# Chapter 15
# The Muskhelishvili-Omnès Problem in Coupled-Channel form Factors

The basic problem that we consider in this section is to find the possible solutions for a set of form factors $F_i(s)$, $i = 1 \ldots n$, ordered in increasing value of their thresholds $s_{\text{th};i}$. Each of the $F_i(s)$ has LHC for $s < s_L$ and RHC for $s > s_{\text{th}}$, where $s_{\text{th}}$ is the lightest of all the thresholds $s_{\text{th};i}$ involved and $s_L$ was defined above.

Along the RHC the imaginary part of $F_i(s)$ is given by the unitarity relation expressed in Eq. (13.4). The latter also allows us to know the discontinuities of these form factors along the RHC because they fulfill the Schwarz reflection principle,

$$F_i(s^*) = F_i(s)^* \;, \tag{15.1}$$

since the form factors are real in the interval $s_L < s < s_{\text{th}}$. As a result, the discontinuity of $F_i(s)$ along the RHC obeys

$$\Im F_i(s + i\varepsilon) - \Im F_i(s - i\varepsilon) = 2i\Im F_i(s + i\varepsilon) \;,\; s > s_{\text{th}} \;. \tag{15.2}$$

On the other hand, the discontinuity of these functions along the LHC is assumed to be given, cf. the example of Eq. (14.20), and it is denoted in the following by $\Delta_L F_i(s)$. Namely,

$$F_i(s + i\varepsilon) - F_i(s - i\varepsilon) = \Delta_L F_i(s) \;,\; s < s_L \;. \tag{15.3}$$

We already have shown in Eq. (13.14) that the $n \times 1$ column vector $F(s)$ of matrix elements $F_i(s)$ can be expressed as the product of the inverse of the $n \times n$ matrix $D(s)$, whose matrix elements are the functions $D_{ij}(s)$, times $L(s)$. The latter is an $n \times 1$ vector column of $n$ analytical functions in the cut complex $s$ plane, $L_i(s)$, which do not have RHC. They could have only LHC (if any) [the possible bound-state poles in $F(s)$ would correspond to zeroes in the $\det D(s)$].

To characterize the different solutions for $F(s)$ it is convenient to introduce an $n \times n$ matrix $\mathcal{S}(s)$ defined as



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$$\mathcal{S}(s) = I + T(s)2i\rho(s) \ . \tag{15.4}$$

Notice that although $T(s)$ is a symmetric this is not the case in general for $\mathcal{S}(s)$. From Eq. (15.4) and the unitarity relation for $T(s)$, Eq. (2.50), it is straightforward to prove that for $s > s_{\text{th};n}$ this matrix satisfies the property,

$$\mathcal{S}(s)\mathcal{S}(s)^* = \mathcal{S}(s)^*\mathcal{S}(s) = I \ . \tag{15.5}$$

To avoid any confusion let us indicate that the asterisk refers to complex conjugation and not to the Hermitian conjugate of the matrix $\mathcal{S}(s)$. Since $\mathcal{S}(s)$ is not symmetric they are not equivalent. Note also the while the $S$ matrix in partial waves, defined in Eq. (2.52), is symmetric and unitary neither of these properties hold in general for $\mathcal{S}(s)$ when $n > 1$.

If we use the $N/D$ method to express $T(s) = D(s)^{-1}N(s)$, we notice that $\mathcal{S}(s)$ can also be written as

$$\begin{aligned}\mathcal{S}(s) &= I + 2iD(s)^{-1}N(s)\rho(s) = D(s)^{-1}\left[D(s) + 2iN(s)\rho(s)\right] \\ &= D(s)^{-1}D(s)^* \ ,\end{aligned} \tag{15.6}$$

an expression valid in the whole complex $s$ plane.

