Chapter 14
The Omnès Solution. Reasoned Warnings on the Use of the Omnès Function

We consider along this section that the uncoupled unitarity relation of Eq. (13.5) can be applied, at least in good approximation, and assume that the strong interacting PWA is known. Given a form factor with only RHC, like for example the vector or scalar form factors of two hadrons, the unitarity relation, Eq. (13.5), provides us with its discontinuity along this cut. The phase of the form factor $F(s)$, $\varphi(s)$, is the same as the phase of the PWA $T(s)$ [also denoted then by $\varphi(s)$] because of the Watson final-state theorem (we are here suppressing any subscript).[1] In the strict elastic region $\varphi(s) = \delta(s)$ but, as we show below, it might be that still the phase of the form factor corresponds approximately to that of the PWA, while the latter departs strongly from $\delta(s)$ in a region with marked inelasticity. The reason is that the form factor mostly couples to a given eigen-channel that diagonalizes the S matrix, for which the elastic treatment holds.

The solution for an analytical function in the cut complex s plane, with a branch point singularity at s_{th} associated with a RHC, along which its phase is known, can be written in terms of the so-called Omnès function. The idea is relatively straightforward and can be implemented in two steps.

First, by the knowledge of $\varphi(s)$ we construct an analytical function with a RHC and branch point discontinuity at s_{th} by writing down the DR

$$\omega(s) = \sum_{i=0}^{n-1} a_i s^i + \frac{s^n}{\pi} \int_{s_{\text{th}}}^{\infty} \frac{\varphi(s')ds'}{(s')^n (s'-s)} , \tag{14.1}$$

where we have assumed that $\varphi(s)$ does not diverge stronger than s^{n-1} for $s \to \infty$, with n a finite integer. We have introduced n subtraction constants so that the result is independent of the subtraction point. Along the RHC this function fulfills that $\omega(s + i\varepsilon) - \omega(s - i\varepsilon) = 2i\varphi(s)$. Second, we next define the Omnès function, $\Omega(s)$, as

[1] If there is a difference between these two phases of π then just take $-F(s)$.

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$$\Omega(s) = \exp \omega(s) \; . \tag{14.2}$$

We always have the freedom to normalize the Omnès function such that $\Omega(0) = 1$, which fixes $a_0 = 1$. It follows also that the combination

$$R(s) = \frac{F(s)}{\Omega(s)} \; , \tag{14.3}$$

is real for $s > s_{\rm th}$ and it has no cuts, so that it is a meromorphic function of s in the first RS of the whole complex s plane.

Let us consider first that $\omega(s)$ is finite along the RHC, so that $0 < |\Omega(s)| < \infty$,[2] and there are no bound states (i.e., $F(s)$ has no poles). It is known in complex analysis that any function that is analytic in the whole complex s plane is constant or unbounded. If we apply this theorem to $R(s)$, we learn then that $F(s)$ diverges as much as or stronger than $\Omega(s)$ for $s \to \infty$. As result we conclude in this case that we can express $F(s)$ as

$$F(s) = R(s)\Omega(s) \; , \tag{14.4}$$

with $R(s)$ a constant or an analytical function which is unbounded at infinity. Indeed, we can expect exponential divergences in $\Omega(s)$ from Eq. (14.2) when the DR for $\omega(s)$ requires for convergence more than one subtraction. The conclusion follows by a similar analysis as the one performed between Eqs. (4.10) and (4.12) in relation with the Sugawara–Kanazawa theorem. Thus, if $\varphi(s)/s^{n-1}$ ($n \geq 2$) were not vanishing for $s \to \infty$, one would have logarithmic divergences like $s^{n-1} \log s$ (here there is only RHC). These divergences could not be canceled by the $a_n s^{n-1}$ term. Therefore, $R(s)$ would be an exponential function so as to guarantee that $F(s)$ does not diverge stronger than a power of s for $s \to \infty$ (and it is then amenable for a DR). Regarding this point, one would expect that a hadronic form factor would typically vanish for $s \to \infty$ because of the finiteness of the non-perturbative scale of QCD, Λ_{QCD}, as also suggested by the quark counting rules [77–79], and then being amenable to a DR. By the same token, one would also expect intuitively that the phase of the form factor tends to a constant limit for $s \to \infty$. However, these expectations could fail in the case of singular interactions at the origin.

