

We can derive the equations satisfy by $G(E)$ and $\Theta(\mathbf{p},E)$ by taking $E\to E_0$ and $E\to E_0-G(E)$ in the previous equation. In order, we are then left with

$$\Theta(\mathbf{p}',E)\frac{-1}{G(E)}\int\frac{d^3q}{(2\pi)^3}\frac{f(\mathbf{q})\Theta(\mathbf{q},E)}{q^2/(2\mu)-E-i\varepsilon} = f(\mathbf{p}') + \int\frac{d^3q}{(2\pi)^3}\frac{f(\mathbf{q})T_V(\mathbf{q},\mathbf{p}',E)}{q^2/(2\mu)-E-i\varepsilon}\,. \quad (16.33)$$

$$\Theta(\mathbf{p},E) = -\frac{f(\mathbf{p})}{G(E)}\int\frac{d^3q}{(2\pi)^3}\frac{f(\mathbf{q})\Theta(\mathbf{q},E)}{q^2/(2\mu)-E-i\varepsilon} + \int\frac{d^3q}{(2\pi)^3}\frac{V(\mathbf{p},\mathbf{q})\Theta(\mathbf{q},E)}{q^2/(2\pi)-E-i\varepsilon}\,.$$

These two equations can be made equivalent by identifying[1]

$$G(E) = -\int\frac{d^3q}{(2\pi)^3}\frac{f(\mathbf{q})\Theta(\mathbf{q},E)}{q^2/(2\mu)-E-i\varepsilon}\,, \quad (16.34)$$

and requiring that $\Theta(\mathbf{p},E)$ satisfies the inhomogeneous IE

$$\Theta(\mathbf{p}',E) = f(\mathbf{p}') + \int\frac{d^3q}{(2\pi)^3}\frac{f(\mathbf{q})T_V(\mathbf{q},\mathbf{p}',E)}{q^2/(2\mu)-E-i\varepsilon}\,. \quad (16.35)$$

Let us notice that this IE can also be rewritten as

$$\Theta(\mathbf{p}',E) = f(\mathbf{p}') + \int\frac{d^3q}{(2\pi)^3}\frac{T_V(\mathbf{p}',\mathbf{q},E)f(\mathbf{q})}{q^2/(2\mu)-E-i\varepsilon} = f(\mathbf{p}') + \int\frac{d^3q}{(2\pi)^3}\frac{V(\mathbf{p}',\mathbf{q})\Theta(\mathbf{q},E)}{q^2/(2\mu)-E-i\varepsilon}\,. \quad (16.36)$$

The three IEs in Eqs. (16.35) and (16.36) are equivalent as it is clear by performing the Neumann series expansion of $T_V(\mathbf{q},\mathbf{p}',E)$ from Eq. (16.35), and by solving iteratively the last IE for $\Theta(\mathbf{p}',E)$ in Eq. (16.36). It is straightforward to show that Eq. (16.32) is fulfilled once Eqs. (16.34) and (16.35) are satisfied. For instance, by inserting Eq. (16.35) in Eq. (16.32), we can combine the first and third terms on the rhs of this equation as $f(\mathbf{p})\Theta(\mathbf{p}',E)/(E-E_0)$. Therefore, we can simplify the factor $\Theta(\mathbf{p}',E)$ on both sides of the resulting equation, which then reads

[1] In Eq. (16.33) we have renamed E_0 as E because there is nothing special on E_0, so that it can also be considered as a variable energy.