Now, let us assume that we have found an $n \times n$ matrix $D(s)$ with only RHC that satisfies Eq. (15.6). From the previous equation, and taking into account that $D(s)^* = D(s^*)$ it also follows the discontinuity relation

$$D(s)^{-1} = \mathcal{S}(s)D(s^*)^{-1} \ . \tag{15.7}$$

Multiplying both sides by $L(s)$, cf. Eq. (13.14), and taking into account that $L(s)$ is real along the RHC, we then have an analogous relation for the form factors

$$F(s) = \mathcal{S}(s)F(s^*) \ . \tag{15.8}$$

As stated, $L(s) = D(s)F(s)$ has only LHC and its discontinuity along this cut is given by

$$\Delta_L L(s) = D(s)\Delta_L F(s) \ , \tag{15.9}$$

since $D(s)$ is regular along the LHC because of extended unitarity. Assuming that $L(s)$ diverges for $s \to \infty$ less strongly than $s^m$ for some integer $m \geq 0$,[1] we can write the following $m$-times subtracted DR

[1] More rigorously we should say that $L(s)$ diverges less strong than $s^{m-1}$, $m \geq 1$, to avoid just a logarithmic vanishing of $L(s)/s^m$. However, for the statement above we always have in mind a power-like vanishing, $|L(s)/s^m| < |s|^{-\gamma}$, $\gamma > 0$, for $s \to \infty$.

$$L(s) = \sum_{i=0}^{m-1} a_i s^i + \frac{s^m}{\pi} \int_{-\infty}^{s_L} \frac{D(s')\Delta_L F(s') ds'}{(s')^m (s'-s)} , \tag{15.10}$$

such that if $m = 0$ there is no subtractive polynomial. The latter equation allows to write the following DR representation for $F(s)$,

$$F(s) = D(s)^{-1} \sum_{i=0}^{m-1} a_i s^i + \frac{s^m}{\pi} \int_{-\infty}^{s_L} \frac{D(s)^{-1} D(s')\Delta_L F(s') ds'}{(s')^m (s'-s)} . \tag{15.11}$$

Now, for a given PWA $T(s)$ we can work out $\mathcal{S}(s)$, Eq. (15.5). The problem of finding an $n \times n$ matrix $D(s)$ of functions $D_{ij}(s)$ with only RHC that allows one to write $\mathcal{S}(s) = D(s)^{-1} D(s)^*$ as in Eq. (15.6) is called the Hilbert problem. From Eq. (15.7) it is clear that each column of $D(s)^{-1}$ satisfies the same discontinuity relation as Eq. (15.8) for the form factors along the RHC. Therefore, every column of $D(s)^{-1}$ is itself a form factor with RHC only. The final form factors $F_i(s)$ are obtained by a linear combination of the columns $D(s)^{-1}$, where the coefficients in this linear superposition are the $L_i(s)$ functions that comprise the possible LHC, cf. Eq. (15.10).

First, let us notice that the determinant of the $S$ matrix, $S(s)$, and that of $\mathcal{S}(s)$ are the same,

$$\det S(s) = \det \mathcal{S}(s) . \tag{15.12}$$

This is clear if we consider that

$$\det S = \det\left(I + 2i\rho^{\frac{1}{2}} T \rho^{\frac{1}{2}}\right) = \det\left(\rho^{\frac{1}{2}}\left[\rho^{-\frac{1}{2}} + 2iT\rho^{\frac{1}{2}}\right]\right) = \det\left(I + 2iT\rho\right) = \det \mathcal{S} . \tag{15.13}$$

Notice also that $\det S$ is given by the sum of the eigen-phase shifts $\varphi_i(s)$ as

$$\det S = \exp 2i \sum_{i=1}^{n} \varphi_i(s) = \det \mathcal{S} . \tag{15.14}$$

For the important two-coupled channel case the sum of the eigen-phase shifts is the sum of the phase shifts, as it is clear from Eq. (13.15).

The fact that $\mathcal{S}(s) = D(s)^{-1} D(s)^*$, Eq. (15.6), allows us to write an Omnès representation for $\det D^{-1}$. The point is that $\det D^{-1}$ has only RHC (as the function $D$ itself) and from Eq. (15.6) we learn that the phase of $\det D^{-1}$ is half the phase of $\det S$, which is denoted in the following as $\Phi(s)$, $\Phi(s) = 2\sum_i \varphi_i(s)$.[2] In addition $\det D^{-1}$ could have zeroes and poles (the former are the generalization of the CDD poles to the coupled-channel case). Out of the zeros and poles of $\det D(s)^{-1}$ we make up the

[2]The number of open channels changes. However, $\Phi(s)$ is a continuous function of $s$ along the RHC.