We can say more about $R(s)$. Let $P(s)$ be the polynomial made out of the possible zeroes of $F(s)$ (if any), and let $Q(s)$ be another polynomial whose zeroes are the possible poles (if there exists any bound state) of $F(s)$. Next, we multiply $F(s)$ by the rational function $Q(s)/P(s)$ and perform a DR of the analytical function $\log\left[F(s)Q(s)/P(s)\right]$ in the cut complex s plane circumventing the RHC. The discontinuity of this function along the RHC is $2i[\delta(s+i\varepsilon) - \delta(s-i\varepsilon)]$, the one corresponding to function $\omega(s)$. Of course, we are assuming also here that $F(s)$ has a finite number of zeroes and bound states (these numbers are p and q, respectively) and that $\log F(s)Q(s)/P(s)$ is amenable to a DR treatment. There, this procedure implies

[2] Later we discuss a specific situation when this is not the case.

that $F(s)$ can be written as in Eq. (14.4) with $R(s) = Q(s)/P(s)$. In the subsequent we further require that $F(s)$ resulting from this analysis does not grow exponentially, so that we are driven to admit that a once-subtracted DR is possible for $\omega(s)$ in Eq. (14.1). In other terms, we assume that for $s \to \infty$ the ratio $|\varphi(s)/s| < s^{-\gamma}$ for some $\gamma > 0$, because otherwise we could apply the analysis above below Eq. (14.4), and based on the process followed for the demonstration of the Sugawara–Kanazawa theorem.[3] We then arrive to the following expression for $F(s)$ that we consider in the following:

$$F(s) = \frac{P(s)}{Q(s)}\Omega(s) , \tag{14.5}$$

$$\Omega(s) = \exp\omega(s) , \tag{14.6}$$

$$\omega(s) = \frac{s}{\pi}\int_{s_{\rm th}}^{\infty}\frac{\varphi(s')ds'}{s'(s'-s)} . \tag{14.7}$$

Here, $P(s)$ absorbs the required normalization constant to permit our choice $\Omega(0) = 1$ without loss of generality.

Let us work out the behavior of $\Omega(s)$ in the limit $s \to \infty$ by taking for granted the existence of the limit $\varphi(\infty) < \infty$. For that we again proceed as in Eq. (4.10) and then we decompose $\omega(s)$ in Eq. (14.7) as

$$\omega(s) = \varphi(\infty)\frac{s}{\pi}\int_{s_{\rm th}}^{\infty}\frac{ds'}{s'(s'-s)} + \frac{s}{\pi}\int_{s_{\rm th}}^{\infty}\frac{\varphi(s')-\varphi(\infty)}{s'(s'-s)}ds' . \tag{14.8}$$

Thus, for $s \to \infty$ we have

$$\omega(s+i\varepsilon) \xrightarrow[s\to\infty]{} -\frac{\varphi(\infty)}{\pi}\log\frac{s}{s_{\rm th}} + i\varphi(\infty) - \frac{1}{\pi}\int_{s_{\rm th}}^{\infty}\frac{\varphi(s')-\varphi(\infty)}{s'}ds' , \tag{14.9}$$

and the logarithmic divergence is the one that dominates for $s \to \infty$. The other two terms in the previous equation are constant ones, the first one is purely imaginary and gives the phase of $\Omega(s)$ while the latter is a constant stemming from the second integral in Eq. (14.8) that renormalizes $P(s)$ in the considered limit of $s \to \infty$. As a result, we have for $\Omega(s)$ the limit behavior

[3] In nonrelativistic scattering we know from the Levinson theorem [17, 80] that $\delta(0) - \delta(\infty) = (n + q/2)\pi$, where n is the number of bound states in the problem and q only applies to S-wave ($\ell = 0$), being the number of zero energy S-wave resonances. For the precise condition of this later case consider Eq. (95) of Ref. [17].

$$\Omega(s) \xrightarrow[s\to\infty]{} \mathcal{C}_\Omega \, e^{i\varphi(\infty)} \times \left(\frac{s_{\text{th}}}{s}\right)^{\frac{\varphi(\infty)}{\pi}} \, . \tag{14.10}$$

This translates into the form factor $F(s)$, Eq. (14.5), as

$$F(s) \xrightarrow[s\to\infty]{} \mathcal{C}_F \, e^{i\varphi(\infty)} \times s^{p-q-\frac{\varphi(\infty)}{\pi}} \, , \tag{14.11}$$

where C_Ω and C_F are constants.