$$\begin{aligned}\frac{\Theta(\mathbf{p},E)}{E-E_0+G(E)} &= \frac{f(\mathbf{p})}{E-E_0}+\int\frac{d^3q}{(2\pi)^3}\frac{1}{q^2/(2\mu)-E-i\varepsilon}\left[\frac{V(\mathbf{p},\mathbf{q})\Theta(\mathbf{q},E)}{E-E_0+G(E)}\right.\\ &\left.+\frac{f(\mathbf{p})f(\mathbf{q})\Theta(\mathbf{q},E)}{(E-E_0)(E-E_0+G(E))}\right] \\ &= \frac{f(\mathbf{p})}{E-E_0}+\frac{1}{E-E_0+G(E)}\int\frac{d^3q}{(2\pi)^3}\frac{V(\mathbf{p},\mathbf{q})\Theta(\mathbf{q},E)}{q^2/(2\mu)-E-i\varepsilon}-\frac{f(\mathbf{p})G(E)}{(E-E_0)(E-E_0+G(E))}\\ &= \frac{f(\mathbf{p})}{E-E_0+G(E)}+\int\frac{d^3q}{(2\pi)^3}\frac{1}{q^2/(2\mu)-E-i\varepsilon}\frac{V(\mathbf{p},\mathbf{q})\Theta(\mathbf{q},E)}{E-E_0+G(E)}\,,\end{aligned} \tag{16.37}$$

which is fulfilled in virtue of Eq. (16.36).

Now, let us give a more general derivation of the separation of the total T-matrix $T(\mathbf{p},\mathbf{p}',E)$ as in Eq. (16.31). Let the full Hamiltonian H be split as in Chap. 2 in the free part H_0 and the potential V, $H=H_0-V$, and let $|0\rangle$ be an eigenstate of H_0, $H_0|0\rangle=E_0|0\rangle$. Let $T_1(E)$ be the T matrix that fulfills a LS equation without the discrete intermediate state $|0\rangle$, namely,

$$\begin{aligned}T_1(E) &= V+\sum_n\int dW\, V|W_n\rangle(W_n-E)^{-1}\langle W_n|T_1(E)\\ &= V+V(H_0-E)^{-1}\theta T_1(E)\,,\end{aligned} \tag{16.38}$$

where $\theta|0\rangle=0$ and this state is then excluded in the sum over intermediate states. In the previous equation we have also used a compressed notation for the sum over discrete states and indexes, represented by n ($n\neq 0$), and integration over the continuum ones, represented by W. The corresponding intermediate state is then indicated by $|W_n\rangle$. The Eq. (16.38) can be recast as the IE for the resolvent of the kernel of a linear IE. For that, we multiply this equation to the right by $(H_0-E)^{-1}\theta$, which then reads

$$T_1(E)(H_0-E)^{-1}\theta = V(H_0-E)^{-1}\theta+V(H_0-E)^{-1}\theta T_1(E)(H_0-E)^{-1}\theta\,. \tag{16.39}$$

The kernel of this IE is $V(H_0-E)^{-1}\theta$ and its resolvent $K_1(E)$ is therefore [18]

$$K_1(E)=T_1(E)(H_0-E)^{-1}\theta\,. \tag{16.40}$$

Now, we take into account that we can formally write from the LS that $T(E)=V(I-(H_0-E)^{-1}V)^{-1}=V(H-E)^{-1}(H_0-E)$. It follows then that the resolvent of the kernel of the LS equation, as identified in Eq. (16.40), is $T(E)(H_0-E)^{-1}=V(H-E)^{-1}$, cf. Eq. (2.64). The inclusion of θ in Eq. (16.40) is just a projection in the subspace orthogonal to $|0\rangle$.

The full T matrix $T(E)$ satisfies a LS equation in which the state $|0\rangle$ contributes as intermediate state. Thus,

$$T(E) = V + V|0\rangle(E_0 - E)^{-1}\langle 0|T(E) + \sum_n \int dW V|W_n\rangle(W_n - E)^{-1}\langle W_n|T(E) \tag{16.41}$$
$$= V + V|0\rangle(E_0 - E)^{-1}\langle 0|T(E) + V(H_0 - E)^{-1}\theta T(E) \ .$$