polynomials $P(s)$ and $Q(s)$, respectively, cf. Eq. (14.5). To simplify the notation we further introduce the symbols

$$\Delta(s) = \det D(s)^{-1} \,, \quad (15.15)$$
$$s_R = s_{\text{th};1} \,.$$

We take the function $e^{-i\frac{\Phi(s_R)}{2}} Q(s)\Delta(s)/P(s)$, which is amenable to an Omnès representation in the form,

$$\Delta(s) = \frac{P(s)}{Q(s)} \exp \omega(s) \,, \quad (15.16)$$
$$\omega(s) = \frac{\Phi(s_R)}{2} + \frac{s - s_R}{2\pi} \int_{s_R}^{\infty} \frac{\Phi(s') - \Phi(s_R)}{(s' - s_R)(s' - s)} ds' \,. \quad (15.17)$$

We have multiplied $\Delta(s)$ by $\exp(-i\Phi(s_R)/2)$ so that the resulting function has a zero phase at $s_R$, which allows the integral in the DR of the previous equation to stay finite (even if $\Phi(s_R)$ is not zero).

Taking into account that the asymptotic behavior of an Omnès function for $s \to \infty$ is given by the asymptotic phase $\Phi(\infty)$, cf. Eq. (14.10), we then have from Eqs. (15.16) and (15.17) the following limit behavior for $\Delta(s)$,[3]

$$\Delta(s) \xrightarrow[s\to\infty]{} s^{p-q-\frac{\Phi(\infty)-\Phi(s_R)}{2\pi}} \,. \quad (15.18)$$

Which is the relativistic coupled-channel version of the Levinson theorem, cf. footnote 3.

An interesting result in connection with Eq. (15.18) is that it relates the asymptotic behavior of $\Delta(s)$ with the leading power behavior in $s$ of the columns in $D(s)^{-1}$ [75, 91, 92]. Let $\phi_i(s)$ be the $i_{\text{th}}$ column of $D(s)^{-1}$ which, as follows from Eq. (15.7), satisfies the same discontinuity linear relation as a form factor,

$$\mathcal{S}(s)\phi_i(s)^* = \phi_i(s) \,, \; s > s_R \,. \quad (15.19)$$

Assuming as in Refs. [75, 92] that $\mathcal{S}(s) \to I$ for $s \to \infty$ it is clear that the leading behavior of $\phi_i(s)$ should be integer-power like (no cut remains in this limit and we always assume that all these functions are amenable to a DR treatment). Furthermore, by appropriate linear combinations we can always choose these $\phi_i(s)$ such that if $\chi_i$ is the leading degree in $s$ of $\phi_i(s)$ [which corresponds to the degree in $s$ of the dominant component among all the components of $\phi_i(s)$] then

$$\Delta(s) \xrightarrow[s\to\infty]{} s^{\chi_1+\chi_2+\cdots+\chi_n} \,. \quad (15.20)$$

[3]We assume that the zeroes and poles of $\Delta(s)$ do not occur at the threshold $s_R$.

To see this result let us discuss first the two-coupled channel case and to fix ideas let us assume that $\chi_2 \geq \chi_1$. If the leading behavior for $\phi_2$ gives rise to a column vector linearly independent to the leading one for $\phi_1$, then the result of Eq. (15.20) is clear. However, if the leading-components vector $\phi_2$ is linearly dependent with the leading one from $\phi_1$, then multiply $\phi_1$ by $s^{\chi_2-\chi_1}$ times a constant and remove it to $\phi_2$, which is then the new $\phi_2$. In this way (iterated if needed), the leading behavior for $\phi_2(s)$ is now a linearly independent vector to $\phi_1$. On the other hand, if $\chi_1 > \chi_2$ we would proceed analogously exchanging $1 \leftrightarrow 2$. It is clear that this process can be further iterated to treat the case with $n$ coupled PWAs, and then Eq. (15.20) results for an appropriately built matrix $D(s)^{-1}$. Notice also that the exponent in the rhs of Eq. (15.20) must match with the one in Eq. (15.18). Thus, we also have that

$$\chi_1 + \chi_2 + \ldots + \chi_n = p - q - \frac{\Phi(\infty) - \Phi(s_R)}{2\pi} \,. \tag{15.21}$$

These results were applied in Ref. [76] to study the strangeness-changing scalar form factors for $K\pi(1)$, $K\eta(2)$ and $K\eta'(3)$, following an analogous set up as in Ref. [83] for the calculation of the $\pi\pi$ and $K\bar{K}$ isoscalar scalar form factors (this latter problem was addressed also by Ref. [84] with a similar approach). The strangeness-changing or $\Delta S = 1$ scalar form factors are defined by

$$\begin{aligned}\langle 0|\partial^\mu(\bar{s}\gamma_\mu u)(0)|K\phi_K\rangle &= -i\sqrt{\frac{3}{2}}\Delta_{K\pi}F_k(s) \,, \\ \Delta_{K\pi} &= m_K^2 - m_\pi^2 \,.\end{aligned} \tag{15.22}$$

The state $|K\pi\rangle$ is in the isospin basis so that its form factor is $\sqrt{3}$ that of $|K^+\pi^0\rangle$, and $|0\rangle$ is the vacuum state.