The Eq. (14.11) offers interesting corollaries
(i) If the high-energy behavior of $F(s)$ is considered to be known and it is of the form s^ν, then we have from this equation that

$$p - q - \frac{\varphi(\infty)}{\pi} = \nu \, , \tag{14.12}$$

which is a kind of relativistic Levinson theorem for the form factor.
(ii) When modeling interactions with limited information, so that we are able to achieve some partial control on the PWA and form factor, we should keep constant under variation of the parameters the relation of Eq. (14.12). Since ν is fixed then we would require that

$$p - q - \frac{\varphi(\infty)}{\pi} = \text{fixed} \tag{14.13}$$

as the parameters vary. In this way, if, e.g., $\varphi(\infty)/\pi$ decreases by one unit and there are no bound states in the system then we should introduce an extra zero in the form factor, so that p increases by one compensating unit. A similar logic would apply for other possible situations.

(iii) We should stress that while we can compensate for the strong-model effects discussed in (ii), by increasing/decreasing p, q and $\varphi(\infty)/\pi$, this is not possible for $\Omega(s)$, which then could be driven into a very troublesome behavior. That is, $\Omega(s)$ is expected to be more strongly dependent on fine details of the hadronic model and it should be used with care, e.g., within a formula like that for $F(s)$ in Eq. (14.5).

As an important example that illustrates the previous points (i)–(iii), we refer to the pion scalar form factor, associated with the light-quark scalar source $\bar{u}u + \bar{d}d$, which is defined as

$$F(s) = \int d^4x e^{i(p+p')x} \langle 0|m_u \bar{u}(x)u(x) + m_d \bar{d}(x)d(x)|0\rangle \, . \tag{14.14}$$

Here u and d are the up and down quarks, m_u and m_d are the masses of these quarks, in order, and $s = (p + p')^2$. In the following we consider the isospin limit (nominally, $m_u = m_d$).

The FSI for this form factor are driven by the isoscalar scalar $\pi\pi$ interaction, which was discussed for low energies with its salient feature of the appearance of the

$f_0(500)$ or σ meson in Chap. 8. Another phenomenologically relevant channel is the $K\bar{K}$ one, with a threshold at 991.4 MeV [34]. Apart from the $f_0(500)$ or σ resonance there is also the $f_0(980)$ resonance, which is relatively narrow [34] and it manifests as a steep rise of the isoscalar scalar phase shifts at around the two-kaon threshold. This resonance couples much more strongly to $K\bar{K}$ than to $\pi\pi$ [81], which causes that as soon as the $K\bar{K}$ is open there is an active conversion of pionic flux in a kaonic one. As a result, the inelasticity parameter η_1 rapidly drops from 1 below the $K\bar{K}$ threshold to much smaller values for $\sqrt{s} > 2m_K$. The aforementioned rapid rise of the phase of the isoscalar scalar $\pi\pi$ PWA $T(s)$, $\varphi(s)$, could be abruptly interrupted at the $K\bar{K}$ before it reached π degrees. All depends on whether $\delta(s)$ at $s = s_K = (2m_K)^2$ is larger or smaller than π, which might be easily changed within the parameters of the hadronic model, being both situations compatible with the present experimental phase shifts at around $s = 1$ GeV^2. As a result, the Omnès function for this case would have two dramatically different behaviors under tiny changes of the parameters, depending on whether $\delta(s_K)$ is larger or smaller than π. In the former case $\Omega(s)$ is huge at the point where $\delta(s) = \pi$ (this point is below s_K), while for the later case $\Omega(s)$ is nearly zero just below the $K\bar{K}$ threshold. This pathological situation was discussed in great detail in Ref. [82]. We also refer to Refs. [10, 81] for explicit accounts of the mentioned experimental data for the isoscalar scalar meson–meson interactions.