We can clearly identify from the previous equation the same kernel, $V(H_0 - E)^{-1}\theta$, as in the IE for $T_1(E)$, cf. Eq. (16.38). Thus, by considering $V + V|0\rangle(E_0 - E)^{-1}\langle 0|T(E)$ as the new independent term, we can write that a solution for $T(E)$ must satisfy

$$T(E) = V + V|0\rangle(E_0 - E)^{-1}\langle 0|T(E) + K_1(E)\Big[V + V|0\rangle(E_0 - E)^{-1}\langle 0|T(E)\Big]$$
$$= T_1(E) + T_1(E)|0\rangle(E - E_0)^{-1}\langle 0|T(E) \ . \tag{16.42}$$

We now multiply the previous IE to the left by $\langle 0|$, so as to express $\langle 0|T(E)$ in terms of known matrix elements. It then results that

$$\langle 0|T(E) = \langle 0|T_1(E) + \langle 0|T_1(E)|0\rangle(E - E_0)^{-1}\langle 0|T(E) \ , \tag{16.43}$$

and then

$$\langle 0|T(E) = \Big[1 - \langle 0|T_1(E)|0\rangle(E_0 - E)^{-1}\Big]^{-1}\langle 0|T_1(E) \ . \tag{16.44}$$

Substituting the previous result into Eq. (16.42) we arrive to the final expression for $T(E)$,

$$T(E) = T_1(E) + T_1(E)|0\rangle\Big[E - E_0 - \langle 0|T_1(E)|0\rangle\Big]^{-1}\langle 0|T_1(E) \ . \tag{16.45}$$

From here we read the full propagator

$$\Delta(E) = \frac{1}{E - E_0 - \langle 0|T_1(E)|0\rangle} \ , \tag{16.46}$$

and the coupling squared operator $T_1(E)|0\rangle\langle 0|T_1(E)$. The latter when acting over the states in the continuum gives rise to the coupling function

$$\Theta(\mathbf{p}_n, E) = \langle \mathbf{p}_n|T_1(E)|0\rangle \ . \tag{16.47}$$

The Eq. (16.45) is the general expression of the scattering matrix $T(E)$ in terms of the reduced one $T_1(E)$, which results after a bare state $|0\rangle$ is removed from the sum over the intermediate states. In particular, it is clear that $T_1(E)$, cf. Eq. (16.38), corresponds to panel (a) of Fig. 16.1, $\Delta(E)$ in Eq. (16.46) arises from the Dyson resummation depicted in the panel (b), and the coupling function $\Theta(\mathbf{p}, E)$ of Eq. (16.47) originates from the FSI of the continuum states, drawn in the panel of (c) of the same figure.

The Eq. (16.31) is a particular case of Eq. (16.45) when projected over states in the continuum. The latter equation can be found in Ref. [100], but not its derivation, which has been offered here in detail for completeness and also for pedagogical reasons.

In particular, let us compare the expression in Eq. (16.5), given in terms of a subtraction constant and a CDD pole, with the model of Ref. [101] that results by applying Eq. (16.31) with $T_V(\mathbf{p}, \mathbf{p}', E)$ corresponding to the plain scattering length approximation,

$$T_V(\mathbf{p}, \mathbf{p}', E) = \frac{2\pi}{\mu} \frac{1}{-\frac{1}{a_V} - ik(E)} \,, \qquad (16.48)$$
$$k(E) = \sqrt{2\mu E} \,.$$

For this particular case indeed $T_V(\mathbf{p}, \mathbf{p}', E)$ only depends on the energy and we better denote it simply as $T_V(E)$. As a result, the evaluation of the self-energy $G(E)$ and the dressed coupling $\Theta(\mathbf{p}, E)$, cf. Eqs. (16.34) and (16.35), respectively, is straightforward once $f(\mathbf{p})$ is known. Nonetheless, Ref. [101] argues that, since one is focusing in the low-energy region so that $k\alpha \ll 1$, with α the typical range of the interaction involved, one could parameterize the whole $f(\mathbf{p})$ by $f_0 = f(0)/(2\pi)$ and then, the diverging integrals from Eqs. (16.34) and (16.35) are regularized by naive dimensional analysis as