The $I = 1/2$ scalar $K\pi$, $K\eta'$ PWAs of Ref. [93] were used for driving the FSI. Reference [76] also checked that the results barely change when considering the $K\eta$ channel as well, so that we disregard it in the following and concentrate in the two-coupled channel problem of $K\pi$ and $K\eta'$ scattering. It was further taken for granted in Ref. [76] that the scalar form factors vanish for $s \to \infty$ because the hadrons are composite objects. This is also in agreement with expectations from QCD counting rules [77–79]. As a result, the following unsubtracted DRs were written for the hadronic form factors $F_1(s)$ and $F_3(s)$,

$$\begin{aligned}F_1(s) &= \frac{1}{\pi}\int_{s_{\text{th};1}}^\infty \frac{\rho_1(s')F_1(s')T_{11}(s')^* ds'}{s'-s} + \frac{1}{\pi}\int_{s_{\text{th};3}}^\infty \frac{\rho_3(s')F_3(s')T_{13}(s')^* ds'}{s'-s} \,, \\ F_3(s) &= \frac{1}{\pi}\int_{s_{\text{th};1}}^\infty \frac{\rho_1(s')F_1(s')T_{13}(s')^* ds'}{s'-s} + \frac{1}{\pi}\int_{s_{\text{th};3}}^\infty \frac{\rho_3(s')F_3(s')T_{33}(s')^* ds'}{s'-s} \,.\end{aligned} \tag{15.23}$$

These coupled linear IEs were solved numerically in Ref. [93] by iteration. The numerical iterative method developed in this reference is summarized in the appendix.

The PWAs considered in Ref. [93] have no bound states, $q = 0$, and the $D^{-1}$ matrix has no CDD poles [they were reabsorbed in the function $N(s)$], $p = 0$. Furthermore, $\Phi(s_R) = 0$. As a result, the rhs of Eq. (15.18) reads

$$\Delta(s) \xrightarrow[s\to\infty]{} s^{-\frac{\Phi(\infty)}{2\pi}} \, . \tag{15.24}$$

The first set of $T$ matrices used in Ref. [93], and derived in Ref. [76], give rise to $\Phi(\infty) = 2\pi$ ($\delta_1(\infty) = \pi$ and $\delta_3(\infty) = 0$). It follows then from Eq. (15.21) that

$$\chi_1 + \chi_2 = -1 \, . \tag{15.25}$$

Since it is not possible that simultaneously $\chi_1$ and $\chi_2$ are negative integers, we then conclude that there is only one linearly independent solution that vanishes at infinity with $\chi_1 = -1$. This is the solution obtained by solving numerically Eq. (15.23) employing the PWAs from the fits (6.10) and (6.11) of Ref. [76]. As starting input Ref. [93] takes for $F_1(s)$ its solution according to an Omnès representation, cf. Eq. (14.5), with constant $P(s)$ and $Q(s)$, while $F_3(s)$ is taking zero initially. The normalization factor corresponds to $F_{K\pi}(0)$ according to its value calculated at NLO in ChPT [94].

Next, Ref. [93] also matched smoothly the unitarized ChPT PWAs of Ref. [76] with a $K$-matrix ansatz at an energy around $\sqrt{s} = 1.75$ GeV. The point is to improve the reproduction of the experimental data of Ref. [95] on $K\pi$ scattering, that was somewhat deficiently accomplished by the PWAs of Ref. [76] for energies above 1.9 GeV. Once this is done Ref. [93] could consider the transition from $\Phi(\infty) = 2\pi$ to $\Phi(\infty) = 4\pi$ by changing some suitable parameters in the $K$-matrices employed, while reproducing satisfactorily the experimental data up to the largest energy available in Ref. [95] ($\sqrt{s} = 2.5$ GeV). For the case $\Phi(\infty) = 4\pi$ we then have from Eq. (15.21) that ($q = p = \Phi(s_R) = 0$)