Let us exemplify this situation by performing an explicit calculation by identifying $\varphi(s)$ with the phase of the PWA $T(s)$ along the RHC. For the numerical evaluation of the DR for $\omega(s)$, Eq. (14.7), it is convenient to rewrite it so as to avoid the explicit numerical calculation of the Cauchy principal value of the integral involving $\varphi(s')$. We then have

$$\omega(s) = \varphi(s)\frac{s}{\pi}\int_{s_{\rm th}}^{\infty}\frac{ds'}{s'(s'-s)} + \frac{s}{\pi}\int_{s_{\rm th}}^{\infty}\frac{\varphi(s')-\varphi(s)}{s'(s'-s)}ds' \,, \tag{14.15}$$

and the former integral can be evaluated algebraically.

The situation in which $\delta(s_K) \to \pi$ drives to a singularity in the Omnès function $\Omega(s)$. When this happens, with a subtraction constant around -2.45, the phase of the strong PWA becomes discontinuous for s above s_K. We plot $\delta(s)$ in the left top panel of Fig. 14.1 and $\varphi(s)$ in the right top one. The PWA $T(s)$ in terms of the phase shifts $\delta(s)$ is given by

$$T(s) = |T(s)|e^{i\varphi(s)} = \frac{1}{2\rho}\left[\eta\sin 2\delta + i(1-\eta\cos 2\delta(s))\right] \,. \tag{14.16}$$

Thus, when $\delta(s_K) < \pi$ we have that above s_K the real part of $T(s)$ changes sign (since $\delta(s)$ keeps growing), and then the phase of $T(s)$ experiences a rapid decrease from values near π below s_K to values in the interval $[0, \pi/2]$ (the imaginary part of $T(s)$ is always positive because of unitarity, $\eta \le 1$). This transition in $\varphi(s)$ from values near to π to others below $\pi/2$ becomes more abrupt as $\delta(s_K) \to \pi^-$, and in reaching this limit the phase $\varphi(s)$ becomes discontinuous at s_K. On the other hand, when $\delta(s_K) > \pi$ the function $\varphi(s)$ keeps growing because the real part of $T(s)$ does not change sign and when η becomes small then it is clear that the imaginary part of $T(s)$ becomes larger than the real part [$\varphi(s) > \pi$ for $s > s_K$]. The presence of such a discontinuity in $\varphi(s)$ at s_K by an amount of $\pi/2$ drives to a singularity in $\omega(s)$ at $s = s_K$. This singularity is of the end-point type, as it is clear by splitting the integral for $\omega(s)$ in two parts, from s_π ($s_\pi = 4m_\pi^2$) to s_K and from the latter to ∞, with $\varphi(s_K - \varepsilon) - \varphi(s_K + \varepsilon) = \pm\pi/2$. In the latter expression, the plus sign applies when $\delta(s_K - \varepsilon) \to \pi^-$ and the minus sign when $\delta(s_K - \varepsilon) \to \pi^+$. The resulting logarithmic singularity in $\omega(s)$ stems then from the fact that the Cauchy's principal value of the integral around $s = s_K$ does not get rid of the pole singularity in the integrand from the factor $1/(s' - s)$. Thus, we are driving to the divergence

$$\frac{1}{\pi}\left[\int^{s_K-\Delta} \frac{\varphi(s_K - \varepsilon)ds'}{s' - s_K} + \int_{s_K+\Delta} \frac{\varphi(s_K + \varepsilon)ds'}{s' - s_K}\right]$$
$$\to \frac{1}{\pi}\left[\varphi(s_K - \varepsilon) - \varphi(s_K + \varepsilon)\right] \log \Delta = \pm\frac{1}{2} \log \Delta \,, \quad (14.17)$$

with $\Delta \to 0^+$ and $\delta(s_K - \varepsilon) \to \pi^\mp$, in order. In this way, when exponentiating $\omega(s)$ to get $\Omega(s)$ this divergent contribution in the exponent gives rise to $(\sqrt{\Delta})^{\pm 1}$. Therefore, $\Omega(s)$ has finally a pole when $\delta(s_K) \to \pi^+$ and a zero when $\delta(s_K) \to \pi^-$. This behavior is represented in the right bottom panel in Fig. 14.1.