$$\widetilde{g}_1(E) = \int \frac{d^3q}{(2\pi)^3} \frac{f(\mathbf{p})^2}{q^2/(2\mu) - E - i\varepsilon} = f_0^2(R + \mu ik) \,, \qquad (16.49)$$
$$\widetilde{g}_2(E) = \int \frac{d^3q}{(2\pi)^3} \frac{f(\mathbf{p})}{q^2/(2\mu) - E - i\varepsilon} = f_0(R' + \mu ik) \,,$$

where one expects that the constants R and R' take values of $\mathcal{O}(\mu/\alpha)$. It is just a matter of simple algebra to deduce from Eqs. (16.34), (16.35) and (16.31) the following expression for the on-shell T matrix ($|\mathbf{p}| = |\mathbf{p}'| = k$) [101],

$$t(E) = -\frac{2\pi}{\mu} \frac{E - E_f + \frac{1}{2} g_f \gamma_V}{(E - E_f)(\gamma_V + ik) + i\frac{1}{2} g_f \gamma_V k} \,, \qquad (16.50)$$

where $\gamma_V = 1/a_V$, while g_f and E_f are functions of the original parameters R, R' and E_0 of the model (the interested reader can consult Ref. [101] for the relations). Notice that g_f has the meaning of a bare coupling and E_f is the energy at which the real part of the denominator in Eq. (16.50) vanishes. Redefining the normalization multiplying $t(E)$ by $\mu/(2\pi)$, we end with a particular case of Eq. (16.5), previously obtained making use of general analytic and unitarity principles. The parameters in Eq. (16.5) are related to those in Eq. (16.50) by

$$\begin{aligned}\beta &= -\gamma_V \,, \\ \gamma &= \frac{1}{2} g_f \gamma_V^2 \,, \\ M_{\text{CDD}} &= E_f - \frac{1}{2} g_f \gamma_V \,.\end{aligned} \tag{16.51}$$

However, the reverse is not true and Eq. (16.5) is not a particular case of Eq. (16.50) from the scattering model of Ref. [101]. As a proof of this statement, let us notice that from Eq. (16.50) the resulting effective range r can only be negative [98]. Applying Eq. (16.6) with the particular values of Eq. (16.51) we have

$$r = -\frac{g_f \gamma_V^2}{2\mu(E_f - g_f \gamma_V/2)^2} \leq 0 \,, \tag{16.52}$$

because $g_f = 2\mu f_0^2 (R - R_V)^2 / R_V^2$ and $R_V = \mu \gamma_V$ [101].

The situation described in this section is particularly interesting for the scattering of heavy-quark mesons near their thresholds where several states with rather exotic properties have been found that go beyond well-established quarkonium spectroscopy [34, 102]. In these systems the coupling to the pion is relatively suppressed compared to that in the light-quark sector. For instance, for the $P^* P$ potential worked out in Ref. [103] (where P^* is a heavy-quark vector-meson resonance and P is a heavy-quark pseudoscalar, with the heavy quark being the c or the b), one has that the strength of the central and tensor components of the one-pion mediated interaction is weaker by around a factor $g^2/(2g_A^2) \simeq 0.06$ compared to that for the NN interactions. Here g is the coupling for $P^* P \pi$ which is around 0.5 [103]. This makes that even though there would be a LHC due to pion exchanges, this can be treated perturbatively and one could neglect its effects in a first approximation. In such a situation we can then apply the results presented in this section [96, 98].

We have only introduced by pass the interesting matter of quantifying the compositeness and elementariness of a pole in the S matrix, since a full discussion on it should imply to abandon the strict realm of DRs and enter in QFT developments. For more discussions the interested reader can consult Ref. [14]. For earlier results one has, e.g., Refs. [96, 98–100, 104–108].