$$\chi_1 + \chi_2 = -2 \, . \tag{15.26}$$

In this case we can then have two linearly independent solutions with negative $\chi_i$ for $\chi_1 = \chi_2 = -1$. This second linearly independent solution was found in Ref. [93] by solving Eq. (15.23) with different input values for the form factors at the origin. Apart from a global normalization another piece of information is needed, since now there are two linearly independent solutions. The Ref. [93] uses the value of the $K\pi$ form factor at the Callan-Treiman point, where $s = \Delta_{K\pi}$, because it can be related quite accurately with the ratio of the weak decay constants of the pseudoscalar kaons ($f_K$) and pions ($f_\pi$). The precise relation is

$$F_{K\pi}(\Delta_{K\pi}) = \frac{f_K}{f_\pi} + \Delta_{CT} \, , \tag{15.27}$$

with $\Delta_{CT}$ estimated as $-3 \times 10^{-3}$ [94], while $f_K/f_\pi$ is taken in Ref. [93] as $1.22 \pm 0.01$, according to the phenomenological information then available. It is worth emphasizing that once the chiral unitary amplitudes of Ref. [76] were implemented with the $K$-matrix ansätze, independently of whether $\Phi(\infty) = 2\pi$ (one linearly independent solution) or $4\pi$ (two linearly independent solutions) the value of $F_{K\pi}(\Delta_{K\pi})$ is in both cases compatible, which indicates the great stability of the results. For the case with only one linearly independent solution it turns out that $F_{K\pi}(\Delta_{K\pi}) = 1.219 - 1.22$ in impressive agreement with Eq. (15.27).

Let us finish this section by connecting with the use of the function $\mathcal{N}(s)$ to express the form factors $F_i(s)$ as in Eq. (13.11), by using the matrix of functions $[I + \mathcal{N}(s)g(s)]^{-1}$. In the special case in which $\mathcal{N}(s)$ is modeled without LHCs, as discussed in Chap. 6, we could end with explicit expressions for $\Omega(s)$ (in the uncoupled case) and for $D(s)$ in the coupled case. For the former case we would have

$$\Omega(s) = \frac{\prod_{i=1}^{q}(s - s_{P;i})}{\prod_{j=1}^{p}(s - s_{Z;j})} \frac{1}{1 + \mathcal{N}(s)g(s)} \,, \tag{15.28}$$

with the subscripts $P$ and $Z$ referring to the poles and zeroes of $1/[1 + \mathcal{N}(s)g(s)]$, which are removed by multiplying this function by the appropriate rational function. For the case of coupled channels, we can identify the matrix $D(s)$ in Eq. (15.7) with

$$D(s) = [I + \mathcal{N}(s)g(s)] \,. \tag{15.29}$$

We can also introduce like in Ref. [92] the analogous of $\Omega(s)$ in coupled channels, denoted by $\mathcal{D}^{-1}(s)$, so that $\mathcal{D}^{-1}(s)$ satisfies Eq. (15.7), it is holomorphic and nonsingular in the cut complex $s$ plane. Given the function $D(s)$ in Eq. (15.29) we notice that the product $\mathcal{D}(s)D(s)^{-1}$ has no cuts because from Eq. (15.7)

$$\begin{aligned} &\mathcal{D}(s + i\varepsilon)D(s + i\varepsilon)^{-1} - \mathcal{D}(s - i\varepsilon)D(s - i\varepsilon)^{-1} \\ &= \mathcal{D}(s - i\varepsilon)\mathcal{S}(s - i\varepsilon)\mathcal{S}(s + i\varepsilon)D(s - i\varepsilon)^{-1} - \mathcal{D}(s - i\varepsilon)D(s - i\varepsilon)^{-1} = 0 \,, \end{aligned} \tag{15.30}$$

taking also into account Eq. (15.5) and that $\mathcal{D}(s^*) = \mathcal{D}(s)^*$. The same property would also hold for $D(s)\mathcal{D}(s)^{-1}$. Therefore, the product $\mathcal{D}(s)D(s)^{-1}$ is a rational function $R(s)$ and we can write [92]

$$D(s)^{-1} = \mathcal{D}(s)^{-1}R(s) \,. \tag{15.31}$$

Of course, this result applies to any possible matrix of functions $D(s)$ satisfying Eq. (15.6), independently of the modeling of $\mathcal{N}(s)$.