This pathological situation has a reflection in the condition expressed in Eq. (14.13), because there is a jump by one in $\varphi(\infty)/\pi$ between the two situations $\delta(s_K - \varepsilon) \to \pi^\pm$. Thus, imposing continuity in the transition $\delta(s_K - \varepsilon) < \pi$ to $> \pi$ requires that p increases by 1, that is, there should be one more zero for $\delta(s_K) > \pi$ as compared with the opposite situation. If we would require the continuity from $\delta(s_K) > \pi$ to $< \pi$ we would have to increase q by one and had bound sate (a pole in the first Riemann sheet). This latter situation can be ruled out in pion physics. It follows then that an Omnès representation of the isoscalar scalar $\pi\pi$ PWA in the case $\delta(s_K) > \pi$ requires the function $\Omega(s)$ to have a zero at the point at which $\Im T(s) = 0$ for $s < s_K$. Well, applying an Omnès representation for $T(s)$ itself this is also a consequence of unitarity because $T(s) = e^{i\delta}\sin(\delta)/\rho$ in the elastic region below the $K\bar{K}$ threshold.

Similar reasoning was applied in Ref. [82] to the pion scalar form factor $F(s)$ which follows (in good approximation) the phase of the isoscalar scalar $\pi\pi$ PWA $T(s)$ (even somewhat above the $K\bar{K}$ threshold). This is shown by explicit calculations of $F(s)$ within other approaches [11, 83, 84]. Indeed, such a situation might be expected by realizing that the $f_0(980)$ couples much more strongly to kaons than to pions, e.g., Ref. [81] reports that the coupling to kaons is larger by a factor 3. As a result, the admixture between the pion and kaon channels is suppressed and both of them follow their own eigen-channel of the isoscalar scalar meson–meson PWAs. We refer to Refs. [82, 85] for detailed discussions that provide the explicit expression for the eigen-channels and eigen-phases.

We would also like to mention that one can precisely determine the point $s = s_1$ at which the form factor has a zero when $\delta(s_K) > \pi$ as find out in Ref. [82]. This reference writes down a twice-subtracted DR for the form factor,

$$F(s) = F(0) + \frac{1}{6}\langle r^2\rangle_s^\pi s + \frac{s^2}{\pi}\int_{s_\pi}^\infty \frac{\Im F(s')ds'}{(s')^2(s'-s)} \ . \tag{14.18}$$

In this expression, $\langle r^2\rangle_s^\pi$ is the quadratic scalar radius of the pion. Indeed one expects from asymptotic QCD [86] that $F(s)$ vanishes at infinity so that the written DR should converge fast, which is of particular interest for relatively low energies. It is then clear from the integral representation of $F(s)$ in Eq. (14.18) that the only point at which $F(s)$ can vanish for $s < s_k$ is where $\Im F(s) = 0$ (since the subtraction polynomial in Eq. (14.18) is real). The latter fact can only occur when $\delta(s) = \pi$ since there is only one zero at such energies and $|\Im F(s)| = |F(s)\sin\delta(s)|/\rho(s)$ in the elastic region, $s < s_K$, and $\delta(s_K) > \pi$. This in turns fixes the first order polynomial that should multiply the Omnès function $\Omega(s)$ so as to achieve a continuous transition for $\delta(s_K)$ greater or smaller than π.

In summary, one should better use the function

$$\Omega(s) = \begin{cases} \exp\omega(s) & , \ \delta(s_K) < \pi \ , \\ \frac{s_1 - s}{s_1}\exp\omega(s) & , \ \delta(s_K) > \pi \ . \end{cases} \tag{14.19}$$

A clear lesson that follows from the discussion in this section is that one should use an Omnès function with great care when employing it while doing fits to data. The latter requires varying the parameters of the theory and one should avoid possible instable behaviors associated with rapid movements in the phases integrated that could strongly affect an Omnès function. As we have seen, nonsense results could arise by a nearby discontinuity in the space of parameters. The fulfillment of the requirement in Eq. (14.13) should then be pursuit, and for the phase of the isoscalar scalar $\pi\pi$ PWA one should use the function in Eq. (14.19) instead of a pure Omnès function, cf. Eq. (14.6).

Given a form factor which also involves LHC,[4] e.g., that for $\gamma\gamma \to \pi\pi$, we could also define the function $R(s)$ as in Eq. (14.3), although now this function also contains LHC, and then we denote it by $L(s)$ [analogously to Eq. (13.14)]. Nonetheless, the introduction of this function allows a clear splitting between the RHC and LHC contributions that is also exploited in the literature. One typically writes down a DR for $L(s)$ along the LHC,

$$\begin{aligned} L(s) &= \sum_{i=1}^{n-1} a_i s^i + \frac{s^n}{\pi}\int_{-\infty}^{s_L} \frac{\Im L(s')ds'}{(s')^n(s'-s)} \ , \\ F(s) &= \Omega(s)L(s) \ , \\ \Im L(s) &= \Omega(s)^{-1}\Im F(s) \ , \ s < s_L \ , \end{aligned} \tag{14.20}$$

[4] Maybe some readers are used to consider that the form factors should only have RHC. Here we use the notation introduced in Chap. 13.

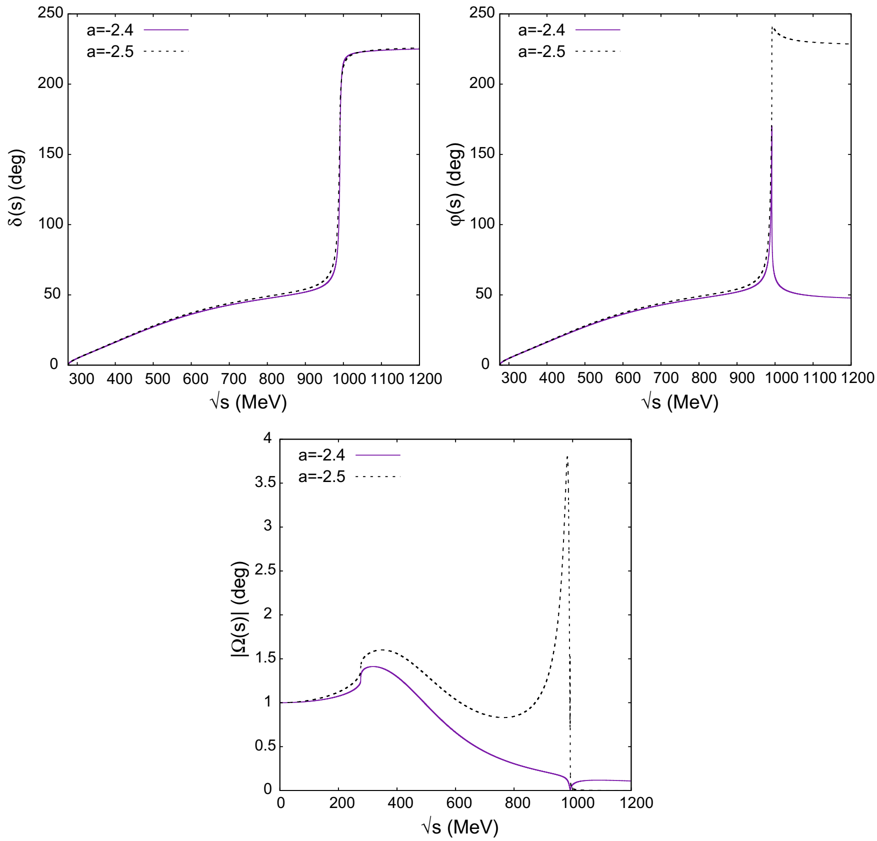

Fig. 14.1 From left to right and top to bottom: Phase shifts, $\delta(s)$, phase of $T(s)$, $\varphi(s)$, and the Omnès function, $|\Omega(s)|$. The solid line corresponds to the subtraction constant a of the $g(s)$ function with the value $a = -2.4$ and the dashed line to $a = -2.5$. We have modeled the $\pi\pi$ and $K\bar{K}$ channels with $I = \ell = 0$ by unitarizing the lowest order ChPT amplitudes. We use Eq. (8.29) and the 2×2 matrix $\mathcal{N}(s)$ is identified with the leading ChPT amplitudes, Eq. (8.30)

with s_L the upper limit for the LHC. In this way, the Omnès function can be known (at least partially) from the knowledge of the strong PWAs along the RHC and then one needs to know $\Im F(s)$ along the LHC. Of course, in the pure elastic case the phase of the Omnès function is the phase of the strong PWA $T(s)$ and we could proceed as discussed above in this section. For the particular case of $\gamma\gamma \to \pi^0\pi^0$ its S wave contribution is discussed in Refs. [87, 88]. The subtraction constants can be adjusted by employing the Low's theorem, which implies that for $s \to 0$ the total $F(s)$ tends to its renormalized Born term contribution (involving the values of the physical couplings and masses). The other subtraction constant is fixed by matching with the one-loop ChPT calculation of Refs. [89, 90]. One could approach $\Im F(s)$ by the contributions from the Born terms and the crossed exchanges of the J^{PC}

resonance multiplets 1^{--} and 1^{++} as in Ref. [88], where explicit formulas for the resonance-exchange tree-level amplitudes can be found.

Incidentally, Refs. [87, 88] use a somewhat different unitarization procedure than Eq. (14.20) to calculate the low-energy cross section for $\gamma\gamma \to \pi^0\pi^0$. These references consider only S wave ($\pi^0\pi^0$ does not have P wave because of Bose–Einstein symmetry) and two isospin channels are possible, the isoscalar and the isotensor ones. The first step is to build up a function with only RHC by subtracting to $F_I(s)$ a function $\widetilde{L}_I(s)$ that contains its LHC. Namely, the new function is

$$\mathcal{F}_I(s) = \frac{F_I(s) - \widetilde{L}_I(s)}{\Omega_I(s)} \,. \tag{14.21}$$

Next, Refs. [87, 88] perform a twice-subtracted DR for the latter, in terms of which $F_I(s)$ reads

$$F_I(s) = \widetilde{L}_I(s) + a_I \Omega_I(s) + c_I s \Omega_I(s) + \Omega_I(s) \frac{s^2}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\widetilde{L}_I(s') \sin\varphi_I(s') ds'}{(s')^2 (s'-s)|\Omega_I(s')|} \,. \tag{14.22}$$

The subtraction constants are fixed as explained above by considering the Low's theorem and matching with the one-loop ChPT calculations of $\gamma\gamma \to \pi\pi$ in Refs. [89, 90]. At the practical level $\widetilde{L}_I(s)$ is also approximated in Refs. [87, 88] by the tree-level amplitudes including the Born terms and the exchange of the 1^{--} and 1^{++} multiplets of vector and axial resonances, in order. In this way, the Low's theorem requires that

$$\lim_{s\to 0} \left[F_I(s) - \widetilde{L}_I(s) \right] = \mathcal{O}(s) \,, \tag{14.23}$$

from which it follows that $a_I = 0$ in Eq. (14.22). Indeed, the contributions of the 1^{++} axial resonances are more important at low energies than that of the 1^{--}. Actually, the former appear one order lower in the chiral expansion than the latter. Despite that the explicit axial exchanges are neglected in Ref. [87], while they are taken into account in Ref. [88]. The calculations performed in this reference confirm that these contributions are phenomenologically relevant and should not be neglected since their contributions are around a 30% of the full result. A major step forward of Ref. [88] compared to Ref. [87] is to use the stable $\Omega_0(s)$ function as defined in Eq. (14.19), instead of just a pure Omnès function. In this way, the output at low energies is much more stable under changes of the parameterizations used for the isoscalar scalar $\pi\pi$ phase shifts in the region of the $f_0(980)$ resonance, accomplishing a reduction of about a factor of 2 in the uncertainty of the cross section for $\gamma\gamma \to \pi^0\pi^0$ at around the mass of the $\rho(770)$, and about a 25% already at around $\sqrt{s} = 500$ MeV. Notice, that even if for $\delta_0(s_K) > \pi$ one has a zero in the denominator because $\Omega_0(s_1) = 0$, as defined in Eq. (14.19), the ratio $\sin\varphi_0(s')/|\Omega_0(s')|$ in the integrand of Eq. (14.22) is well defined because the zero of $\Omega_0(s')$ happens at the same point s_1 at which $\varphi_0(s_1) = \pi$, cf. Eq. (14.18).